# POLITECNICO DI TORINO 

## SCUOLA DI DOTTORATO

Dottorato in Ingegneria Aerospaziale - XXV ciclo

Tesi di Dottorato

## Advanced higher-order one-dimensional models for fluid-structure interaction analysis



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Ai miei splendidi genitori:
mimina Marisa e pappo Gianpaolo

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## Abstract

The aim of this work is the development of a refined reduced order model suitable for numerical applications in solid and fluid mechanics with a remarkable reduction in computational cost. Nowadays, numerical reduced order models are widely exploited in many areas, such as aerospace, mechanical and biomechanical engineering for structural analysis, fluid dynamic analysis and coupled (aeroelastic) fluid-structure interaction analysis.
One-dimensional (1D) structural models, commonly known as beams, are for instance used in many applications to analyze the structural behavior of slender bodies, such as columns, arches, blades, aircraft wings, bridges, skyscrapers, rotor and wind turbine blades. One-dimensional structural elements are simpler and computationally more efficient than 2D (plate/shell) and 3D (solid) elements. This feature makes beam theories still very attractive for the static, dynamic response, free vibration and aeroelastic analyses, despite the approximations which they introduce in the simulation.
Recently, 1D models are intensively exploited for the simulation of the human cardiovascular system under either physiological or pathological conditions. As it is easily comprehensible, fluid flows in pipes, channel, capillaries or even arteries are particularly suitable for the application of one-dimensional models also to fluid dynamics. Typically, one-dimensional models for fluid dynamics and fluid-structure interaction (FSI) problems are again remarkably more efficient than three-dimensional methods in terms of computational cost.
A key point for reduced order models is the capability in simulating in an accurate way the investigated physical problem. For instance, in last decades the growing use of advanced composite and sandwich materials in thin-walled beam-like structures has revealed that 1D theories have to be refined in order to predict the behavior of such complex structures with high fidelity. For this purpose, a higher-order one-dimensional method is introduced in this work and its capabilities are highlighted and discussed. The present work is subdivided into three fundamental parts corresponding to the physical fields the proposed refined model is applied to.
Firstly, a structural part presents the formulation of a displacement-based higher-order one-dimensional model for the analysis of beam-like structures. Classical beam theories (Euler-Bernoulli and Timoshenko) have intrinsic limitations which preclude their applications for the analysis of a wide class of engineering problems. The Carrera Unified Formulation (CUF) is employed to introduce a hierarchical modeling with a variable order of expansion for the displacement unknowns over the beam cross-section. The finite element method (FEM) is used to handle arbitrary geometries and loading conditions. The influence of higher-order effects over the cross-section deformation, not detectable by classical and low-order beam theories, on the static, free vibration and time-dependent response of several structures with arbitrary cross-section geometries and made of arbitrary materials is remarked through the numerical results presented.
Secondly, an aeroelastic part describes the extension of the refined structural model to the static aeroelastic analysis of lifting surfaces made of metallic and composite materials. A


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coupled aeroelastic computational model based on the Vortex Lattice aerodynamic Method and the finite element method (FEM) is formulated. A refined aeroelastic approach is also presented by replacing the Vortex Lattice aerodynamic Method with the more powerful 3D Panel Method. Comparison with results obtained by existing plate/shell aeroelastic models shows that the present 1D model could result less expensive from the computational point of view with respect to shell cases with same accuracy. The effect of the cross-section deformation on the aeroelastic static response and on the critical wing divergence velocity is evaluated for different wing configurations. The beneficial effects of aeroelastic tailoring in the case of wings made of composite anisotropic materials are also confirmed by using the present model. Finally, a third part concerning the use of the refined one-dimensional CUF model for fluid dynamic problems is presented. The basic partial differential equations (PDEs) of fluid mechanics (Navier-Stokes and Stokes equations) are faced and 1D refined models with variable velocity-pressure accuracy are presented on the basis of the one-dimensional Carrera Unified Formulation and the finite element method. The application of these higher-order models to describe the three-dimensional fluid flow evolution on a computational domain is formulated for the Stokes problem. The present approach reveals its capabilities in predicting accurately, with a reduced computational cost with respect to more consuming two-dimensional or three-dimensional methods, nonclassical and complex fluid flows. Moreover, the numerical results show the promising potentiality of such an approach to the future extension of fluid-structure CUF-CUF models, i.e. the coupling of CUF models used for both structural and fluid dynamic analyses.


Keywords: one-dimensional refined models, higher-order terms, Carrera Unified Formulation, Finite Element Method, Vortex Lattice Method, 3D Panel Method, structural dynamics, fluid dynamics, aeroelasticity, composite materials.

## Sommario

Questa tesi di dottorato ha come obiettivo lo sviluppo di avanzati modelli di ordine ridotto per applicazioni numeriche di meccanica dei solidi e dei fluidi caratterizzati da un contenuto costo computazionale. Oggigiorno, numerosi modelli numerici di ordine ridotto sono ampiamente sfruttati in svariati campi dell'ingegneria, come per esempio quella aerospaziale, meccanica e biomedica, per effettuare analisi strutturali, analisi fluidodinamiche e analisi accoppiate (aeroelastiche) di iterazioni fluido-struttura.

Modelli strutturali unidimensionali (1D), comunemente noti come travi, sono ad esempio impiegati per valutare il comportamento strutturale di corpi allungati, quali palette di turbina, ali di aereo, pale eoliche, travi da costruzione edile, ponti e grattacieli. La particolarità vantaggiosa di questi modelli strutturali unidimensionali è individuata nella loro semplicità e nel ridotto costo computazionale se confrontati con modelli bidimensionali (2D), denominati tipicamente piastre e gusci, o tridimensionali (3D), i cosiddetti solidi. Questa caratteristica rende le teorie trave molto appetibili per l'analisi statica, aeroelastica, vibrazionale senza forzante e di risposta dinamica, nonostante le approssimazioni che esse introducono nella simulazione.

Recentemente, modelli 1D sono addirittura impiegati nella simulazione del sistema cardiovascolare umano sottoposto a condizioni fisiologiche o patologiche. Come è facile immaginare, flussi di fluidi in condotti, canali, capillari o anche arterie rappresentano applicazioni particolarmente adatte all'uso di modelli unidimensionali nel campo della fluidodinamica. Anche nei problemi di questo campo come in quelli di iterazione fluidostruttura (FSI), dove è studiato l'accoppiamento tra fenomeni strutturali e fluidinamici, i modelli ridotti 1 D risultano più efficienti in termini di costo computazionale rispetto a soluzioni tridimensionali.

Uno dei punti chiave per i modelli di ordine ridotto è la capacità di simulare correttamente il problema fisico investigato. Giusto per citare un caso di esempio, negli ultimi decenni l'uso crescente di materiali compositi e di tipo sandwich in strutture a parete sottile di tipo beam-like (allungate) ha rivelato che le teorie 1D devono essere sufficientemente accurate in modo da predirre con precisione il comportamento di strutture complesse. A questo scopo, un metodo unidimensionale di ordine superiore è introdotto in questa tesi di dottorato e le sue vantaggiose peculiarità sono evidenziate e discusse durante l'esposizione dell'elaborato. Il presente lavoro è suddiviso in tre parti distinte, corrispondenti ai campi di studio nei quali il suddetto modello avanzato è stato applicato.

In primo luogo, una parte strutturale espone la formulazione di un modello unidimensionale di ordine superiore basato agli spostamenti per l'analisi di strutture beam-like. Le teorie delle travi classiche (come quelle di Eulero-Bernoulli e Timoshenko) hanno limitazioni intrinseche che precludono la loro applicazione per una gran varietà di problemi ingegneristici. La Carrera Unified Formulation (CUF) è impiegata per introdurre una modellazione numerica di tipo gerarchico, la cui caratteristica è la possibilità di avere un variabile ordine di espansione per le incognite di spostamento lungo la sezione trasversale (cross-section).

Si fa uso del metodo agli elementi finiti (FEM) per gestire geometrie e condizioni di carico arbitrarie. L'influenza degli effetti di ordine superiore sulla deformazione della sezione trasversale, non percettibile dalle teorie classiche, per risposta statica, vibrazionale senza forzante e dinamica nel tempo di una vasta gamma di strutture con arbitrarie geometrie di sezione e materiali arbitrari è messa in risalto dai risultati numerici presentati.

Una seconda parte aeroelastica descrive l'estensione del modello strutturale all'analisi aeroelastica statica di superfici portanti realizzate in materiale metallico e composito. La formulazione di un modello computazionale aeroelastico accoppiato basato sul metodo aerodinamico Vortex Lattice Method e sul metodo agli elementi finiti (FEM) è qui esposta. Un avanzato approccio aeroelastico viene inoltre presentato sostituendo il più sofisticato 3D Panel Method al metodo aerodinamico Vortex Lattice Method. Confronti con risultati ottenuti da modelli aeroelastici commerciali esistenti (di tipo plate/shell) mostrano che il modello 1D proposto può risultare meno oneroso dal punto di vista di costo computazionale a parità di accuratezza. L'effetto della deformazione della sezione trasversale non solo sulla risposta aeroelastica statica ma anche sulla velocità critica di divergenza caratteristica dell'ala è studiato per svariate configurazioni alari. Gli effetti benefici del tailoring aeroelastico nel caso di ali realizzate in materiale composito sono inoltre rilevati e confermati dall'uso del qui proposto modello unidimensionale.

Per finire, una terza ed ultima parte della tesi discute l'uso dell'avanzato modello unidimensionale CUF nel campo della fluidodinamica. Sono qui affrontate le tipiche equazioni alle derivate parziali (PDEs) della meccanica dei fluidi (equazioni di NavierStokes ed equazioni di Stokes) e modelli 1D di ordine superiore con un accuratezza variabile sulle incognite di velocità e pressione vengono presentati sulla base della Carrera Unified Formulation e del metodo agli elementi finiti. L'applicazione di questi modelli gerarchici per descrivere l'evoluzione tridimensionale del flusso di fluido all'interno di un dominio computazionale è formulata per il problema di Stokes. L'approccio proposto evidenzia le sue vantaggiose peculiarità per predirre accuratamente, e con un costo computazionale limitato rispetto a più dispendiosi metodi bidimensionali o tridimensionali, flussi di fluido non classici e complessi. Inoltre, i risultati numerici mostrano le promettenti potenzialità di questo approccio per l'estensione futura a modelli fluido-struttura di tipo CUF-CUF, ovvero l'accoppiamento di modelli CUF utilizzati sia per l'analisi strutturale sia per quella fluidodinamica.

Parole chiave: modelli unidimensionali avanzati, termini di ordine superiore, Carrera Unified Formulation, metodo agli elementi finiti, Vortex Lattice Method, 3D Panel Method, dinamica strutturale, fluidodinamica, aeroelsticità, materiali compositi.

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## Chapter 1

## Introduction

### 1.1 Present outline

Civil transport and cargo traffic are expected to significantly increase in the next decade, especially along medium and long large routes world-wide. The study of programs for the realization of aircrafts which will know how to improve efficiency and cost reduction performances in consideration of always increasing dimensions, is needed for the future prospects of aerial transport development.

The report "A Vision for 2020", emitted by Advisory Council for Aeronautic Research in Europe in 2002, sets up a list of five challenges for aircraft industry to meet at the horizon 2020: quality and affordability, environment, safety, security and air transport system efficiency. The ambitious targets for the next generation air transport system could be summarized in the following points:

- $30 \%$ Reduction of DOCs (Direct Operative Costs);
- Cutting of noise and cabin noise level;
- $80 \%$ Reduction of pollution in the atmosphere and noxious emissions by propulsors;
- More available space and confort for passengers;
- 10-12 \% Time reduction for boarding and debarkation of passengers and luggage;
- Improvement of cargo capacity;
- Possibility of operating from present runways and airports;
- Improvement of the operative life;
- Cut of initial investment and maintenance costs;
- Mach 0.85 as minimum cruise speed;
- Increased level of survivability to accidents in takeoff and landing (design against crash and fire, fuel tanks, new materials, evacuation system, etc.);
- Approach and landing wake vortex turbulence separations smaller than the present one.

The problem of reducing DOCs, noise and emissions can be faced by using technology advancements (new materials for structures and engines, reduction of production and maintenance costs, etc...). These advancements can produce only long term benefits and the trend says that a reduction of $30 \%$ or more will not be practicable in the next decade.

Increasing aircraft capacity is another way to reduce unit costs, but on short routes this solution is not applicable and, on long routes, airport rules limit to an $80 \times 80 \mathrm{~m}$ horizontal square the largest area that an aircraft can fill. So the advantage of increasing dimensions came to its end with the A380 aircraft.

Concerning to emissions and consumption reduction problems, a solution could be find out in a more accurate aerodynamic study. In fact a $1 \%$ reduction of drag for a large transport aircraft could save about 400000 liters of fuel and, consequently, 5000 kg of noxious emissions per year [4]. All the future requirements listed will not be satisfied without a significant improvement of aerodynamic design against drag. The improvement of the low speed aerodynamic efficiency is also a challange for reducing noise and noxious emissions close to airport areas.

In a large transport aircraft during cruise flight, friction and induced drags have almost the same influence but the induced drag depends on the lift distribution along wing span, whereas friction drag is proportional to the scale factor of the aircraft. Since the lift distribution of today civil transport aircraft is so optimized that any further significant reduction of induced drag cannot be easily obtained, a possible jump forward in air transportation will come from the introduction of completely new, non-conventional aircrafts.

### 1.2 Non-conventional wing architectures

Several non-conventional wing solutions are summarized in Fig. 1.1.


Figure 1.1: Drag efficiency factors for various wing concepts [1].

### 1.2.1 The joined wing

The joined-wing airplane may be defined as an airplane that incorporates tandem wings arranged to form diamond shapes in both plan and front views. This general concept can take different forms, but common characteristic is the presence of a positive sweep angle for the front wing and a negative one for the rear wing, that is connected to the superior part of the fuselage on the vertical empennage level. In fact, all the models obtain the
desired diamond shaped front view by locating the root of the rear wing at or close to the top of the vertical tail as shown in Fig. 1.2.


Figure 1.2: An example of joined wing plane.

Different configurations are allowed: sometimes the wing-joining members are small bodies, but it is also possible to join the wings using twin fins to support the center section of the front wing [5]. Two joined wing solutions are shown in Fig. 1.3.


Figure 1.3: Typical Joined wing configurations.

Structural and aerodynamic advantages claimed for joined wing configuration are: reduced weight, high stiffness, good transonic area distribution, high trimmed $C_{L \text { max }}$, reduced wetted area and parasite drag, direct lift control capability, direct sideforce control capability, good stability and control in flight. With regard to weight saving, the (b) solution in Fig. 1.3 assures a more contained structural weight than other alternative configurations, even if it is subordinated to the reorganization of wing resistant sections to adeguate themselves to a loaded configuration substantially different from the one with a conventional wing.

According to the studies of Shyu and Miura it is possible to get joined wing configurations with weight included between $65 \%$ and $78 \%$ of the equivalent conventional wing [6]. Weight saving is also due to the geometrical peculiarities of the joined wing configuration adopted and especially to the fact that the rear wing, working as a wind bracing, contributes to reduce stresses in the front wing internal portion thanks to the junction.

Another positive feature is that the joined wings create a structure with high torsional
stiffness, since the torsion on one of the two semi-wings can be beared by the bending of the other one. A growing efficacy of the aileron is established and a global improvement of the aeroelastic behaviour of the building solution is supposed. Like all the closed wing designs, joined wings are the maximum expression of wingtip devices, which aim to eliminate the influence of the wingtip vortices that occur at the tips of conventional wings. These vortices, which are a major component of wake turbulence, are associated with induced drag and then negatively affect the aerodynamic performance in most regimes. The elimination of the aircraft's wingtips, and thus the great reduction or total elimination of wingtip drag, has great implications for the improvement of fuel efficiency in the airline industry.

Furthermore, this is not the only aerodynamical feature that makes joined-wing configuration adoption profitable. With regard to aircraft controllability, the downwash effect of the front wing on the rear one makes possible the fact that this rear wing is still lifting even in case of stall of the first one, amending such a critical situation. The compensation of the opposite effects of the positive and negative sweep allows to avoid the dihedral effect increase as the lift coefficient raises. It is also foreseen a reduction of acoustic pollution thanks to the increased lift-to-drag ratio during takeoff and landing.

At the moment the joined wing configuration is under consideration even for military applications. In recent years there has been a push towards the design and development of unmanned aerial vehicles (UAVs), designed for various missions, including atmospheric sensing, border monitoring, military patrol and combat. One example of unconventional mission is the "sensocraft", designed for long-range, high-altitude, intelligence, surveillance and reconnaissance.

Targeting, tracking and foliage penetration require large antennas and demand of 360 degree coverage. Among the several configurations currently being considered for Sensorcraft mission (see Fig. 1.4), the joined-wing design, while possibly providing weight saving and improved aerodynamic performances over a conventional vehicle, lends itself a continuous 360 -degree coverage.


Figure 1.4: Sensorcraft wing system.

### 1.2.2 The Boxplane

In the 70s Luis Miranda of Lockheed Corporation proposed the project of a transport aircraft with big dimensions which made use of a biplanar wing referable to the joined wing concept. The most important difference was the presence of two vertical fins to join front and rear wings. In frontal projection the wing structure so obtained looked like a rectangular box, see Fig. 1.5. The main property of the box wing shape is the research of the maximum possible reduction of induced drag. This research project was deserted when simulations in wind tunnel showed the poor aerolastic features of the proposed model.


Figure 1.5: Typical example of boxplane.

### 1.2.3 The PrandtlPlane

The Prandtlplane is a new aircraft configuration with a low induced drag, whose study started from an intuition by the German scientist Ludwig Prandtl. Among all the possible wing systems generating a given lift with a given wingspan, the one who minimize the induced drag is a box-like wing called by Prof. Prandtl: "Best Wing System" (BWS) [7]. The BWS is a biplane where the horizontal wing tips are connected together by two vertical wings (bulkheads). The lift distribution over the two horizontal wings, under optimum conditions, results from the superposition of a constant and an elliptical part, and it is equal. Total lift over them is also the same, while a butterfly shaped lift distribution is generated on the bulkheads. When the condition of minimum drag occurs, the velocity induced by the free vortices is constant along the two horizontal wings and identically zero on the vertical side wings (lift distribution on vertical fins symmetrical with respect to their horizontal symmetrical axis, so that their resulting lift force results to be equal to zero). The efficiency of this system grows as the non dimensional gap between the wings increases, because of induced drag decreasing.

In the hypothesis of gap/span ratio between 0.1 and 0.2 , a PrandtlPlane can offer a significative induced drag reduction in comparison with a monoplane with identical wingspan and same lift. Reminding the lift distribution on the horizontal and vertical wings, according to Munk's theorems, the induced drag is independent of sweep angles of the wings, so Prandtl's concept can be applied also to transonic and supersonic aircraft. PrandtlPlane configuration (see Fig. 1.6) can be used to design a complete family of aircraft, ranging from small to very wide bodies (even larger than Airbus A380), fully compatible with present airports. In fact, in the case of an aircraft larger than e.g. A380, the higher efficiency of the configuration could be used to reduce the wingspan inside 80 m , without drag penalty with respect to conventional aircraft. The possibility of improving the PrandtlPlane capacities beyond the largest possible conventional aircraft is one of the probable advantages for reducing drag.

In 2002 five Italian Universities, starting from BWS principle studies and thanks to governmental funds, carried out an innovative project for a very large non conventional transport aircraft(600 seats), giving birth to the first Prandtlplane concept application.

An important result of the research was the solution of the initial conflict between aerodynamic efficiency and stability of flight, obtained with the development of a new configuration. The fusolage is enlarged horizontally, with a single desk for passengers and the bottom one for good and luggage, and the rear wing is positioned over the fusolage
and connected to it by two fins. The aircraft is stable in cruise flight, due to the high aerodynamic efficiency of the rear wing, the margin of stability can be controlled and modified by a proper variation of chords together with sweep and twist angles along the span of both the wings.


Figure 1.6: The PrandtlPlane configuration.
Researches continued at University of Pisa as long as a flying scale model of a ULM version of the Prandtlplane was built and tested in 2006, while a study based on PrandtlPlane concept for a 250-300 seat civil transport aircraft was completed for Airbus Deutschland in 2007. Furthermore, a static model of a PrandtlPlane designed for Bauhaus Luftfahrt has been presented during the Berlin Air Show in May 2008.

Thanks to its maximum aerodynamic efficiency, its stability in flight and high standard qualities, more quickly and easier ground operations, simpler engine integrations and great adaptability to different dimensional ranging of aircraft, the PrandtlPlane represents a significative alternative for aircrafts conceived in order to fulfil the new requirements contained in "European Aeronautics: a vision for 2020" in matter of efficiency, safety, environment, quality and affordability.

### 1.3 Introduction to Aeroelasticity

### 1.3.1 Aeroelastic phenomena and Flutter

Nowadays, in many engineering fields the coupled analysis of all the disciplines involved in the design process plays a more and more fundamental role. Evidently this multidisciplinary approach counts even for the Aerospace Engineering, being itself the combination of many matters itself. The presented non-conventional configurations therefore need such an approach in order to accurately predict their behavior and performance. For that reason the study of aeroelasticity becomes important.

Aeroelasticity could be defined as a science which studies the mutual interaction between aerodynamics and elastic forces, and the influence of this interaction on structural design. Many important aeroelastic phenomena involve inertial forces as well as aerodynamics and elastic forces. In aeroelastic phenomena air flow is completely modified by structural deflection. Problems in aeroelasticity in structural engineering field could be divided into two large categories: static aeroelastic problems like torsional divergence, phenomena of control reversal (aerospace engineering) and dynamic aeroelastic problems such as lock-in, vortex shedding (von Karman vortices separation), flutter, buffeting and galloping.

Flutter is a self-feeding and potentially destructive vibration where aerodynamic forces on an object couple with one or more structure's natural modes of vibration to produce rapid periodic motion. It can happen in any object within a strong fluid flow, under the conditions that a positive feedback occurs between the structure's natural vibration and the aerodynamic forces. In other words, the vibrational movement of the object increases an aerodynamic load which in turn drives the object to move further. If the energy inflow during the period of aerodynamic excitation is larger than the natural damping of the system, the level of vibration will increase. Therefore, flutter is connected to aerodynamic forces variations due to the different orientation that the parts of the structure, because of their own oscillations, assume with regard to the direction of relative wind-flow.

The flutter phenomenon can occur in aeronautical field, in flexible structures of civil engineering and in industrial or mechanical applications: rotating systems, helicopter rotors, airfoils, propellers, turbomachinery blades, rotor blades, axis instability of a pipe crossed by a fluid, turbines (especially the recent ones with high aspect ratio). The classical type of flutter is associated with potential flow and usually involves the coupling of two or more degrees of freedom, while stall flutter (or nonclassical flutter) can be defined as any flutter of a lifting surface in which the airfoil sections are in stalled flow during at least part of each cycle of oscillation. In aircrafts flutter can be prevented by using an automatic control system to limit structural vibration. As said, flutter can also occur on structures other than aircraft. One famous example of flutter phenomenon is the collapse of the original Tacoma Narrows Bridge.

### 1.3.2 Tacoma Narrow Bridge collapse

In civil field, the importance of aeroelastic phenomena increased when, from structures almost exclusively composed by stones, bricks and concrete, constructors passed to others with bigger dimensions, built with a large use of steel. Lower damping associated to higher deformability and to aerodynamic instable sections (skyscrapers, television towers, suspension bridges, cooling towers in thermic power plants, off-shore platforms etc...) were collapse causes of structures which traditional analyses had considered safe. In particular, phenomena strictly related to wing flutter could occur for suspension bridges.

The first Tacoma Narrows Bridge, opened on July 1, 1940, was revolutionary in its design and historic in its collapse on November 7 of the same year. From the beginning this suspension bridge was known as "galloping gentle" because of its considerable vertical oscillations, as depicted in Fig. 1.7. Several strategies were used to reduce the motion of the bridge, such as attachment of tie-down cables to the plate girders and the addition of a pair of inclined cable stays (that connected the main cables to the bridge deck at mid-span). Furthermore, the structure was equipped with hydraulic buffers installed between the towers and the floor system of the deck in order to damp longitudinal motion of the main span. Unfortunately, many of them were ineffective on November 7.

The wind-induced collapse was due to torsional oscillations not enlarged by mechanical resonance but by the aeroelastic instability connected to a stall flutter phenomenon. Aeroelastic instability determined Tacoma Narrows bridge collapse: the wind, whose statical effects were yet foreseen and tolerable, steady at 42 miles per hour ( $68 \mathrm{~km} / \mathrm{h}$ ), blowed for hours, inducing growing torsional oscillations on the central span. Torsional rotation of the floor system gained twist angle higher than $45^{\circ}$ with respect to undeformed configuration, causing the breaking of a cable and immediately modifying the dynamic configuration of the structure, leading to the collapse.

At that time nobody was worried about studying the interactions of aerodynamic forces


Figure 1.7: The galloping gentle.
over the bridge, thinking that they would not have damaged the impressive structure of Tacoma Narrows Bridge. Even if correctly designed with regard to static effects, the bridge was too much deformable to bending and torsional stresses, and also too sensible to wind action and generally to (aero)dynamic effects.

### 1.4 Refined structural and aeroelastic models: state-of-theart

### 1.4.1 Advances in aeroelastic modeling of composite wings

Composite materials are widely used nowadays in a large variety of applications and engineering fields. The advantages related to their spread are becoming so significant that composites are by now a must for state-of-the-art manufacturing technology. The requirements of weight saving and structural efficiency for aerospace systems such as aircraft wings, helicopter rotor blades, and turbine blades are leading to a wide use of structures in the form of composite thin-walled beams. Furthermore, the forthcoming employment of composite materials in next-generation aircraft configurations will be certainly valuable, as proved by the last few years design studies. Among the possible future applications, High-Altitude Long-Endurance aircraft (HALE) [8], strut-braced wings [9], truss-braced wings [10], and C-wing configurations [11] are worth mentioning.

With the advent of composites, the accurate evaluation of the response of deformable lifting bodies (LBs) when subjected to steady and unsteady aerodynamic loadings is an even more challenging issue for the aeroelastic design of aerospace vehicles [12]. The successful construction of the Grumman X-29 forward-swept wing experimental aircraft was the best example of the exceptional interest afforded to this issue [13]. In last decades, a considerable amount of research activity devoted to the aeroelastic analysis and optimization was undertaken since the idea of aeroelastic tailoring to avoid divergence instability of forward-swept wing was suggested many years ago [14]. Valuable contributions have been made by Weisshaar who considered aeroelastic problems of forward-swept wings including spanwise lift redistribution and aileron effectiveness [15, 16]. A discussion on the various techniques adopted in literature to introduce a bending-twist coupling parameter to be
used in tailoring was also carried out by the same author [17]. Housner and Stein [18] investigated the flutter characteristic of box-beam structures with cross-ply symmetrically laminated skins and variable stiffness properties. This parametric study included the effect of filament orientation upon the flutter speed for wings with various sweep, mass ratios, and skin thickness.

A thin-walled anisotropic beam model incorporating non-classical effects was introduced by Librescu and Song [19] to analyze the sub-critical static aeroelastic response and the divergence instability of swept-forward wing structures. A review was carried out by Patil [20], who investigated the variation of aeroelastic critical speeds with composite ply lay-up of box beams via the unsteady Theodorsen's theory. The 2D cross-section was structurally modeled using an asymptotically correct cross-sectional analysis. Qin and Librescu [21] developed an aeroelastic model to investigate the influence of directionally property of composite materials and non-classical effects such as transverse shear and warping restraint on the aeroelastic instability of thin-walled aircraft wings featuring circumferentially asymmetric stiffness lay-up. Among the several composite rotor blades applications, the work done by Jeon et al.[22, 23] concerning the steady equilibrium deflections and aeroelastic modal damping via a large deflection type beam theory with small strains is worth mentioning. A recent investigation on the minimum weight design of composite plate wings subjected to the constraints on flutter and divergence speeds has been conducted by Kameyama and Fukunaga [24] by using a genetic algorithm and a finite element approach. An interesting aeroelastic design optimization of a slender, thin-walled, isotropic unswept wing against divergence has been carried out by Librescu and Maalawi [25].

Detailed structural and aeroelastic models are essential to fully exploit non-classical effects in design of composite beam-like structures due to the properties characterizing advanced composite materials, such as anisotropy, heterogeneity, transverse shear flexibility [26, 27] and torsional warping [28]. As well as the accuracy, the computational cost of a refined model becomes also important especially for aeroelastic analysis where the fluid-structure coupling is addressed. Beam-like components can be analyzed by means of one-dimensional formulations and one main advantage is that 1D models require a lower computational cost compared with 2D plate and shell or 3D solid models. A detailed review of the recent development of refined beam models can be found in [29]. A considerable amount of work was done in trying to improve the global response of classical beam theories [30,31] using appropriate shear correction factors, as described by Timoshenko [31]. El Fatmi [32, 33] improved the displacement field over the beam cross-section by introducing a warping function to refine the description of normal and shear stress of the beam. Generalized beam theories (GBT) originated with Schardt's work [34] and improved classical theories by using a piecewise beam description of thin-walled sections [35, 36]. An asymptotic type expansion in conjunction with variational methods was proposed by Berdichevsky et al.[37], where a commendable review of prior works on beam theory development was given. An alternative approach in formulating refined beam theories based on asymptotic variational methods (VABS) has led to an extensive contribution in last decade by Volovoi, Hodges, Popescu [38, 39], Yu and co-workers [40, 41].

### 1.4.2 Advances in one-dimensional higher-order models

Nowadays different kinds of slender structures are involved in many areas such as aerospace, civil and biomechanical engineering. There are many examples of these one-dimensional (1D) structures such as rotor and wind blades, aircraft wings, bridges and towers, and even
veins. Such beam-like components can be analyzed by means of 1D formulations and one main advantage is that 1D models require a lower computational cost compared with 2D plate and shell or 3D solid models [42].

The 1D models used in early studies of slender structures were based on classical theories. Euler-Bernoulli theory [30] neglected the transverse shear deformation completely. The first shear deformation theory of Timoshenko [43] assumed a constant shear strain across the cross-section. The growing use of advanced composite and sandwich materials in thin-walled beam-like structures has revealed that 1D theories have to be refined in order to predict the behavior of such complex structures in an accurate way. Moreover, refined 1D theories are necessary to cope with arbitrary cross-section geometries, short beams, non-homogenous sections and curved shapes by taking into account effects such as warping and in-plane cross-section deformation. In the past, many theoretical and computational approaches were taken to address these issues. Recently, refined theories such as those based on the 1D Carrera Unified Formulation (CUF) [44, 45] and variational asymptotic methods (VABS) [40] as well as the Generalized Beam Theory (GBT) [35] have presented remarkable advances in static, buckling, and free vibration analysis.

Most beam-like structural systems in physical applications are actually subjected to dynamic loadings of all kinds, for instance, blood flow in veins [46]; lifting systems under the action of unsteady aerodynamic pressures [47]; blast and sonic-boom loadings [48]; interaction between bridges and moving vehicles [49]; impulsive loadings by missile launch or impact on aircraft wings [50]; and the effect of seismic waves on buildings [51]. Hence, an accurate understanding of the dynamic characteristics of a large number of structures is crucial in engineering. The importance of refined 1D models is even more relevant for accurate prediction of the time-dependent response of thin-walled slender structures [52].

A detailed review of several theories for vibrations and wave propagation was presented by Kapania and Raciti [27]. A brief, though not exhaustive, review of refined 1D models introduced in recent decades for the dynamic analysis of beams is here presented. A second-order theory with cross-sectional warping was proposed by Stephen and Levinson [53]. Heyliger and Reddy [54] and Soldatos and Elishakoff [55] proposed a third-order theory with a quadratic variation of the shear strain accross the cross-section. Early fourthorder beam theories were formulated by Levinson [56], Rychter [57] and were extended by Bickford [58] to the dynamic analysis. Kant and Gupta [59] proposed a refined FE higherorder model with quadratic transverse shear strain that was applied to the free vibration analysis of angle-ply laminated, deep sandwich and composite beams [60, 61]. Kant et al. [62] provided an analytical solution to the natural frequency analysis of thick and thin composite beams by accurately describing the cross-section warping. The formulation of two higher-order shear deformation theories by Subramanian [63] satisfied the traction-free surface conditions at the top and bottom beam surfaces. A higher-order FE model based on classical laminated theory presented higher-frequencies analysis capabilities for the vibration response of laminated tapered beams [64, 65]. Recently, Şimşek and Kocatürk [66] highlighted that a third-order shear deformation theory gives significantly better results than classical theories in the case of short beams and high mode numbers.

As far as the dynamic response is concerned, many shear deformable models have been introduced in last decades. Tong et al. [67] offered an analytical solution for free and forced vibrations of stepped generally non-uniform Timoshenko beams. A higher-order shear deformation theory was used by Rao and Ganesan [68] to evaluate the harmonic response of tapered composite wings. Marur and Kant extended their work [60] to the transient dynamic analysis of symmetric and unsymmetric sandwich and composite structures [52]. The efficacy of higher-order terms in predicting displacements and stress resultants in time
was clearly brought out. By involving the action of a moving dynamic load, the importance of third-order shear deformation effect in the strength analysis of cross-ply and angle-ply laminated beams was highlighted even when the slender ratio is not very low [69].

Librescu and Na [70] used a nonclassical beam model which includes transverse shear, secondary warping and heterogeneity to control the bending oscillations of cantilevers subjected to time-dependent excitations. The same authors [71] studied nonuniform anisotropic thin-walled beams incorporating adaptive capabilities through a beam model with transverse shear and warping inhibition which was formulated in [48]. Piovan and Cortínez [72] developed a new theoretical model for the generalized linear analysis of composite thin-walled curved beams with open and closed arbitrary cross-sections, by showing the influence of shear deformability on the mechanics of such complex structures. As a particular case of dynamic response, the third-order shear deformation theory used by Şimşek [73] indicated the importance of higher-order terms in correctly predicting the dynamic behavior of functionally graded beams and thus in tailoring FG material properties.

### 1.4.3 Review on refined thin and thick shell models

In many fields such as aerospace, mechanical, civil and biomechanical engineering different kinds of thin- and thick-walled slender structures are involved nowadays. Typical examples are rotor and wind blades, aircraft wings, pipes, bridges and towers, and even blood vessels.

Most structures in physical applications are actually subjected to dynamic loadings of all kinds, for instance, unsteady aerodynamic pressures on lifting systems [47]; blast and sonic-boom loadings [48]; blood flow in arteries [46]; interaction between bridges and moving vehicles [49]; impulsive loadings by missile launch or impact on aircraft wings [50]; and the effect of seismic waves on buildings [51]. As a consequence, an accurate understanding of the dynamic characteristics of a large number of structures is crucial in engineering. The importance of refined models to discretize thin- and thick-walled slender structures is even more relevant for a proper prediction of the time-dependent response [52]. Typically two-dimensional (2D) plate and shell or three-dimensional (3D) solid models are used for accurately modeling this kind of structures. Nonetheless, these approaches often reveal the disadvantage of a large number of degrees of freedom and hence a high computational cost.

In last decades, a considerable amount of research activity devoted to the dynamic analysis of shells was undertaken since the first classical theories for thin elastic isotropic shells were formulated by Flügge [74], Lur'e [75], Byrne [76], Love [77] and Sanders [78]. An exhaustive review of the recent research advances on the dynamic analysis of homogeneous and composite shells can be found in the works by Qatu and co-workers [79, 80, 81]. Valuable contributions were made by Herrmann and Mirsky, who investigated the axially and nonaxially symmetric motions in a hollow circular cylinder of finite length [82, 83]. Based on the analysis developed in [84], Armenàkas et al. [85] obtained closed form solutions of the governing three-dimensional (3D) elasticity equations for cylindrical shells in terms of Bessel functions as well as in [82, 86].

As far as three-dimensional (3D) analyses of cylindrical shells are concerned, a detailed review of the literature was presented by Soldatos [87]. An iterative approach based on the introduction of fictitious layers along the shell thickness to solve the governing equations of 3D linear elasticity was used in [88]. Bhimaraddi [89] developed a two-dimensional (2D) higher-order shell theory for free vibration response of isotropic circular cylindrical shells. Timarci and Soldatos [90] analyzed the vibrations of angle-ply laminated circular cylindrical

## Chapter 1. Introduction

shells with different edge boundary conditions by using the Love-type version of the unified shear-deformable shell theory developed by the same authors [91]. Utilizing the infinite circular cylinders solution based on the technique of variables separation, Mofakhami et al. [92] developed a general solution to analyze the vibration of finite isotropic circular cylinders with different end boundary conditions. Toorani and Lakis [93] analyzed the free vibrations of nonuniform composite cylindrical shells via a semi-analytical approach which combined hybrid finite elements with a shearable shell theory. More recently, a closed-form formulation of 3D refined higher-order shear deformation theory for the free vibration analysis of isotropic cylindrical shells was presented in [94] by taking into account transverse normal and shear strains as well as in-plane and rotary inertia effects.

## Part I

## Structural Formulation

## Chapter 2

## Preliminaries

This chapter is dedicated to introduce the notation, the beam geometry and to define the displacement, stress, and strain vectors which will be used in the following chapters. The geometrical strain-displacement relations as well as the material constitutive equations are addressed herein in accordance to the adopted notation.

### 2.1 Displacement, Stress, and Strain Vectors

In structural dynamics, a slender structure is often considered and studied as a beam. Indeed, a beam is a structure whose longitudinal axial length $L$ is predominant with respect to the two other orthogonal dimensions. Hence, let the beam longitudinal axis to be defined as the centroidal one. The intersection of the beam with a plane which is perpendicular to its longitudinal axis usually identifies the so-called beam cross-section $\Omega$. As depicted in Fig. 2.1, a cartesian coordinate system composed of $x$ and $z$ axes parallel to the cross-section plane is defined, whereas $y$ represents the out-of-plane coordinate. Nonetheless, the $y$ axis is not necessarily the beam centroidal axis in the hereinafter described formulation. In fact, Fig. 2.1(b) shows that the origin $O$ of the coordinate system can lie outside the contour of the cross-section, which is considered to be constant along the beam axis.


Figure 2.1: Beam cross-section geometry and coordinate system.
More in general, the hereinafter described formulation can take into account a beam arbitrarily oriented in the three-dimensional space. In other words, a beam can be the result of a geometrical extrusion of its cross-section along a direction arbitrarily oriented and in general no more perpendicular with respect to the cross-section-plane, as shown in Fig. 2.2. In the case this direction is not perpendicular, $x$ and $z$ axes are defined parallel to
the cross-section plane, but the $y$ axis is no more parallel to the longitudinal (centroidal) axis in order to respect the definition of a cartesian coordinate system. Otherwise, in the case the direction is perpendicular, the $y$ axis is parallel to the longitudinal (centroidal) axis and the beam is bounded such that $0 \leq y \leq L$. The choice of the cross-section is arbitrary, since it does not affect the following theoretical formulation. This gives high versatility to the present structural beam model. From now on the longitudinal (centroidal) beam axis will be simply referred as the beam axis.


Figure 2.2: Example of an arbitrarily oriented beam.
The notation for the displacement field is:

$$
\mathbf{u}(x, y, z ; t)=\left\{\begin{array}{l}
u_{x}(x, y, z ; t)  \tag{2.1}\\
u_{y}(x, y, z ; t) \\
u_{z}(x, y, z ; t)
\end{array}\right\}
$$

where $u_{x}, u_{y}$, and $u_{z}$ are the displacement components along $x, y$, and $z$ axes respectively. In general they are functions of the spatial coordinates and of time $t$. Stress $\boldsymbol{\sigma}$ and strain $\varepsilon$ vectors contain the components shown in Fig. 2.3, which are:

$$
\begin{gather*}
\boldsymbol{\sigma}=\left\{\begin{array}{llllll}
\sigma_{y y} & \sigma_{x x} & \sigma_{z z} & \sigma_{x z} & \sigma_{y z} & \sigma_{x y}
\end{array}\right\}^{T}  \tag{2.2}\\
\boldsymbol{\varepsilon}=\left\{\begin{array}{llllll}
\varepsilon_{y y} & \varepsilon_{x x} & \varepsilon_{z z} & 2 \varepsilon_{x z} & 2 \varepsilon_{y z} & 2 \varepsilon_{x y}
\end{array}\right\}^{T}
\end{gather*}
$$

where superscript $T$ respresents the transposition operator. The stress components to be considered are six instead of nine thanks to the fact the moment equilibrium (balance of angular moment) yields symmetry of stress given by:

$$
\begin{equation*}
\sigma_{i j}=\sigma_{j i} \quad i \neq j \tag{2.3}
\end{equation*}
$$

Similarly, the strain components are symmetric as will be clarified afterwards by the strain-displacement relations:

$$
\begin{equation*}
\varepsilon_{i j}=\varepsilon_{j i} \quad i \neq j \tag{2.4}
\end{equation*}
$$

The stress and strain components are split into terms $\boldsymbol{\sigma}_{p}, \boldsymbol{\varepsilon}_{p}$ related to the beam crosssection $\Omega$ :

$$
\begin{gather*}
\boldsymbol{\sigma}_{p}=\left\{\begin{array}{lll}
\sigma_{z z} & \sigma_{x x} & \sigma_{x z}
\end{array}\right\}^{T} \\
\boldsymbol{\varepsilon}_{p}=\left\{\begin{array}{llll}
\varepsilon_{z z} & \varepsilon_{x x} & 2 \varepsilon_{x z}
\end{array}\right\}^{T}=\left\{\begin{array}{lll}
\varepsilon_{z z} & \varepsilon_{x x} & \gamma_{x z}
\end{array}\right\}^{T} \tag{2.5}
\end{gather*}
$$

and terms $\boldsymbol{\sigma}_{n}, \varepsilon_{n}$ related to the out-of-plane direction:

$$
\begin{gather*}
\boldsymbol{\sigma}_{n}=\left\{\begin{array}{llll}
\sigma_{y z} & \sigma_{x y} & \sigma_{y y}
\end{array}\right\}^{T}  \tag{2.6}\\
\varepsilon_{n}=\left\{\begin{array}{llll}
2 \varepsilon_{y z} & 2 \varepsilon_{x y} & \varepsilon_{y y}
\end{array}\right\}^{T}=\left\{\begin{array}{lll}
\gamma_{y z} & \gamma_{x y} & \varepsilon_{y y}
\end{array}\right\}^{T}
\end{gather*}
$$

where the 'engineering' shear strains are introduced:

$$
\begin{equation*}
\gamma_{i j}=2 \varepsilon_{j i} \quad i \neq j \tag{2.7}
\end{equation*}
$$



Figure 2.3: Stress (a) and strain (b) components.

### 2.2 Geometrical Relations

In the case of small displacement with respect to the length $L$, the linear relations between strain and displacement components hold. According to Eqs. 2.5 and 2.6, the following relations hold:

$$
\begin{align*}
& \varepsilon_{p}=\left\{\begin{array}{l}
\varepsilon_{z z}=u_{z, z} \\
\varepsilon_{x x}=u_{x, x} \\
\gamma_{x z}=2 \varepsilon_{x z}=u_{x, z}+u_{z, x}
\end{array}\right\} \\
& \varepsilon_{n}=\left\{\begin{array}{l}
\gamma_{y z}=2 \varepsilon_{y z}=u_{y, z}+u_{z, y} \\
\gamma_{x y}=2 \varepsilon_{x y}=u_{x, y}+u_{y, x} \\
\varepsilon_{y y}=u_{y, y}
\end{array}\right\} \tag{2.8}
\end{align*}
$$

Derivatives with respect to the spatial coordinates are represented via a subscript in which comma precedes the spatial coordinate. A compact vectorial notation can be adopted:

$$
\left\{\begin{array}{l}
\varepsilon_{p}=\mathbf{D}_{p} \mathbf{u}  \tag{2.9}\\
\boldsymbol{\varepsilon}_{n}=\mathbf{D}_{n} \mathbf{u}=\mathbf{D}_{n p} \mathbf{u}+\mathbf{D}_{n y} \mathbf{u}
\end{array}\right.
$$

where $\mathbf{D}_{p}, \mathbf{D}_{n p}$ and $\mathbf{D}_{n y}$ are differential matrix operators:

$$
\mathbf{D}_{p}=\left[\begin{array}{ccc}
0 & 0 & \frac{\partial}{\partial z}  \tag{2.10}\\
\frac{\partial}{\partial x} & 0 & 0 \\
\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}
\end{array}\right], \quad \mathbf{D}_{n p}=\left[\begin{array}{ccc}
0 & \frac{\partial}{\partial z} & 0 \\
0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0
\end{array}\right], \quad \mathbf{D}_{n y}=\left[\begin{array}{ccc}
0 & 0 & \frac{\partial}{\partial y} \\
\frac{\partial}{\partial y} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0
\end{array}\right]
$$

### 2.3 Constitutive Equations

Let a material coordinate system $(1,2,3)$ to be defined as shown in Fig. 2.4. In general, this coordinate system is different from the physical coordinate system ( $x, y, z$ ). Similarly to the strain and stress vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, which are defined in Eq. 2.2 and depicted in Fig. 2.3 in the physical coordinate system, the material strain and stress vectors $\boldsymbol{\sigma}_{m}$ and $\varepsilon_{m}$ are introduced in the material coordinate system:

$$
\begin{gather*}
\boldsymbol{\sigma}_{m}=\left\{\begin{array}{llllll}
\sigma_{33} & \sigma_{22} & \sigma_{11} & \sigma_{21} & \sigma_{31} & \sigma_{23}
\end{array}\right\}^{T} \\
\boldsymbol{\varepsilon}_{m}=\left\{\begin{array}{llllll}
\varepsilon_{33} & \varepsilon_{22} & \varepsilon_{11} & 2 \varepsilon_{21} & 2 \varepsilon_{31} & 2 \varepsilon_{23}
\end{array}\right\}^{T} \tag{2.11}
\end{gather*}
$$



Figure 2.4: Physical and material coordinate systems.
Under the hypothesis of linear elastic behaviour of the material beam is made of, the generalized Hooke's law holds. Its compact vectorial form is:

$$
\begin{equation*}
\boldsymbol{\sigma}_{m}=\mathbf{C}_{m} \varepsilon_{m} \tag{2.12}
\end{equation*}
$$

where $\mathbf{C}_{m}$ is the $6 \times 6$ material stiffness matrix of elastic coefficients. When these coefficients are functions of position the material is heterogeneous, whereas when they are constant throughout the material it is homogeneous. In this chapter from here on, the case of homogeneous materials is addressed; the nonhomogeneous (heterogeneous) materials case will be introduced in section 3.4.1. The thirty-six coefficients of $\mathbf{C}_{m}$ are not all independent each other. The number of indipendent constants depends on the material constitutions. First of all, there are only 21 independent elastic coefficients for anisotropic materials due to the symmetry of the material stiffness matrix:

$$
\begin{equation*}
C_{m_{i j}}=C_{m_{j i}} \quad i, j=1, \ldots, 6 \tag{2.13}
\end{equation*}
$$

For the sake of completeness, it is important to clarify that a material is anisotropic when it exhibits material properties that are directionally dependent, i.e. a given material property can have different values in different directions. On the contrary, a material is isotropic if all its material properties at a point are independent of the direction. Some anisotropic materials, such as monoclinic or orthotropic, may possess material symmetries and their constitutive behavior can be described with fewer than 21 constants. In this section, orthotropic and isotropic materials are considered, disregarding the monoclinic materials case.

### 2.3.1 Orthotropic Material

If a material has three mutually perpendicular planes of elastic symmetry is referred as orthotropic material. In this case, the number of coefficients can be reduced to 9. In fact, Hooke's law (Eq. 2.12) becomes:

$$
\begin{gather*}
\boldsymbol{\sigma}_{m}=\mathbf{C}_{m} \boldsymbol{\varepsilon}_{m} \\
\left\{\begin{array}{l}
\sigma_{33} \\
\sigma_{22} \\
\sigma_{11} \\
\sigma_{21} \\
\sigma_{31} \\
\sigma_{23}
\end{array}\right\}=\left[\begin{array}{cccccc}
C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\
C_{23} & C_{22} & C_{12} & 0 & 0 & 0 \\
C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{33} \\
\varepsilon_{22} \\
\varepsilon_{11} \\
2 \varepsilon_{21} \\
2 \varepsilon_{31} \\
2 \varepsilon_{23}
\end{array}\right\} \tag{2.14}
\end{gather*}
$$

Coefficients $C_{m_{i j}}$ are defined as follows (see [95, 96]):

$$
\begin{array}{ccc}
C_{11}=\frac{E_{1}\left(1-\nu_{23} \nu_{32}\right)}{\Delta} ; & C_{12}=\frac{E_{1}\left(\nu_{21}+\nu_{23} \nu_{31}\right)}{\Delta} ; & C_{13}=\frac{E_{1}\left(\nu_{31}+\nu_{21} \nu_{32}\right)}{\Delta} ; \\
C_{22}=\frac{E_{2}\left(1-\nu_{13} \nu_{31}\right)}{\Delta} ; & C_{23}=\frac{E_{2}\left(\nu_{32}+\nu_{12} \nu_{31}\right)}{\Delta} ; & C_{33}=\frac{E_{3}\left(1-\nu_{12} \nu_{21}\right)}{\Delta} ;  \tag{2.15}\\
& C_{44}=G_{21} ; & C_{55}=G_{31} ;
\end{array} c
$$

where:

$$
\begin{equation*}
\Delta=1-\nu_{12} \nu_{21}-\nu_{13} \nu_{31}-\nu_{23} \nu_{32}-\nu_{12} \nu_{23} \nu_{31}-\nu_{13} \nu_{21} \nu_{32} \tag{2.16}
\end{equation*}
$$

Terms $\left\{E_{i}: i=1,2,3\right\}$ are Young's moduli, $\left\{\nu_{i j}: i, j=1,2,3\right\}$ are Poisson's ratios and $\left\{G_{i j}: i=2,3 ; j=1,3\right\}$ are the shear moduli in the material coordinate system. Poisson's ratios are defined as:

$$
\begin{equation*}
\nu_{i j}=-\frac{\varepsilon_{j j}}{\varepsilon_{i i}} \quad i, j=1,2,3 \quad i \neq j \tag{2.17}
\end{equation*}
$$

The symmetry of the material stiffness matrix, expressed by Eq. 2.13, is verified in Eq. 2.18 and ensured by the relations between Young's moduli and Poisson's ratios, see Eq. 2.19.

$$
\begin{gather*}
C_{21}=\frac{E_{2}\left(\nu_{12}+\nu_{13} \nu_{32}\right)}{\Delta}=C_{12} \\
C_{32}=\frac{E_{3}\left(\nu_{23}+\nu_{13} \nu_{21}\right)}{\Delta}=C_{23}  \tag{2.18}\\
C_{31}=\frac{E_{3}\left(\nu_{13}+\nu_{12} \nu_{23}\right)}{\Delta}=C_{13} \\
\frac{\nu_{i j}}{E_{i}}=\frac{\nu_{j i}}{E_{j}} \quad i, j=1,2,3 \quad i \neq j \tag{2.19}
\end{gather*}
$$

So far, the material stiffness matrix has been referred to the material coordinate system. For the orthotropic material case, Fig. 2.4 shows that this reference system is supposed to be aligned with the fibers in a unidirectionally reinforced lamina which lies in the $2-3$ plane. In other words, axis 2 is the fiber longitudinal direction $L$, axis 3 is aligned with the fiber transversal in-plane direction $T$ (in the plane of the lamina) and 1 is the transversal out-of-plane direction $Z$. According to Fig. 2.4, material axes 2 and 3 are rotated by a positive counterclockwise angle $\theta$ about the $z$ axis, coincident to axis 1 , from physical $x$ and $y$ axes. As a consequence, coordinate transformation equations are employed to obtain
the stress vector $\boldsymbol{\sigma}$ in physical coordinates in terms of the stress vector $\boldsymbol{\sigma}_{m}$ in material coordinates. Exploiting a vectorial notation:

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathbf{T} \boldsymbol{\sigma}_{m} \tag{2.20}
\end{equation*}
$$

where $\mathbf{T}$ is the $6 \times 6$ transformation matrix:

$$
\mathbf{T}=\left[\begin{array}{cccccc}
\cos ^{2} \theta & \sin ^{2} \theta & 0 & 0 & 0 & \sin 2 \theta  \tag{2.21}\\
\sin ^{2} \theta & \cos ^{2} \theta & 0 & 0 & 0 & -\sin 2 \theta \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & -\sin \theta & 0 \\
0 & 0 & 0 & \sin \theta & \cos \theta & 0 \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & 0 & 0 & 0 & \cos ^{2} \theta-\sin ^{2} \theta
\end{array}\right]
$$

Similarly, in Eq. 2.22 it is possible to relate the components of the strain vector referred to the material system in terms of its components referred to the physical system:

$$
\begin{equation*}
\varepsilon_{m}=\mathbf{T}^{T} \varepsilon \tag{2.22}
\end{equation*}
$$

where $\mathbf{T}^{T}$ is the transpose of matrix $\mathbf{T}$. For the sake of brevity, more details about the derivation of Eqs. 2.20 and 2.22 are not reported here, but can be found in appendix A. Substituting Eqs. 2.12 and 2.22 into Eq. 2.20, a transformed material stiffness matrix is therefore introduced and referred as $\widetilde{\mathbf{C}}$ :

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathbf{T} \boldsymbol{\sigma}_{m}=\mathbf{T} \mathbf{C}_{m} \boldsymbol{\varepsilon}_{m}=\mathbf{T} \mathbf{C}_{m} \mathbf{T}^{T} \boldsymbol{\varepsilon}=\widetilde{\mathbf{C}} \boldsymbol{\varepsilon} \tag{2.23}
\end{equation*}
$$

The transformed material stiffness matrix contains the elastic coefficients referred to the physical coordinate system:

$$
\begin{equation*}
\widetilde{\mathbf{C}}=\mathbf{T} \mathbf{C}_{m} \mathbf{T}^{T} \tag{2.24}
\end{equation*}
$$

In general, Eq. 2.24 is valid for an arbitrary constitutive matrix $\mathbf{C}_{m}$ (i.e. for orthotropic as well as anisotropic). Of course, $\mathbf{T}$ is the matrix based on the particular transformation depicted in Fig. 2.4 (rotation about a transverse nornal to the lamina).

The stress-strain relations of Eq. 2.23 referred to the physical coordinate system can be written in a vectorial notation as follows [cf. Eq. 2.14 for the material coordinate system]:

$$
\left\{\begin{array}{l}
\sigma_{y y}  \tag{2.25}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\widetilde{C}_{33} & \widetilde{C}_{23} & \widetilde{C}_{13} & 0 & 0 & \widetilde{C}_{36} \\
\widetilde{C}_{23} & \widetilde{C}_{22} & \widetilde{C}_{12} & 0 & 0 & \widetilde{C}_{26} \\
\widetilde{C}_{13} & \widetilde{C}_{12} & \widetilde{C}_{11} & 0 & 0 & \widetilde{C}_{16} \\
0 & 0 & 0 & \widetilde{C}_{44} & \widetilde{C}_{45} & 0 \\
0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{55} & 0 \\
\widetilde{C}_{36} & \widetilde{C}_{26} & \widetilde{C}_{16} & 0 & 0 & \widetilde{C}_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

For the sake of completeness, in Eq. 2.25 it is interesting to note that the transformed material stiffness matrix $\widetilde{\mathbf{C}}$, obtained here for orthotropic materials, has a structure formally analogous to the material stiffness matrix corresponding to monoclinic materials with plane $2-3$ of elastic symmetry. Indeed, a monoclinic material is a material with only one plane of elastic symmetry and so the number of independent elastic coefficients is 13 . In an analogous way, the physical $x-y$ plane (parallel to the $2-3$ plane) is a plane of elastic symmetry for the orthotropic material in Fig. 2.4, whereaes planes $y-z$ and $x-z$ are not (when in general $\theta \neq 0$ ).

Carrying out the matrix multiplications in Eq. 2.24 for the orthotropic case, the elastic coefficients of the transformed material stiffness matrix $\widetilde{\mathbf{C}}$ are expressed as a function of the coefficients of the material stiffness matrix $\mathbf{C}$ and the angle $\theta$ :

$$
\begin{align*}
& \widetilde{C}_{33}=C_{33} \cos ^{4} \theta+2\left(C_{23}+2 C_{66}\right) \sin ^{2} \theta \cos ^{2} \theta+C_{22} \sin ^{4} \theta \\
& \widetilde{C}_{23}=C_{23}\left(\sin ^{4} \theta+\cos ^{4} \theta\right)+\left(C_{33}+C_{22}-4 C_{66}\right) \sin ^{2} \theta \cos ^{2} \theta \\
& \widetilde{C}_{13}=C_{13} \cos ^{2} \theta+C_{12} \sin ^{2} \theta \\
& \widetilde{C}_{36}=\left(-C_{33}+C_{23}+2 C_{66}\right) \sin \theta \cos ^{3} \theta+\left(C_{22}-C_{23}-2 C_{66}\right) \sin ^{3} \theta \cos \theta \\
& \widetilde{C}_{22}=C_{22} \cos ^{4} \theta+2\left(C_{23}+2 C_{66}\right) \sin ^{2} \theta \cos ^{2} \theta+C_{33} \sin ^{4} \theta \\
& \widetilde{C}_{12}=C_{12} \cos ^{2} \theta+C_{13} \sin ^{2} \theta \\
& \widetilde{C}_{26}=\left(-C_{33}+C_{23}+2 C_{66}\right) \sin ^{3} \theta \cos \theta+\left(C_{22}-C_{23}-2 C_{66}\right) \sin \theta \cos ^{3} \theta  \tag{2.26}\\
& \widetilde{C}_{11}=C_{11} \\
& \widetilde{C}_{16}=\left(C_{12}-C_{13}\right) \sin \theta \cos \theta \\
& \widetilde{C}_{44}=C_{44} \cos ^{2} \theta+C_{55} \sin 2 \theta \\
& \widetilde{C}_{45}=\left(C_{44}-C_{55}\right) \sin \theta \cos \theta \\
& \widetilde{C}_{55}=C_{55} \cos ^{2} \theta+C_{44} \sin ^{2} \theta \\
& \widetilde{C}_{66}=\left(C_{33}+C_{22}-2 C_{23}-2 C_{66}\right) \sin ^{2} \theta \cos ^{2} \theta+C_{66}\left(\sin ^{4} \theta+\cos ^{4} \theta\right)
\end{align*}
$$

According to the splitting carried out for strain and stress vectors in Eqs. 2.5 and 2.6, Hooke's law of Eq. 2.23 referred to the physical coordinate system can be rewritten as follows:

$$
\left\{\begin{align*}
\sigma_{p} & =\widetilde{\mathbf{C}}_{p p} \varepsilon_{p}+\widetilde{\mathbf{C}}_{p n} \varepsilon_{n}  \tag{2.27}\\
\boldsymbol{\sigma}_{n} & =\widetilde{\mathbf{C}}_{n p} \varepsilon_{p}+\widetilde{\mathbf{C}}_{n n} \varepsilon_{n}
\end{align*}\right.
$$

where matrices $\widetilde{\mathbf{C}}_{p p}, \widetilde{\mathbf{C}}_{p n}, \widetilde{\mathbf{C}}_{n p}$, and $\widetilde{\mathbf{C}}_{n n}$ derive from matrix $\widetilde{\mathbf{C}}$ :

$$
\begin{gather*}
\widetilde{\mathbf{C}}_{p p}=\left[\begin{array}{ccc}
\widetilde{C}_{11} & \widetilde{C}_{12} & 0 \\
\widetilde{C}_{12} & \widetilde{C}_{22} & 0 \\
0 & 0 & \widetilde{C}_{44}
\end{array}\right] ; \quad \widetilde{\mathbf{C}}_{p n}=\left[\begin{array}{ccc}
0 & \widetilde{C}_{16} & \widetilde{C}_{13} \\
0 & \widetilde{C}_{26} & \widetilde{C}_{23} \\
\widetilde{C}_{45} & 0 & 0
\end{array}\right] ; \\
\widetilde{\mathbf{C}}_{n p}=\widetilde{\mathbf{C}}_{p n}^{T}=\left[\begin{array}{ccc}
0 & 0 & \widetilde{C}_{45} \\
\widetilde{C}_{16} & \widetilde{C}_{26} & 0 \\
\widetilde{C}_{13} & \widetilde{C}_{23} & 0
\end{array}\right] ; \quad \widetilde{\mathbf{C}}_{n n}=\left[\begin{array}{ccc}
\widetilde{C}_{55} & 0 & 0 \\
0 & \widetilde{C}_{66} & \widetilde{C}_{36} \\
0 & \widetilde{C}_{36} & \widetilde{C}_{33}
\end{array}\right] \tag{2.28}
\end{gather*}
$$

In the case of classical theories (i.e. Euler-Bernoulli and Timoshenko beam theories) and first-order approximation model, the transformed elastic coefficients $\widetilde{C}_{i j}$ in Eq. 2.28 have to be corrected by the procedure addressed in appendix B in order to contrast the Poisson's locking effect.

### 2.3.2 Isotropic Material

As previously explained, the material properties of an isotropic material are independent of the direction. As a consequence, there are an infinite number of planes of elastic symmetry (or no preferred directions) and the material properties can be described in terms of only two elastic coefficients. Moreover, there is no need for introducing any material coordinate
system. Hence, Hooke's law for an isotropic material is:

$$
\left\{\begin{array}{c}
\sigma_{y y}  \tag{2.29}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\
C_{23} & C_{22} & C_{12} & 0 & 0 & 0 \\
C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

Let the material stiffness matrix $\mathbf{C}$ to be introduced from Eq. 2.29:

$$
\begin{equation*}
\sigma=\mathbf{C} \varepsilon \tag{2.30}
\end{equation*}
$$

The elastic coefficients in Eq. 2.29 depend only on Young's modulus $E$ and Poisson's ratio $\nu$ :

$$
\begin{gather*}
C_{11}=C_{22}=C_{33}=\frac{(1-\nu) E}{(1+\nu)(1-2 \nu)}=\lambda+2 \mu \\
C_{12}=C_{13}=C_{23}=\frac{\nu E}{(1+\nu)(1-2 \nu)}=\lambda  \tag{2.31}\\
C_{44}=C_{55}=C_{66}=\frac{E}{2(1+\nu)}=G=\mu
\end{gather*}
$$

where $\lambda$ and $\mu$ are the Lamé parameters and $G$ is the shear modulus. Retrieving the stress and strain vectors defined in Eqs. 2.5 and 2.6, Hooke's law for an isotropic material in Eq. 2.30 can be reformulated in an expression similar to Eq. 2.27:

$$
\left\{\begin{array}{l}
\sigma_{p}=\mathbf{C}_{p p} \varepsilon_{p}+\mathbf{C}_{p n} \varepsilon_{n}  \tag{2.32}\\
\boldsymbol{\sigma}_{n}=\mathbf{C}_{n p} \varepsilon_{p}+\mathbf{C}_{n n} \varepsilon_{n}
\end{array}\right.
$$

where matrices $\mathbf{C}_{p p}, \mathbf{C}_{p n}, \mathbf{C}_{n p}$, and $\mathbf{C}_{n n}$ derive from matrix $\mathbf{C}$ :

$$
\begin{array}{cc}
\mathbf{C}_{p p}=\left[\begin{array}{ccc}
C_{11} & C_{12} & 0 \\
C_{12} & C_{22} & 0 \\
0 & 0 & C_{44}
\end{array}\right] ; & \mathbf{C}_{p n}=\left[\begin{array}{ccc}
0 & 0 & C_{13} \\
0 & 0 & C_{23} \\
0 & 0 & 0
\end{array}\right] ; \\
\mathbf{C}_{n p}=\mathbf{C}_{p n}^{T}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
C_{13} & C_{23} & 0
\end{array}\right] ; & \mathbf{C}_{n n}=\left[\begin{array}{ccc}
C_{55} & 0 & 0 \\
0 & C_{66} & 0 \\
0 & 0 & C_{33}
\end{array}\right] \tag{2.33}
\end{array}
$$

As indicated for orthotropic materials, in the case of classical theories (i.e. Euler-Bernoulli and Timoshenko beam theories) and first-order approximation model, the elastic coefficients in Eq. 2.33 have to be corrected by the procedure addressed in appendix B in order to contrast the Poisson's locking effect.

### 2.4 Variable Kinematic 1D models

Since the dimensions of the cross-section are typically negligible with respect to the beam longitudinal axis, in typical structural beam modeling the variation of the main unknowns of the structural model with respect to the directions lying on the cross-section $\Omega$ can be approximated. According to the framework of Carrera Unified Formulation (CUF), the
displacement field is assumed to be an expansion of a certain class of functions $F_{\tau}$, which depend on the cross-section coordinates $x$ and $z$ :

$$
\begin{equation*}
\mathbf{u}(x, y, z ; t)=F_{\tau}(x, z) \mathbf{u}_{\tau}(y ; t) \quad \tau=1, \ldots, N_{u}=N_{u}(N) \tag{2.34}
\end{equation*}
$$

The compact expression is based on Einstein's notation: repeated subscript $\tau$ indicates summation. The vector $\mathbf{u}_{\tau}$ is the $\tau^{t h}$ generalized displacement unknowns vector which contains the components $\left\{\begin{array}{lll}u_{x \tau} & u_{y \tau} & u_{z \tau}\end{array}\right\}^{T}$. The vector $\mathbf{u}_{\tau}$ depends on time $t$ and on the single spatial coordinate $y$ and so Eq. 2.34 represents the key formulation of the onedimensional (1D) displacement-based CUF model. The number of expansion terms $N_{u}$ depends on the expansion order $N$, which is a free parameter of the formulation. The expansion order depends directly on the choice of the cross-section functions $F_{\tau}$, which is arbitrary. Thanks to the hierarchical CUF approach, different higher-order theories with a variable order of expansion for the displacement unknowns and hence with a variable accuracy can be easily developed. In general, Eq. 2.34 describes the three-dimensional behavior of a structure and the accuracy of the axiomatic model in Eq. 2.34 can be freely increased. This implies that the present method can be used also for relatively small aspect ratio structures unlike classical beam theories.

CUF was initially proposed by Carrera for two-dimensional (2D) plate and shell models [97, 98, 99, 100], where it was exploited to describe the displacement field as an expansion in the thickness direction by means of thickness functions $F_{\tau}$. Hence, for plate and shell models these functions depend only on the thickness coordinate $z$ whereas the generalized displacement unknowns $\mathbf{u}_{\tau}$ depend on the reference surface coordinates $x$ and $y$. The equivalent approach for the one-dimensional modeling (Eq. 2.34) which is described in this doctoral work considers multivariate (2D) polynomial functions, depending on the beam cross-section coordinates, as approximation of the cross-section deformation.

As far as the approximation type is concerned, the choice of cross-sections functions $F_{\tau}$ determines the class of the 1D CUF model adopted. Several multivariate (depending on $x$ and $z$ coordinates) polynomial functions can be assumed for $F_{\tau}$. For instance, refined models based on Maclaurin and Lagrange polynomials are herein proposed and addressed.

### 2.4.1 Maclaurin polynomials

The choice of Maclaurin polynomials to approximate the cross-section behavior is inspired straightforwardly by the classical beam models. In this case, the generic $\tau^{t h}$ cross-section function is a multivariate Maclaurin polynomial given by the multiplication of a monomial in $x$ by a monomial in $z$ as described in Eq. 2.35:

$$
\begin{equation*}
F_{\tau}(x, z)=x^{h} z^{k} \quad h, k=0, \ldots, N \quad \tau=1, \ldots, N_{u}=\frac{(N+1)(N+2)}{2} \tag{2.35}
\end{equation*}
$$

The relation between indices $\tau, h$, and $k$ in Eq. 2.35 is presented in Table 2.1.
With the kinematic assumptions in Eq. 2.35, most displacement-based beam theories can be formulated on the basis of the generic kinematic field of Eq. 2.34. Higher-order one-dimensional theories can be described assuming an expansion order $N$ higher than 1. For instance, when $N=2$, the second-order axiomatic displacement field is given by:

$$
\begin{align*}
& u_{x}=u_{x 1}+u_{x 2} x+u_{x 3} z+u_{x 4} x^{2}+u_{x 5} x z+u_{x 6} z^{2} \\
& u_{y}=u_{y 1}+u_{y 2} x+u_{y 3} z+u_{y 4} x^{2}+u_{y 5} x z+u_{y 6} z^{2}  \tag{2.36}\\
& u_{z}=u_{z 1}+u_{z 2} x+u_{z 3} z+u_{z 4} x^{2}+u_{z 5} x z+u_{z 6} z^{2}
\end{align*}
$$

Subsequently, the classical beam models such as Euler-Bernoulli's (EBBM) [30] and Timoshenko's (TBM) [43] are easily derived from the first-order approximation model

Table 2.1: Relation between indices $\tau, h$ and $k$ for different values of the expansion order $N$. Maclaurin polynomials case.

| $N$ | $N_{u}$ | $\tau$ | $h$ | $k$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 0 | 0 |
|  |  | 2 | 1 | 0 |
| 1 | 3 | 3 | 0 | 1 |
|  |  | 4 | 2 | 0 |
| 2 | 6 | 5 | 1 | 1 |
|  |  | 6 | 0 | 2 |
|  |  | 7 | 3 | 0 |
| 3 | 10 | 8 | 2 | 1 |
|  |  | 9 | 1 | 2 |
|  | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

( $N=1$ ). In fact, it is noteworthy that the model $N=1$ is more accurate than the classical beam models mentioned above (EBBM and TBM), since it takes into account a first-order approximation of all the three components of the displacement field. In other words, for $N=1$ the displacement components $u_{x}$ and $u_{z}$ are approximated via a first-order approximation instead of a constant approximation for classical beam theories.

According to Table 2.1, the cross-section functions $F_{\tau}$ depend on $N$ as presented in the scheme of Table 2.2, which reminds Pascal's triangle, also called Tartaglia's triangle. Its particular triangular shape is highlighted in Fig. 2.5, which depicts the multivariate Maclaurin polynomials considered for the approximation of the beam cross-section deformation as the expansion order $N$ increases. For the sake of simplicity, the polynomials in Fig. 2.5 refer to a rectangular domain with dimensions equal to $a$ and $b$ along $x$ and $z$ axes, respectively.

Table 2.2: Cross-section functions $F_{\tau}$ and number of expansion terms $N_{u}$ as functions of the expansion order $N$. Maclaurin polynomials case.

| $N$ | $N_{u}$ | $F_{\tau}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1 | $F_{1}=1$ |  |  |
| 1 | 3 | $F_{2}=x \quad F_{3}=z$ |  |  |
| 2 | 6 | $F_{4}=x^{2} \quad F_{5}=x z \quad F_{6}=z^{2}$ |  |  |
| 3 | 10 | $F_{8}=x^{2} z$ | $F_{9}=x z^{2} \quad F_{10}=z^{3}$ |  |
| $\vdots$ | $\vdots$ |  |  |  |
| $N$ | $\frac{(N+1)(N+2)}{2}$ | $F_{\frac{\left(N^{2}+N+2\right)}{2}}=x^{N}$ | $F_{\frac{\left(N^{2}+N+4\right)}{2}}=x^{N-1} z$ | $\ldots$ |

The structural, aeroelastic, and fluid dynamics formulations developed during this doctoral research are described in the following chapters precisely on the basis of the choice of Maclaurin polynomials as $F_{\tau}$ for the one-dimensional CUF model. For the sake of completeness, it is important to note that from here on this kind of expansion is denoted also as Taylor or Taylor-like expansion.


Figure 2.5: Higher-order Maclaurin polynomials for different values of the expansion order $N$.

### 2.4.2 Lagrange Polynomials

In structural dynamics, different classes of polynomials are widely used as approximation funtions in numerical modeling. The case of Taylor-like expansion for classical and higherorder beam models has been presented in the previous section. A prime example of the use of a different polynomial class is the layer-wise approach used in plate and shell (2D) modeling to describe the mechanical behavior of a laminate structure [101]. In layer-wise plate and shell models, 1D Legendre polynomials approximate the structural unknonws (displacement field for displacement-based models or displacement and stress fields for mixed models) along the thickness and thus they depend only on the thickness coordinate.

Lagrange polynomials are an other important class of functions broadly employed in numerical modeling. In particular, they have been fundamental to develop the finite element method in solid mechanics and fluid dynamics in the last decades. Hence, the choice of
multivariate (2D) Lagrange polynomials can be an appropriate example of approximation of the cross-section deformation for one-dimensional modeling and thus is here faced.

Lagrange polynomials are typically expressed in a natural coordinate system. Hence, for the sake of convenience, the introduction of multivariate Lagrange polynomials herein refers to the natural coordinates $(r, s)$. Let the natural coordinate system to be ranged in a square so that $-1 \leq r \leq 1$ and $-1 \leq s \leq 1$. In general, the coordinate transformation from an arbitrary rectangular cross-section referred to $(x, z)$ coordinates to the natural square $(r, s)$ is trivial and more details can be found in [42, 102].

With multivariate Lagrange polynomials, it is necessary to introduce a lattice of Lagrange points, also called Lagrange nodes, over the natural domain $(r, s)$. Let $\left(N_{h}+1\right)$ to be the number of Lagrange points along the $r$ axis, whereas $\left(N_{k}+1\right)$ is the number of Lagrange points along the $s$ axis. Two sample schemes of Lagrange points distribution are illustrated in Fig.2.6 for the simple case $N_{h}=N_{k}$.


Figure 2.6: Sample schemes of Lagrange points distibution over the natural coordinate system $(r, s)$ for two different values of $N_{h}=N_{k}$.

In the natural domain, the generic $\tau^{\text {th }}$ cross-section function to be used in Eq. 2.34 is a multivariate Lagrange polynomial given by the multiplication of a 1D Lagrange polynomial in $r$ by a 1D Lagrange polynomial in $s$ as described in Eq. 2.37:

$$
\begin{array}{ll}
F_{\tau}(r, s)=F_{h}(r) F_{k}(s) & h=0, \ldots, N_{h}  \tag{2.37}\\
k=0, \ldots, N_{k}
\end{array} \quad \tau=1, \ldots, N_{u}=\left(N_{h}+1\right)\left(N_{k}+1\right)
$$

where:

$$
\begin{align*}
& F_{h}(r)=\prod_{\xi=0, \xi \neq h}^{N_{h}} \frac{\left(r-r_{\xi}\right)}{\left(r_{h}-r_{\xi}\right)}  \tag{2.38}\\
& F_{k}(s)=\prod_{\zeta=0, \zeta \neq k}^{N_{k}} \frac{\left(s-s_{\zeta}\right)}{\left(s_{k}-s_{\zeta}\right)} \tag{2.39}
\end{align*}
$$

The relation between indices $\tau, h$, and $k$ in Eq. 2.37 is presented in Table 2.3, where the natural coordinates of the generic $\tau^{t h}$ Lagrange point $\left(r_{h}, s_{k}\right)$, i.e. $\left(r_{\tau}, s_{\tau}\right)$, are summarized. $N_{h}$ represents the order of the 1D Lagrange polynomial $F_{h}(r)$ whereas $N_{k}$ represents the order of the 1D Lagrange polynomial $F_{k}(s)$, see Eqs. 2.38 and 2.39. It is noteworthy that
the number of Lagrange points $\left(N_{h}+1\right) \times\left(N_{k}+1\right)$ is the number of Lagrange polynomials involved and therefore the number of expansion terms $N_{u}$.

Table 2.3: Relation between indices $\tau, h$ and $k$ for different values of the expansion order $N$. Lagrange polynomials case with $N_{h}=N_{k}=N$.

| $N$ | $N_{u}$ | $\tau$ | $h$ | $k$ | $r_{h}$ | $s_{k}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 1 | 0 | 0 | -1 | -1 |
| 1 | 4 | 2 | 1 | 0 | 1 | -1 |
|  |  | 3 | 1 | 1 | 1 | 1 |
|  |  | 4 | 0 | 1 | -1 | 1 |
|  |  | 1 | 0 | 0 | -1 | -1 |
|  |  | 2 | 1 | 0 | 0 | -1 |
|  |  | 3 | 2 | 0 | 1 | -1 |
|  |  | 4 | 2 | 1 | 1 | 0 |
| 2 | 9 | 5 | 2 | 2 | 1 | 1 |
|  |  | 6 | 1 | 2 | 0 | 1 |
|  |  | 7 | 0 | 2 | -1 | 1 |
|  |  | 8 | 0 | 1 | -1 | 0 |
|  |  | 9 | 1 | 1 | 0 | 0 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Lagrange polynomials have the feature to be equal to 1 on the corresponding Lagrange point and to be equal to 0 on the other Lagrange points:

$$
F_{\tau}\left(r_{\lambda}, s_{\lambda}\right)= \begin{cases}1 & \lambda=\tau  \tag{2.40}\\ 0 & \lambda \neq \tau\end{cases}
$$

Thank to this feature, the generalized displacement unknown vector $\mathbf{u}_{\tau}$ in Eq. 2.34 is the actual three-dimensional displacement of the Lagrange point with natural coordinates $\left(r_{\tau}, s_{\tau}\right)$. In other words, Lagrange polynomials do not involve the derivatives of the problem unknowns as degree of freedom in the governing equations. Thus, they are particularly suited to be assumed in case of approximation above the cross-section subdomain, since congruency and equilibrium conditions can be easily satisfied.

For the sake of simplicity, the case $N_{h}=N_{k}=N$ is here considered and $N$ is referred as the expansion order of the model. It means that the lattice of Lagrange points over the natural coordinate system is equally distributed along the coordinates $r$ and $s$. This assumption simplifies Eq. 2.37 as follows:

$$
\begin{equation*}
F_{\tau}(r, s)=F_{h}(r) F_{k}(s) \quad h, k=0, \ldots, N \quad \tau=1, \ldots, N_{u}=(N+1)^{2} \tag{2.41}
\end{equation*}
$$

According to Table 2.3, in the case $N=1$ four Lagrange points in the natural coordinate system are taken into account and coincident to the square corner points as shown in Fig. 2.6(a). Lagrange polynomials of Eq. 2.41 become:

$$
\begin{equation*}
F_{\tau}(r, s)=\frac{1}{4}\left(1+r r_{\tau}\right)\left(1+s s_{\tau}\right) \quad \tau=1,2,3,4 \tag{2.42}
\end{equation*}
$$

As can be shown in Fig.2.7, for $N=1$ the cross-section functions $F_{\tau}$ have first-order approximation with respect to $r$ and first-order of approximation to $s$.


Figure 2.7: Higher-order Lagrange polynomials for $N_{h}=N_{k}=N=1$.


Figure 2.8: Higher-order Lagrange polynomials for $N_{h}=N_{k}=N=2$.

When the expansion order $N$ increses, higher-order Lagrange polynomials are considered
to describe the cross-section deformation. In the sample case of $N=2$ model, the crosssection functions of Eq. 2.41 become:

$$
\begin{gather*}
F_{\tau}(r, s)=\frac{1}{4}\left(r^{2}+r r_{\tau}\right)\left(s^{2}+s s_{\tau}\right) \quad \tau=1,3,5,7 \\
F_{\tau}(r, s)=\frac{1}{2} r_{\tau}^{2}\left(r^{2}+r r_{\tau}\right)\left(1-s^{2}\right)+\frac{1}{2} s_{\tau}^{2}\left(1-r^{2}\right)\left(s^{2}+s s_{\tau}\right) \quad \tau=2,4,6,8 \\
F_{\tau}(r, s)=\left(1-r^{2}\right)\left(1-s^{2}\right) \quad \tau=9 \tag{2.43}
\end{gather*}
$$

where the natural coordinates $\left(r_{\tau}, s_{\tau}\right)$ of the $\tau^{\text {th }}$ Lagrange point refers to the coordinates $r_{h}$ and $s_{k}$ presented in Table 2.3. For $N=2$, the axiomatic structural model (Eq. 2.34) interpolates the cross-section deformation as a linear combination of the polynomial functions depicted in Fig. 2.8 and the coefficients (weights) of this combination are the unknown values of displacement at the nine Lagrange points in Fig. 2.6(b).

The present discussion about the choice of Lagrange polynomials as cross-section functions has been reported here for the sake of completeness. In fact, the structural, aeroelastic, and fluid dynamics formulations described in the following chapters base on Maclaurin polynomials instead of Lagrange polynomials. Nonetheless, these formulations can be easily extended to the Lagrange polynomials case and more details about their use for one-dimensional CUF approximation can be found in [103].

## Chapter 3

## 1D CUF Finite Element Formulation

This chapter focuses on the solution of the mathematical equations governing the structural problem and obtained by means of the Principle of Virtual Displacements (PVD), which is a widely used variational statement in solid mechanics.

In past research works, closed form analytical solutions of the structural problem introduced in this chapter were obtained and more details about these exact solutions can be found in [44]. Unfortunately, analytical solutions are only possible for a few particular cases which frequently represent coarse simplifications of reality. For this reason, in the present work a numerical method is employed in order to provide a solution, in form of a set of numbers, to the mathematical governing equations. The aim is to transform the mathematical expressions into a set of algebraic equations which depend on a finite set of parameters. In particular, the Finite Element Method (FEM) is described in this chapter and the typical finite element matrices are computed according to Carrera Unified Formulation (CUF) introduced in section 2.4.

### 3.1 The Finite Element Method for 1D CUF

A finite element can be visualized as a small portion of a continuum. The term "finite" distinguishes such a portion from the infinitesimal elements of differential calculus. The geometry of the continuum is considered to be represented by the assembly of a collection of non-overlapping domains with simple geometry referred as finite elements. It is usually said that a mesh of finite elements discretizes the continuum [104].

The principal idea of the displacement-based finite element formulation is to express the element displacements at any point as a combination of the element nodal displacements through the use of interpolation functions (also called shape functions). Moreover, as reported in [42], the interpolation of the element coordinates and element displacements using the same shape functions, which are defined in a natural coordinate system, is the basis of the isoparametric finite element displacement-based formulation.

In the present doctoral research, isoparametric one-dimensional (1D) finite elements are developed in order to approximate the displacement field along the direction out of the cross-section plane ( $\mathbf{u}_{\tau}(y ; t)$ in Eq. 2.34). Let the generic 1D finite element to be introduced in Fig. 3.1. The number of nodes of the finite element $N_{N}$ can be arbitrary and defines the type of the 1D element. For any type of 1D finite element here developed, Node 1 lies on the left-hand terminal point of the element, whereas Node 2 lies on the right-hand
terminal point, exactly as depicted in Fig. 3.1.


Figure 3.1: Natural coordinate system for the generic one-dimensional finite element.
Let the natural axis $r$ to be ranged so that $-1 \leq r \leq 1$, see Fig. 3.1. The coordinate transformation from the physical coordinate $y$ along the finite element to the natural coordinate $r$ is:

$$
\begin{equation*}
r=\frac{\left(y-y_{1}\right)+\left(y-y_{2}\right)}{\left(y_{2}-y_{1}\right)}=\frac{2\left(y-y_{1}\right)-L_{\mathrm{EL}}}{L_{\mathrm{EL}}} \tag{3.1}
\end{equation*}
$$

where $y_{1}$ and $y_{2}$ are the physical coordinate of Nodes 1 and 2 of the finite element, respectively. The length of the element $L_{\mathrm{EL}}$ is equal to $\left(y_{2}-y_{1}\right)$. For an arbitrary number of nodes $N_{N}$, let the natural and physical coordinates of the generic $i^{\text {th }}$ node to be indicated as $r_{i}$ and $y_{i}$, respectively.

In finite element modeling, the physical coordinate at any point of the element is interpolated as a linear combination of the physical coordinates of the $N_{N}$ element nodes through the shape functions $N_{i}$ :

$$
\begin{equation*}
y=N_{i}(r) y_{i} \quad i=1, \ldots, N_{N} \tag{3.2}
\end{equation*}
$$

where repeated subscript $i$ indicates summation based on Einstein's notation. As mentioned before, the shape functions are defined in the natural coordinate system $r$ and this fact is highlighted in Eq. 3.2. In order to be consistent with the fact that Eq. 3.2 has to become $y=y_{i}$ when $y=y_{i}$, the $i^{t h}$ shape function $N_{i}$ must have the feature to be equal to 1 on the corresponding $i^{t h}$ node and equal to 0 on all the other nodes:

$$
N_{i}\left(r_{\lambda}\right)= \begin{cases}1 & \lambda=i  \tag{3.3}\\ 0 & \lambda \neq i\end{cases}
$$

As far as the choice of shape functions is concerned, a rather natural class of functions to be used for approximating element displacements are polynomials because they are commonly employed to approximate unknown functions, and the higher the degree of the polynomial, the better the approximation that is expected. In addition, polynomials are easy to differentiate; i.e., if the polynomials approximate the displacements of the structure, the strains can be evaluated with relative ease. For this reason, in this case the class of 1D Lagrange polynomials represents an appropriate candidate as shape functions since they satisfy the condition in Eq. 3.3. Hence:

$$
\begin{equation*}
N_{i}(r)=\prod_{\xi=1, \xi \neq i}^{N_{N}} \frac{\left(r-r_{\xi}\right)}{\left(r_{i}-r_{\xi}\right)} \tag{3.4}
\end{equation*}
$$

Equation 3.4 is consistent with Eq. 2.38, which has been introduced to define multivariate Lagrange polynomials. The natural coordinates of 1D element nodes thus coincide with the natural coordinates of 1D Lagrange points, according to the discussion in section 2.4.2.

As mentioned, in general the number of nodes $N_{N}$ is arbitrary and defines the type of the element. In fact, the order of the shape functions of an element is equal to $\left(N_{N}-1\right)$. In the present dissertation, finite elements with a number of nodes $N_{N}$ equal to 2,3 , and 4 are formulated in the present chapter and named $B 2, B 3$, and $B 4$, respectively. The values of the natural coordinates $r_{i}$ are reported in Table 3.1 for these types of finite elements. The 1D Lagrange polynomials used as shape functions for $B 2, B 3$, and $B 4$ elements are separately computed in the following sections.

Table 3.1: Natural coordinates of nodes for $B 2, B 3$, and $B 4$ one-dimensional finite elements.

| Element type | $N_{N}$ | $i$ | $r_{i}$ |
| :---: | :---: | ---: | ---: |
| $B 2$ | 2 | 1 | -1 |
|  |  | 2 | 1 |
|  |  | 1 | -1 |
| $B 3$ | 3 | 2 | 0 |
|  |  | 3 | 1 |
|  |  | 1 | -1 |
| $B 4$ | 4 | 2 | $-1 / 3$ |
|  | 3 | $1 / 3$ |  |
|  |  | 4 | 1 |

### 3.1.1 B2 Element

The 1D finite element with two nodes ( $N_{N}=2$ ) is referred as $B 2$ element, see Fig. 3.2. The order of each shape function is therefore $\left(N_{N}-1\right)=1$. Here it is proved that first-order Lagrange polynomials (based on two Lagrange points) are precisely the class of first-order polynomials which satisfy the fundamental conditions for shape functions in Eq. 3.3.


Figure 3.2: B2 element (2 nodes) in the natural coordinate system.
In order to satisfy the second condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=0\right.$ for $\left.\lambda \neq i\right)$, the shape functions for $B 2$ element are:

$$
\begin{align*}
& N_{1}=c_{21}(r-1) \\
& N_{2}=c_{22}(r+1) \tag{3.5}
\end{align*}
$$

where $c_{21}$ and $c_{22}$ can be arbitrary coefficients. In order to satisfy also the first condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=1\right.$ for $\left.\lambda=i\right)$, it can be easily verified that these two coefficients must be as follows:

$$
\begin{equation*}
c_{21}=-\frac{1}{2} \quad c_{22}=+\frac{1}{2} \tag{3.6}
\end{equation*}
$$

Finally, substituting Eq. 3.6 into Eq. 3.5, the shape functions for $B 2$ element exactly correspond to Lagrange polynomials obtained by Eq. 3.4 for $N_{N}=2$ and are depicted in

Fig. 3.3:

$$
\begin{align*}
& N_{1}=-\frac{1}{2} r+\frac{1}{2} \\
& N_{2}=+\frac{1}{2} r+\frac{1}{2} \tag{3.7}
\end{align*}
$$



Figure 3.3: 1D Lagrange polynomials as shape functions of $B 2$ element.

### 3.1.2 B3 Element

The 1D finite element with three nodes $\left(N_{N}=3\right)$ is referred as $B 3$ element, see Fig. 3.4. The order of each shape function is therefore $\left(N_{N}-1\right)=2$. Here it is proved that secondorder Lagrange polynomials (based on three Lagrange points) are precisely the class of second-order polynomials which satisfy the fundamental conditions for shape functions in Eq. 3.3.


Figure 3.4: B3 element (3 nodes) in the natural coordinate system.
In order to satisfy the second condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=0\right.$ for $\left.\lambda \neq i\right)$, the shape functions for $B 3$ element are:

$$
\begin{align*}
& N_{1}=c_{31} r(r-1) \\
& N_{2}=c_{32} r(r+1)  \tag{3.8}\\
& N_{3}=c_{33}(r+1)(r-1)
\end{align*}
$$

where $c_{31}, c_{32}$, and $c_{33}$ can be arbitrary coefficients. In order to satisfy also the first condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=1\right.$ for $\left.\lambda=i\right)$, it can be easily verified that these three coefficients must be as follows:

$$
\begin{equation*}
c_{31}=\frac{1}{2} \quad c_{32}=\frac{1}{2} \quad c_{33}=-1 \tag{3.9}
\end{equation*}
$$

Finally, substituting Eq. 3.9 into Eq. 3.8, the shape functions for $B 3$ element exactly correspond to Lagrange polynomials obtained by Eq. 3.4 for $N_{N}=3$ and are depicted in Fig. 3.5:

$$
\begin{align*}
& N_{1}=+\frac{1}{2} r^{2}-\frac{1}{2} r \\
& N_{2}=+\frac{1}{2} r^{2}+\frac{1}{2} r  \tag{3.10}\\
& N_{3}=-r^{2}+1
\end{align*}
$$



Figure 3.5: 1D Lagrange polynomials as shape functions of $B 3$ element.

### 3.1.3 B4 Element

The 1D finite element with four nodes $\left(N_{N}=4\right)$ is referred as $B 4$ element, see Fig. 3.6. The order of each shape function is therefore $\left(N_{N}-1\right)=3$. Here it is proved that third-order Lagrange polynomials (based on four Lagrange points) are precisely the class of third-order polynomials which satisfy the fundamental conditions for shape functions in Eq. 3.3.


Figure 3.6: $B 4$ element (4 nodes) in the natural coordinate system.

In order to satisfy the second condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=0\right.$ for $\left.\lambda \neq i\right)$, the shape
functions for $B 4$ element are:

$$
\begin{align*}
& N_{1}=c_{41}\left(r+\frac{1}{3}\right)\left(r-\frac{1}{3}\right)(r-1) \\
& N_{2}=c_{42}(r+1)\left(r+\frac{1}{3}\right)\left(r-\frac{1}{3}\right)  \tag{3.11}\\
& N_{3}=c_{43}(r+1)\left(r-\frac{1}{3}\right)(r-1) \\
& N_{4}=c_{44}(r+1)\left(r+\frac{1}{3}\right)(r-1)
\end{align*}
$$

where $c_{41}, c_{42}, c_{43}$, and $c_{44}$ can be arbitrary coefficients. In order to satisfy also the first condition of Eq. $3.3\left(N_{i}\left(r_{\lambda}\right)=1\right.$ for $\left.\lambda=i\right)$, it can be easily verified that these four coefficients must be as follows:

$$
\begin{equation*}
c_{41}=-\frac{9}{16} \quad c_{42}=+\frac{9}{16} \quad c_{43}=+\frac{27}{16} \quad c_{44}=-\frac{27}{16} \tag{3.12}
\end{equation*}
$$

Finally, substituting Eq. 3.12 into Eq. 3.11, the shape functions for $B 4$ element exactly correspond to Lagrange polynomials obtained by Eq. 3.4 for $N_{N}=4$ and are depicted in Fig. 3.7:


$$
N_{1}=-\frac{9}{16} r^{3}+\frac{9}{16} r^{2}+\frac{1}{16} r-\frac{1}{16}
$$

$$
\begin{equation*}
N_{2}=+\frac{9}{16} r^{3}+\frac{9}{16} r^{2}-\frac{1}{16} r-\frac{1}{16} \tag{3.13}
\end{equation*}
$$

$$
N_{3}=+\frac{27}{16} r^{3}-\frac{9}{16} r^{2}-\frac{27}{16} r+\frac{9}{16}
$$

$$
N_{4}=-\frac{27}{16} r^{3}-\frac{9}{16} r^{2}+\frac{27}{16} r+\frac{9}{16}
$$

Figure 3.7: 1D Lagrange polynomials as shape functions of $B 4$ element.

### 3.1.4 Variable kinematic 1D CUF FE model

As previously mentioned, the basic procedure in the isoparametric finite element displacementbased formulation is to express both the element coordinates and the element displacements
in the form of interpolations (shape functions) using the natural coordinate system of the element. The same shape functions $N_{i}$ introduced in the previous section for the element coordinate interpolation are therefore used to approximate the displacement unknowns. According to CUF formulation in Eq. 2.34 and finite element coordinates approximation in Eq. 3.2, generic displacements $\mathbf{u}_{\tau}$ lying on the beam axis are expressed as:

$$
\begin{equation*}
\mathbf{u}_{\tau}(y ; t)=N_{i}(y) \mathbf{q}_{\tau i}(t) \quad i=1, \ldots, N_{N} \tag{3.14}
\end{equation*}
$$

where again repeated subscript $i$ indicates summation based on Einstein's notation. The generic nodal displacement vector $\mathbf{q}_{\tau i}$ contains the degrees of freedom of the generic $\tau^{\text {th }}$ expansion term corresponding to the $i^{t h}$ element node. The dimensions of this vector are $3 \times 1$ and its components are:

$$
\mathbf{q}_{\tau i}(t)=\left\{\begin{array}{l}
q_{u_{x} \tau i}(t)  \tag{3.15}\\
q_{u_{y} \tau i}(t) \\
q_{u_{z} \tau i}(t)
\end{array}\right\}
$$

Although the finite element formulation for displacements in Eq. 3.14 contains the term $N_{i}(y)$, i.e. shape functions referred to the physical coordinate $y$, these shape functions are exactly the same as those $\left(N_{i}(r)\right)$ used for element coordinates and referred to the natural coordinate $r$, according to the isoparametric FE formulation. A coordinate transformation is required. In fact:

$$
\begin{equation*}
N_{i}(y)=N_{i}(y(r)) \Rightarrow N_{i}(r) \tag{3.16}
\end{equation*}
$$

where the coordinate transformation from the natural coordinate $r$ to the physical coordinate $y$ is computed inverting Eq. 3.1:

$$
\begin{equation*}
y=\frac{y_{1}}{2}(1-r)+\frac{y_{2}}{2}(1+r)=y_{1}+\frac{L_{\mathrm{EL}}}{2}(1+r) \tag{3.17}
\end{equation*}
$$

On the contrary:

$$
\begin{equation*}
N_{i}(r)=N_{i}(r(y)) \Rightarrow N_{i}(y) \tag{3.18}
\end{equation*}
$$

where the relation between $r$ and $y$ is here retrieved (see Eq. 3.1):

$$
\begin{equation*}
r=\frac{\left(y-y_{1}\right)+\left(y-y_{2}\right)}{\left(y_{2}-y_{1}\right)}=\frac{2\left(y-y_{1}\right)-L_{\mathrm{EL}}}{L_{\mathrm{EL}}} \tag{3.19}
\end{equation*}
$$

The expression of shape functions $N_{i}(y)$ in terms of the physical coordinate $y$ can be easily obtained combining Eqs. 3.4 and 3.19. Nonetheless, this procedure is here disregarder for the sake of brevity. As it will be clear in appendix C, the above coordinate transformations have to be taken into account in the integration of shape functions over the finite element domain.

Combining the finite element approximation in Eq. 3.14 and CUF formulation in Eq. 2.34, the displacement field described by the present one-dimensional models becomes:

$$
\mathbf{u}(x, y, z ; t)=F_{\tau}(x, z) N_{i}(y) \mathbf{q}_{\tau i}(t) \quad \begin{align*}
& \tau=1, \ldots, N_{u}=N_{u}(N)  \tag{3.20}\\
& i=1, \ldots, N_{N}
\end{align*}
$$

Thanks to the hierarchical CUF approach, by means of Eq. 3.20 different higher-order 1D CUF finite element models with variable accuracy can be developed and employed to solve the structural problem which will be derived afterwards. In fact, the accuracy of the model is a free parameter of the analysis and depends directly on the expansion order $N$ and the
type of finite elements, i.e. $N_{N}$, selected. In this sense, the term variable kinematic model is adopted to define the present one-dimensional formulation.

At first, let a single finite element to be considered. Later, the typical finite element assembly procedure will be considered. $N_{u}$ generic nodal displacement vectors $\mathbf{q}_{\tau i}$ correspond to the generic $i^{\text {th }}$ node. Since the single element has $N_{N}$ nodes, thus ( $N_{u} \times N_{N}$ ) generic nodal displacement vectors $\mathbf{q}_{\tau i}$ correspond to a single element. The number of degrees of freedom related to the single element is referred as $\mathrm{DOFs}_{\mathrm{EL}}$ and its numerical value is:

$$
\begin{equation*}
\mathrm{DOFs}_{\mathrm{EL}}=3 N_{u} N_{N} \tag{3.21}
\end{equation*}
$$

since $\mathbf{q}_{\tau i}$ contains three components, see Eq. 3.15.
Following the standard finite element method, the structural domain is discretized by a mesh, which constitutes a series of connected (non-overlapping) one-dimensional finite elements. An example of two $B 3$ elements assembled is illustrated in Fig. 3.8. The connection of the finite elements is verified by the fact that Node $2^{\text {EL } 1}$ of Element 1 and Node $1^{\mathrm{EL} 2}$ of Element 2 are coincident. The numerical method which achieves this connection is the typical finite element assembly procedure, which is described in details in section 3.4.


Figure 3.8: Example of two $B 3$ finite elements assembled (Node $2^{\text {EL } 1} \equiv$ Node $1^{\text {EL } 2}$ ).
The number of finite elements the mesh is made of is indicated as $N_{\text {EL }}$. Given a mesh of $N_{\mathrm{EL}}$ connected elements with $N_{N}$ nodes per element, the total number of nodes $N_{N \text { TOT }}$ of the FE mesh is computed as:

$$
\begin{equation*}
N_{N \text { TOT }}=\left(N_{N}-1\right) N_{\mathrm{EL}}+1 \tag{3.22}
\end{equation*}
$$

This formula takes into account the fact that two connected finite elements share the same node. From Eq. 3.22 it is easily verified that $N_{N \text { tot }}$ is equal to 5 for the sample case of two connected $B 3$ elements depicted in Fig. 3.8, where ( $N_{N}=2, N_{\text {EL }}=2$ ). All the $N_{u}$ generic nodal displacement vectors $\mathbf{q}_{\tau i}$ for each of the $N_{N \text { тот }}$ nodes of the mesh will be collected in the nodal displacement vector $\mathbf{q}$ for all values of $\tau\left(\tau=1, \ldots, N_{u}\right)$ and $i$ $\left(i=1, \ldots, N_{N}\right)$ of each element, see section 3.3. As a consequence, vector $\mathbf{q}$ contains all the nodal degrees of freedom of the present structural model, which are commonly referred as DOFs. Its dimensions are $\left(3 N_{u} N_{N \text { тот }}\right) \times 1$, therefore:

$$
\begin{equation*}
\mathrm{DOFs}=\mathrm{DOFs}_{N} N_{N \text { тот }}=3 N_{u} N_{N \text { тот }} \tag{3.23}
\end{equation*}
$$

where $\mathrm{DOFs}_{N}$ is the number of degrees of freedom related to the single element node. It is reminded that the number of expansion terms $N_{u}$ depends on the class of polynomial functions chosen as cross-section functions $F_{\tau}$, as discussed in section 2.4. In particular, $N_{u}$ depends directly on the expansion order $N$ according to Eq. 2.35 for Maclaurin polynomials and Eq. 2.37 for Lagrange polynomials. As mentioned at the end of section 2.4.2, the formulation developed in the present dissertation bases on the choice of Maclaurin
polynomials instead of Lagrange polynomials. Hence, the total number of degrees of freedom of the model, in case of Maclaurin polynomials, is:

$$
\begin{equation*}
\mathrm{DOFs}=3 N_{u} N_{N \text { TOT }}=3 \frac{(N+1)(N+2)}{2}\left[\left(N_{N}-1\right) N_{\mathrm{EL}}+1\right] \tag{3.24}
\end{equation*}
$$

In conclusion, it is important to note that DOFs depend on the expansion order $N$, the finite element type used in the discretization $\left(N_{N}\right)$, and the number of finite elements in the mesh used for the model discretization.

From here on, Maclaurin polynomials will considered as cross-section functions $F_{\tau}$. As far as the number of DOFs is concerned, some example are presented for the sake of completeness. $N=1$ model involves 3 unknowns for each displacement component $u_{x}, u_{y}$, $u_{z}$ and then 9 degrees of freedom per node $\mathrm{DOFs}_{N}$, since:

$$
\begin{equation*}
\mathrm{DOFs}_{N}=3 N_{u}=3 \frac{(N+1)(N+2)}{2} \tag{3.25}
\end{equation*}
$$

Following Eq. 3.25, the fourth-order model involves 15 unknowns per displacement component and hence 45 DOFs per node. This is in consistent with Table 2.2 and Fig. 2.5. As a consequence, for the sample case of two connected $B 3$ elements $\left(N_{N \text { тот }}=5\right)$ depicted in Fig. 3.8 the number of degrees of freedom DOFs is equal to $9 \times 5=45$ for $N=1$ and $45 \times 5=225$ for $N=4$.

Using 1D CUF FE approximation in Eq. 3.20, the strain vectors, which in Eq. 2.9 are expressed in a compact vectorial notation in terms of the displacement unknowns, can be directly related to the nodal displacement vector $\mathbf{q}_{\tau i}$ :

$$
\left\{\begin{array}{l}
\varepsilon_{n}=\left(\mathbf{D}_{n p} F_{\tau} \mathbf{I}\right) N_{i} \mathbf{q}_{\tau i}+F_{\tau}\left(\mathbf{D}_{n y} N_{i} \mathbf{I}\right) \mathbf{q}_{\tau i}  \tag{3.26}\\
\varepsilon_{p}=\left(\mathbf{D}_{p} F_{\tau} \mathbf{I}\right) N_{i} \mathbf{q}_{\tau i}
\end{array}\right.
$$

where $\mathbf{I}$ is the identity matrix. Parentheses are introduced to highlight the effect of the differential matrix operators. Cross-section functions $F_{\tau}$ are independent of $y$ coordinate and shape functions $N_{i}$ do not depend on $x$ and $z$ coordinates. As a consequence, the differential matrix operators $\mathbf{D}_{p}$ and $\mathbf{D}_{n p}$ have effect only on $F_{\tau}$, whereas $\mathbf{D}_{n y}$ has effect only on $N_{i}$.

### 3.2 Principle of Virtual Displacements for 1D CUF

In order to introduce the governing equations of the elasticity problem, a brief overview on the three alternative forms of a mathematical model is here presented. For more details, a very clear and comprehensive introduction to variational problems is made by Felippa [105] and Bathe [42].

### 3.2.1 Strong, weak and variational forms in elasticity

A mathematical model can be described in three alternative forms.
The strong form (SF) of a mathematical problem is presented as a system of ordinary or partial differential equations in space and/or time, complemented by appropriate boundary conditions. Occasionally this form may be presented in integro-differential form, or reduce to algebraic equations.

The weak form (WF) is presented as a weighted integral equation that "relaxes" the strong form into a domain-averaging statement.

The variational form (VF) is presented as a functional whose stationary conditions generate the weak and strong forms.

It is important to note that the above three forms of the analysis problem are totally equivalent. Standard Variational Calculus (SVC) and Extended Variational Calculus (EVC) comprise a set of rules and techniques by which it is possible to pass from one of these forms to another.

From the strong form, several weak forms can be obtained by selectively "weakening" strong connections. As a consequence, the space of functions used for the solution of the weak form is larger than the space of functions used for the solution of the strong form. In the weak form, a residual function associated with the strong form is introduced. For the exact solution of the governing problem, the residual function is equal to zero. The weighted residual form is obtained multiplying the residual function by weight functions, integrating over the mathematical domain and taking into account the contributions of the boundary conditions. If the weighted functions are written as variations of arbitrary functions called test functions, a variational statement is obtained from the weighted residual form. The concept of variation is described afterwards. Homogenizating the variations of the variational statement and through a further integration, the variational form is achieved and the transformation from the weak form to the variational form of the mathematical model is completed. Similarly, it is possible to pass from the weak form to the strong form enforcing all relations pointwise [105].

As far as the analysis of solids and structures is concerned, several functional have been defined in literature during the last century of research. Among the others, the Total Potential Energy $\Pi$ is introduced as functional of the variational formulation. The definition of $\Pi$ is:

$$
\begin{equation*}
\Pi=U-W \tag{3.27}
\end{equation*}
$$

where $U$ is the total internal energy and $W$ is the external energy due to applied mechanical loads. The Total Potential Energy depends only on the displacement $\mathbf{u}$, which in variational calculus is called the primary variable of the functional. It is noteworthy that only the primary variable(s) of a functional may be varied. Since the concept of variation is fundamental in variational calculus, it is introduced here and referred with symbol $\delta$. Suppose that $\mathbf{u}(x, y, z ; t)$ is changed to $\mathbf{u}(x, y, z ; t)+\delta \mathbf{u}(x, y, z ; t)$. The variation $\delta \mathbf{u}(x, y, z ; t)$ should not be confused with the ordinary differential du $(x, y, z ; t)$ since on taking the variation the independent variables $(x, y, z ; t)$ are frozen; that is, $\delta x=0, \delta y=0$, $\delta z=0, \delta t=0$. The functional changes from $\Pi$ to $\Pi+\delta \Pi$. A displacement variation $\delta \mathbf{u}(x, y, z ; t)$ is said to be admissible when both $\mathbf{u}(x, y, z ; t)$ and $\mathbf{u}(x, y, z ; t)+\delta \mathbf{u}(x, y, z ; t)$ are kinematically admissible. A kinematically displacement is admissible when it is continuous and satisfies any displacement boundary condition.

Once the functional Total Potential Energy $\Pi$ has been defined as functional, the Mimimum Potential Energy principle of classical elasticity states that the actual displacement solution $\mathbf{u}^{\star}(x, y, z ; t)$ that satisfies the governing equations is that which makes $\Pi$ stationary:

$$
\begin{equation*}
\delta \Pi=\delta U-\delta W=0 \quad \text { if and only if } \mathbf{u}=\mathbf{u}^{\star} \tag{3.28}
\end{equation*}
$$

with respect to admissible variations $\mathbf{u}=\mathbf{u}^{\star}+\delta \mathbf{u}$ of the exact displacement field $\mathbf{u}^{\star}$. This stationary equation leads to the Principle of Virtual Displacements (PVD) (also called Principle of Virtual Work), which is the weak form of the Minimum Potential Energy principle. In particular, PVD is the variational statement where the weight functions have the meaning of "virtual" variations of displacement $\delta \mathbf{u}(x, y, z ; t)$, also known as virtual displacements. The Principle of Virtual Displacements states that the equilibrium of the
body requires that for any admissible virtual displacements imposed on the body in its state of equilibrium, the internal virtual work is equal to the external virtual work [42]:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta L_{\mathrm{ext}} \tag{3.29}
\end{equation*}
$$

where $L_{\text {int }}$ is the internal work (also called strain energy) and $L_{\text {ext }}$ is the external work due to external loads. The external work can be split into the contributions of the work carried out by the point loadings $L_{P}$, the work carried out by line loadings $L_{l}$, the work carried out by surface loadings $L_{s}$ and the work carried out by volume loadings $L_{V}$. These four terms constitute the work of applied loadings $L_{\mathrm{al}}$. In addition to these terms, using the d'Alembert principle, the inertia forces can be simply included as part of the body forces [42]. The contribution of virtual work of inertial loadings $L_{\text {ine }}$ is thus taken into account in the external virtual work, in addition to $L_{\text {al }}$. Eq. 3.29 becomes:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta L_{\mathrm{al}}-\delta L_{\mathrm{ine}}=\delta L_{P}+\delta L_{l}+\delta L_{s}+\delta L_{V}-\delta L_{\mathrm{ine}} \tag{3.30}
\end{equation*}
$$

The Principle of Virtual Displacements (Eqs. 3.29 and 3.30) is equivalent to the Mimimum Potential Energy principle and, referring to Eq. 3.28, establishes that any displacement allowed by the equilibrium configuration causes an increment of the system's Total Potential Energy. In fact, homogenizing the variations and integrating the PVD, the Total Potential Energy functional can be obtained. The Principle of Virtual Displacements can be also derived from the differential equations of equilibrium (strong form), see Washitzu [106] and Felippa [107].

For the sake of completeness, alternative weak forms of the structural problem could be derived by defining functionals different from the Total Potential Energy. For instance, this is the case of Hellinger-Reissner (HR) stress-displacement functional. The weak form of the Hellinger-Reissner principle of classical elasticity is often used for mixed formulations, whereas the Principle of Virtual Displacements is suitable for displacementbased formulations. Several other functionals have been developed in classical linear elastostatics literature such as Complementalry Energy (CE), Strain-Displacement Reissner (SDR), or Hu-Washitzu (HW) functionals. These functionals are defined as canonical functionals according to Oden and Reddy [108, 109].

The element matrices corresponding to the required degrees of freedom of the structural problem are directly obtained through the Principle of Virtual Displacements in Eq. 3.30. The contribution of each term to the governing equations is separately considered in the following sections leading to the construction of the Structural Stiffness Matrix, the Mass Matrix and the Vector of Equivalent Nodal Forces. It is important to note that, for the sake of simplicity, the formulation of matrices and vectors here presented corresponds to the local degrees of freedom of the single finite element, since Eq. 3.20 (1D CUF FE approximation) refers to the single finite element. Additional considerations will be necessary in the finite element assembly procedure (see sections 3.4.1, 3.4.2 and 3.4.3) and when the elements are arbitrarily oriented in the three-dimensional space, as will be clear in section 9.3.

### 3.2.2 Strain Energy

The first term of Eq. 3.30 is the virtual strain energy $\delta L_{\text {int }}$ (also known as internal virtual work). Following the classical procedure in [106] or [107] to obtain Principle of Virtual Displacements starting from the equilibrium differential equations, the expression of $\delta L_{\mathrm{int}}$ depends on stress and virtual strain components:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\int_{V}\left(\delta \varepsilon_{x x} \sigma_{x x}+\delta \varepsilon_{y y} \sigma_{y y}+\delta \varepsilon_{z z} \sigma_{z z}+\delta \gamma_{x y} \sigma_{x y}+\delta \gamma_{x z} \sigma_{x z}+\delta \gamma_{y z} \sigma_{y z}\right) \mathrm{d} V \tag{3.31}
\end{equation*}
$$

where the virtual strains corresponds to the components of virtual displacements as follows:

$$
\begin{array}{ll}
\delta \varepsilon_{x x}=\frac{\partial \delta u_{x}}{\partial x} & \delta \gamma_{x y}=2 \delta \varepsilon_{x y}=\frac{\partial \delta u_{x}}{\partial y}+\frac{\partial \delta u_{y}}{\partial x} \\
\delta \varepsilon_{y y}=\frac{\partial \delta u_{y}}{\partial y} & \delta \gamma_{x z}=2 \delta \varepsilon_{x z}=\frac{\partial \delta u_{x}}{\partial z}+\frac{\partial \delta u_{z}}{\partial x}  \tag{3.32}\\
\delta \varepsilon_{z z}=\frac{\partial \delta u_{z}}{\partial z} & \delta \gamma_{y z}=2 \delta \varepsilon_{y z}=\frac{\partial \delta u_{y}}{\partial z}+\frac{\partial \delta u_{z}}{\partial y}
\end{array}
$$

The components of stress vectors in Eq. 3.31 represent stresses in equilibrium with the applied loads which contribute to the external virtual work term. In other words, these stress components are the unique stresses that exacty balance the applied loads. Referring to the stress and strain vectors defined in Eqs. 2.5 and 2.6, the virtual strain energy can be considered as the summation of two contributes:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\int_{V}\left(\delta \varepsilon_{n}^{T} \boldsymbol{\sigma}_{n}+\delta \varepsilon_{p}^{T} \boldsymbol{\sigma}_{p}\right) \mathrm{d} V=\int_{l} \int_{\Omega}\left(\delta \varepsilon_{n}^{T} \boldsymbol{\sigma}_{n}+\delta \varepsilon_{p}^{T} \boldsymbol{\sigma}_{p}\right) \mathrm{d} \Omega \mathrm{~d} y \tag{3.33}
\end{equation*}
$$

As mentioned in the previous section, it is reminded that the present procedure has to be followed to obtain the finite element matrices related to a single finite element; i.e. here the assembly procedure is not yet considered. As a consequence, the integration in Eq. 3.33 is performed over the volume corresponding to the domain of a single finite element. By definition, the cross-section related to the single one-dimensional finite element is considered to be constant over the element length. Therefore, the integral over the volume in Eq. 3.33 and in the following equations is split into the integral over the cross-section and the integral along the axis of the one-dimensional finite element, which has length $L_{\mathrm{EL}}$. For the sake of completeness, it is notified that the integration in Eq. 3.31 is performed over the original volume of the body, unaffected by the imposed virtual displacements.

By means of PVD, the governing equations are obtained in terms of the displacement vector components and their derivatives. The aim of the finite element method is to express the governing equations in terms of nodal displacement unknowns. For this purpose, constitutive equations are involved to express the stress components in terms of strain components and then geometrical relations are considered in order to express strains (and virtual strains) in terms of displacements (and virtual displacements).

The relation between the vectors of virtual strains and the vector of virtual nodal displacements for 1D CUF FE model is computed introducing the virtual variation $\delta$ in Eq. 3.26:

$$
\left\{\begin{array}{l}
\delta \varepsilon_{n}=\left(\mathbf{D}_{n p} F_{\tau} \mathbf{I}\right) N_{i} \delta \mathbf{q}_{\tau i}+F_{\tau}\left(\mathbf{D}_{n y} N_{i} \mathbf{I}\right) \delta \mathbf{q}_{\tau i}  \tag{3.34}\\
\delta \varepsilon_{p}=\left(\mathbf{D}_{p} F_{\tau} \mathbf{I}\right) N_{i} \delta \mathbf{q}_{\tau i}
\end{array}\right.
$$

For the sake generality, the procedure is presented here for an orthotropic material. The constitutive equations and material stiffness matrices described in Eqs. 2.27 and 2.28 (see section 2.3.1) are thus involved here. The isotropic material case can be easily derived by involving Eqs. 2.32 and 2.33 of section 2.3.2. In fact, as can be seen comparing the material stiffness matrices of orthotropic and isotropic cases, isotropic matrices can be seen as a particular case of the orthotropic ones. Substituting Eq. 3.34 into Eq. 3.33, and employing the property of transponse for the product of matrices (the factors reverse):

$$
\begin{align*}
& {\left[\left(\mathbf{D}_{n p} F_{\tau} \mathbf{I}\right) N_{i} \delta \mathbf{q}_{\tau i}\right]^{T}=\delta \mathbf{q}_{\tau i}^{T} N_{i}\left(\mathbf{I}^{T} \mathbf{D}_{n p}^{T} F_{\tau}\right)=\delta \mathbf{q}_{\tau i}^{T} N_{i}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right)}  \tag{3.35}\\
& {\left[F_{\tau}\left(\mathbf{D}_{n y} N_{i} \mathbf{I}\right) \delta \mathbf{q}_{\tau i}\right]^{T}=\delta \mathbf{q}_{\tau i}^{T}\left(\mathbf{I}^{T} \mathbf{D}_{n y}^{T} N_{i}\right) F_{\tau}=\delta \mathbf{q}_{\tau i}^{T}\left(\mathbf{D}_{n y}^{T} N_{i} \mathbf{I}\right) F_{\tau}} \tag{3.36}
\end{align*}
$$

$$
\begin{equation*}
\left[\left(\mathbf{D}_{p} F_{\tau} \mathbf{I}\right) N_{i} \delta \mathbf{q}_{\tau i}\right]^{T}=\delta \mathbf{q}_{\tau i}^{T} N_{i}\left(\mathbf{I}^{T} \mathbf{D}_{p}^{T} F_{\tau}\right)=\delta \mathbf{q}_{\tau i}^{T} N_{i}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \tag{3.37}
\end{equation*}
$$

the two terms of the virtual strain energy become:

$$
\begin{align*}
\delta L_{\mathrm{int}}= & \delta \mathbf{q}_{\tau i}^{T} \int_{l} \int_{\Omega}\left\{[ N _ { i } ( \mathbf { D } _ { n p } ^ { T } F _ { \tau } \mathbf { I } ) + ( \mathbf { D } _ { n y } ^ { T } N _ { i } \mathbf { I } ) F _ { \tau } ] \left[\widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) N_{j}+\right.\right. \\
& \left.\widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) N_{j}+\widetilde{\mathbf{C}}_{n n} F_{s}\left(\mathbf{D}_{n y} N_{j} \mathbf{I}\right)\right]+  \tag{3.38}\\
& N_{i}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right)\left[\widetilde{\mathbf{C}}_{p p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) N_{j}+\widetilde{\mathbf{C}}_{p n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) N_{j}+\right. \\
& \left.\left.\widetilde{\mathbf{C}}_{p n} F_{s}\left(\mathbf{D}_{n y} N_{j} \mathbf{I}\right)\right]\right\} \mathrm{d} \Omega \mathrm{~d} y \mathbf{q}_{s j}
\end{align*}
$$

The dependence of the virtual strain energy on the nodal displacement unknowns is clear in Eq. 3.38. The generic virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}$ contains the virtual degrees of freedom of the generic $\tau^{\text {th }}$ expansion term corresponding to the $i^{t h}$ element node. So indices $\tau$ and $i$ are used for the terms coming from the virtual strain vectors in Eq. 3.33. For the sake of clarity, different indices $s$ and $j$ are instead used for the terms coming from the stress vectors in Eq. 3.33, which are the actual stresses in equilibrium with the applied loads. In fact, the generic nodal displacement vector $\mathbf{q}_{s j}$ contains the actual degrees of freedom of the generic $s^{t h}$ expansion term corresponding to the $j^{t h}$ element node.

It is important to remind that the shape functions of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$, see Eq. 3.20. Hence, they can be taken out of the integral over the cross-section $\Omega$. Separating the single terms of Eq. 3.38, the virtual strain energy is split into nine contributions:

$$
\begin{align*}
\delta L_{\text {int }}= & \delta \mathbf{q}_{\tau i}^{T}\left\{\int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+\right. \\
& \int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+ \\
& \int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n n} F_{s} \mathrm{~d} \Omega\right] \mathbf{I}_{\Omega y} N_{j, y} \mathrm{~d} y+ \\
& \int_{l} N_{i, y} \mathbf{I}_{\Omega y}^{T}\left[\int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+ \\
& \int_{l} N_{i, y} \mathbf{I}_{\Omega y}^{T}\left[\int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+  \tag{3.39}\\
& \int_{l} N_{i, y} \mathbf{I}_{\Omega y}^{T}\left[\int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n n} F_{s} \mathrm{~d} \Omega\right] \mathbf{I}_{\Omega y} N_{j, y} \mathrm{~d} y+ \\
& \int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+ \\
& \int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega\right] N_{j} \mathrm{~d} y+ \\
& \left.\int_{l} N_{i}\left[\int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p n} F_{s} \mathrm{~d} \Omega\right] \mathbf{I}_{\Omega y} N_{j, y} \mathrm{~d} y\right\} \mathbf{q}_{s j}
\end{align*}
$$

where matrices $\mathbf{I}_{\Omega y}$ and $\mathbf{I}_{\Omega y}^{T}$ are introduced according to matrices $\mathbf{D}_{n y}$ and $\mathbf{D}_{n y}^{T}$ :

$$
\mathbf{I}_{\Omega y}=\left[\begin{array}{ccc}
0 & 0 & 1  \tag{3.40}\\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right] \quad \mathbf{I}_{\Omega y}^{T}=\left[\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right]
$$

As far as the material properties are concerned (stress components depend of course on material properties, see Eqs. 2.27 and 2.32 for orthotropic and isotropic materials, respectively), the material stiffness matrices expressed in Eqs. 2.28 and 2.33 are considered to be constant along the single finite element axis. This is a typical assumption used in finite element modeling. In case of possible variation of the material properties along the longitudinal axis of the structure, a refined mesh would be employed and, practically speaking, the material properties variation would be approximated as a piecewise constant function, with constant material properties in the single finite element domain.

By definition in Eq. 2.34, cross-section functions $F_{\tau}$ depend only on the cross-section coordinates. Their derivatives, through the differential matrix operators $\mathbf{D}_{p}$ and $\mathbf{D}_{n p}$, can be therefore taken out of the integral along the element length, as well as the material stiffeness matrices. For the reasons elucidated above, it is possible to split the integral along the element length and the integral over the element cross-section into two different contributions to be multiplied:

$$
\begin{align*}
\delta L_{\mathrm{int}}= & \delta \mathbf{q}_{\tau i}^{T}\left\{\int_{l} N_{i} N_{j} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+\right. \\
& \int_{l} N_{i} N_{j} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+ \\
& \int_{l} N_{i} N_{j, y} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n n} F_{s} \mathrm{~d} \Omega \mathbf{I}_{\Omega y}+ \\
& \int_{l} N_{i, y} N_{j} \mathrm{~d} y \mathbf{I}_{\Omega y}^{T} \int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+ \\
& \int_{l} N_{i, y} N_{j} \mathrm{~d} y \mathbf{I}_{\Omega y}^{T} \int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+  \tag{3.41}\\
& \int_{l} N_{i, y} N_{j, y} \mathrm{~d} y \mathbf{I}_{\Omega y}^{T} \int_{\Omega} F_{\tau} \widetilde{\mathbf{C}}_{n n} F_{s} \mathrm{~d} \Omega \mathbf{I}_{\Omega y}+ \\
& \int_{l} N_{i} N_{j} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+ \\
& \int_{l} N_{i} N_{j} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right) \mathrm{d} \Omega+ \\
& \left.\int_{l} N_{i} N_{j, y} \mathrm{~d} y \int_{\Omega}\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p n} F_{s} \mathrm{~d} \Omega \mathbf{I}_{\Omega y}\right\} \mathbf{q}_{s j}
\end{align*}
$$

The expression in curly brackets in Eq. 3.41 represents a matrix which is pre-multiplied by the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ and multiplied by the nodal displacement vector $\mathbf{q}_{s j}$. The virtual strain energy can be rewritten in a compact vectorial notation:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{K}^{\tau s i j} \mathbf{q}_{s j} \tag{3.42}
\end{equation*}
$$

where $\mathbf{K}^{\tau s i j}$ is the fundamental nucleus of the Structural Stiffness Matrix of the 1D CUF FE model. This nucleus is a matrix with dimensions $3 \times 3$ and, in some sense, it can be seen as the stiffness of the system in equilibrium with the applied loads for the generic $\tau^{\text {th }}$ expansion term of CUF formulation corresponding to the $i^{t h}$ node of the single finite element, see Eq. 3.20. The fundamental nucleus will be expanded with respect to the indices $\tau, s, i$, and $j$ in order to build the Structural Stiffness Matrix of the single finite element. The expression of the fundamental nucleus of the Structural Stiffness Matrix can
be rewritten in a more compact form:

$$
\begin{align*}
\mathbf{K}^{\tau s i j}= & E_{j}^{i} \triangleleft\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right)\left[\widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right)+\widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right)\right]+ \\
& \left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right)\left[\widetilde{\mathbf{C}}_{p p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right)+\widetilde{\mathbf{C}}_{p n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right)\right] \triangleright{ }_{\Omega}+ \\
& E_{j, y}^{i} \triangleleft\left[\left(\mathbf{D}_{n p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{n n}+\left(\mathbf{D}_{p}^{T} F_{\tau} \mathbf{I}\right) \widetilde{\mathbf{C}}_{p n}\right] F_{s} \triangleright{ }_{\Omega} \mathbf{I}_{\Omega y}+  \tag{3.43}\\
& E_{j}^{i, y} \mathbf{I}_{\Omega y}^{T} \triangleleft F_{\tau}\left[\widetilde{\mathbf{C}}_{n p}\left(\mathbf{D}_{p} F_{s} \mathbf{I}\right)+\widetilde{\mathbf{C}}_{n n}\left(\mathbf{D}_{n p} F_{s} \mathbf{I}\right)\right] \triangleright{ }_{\Omega}+ \\
& E_{j, y}^{i, y} \mathbf{I}_{\Omega y}^{T} \triangleleft F_{\tau} \widetilde{\mathbf{C}}_{n n} F_{s} \triangleright{ }_{\Omega} \mathbf{I}_{\Omega y}
\end{align*}
$$

where the integrals of the products of shape functions along the element length in Eq. 3.41 are collected in the following terms:

$$
\begin{array}{ll}
E_{j}^{i}=\int_{l} N_{i} N_{j} \mathrm{~d} y & E_{j, y}^{i}=\int_{l} N_{i} N_{j, y} \mathrm{~d} y \\
E_{j}^{i, y}=\int_{l} N_{i, y} N_{j} \mathrm{~d} y & E_{j, y}^{i, y}=\int_{l} N_{i, y} N_{j, y} \mathrm{~d} y \tag{3.44}
\end{array}
$$

These integrals along the element length can be performed through full, reduced or selective integration. More details can be found in appendix C. The symbol $\triangleleft \ldots \triangleright_{\Omega}$ in Eq. 3.43 indicates integration over the cross-section and is introduced for the sake of simplicity of notation:

$$
\begin{equation*}
\triangleleft \ldots \triangleright_{\Omega}=\int_{\Omega} \ldots \mathrm{d} \Omega \tag{3.45}
\end{equation*}
$$

In general, the integration over $\Omega$ can be performed numerically over an arbitrary crosssection. For classical cross-sections such as rectangular or circular sections, the integration can be performed analytically. For the sake of brevity, more details are not reported here, but can be found in [45]. Moreover, the same analytical procedure can be followed for cross-sections composed of rectangular or circular subsections, such as for instance T-shape, C-shape or layered circular sections.

The expressions of the components of $\mathbf{K}^{\tau s i j}$ will be explicitly computed in section 3.4.1. In that section, the fundamental nucleus will lead to the construction of the Structural Stiffness Matrix.

### 3.2.3 Work of Inertial Loadings

In this section, the virtual variation of the work of inertial loadings term $\delta L_{\text {ine }}$ in the Principle of Virtual Displacements (Eq. 3.30) is addressed. The expression of the virtual variation of the work of inertial loadings, typically abbreviated as virtual work of inertial loadings, is:

$$
\begin{equation*}
\delta L_{\text {ine }}=\int_{V} \delta \mathbf{u}^{T} \rho \ddot{\mathbf{u}} \mathrm{~d} V=\int_{l} \int_{\Omega} \delta \mathbf{u}^{T} \rho \ddot{\mathbf{u}} \mathrm{~d} \Omega \mathrm{~d} y \tag{3.46}
\end{equation*}
$$

where $\rho$ is the material density and $\ddot{\mathbf{u}}$ is the acceleration vector. As can be seen in Eq. 3.46, $\delta L_{\text {ine }}$ is directly written in terms of the virtual displacement unknowns $\delta \mathbf{u}$. As done for the strain energy (Eq. 3.33), the integration in Eq. 3.46 is performed over the volume corresponding to the domain of a single finite element. Assuming again the cross-section of the single one-dimensional finite element to be constant over the element length, the integral over the volume in Eq. 3.46 is split into the integral over the cross-section and the integral along the axis of the one-dimensional finite element, which has length $L_{\mathrm{EL}}$.

Retrieving the 1D CUF FE approximation of the displacement field (Eq. 3.20), the virtual displacement unknowns $\delta \mathbf{u}$ can be written in terms of the vector of virtual nodal displacements $\delta \mathbf{q}_{\tau i}$ :

$$
\delta \mathbf{u}(x, y, z ; t)=F_{\tau}(x, z) N_{i}(y) \delta \mathbf{q}_{\tau i}(t) \quad \begin{align*}
& \tau=1, \ldots, N_{u}=N_{u}(N)  \tag{3.47}\\
& i=1, \ldots, N_{N}
\end{align*}
$$

According to the definition of variation, the independent variables $(x, y, z ; t)$ are frozen; that is, $\delta x=0, \delta y=0, \delta z=0, \delta t=0$. Assuming that the accelerations are approximated in the same way as the displacements, the 1D CUF FE approximation (Eq. 3.20) is considered also to model the acceleration vector:

$$
\ddot{\mathbf{u}}(x, y, z ; t)=F_{\tau}(x, z) N_{i}(y) \ddot{\mathbf{q}}_{\tau i}(t) \quad \begin{align*}
& \tau=1, \ldots, N_{u}=N_{u}(N)  \tag{3.48}\\
& i=1, \ldots, N_{N}
\end{align*}
$$

where the generic nodal acceleration vector $\ddot{\mathbf{q}}_{\tau i}$ contains the acceleration components related to the generic $\tau^{t h}$ expansion term corresponding to the $i^{t h}$ element node. Substituting Eqs. 3.47 and 3.48 into Eq. $3.46, \delta L_{\text {ine }}$ can be rewritten in terms of virtual nodal displacements and nodal accelerations as follows:

$$
\begin{equation*}
\delta L_{\mathrm{ine}}=\delta \mathbf{q}_{\tau i}^{T}\left\{\int_{l} N_{i}\left[\int_{\Omega} \rho F_{\tau} F_{s} \mathrm{~d} \Omega \mathbf{I}\right] N_{j} \mathrm{~d} y\right\} \ddot{\mathbf{q}}_{s j} \tag{3.49}
\end{equation*}
$$

where the $3 \times 3$ identity matrix $\mathbf{I}$ is introduced in order to build later a $3 \times 3$ matrix to be considered as CUF nucleus. According to the procedure followed for the strain energy, indices $\tau$ and $i$ are used for the displacement unknowns in Eq. 3.33. Different indices $s$ and $j$ are instead used for the acceleration unknowns. It is important to remind that the shape functions of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$, see Eq. 3.20. Hence, in Eq. 3.49 the shape functions are taken out of the integral over the cross-section $\Omega$.

Coherently with the procedure in section 3.2 .2 , the material properties, are assumed to be constant along the single finite element axis. By definition in Eq. 2.34, cross-section functions $F_{\tau}$ depend only on the cross-section coordinates. Hence, the material density $\rho$ can be integrated only over $\Omega$. As a consequence, it is possible to split the integral along the element length and the integral over the element cross-section into two different contributions to be multiplied:

$$
\begin{equation*}
\delta L_{\mathrm{ine}}=\delta \mathbf{q}_{\tau i}^{T}\left\{\int_{l} N_{i} N_{j} \mathrm{~d} y \int_{\Omega} \rho F_{\tau} F_{s} \mathrm{~d} \Omega \mathbf{I}\right\} \ddot{\mathbf{q}}_{s j} \tag{3.50}
\end{equation*}
$$

The expression in curly brackets in Eq. 3.50 represents a matrix which is pre-multiplied by the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ and multiplied by the nodal acceleration vector $\ddot{\mathbf{q}}_{s j}$. The virtual strain energy can be rewritten in a compact vectorial notation:

$$
\begin{equation*}
\delta L_{\mathrm{ine}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j} \tag{3.51}
\end{equation*}
$$

where $\mathbf{M}^{\tau s i j}$ is the fundamental nucleus of the Mass Matrix of the 1D CUF FE model. This nucleus is a matrix with dimensions $3 \times 3$ and model the inertial properties of the structure, which is modeled throug FEM, for the generic $\tau^{\text {th }}$ expansion term of CUF formulation corresponding to the $i^{t h}$ node of the single finite element, see Eq. 3.20. The fundamental nucleus will be expanded with respect to the indices $\tau, s, i$, and $j$ in order
to build the Mass Matrix of the single finite element. The expression of the fundamental nucleus of the Mass Matrix can be rewritten in a more compact form:

$$
\begin{equation*}
\mathbf{M}^{\tau s i j}=E_{j}^{i} \triangleleft \rho F_{\tau} F_{s} \triangleright_{\Omega} \mathbf{I} \tag{3.52}
\end{equation*}
$$

where the term $E_{j}^{i}$ has been already defined in Eq. 3.44:

$$
\begin{equation*}
E_{j}^{i}=\int_{l} N_{i} N_{j} \mathrm{~d} y \tag{3.53}
\end{equation*}
$$

Equation 3.52 shows that the fundamental nucleus of the Mass Matrix is a diagonal matrix. Once again, the symbol $\triangleleft \ldots \triangleright_{\Omega}$ indicates integration over the cross-section. The discussion about the numerical or analytical method to perform this integration is not repeated here, but more details can be found in [45].

The expressions of the components of $\mathbf{M}^{\tau s i j}$ will be explicitly computed in section 3.4.2. In that section, the fundamental nucleus will lead to the construction of the Mass Matrix.

### 3.2.4 Work of Applied Loadings

In this section, the virtual variation of the work of applied loadings term $L_{\mathrm{al}}$ in the Principle of Virtual Displacements (Eq. 3.30) is addressed on the basis of the 1D CUF FE approximation (Eq. 3.20). As mentioned in section 3.2.1, it is the sum of four terms corresponding to point loadings, line loadings, surface loadings, and volume loadings:

$$
\begin{equation*}
\delta L_{\mathrm{al}}=\delta L_{P}+\delta L_{l}+\delta L_{s}+\delta L_{V} \tag{3.54}
\end{equation*}
$$

The contributions of each term are separately considered in the following sections, where fundamental nuclei of the vector of nodal forces are introduced, and then added together. The expressions of the components of these nuclei and the procedure to build the Vector of Equivalent Nodal Forces will be addressed in section 3.4.3.

## Point Loadings

Point loads can be applied to the structure at any point of the three-dimensional domain. This section focuses on the expression of the virtual work of a single point loading based on the present 1D CUF FE model. In case of more point loads applied, the same procedure has to be followed for each of the point loads. Let the generic point load $\mathbf{P}=\left\{\begin{array}{lll}P_{u_{x}} & P_{u_{y}} & P_{u_{z}}\end{array}\right\}^{T}$ acting on the load application point $\left(x_{P}, y_{P}, z_{P}\right)$ to be considered.

Using Eq. 3.20, the expression of the virtual work of point loadings, due to a single point load, is:

$$
\begin{equation*}
\delta L_{P}=\delta \mathbf{u}^{T} \mathbf{P}=\delta \mathbf{q}_{\tau i}^{T} N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) \mathbf{P}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{P} \tag{3.55}
\end{equation*}
$$

where it is explicitly indicated that $F_{\tau}$ is evaluated in $\left(x_{P}, z_{P}\right)$ and $N_{i}$ is computed in $y_{P}$. As a consequence, terms $F_{\tau}$ and $N_{i}$ in Eq. 3.55 are known coefficients, as well as the components of the point load vector $\mathbf{P}$. In Eq. 3.55 the term $\mathbf{F}_{\tau i}^{P}$ has been introduced. In fact, the virtual work of point loadings can be rewritten in a compact vectorial notation in terms of the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ :

$$
\begin{equation*}
\delta L_{P}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{P} \tag{3.56}
\end{equation*}
$$

where $\mathbf{F}_{\tau i}^{P}$ is the fundamental nucleus of the vector of nodal forces equivalent to the point loadings of the 1D CUF FE model. This nucleus is a vector with dimensions $3 \times 1$ and, in
some sense, it can be seen as the vector of point loadings applied to the element nodes which are energetically equivalent (consistent with the 1D FE discretization) to the actual point loadings. According to Eq. 3.55, the nucleus is:

$$
\mathbf{F}_{\tau i}^{P}=\left\{\begin{array}{l}
F_{x \tau i}^{P}=N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{x}}  \tag{3.57}\\
F_{y \tau i}^{P}=N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{y}} \\
F_{z \tau i}^{P}=N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{z}}
\end{array}\right\}
$$

## Line Loadings

The work of line loadings is obvioulsy due to line loadings applied along lines lying on the external surface of the structure. For the sake of simplicity, let a structure with a rectangular cross-section to be considered. For this sample scheme, the line loadings can be applied along lines lying on the top, bottom, left, and right sides of the structure, as sketched in Fig. 3.9. For the sake of simplicity, these lines are parallel to the longitudinal axis. For instance, a point lying on the line on the top of the structure has coordinates $\left(x_{z}^{+}, y, z_{z}^{+}\right)$, where $x_{z}^{+}$and $z_{z}^{+}$are known quantities. The same consideration counts for the other lines along which the line loadings are applied.


Figure 3.9: Line loadings.

Using Eq. 3.20, the expression of the virtual work of line loadings is:

$$
\begin{equation*}
\delta L_{l}=\delta L_{l}^{p_{x x}^{l \pm}}+\delta L_{l}^{p_{x y}^{l \pm}}+L_{l}^{p_{x z}^{l \pm}}+\delta L_{l}^{p_{z x}^{l \pm}}+\delta L_{l}^{p_{z y}^{l \pm}}+\delta L_{l}^{p_{z z}^{l \pm}} \tag{3.58}
\end{equation*}
$$

Each term of Eq. 3.58 can be computed according to the lines along which the loadings are
applied in Fig. 3.9:

$$
\begin{align*}
\delta L_{l}^{p_{x x}^{l \pm}} & =\int_{L_{x}^{ \pm}} \delta u_{x} p_{x x}^{l \pm} \mathrm{d} y=\delta q_{u_{x} \tau i} \int_{L_{x}^{ \pm}} N_{i} p_{x x}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{x}^{l \pm}} \\
\delta L_{l}^{p_{x y}^{l \pm}} & =\int_{L_{x}^{ \pm}} \delta u_{y} p_{x y}^{l \pm} \mathrm{d} y=\delta q_{u_{y} \tau i} \int_{L_{x}^{ \pm}} N_{i} p_{x y}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{x y}^{l \pm}} \\
\delta L_{l}^{p_{x z}^{l \pm}} & =\int_{L_{x}^{ \pm}} \delta u_{z} p_{x z}^{l \pm} \mathrm{d} y=\delta q_{u_{z} \tau i} \int_{L_{x}^{ \pm}} N_{i} p_{x z}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{x}^{l \pm}}  \tag{3.59}\\
\delta L_{l}^{p_{z x}^{l \pm}} & =\int_{L_{z}^{ \pm}} \delta u_{x} p_{z x}^{l \pm} \mathrm{d} y=\delta q_{u_{x} \tau i} \int_{L_{z}^{ \pm}} N_{i} p_{z x}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{z x}^{l \pm}} \\
\delta L_{l}^{p_{z y}^{l \pm}} & =\int_{L_{z}^{ \pm}} \delta u_{y} p_{z y}^{l \pm} \mathrm{d} y=\delta q_{u_{y} \tau i} \int_{L_{z}^{ \pm}} N_{i} p_{z y}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{z i}^{l \pm}} \\
\delta L_{l}^{p_{z z}^{l \pm}} & =\int_{L_{z}^{ \pm}} \delta u_{z} p_{z z}^{l \pm} \mathrm{d} y=\delta q_{u_{z} \tau i} \int_{L_{z}^{ \pm}} N_{i} p_{z z}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{z z}^{l \pm}}
\end{align*}
$$

It is important to remind that the cross-section functions $F_{\tau}$ of the present 1D CUF model depends on cross-section coordinates $x$ and $z$. Since the lines along which the line loadings are applied are assumed to be parallel to the longitudinal axis, the values of $F_{\tau}$ at coordinates $\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)$and $\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)$are thus known coefficients, which can be taken out of the integral. In Eq. 3.59 the terms $F_{x \tau i}^{p_{x x}^{l \pm}}, F_{y \tau i}^{p_{x y}^{l y}}, F_{z \tau i}^{p_{x z}^{l \pm}}, F_{x \tau i}^{p_{z x}^{l \pm}}, F_{y \tau i}^{p_{z y}^{l \pm}}, F_{z \tau i}^{p_{z z}^{l \pm}}$ have been introduced. In fact, the virtual work of line loadings can be rewritten in a compact vectorial notation in terms of the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ :

$$
\begin{equation*}
\delta L_{l}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{l} \tag{3.60}
\end{equation*}
$$

where $\mathbf{F}_{\tau i}^{l}$ is the fundamental nucleus of the vector of nodal forces equivalent to the line loadings of the 1D CUF FE model. This nucleus is a vector with dimensions $3 \times 1$ and, in some sense, it can be seen as the vector of line loadings energetically consistent to the FE discretization. According to Eq. 3.59, the nucleus is:

$$
\mathbf{F}_{\tau i}^{l}=\left\{\begin{array}{l}
F_{x \tau i}^{l}=F_{x \tau i}^{p_{x \tau}^{l}}+F_{x \tau i}^{p_{z i}^{l}}  \tag{3.61}\\
F_{y \tau i}^{l}=F_{y \tau i}^{p_{y y}^{t}}+F_{y \tau i}^{p_{z i}^{l}} \\
F_{z \tau i}^{l}=F_{z \tau i}^{p_{x \tau}^{l}}+F_{z \tau i}^{p_{z \tau}^{l \pm}}
\end{array}\right\}
$$

For the sake of completeness, the case of line loadings constant along the single finite element lenght $L_{\mathrm{EL}}$ is here considered. In this case, it is possible to take the line loading out of the integral along $L_{x}^{ \pm}$or $L_{z}^{ \pm}$, which becomes the integral along the single element
length. Hence:

$$
\begin{align*}
& \delta L_{l}^{p_{x x}^{l \pm}}=\delta q_{u_{x} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x x}^{l \pm}=E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x x}^{l \pm}=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{x x i}^{l \pm}} \\
& \delta L_{l}^{p_{x y}^{l \pm}}=\delta q_{u_{y} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x y}^{l \pm}=E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x y}^{l \pm}=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{x y}^{l \pm}} \\
& \delta L_{l}^{p_{x}^{l \pm}}=\delta q_{u_{z} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x z}^{l \pm}=E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x z}^{l \pm}=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{x z}^{l \pm}} \\
& \delta L_{l}^{p_{z x}^{l \pm}}=\delta q_{u_{x} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z x}^{l \pm}=E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z x}^{l \pm}=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{z x}^{l \pm}} \\
& \delta L_{l}^{p_{z y}^{l \pm}}=\delta q_{u_{y} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z y}^{l \pm}=E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z y}^{l \pm}=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{z y}^{l \pm}} \\
& \delta L_{l}^{p_{z}^{l \pm}}=\delta q_{u_{z} \tau i} \int_{l} N_{i} \mathrm{~d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z z}^{l \pm}=E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z z}^{l \pm}=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{z z}^{l \pm}} \tag{3.62}
\end{align*}
$$

where:

$$
\begin{equation*}
E^{i}=\int_{l} N_{i} \mathrm{~d} y \tag{3.63}
\end{equation*}
$$

## Surface Loadings

The work of surface loadings is obvioulsy due to pressure loadings applied on the external surface of the structure. For the sake of simplicity, let a structure with a rectangular cross-section to be considered. For this sample scheme, the pressure loads can be applied on surfaces lying on the top, bottom, left, and right sides of the structure, as sketched in Fig. 3.10.


Figure 3.10: Surface loadings.

Using Eq. 3.20, the expression of the virtual work of surface loadings is:

$$
\begin{equation*}
\delta L_{s}=\delta L_{s}^{p_{x x}^{ \pm}}+\delta L_{s}^{p_{x y}^{ \pm}}+L_{s}^{p_{x z}^{ \pm}}+\delta L_{s}^{p_{z x}^{ \pm}}+\delta L_{s}^{p_{z y}^{ \pm}}+\delta L_{s}^{p_{z z}^{ \pm}} \tag{3.64}
\end{equation*}
$$

Each term of Eq. 3.64 can be computed according to the surfaces on which the pressures
are applied in Fig. 3.10:

$$
\begin{align*}
& \delta L_{s}^{p_{x x}^{ \pm}}=\int_{S_{x}^{ \pm}} \delta u_{x} p_{x x}^{ \pm} \mathrm{d} z \mathrm{~d} y=\delta q_{u_{x} \tau i} \int_{l} N_{i}\left[\int_{b^{ \pm}} F_{\tau} p_{x x}^{ \pm} \mathrm{d} z\right] \mathrm{d} y=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{x x}^{ \pm}} \\
& \delta L_{s}^{p_{x y}^{ \pm}}=\int_{S_{x}^{ \pm}} \delta u_{y} p_{x y}^{ \pm} \mathrm{d} z \mathrm{~d} y=\delta q_{u_{y} \tau i} \int_{l} N_{i}\left[\int_{b^{ \pm}} F_{\tau} p_{x y}^{ \pm} \mathrm{d} z\right] \mathrm{d} y=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{x y}^{ \pm}} \\
& \delta L_{s}^{p_{x z}^{ \pm}}=\int_{S_{x}^{ \pm}} \delta u_{z} p_{x z}^{ \pm} \mathrm{d} z \mathrm{~d} y=\delta q_{u_{z} \tau i} \int_{l} N_{i}\left[\int_{b^{ \pm}} F_{\tau} p_{x z}^{ \pm} \mathrm{d} z\right] \mathrm{d} y=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{x z}^{ \pm}} \\
& \delta L_{s}^{p_{z x}^{ \pm}}=\int_{S_{z}^{ \pm}} \delta u_{x} p_{z x}^{ \pm} \mathrm{d} x \mathrm{~d} y=\delta q_{u_{x} \tau i} \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z x}^{ \pm} \mathrm{d} x\right] \mathrm{d} y=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{z x}^{ \pm}}  \tag{3.65}\\
& \delta L_{s}^{p_{z y}^{ \pm}}=\int_{S_{z}^{ \pm}} \delta u_{y} p_{z y}^{ \pm} \mathrm{d} x \mathrm{~d} y=\delta q_{u_{y} \tau i} \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z y}^{ \pm} \mathrm{d} x\right] \mathrm{d} y=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{z y}^{ \pm}} \\
& \delta L_{s}^{p_{z z}^{ \pm}}=\int_{S_{z}^{ \pm}} \delta u_{z} p_{z z}^{ \pm} \mathrm{d} x \mathrm{~d} y=\delta q_{u_{z} \tau i} \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z z}^{ \pm} \mathrm{d} x\right] \mathrm{d} y=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{z z}^{ \pm}}
\end{align*}
$$

where it is reminded that the shape functions $N_{i}$ of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$. In Eq. 3.65 the terms $F_{x \tau i}^{p_{x x}^{ \pm}}, F_{y \tau i}^{p_{x y}^{ \pm}}, F_{z i}^{p_{x z}^{ \pm}}, F_{x \tau i}^{p_{z x}^{ \pm}}, F_{y \tau i}^{p_{z y}^{ \pm}}, F_{z i i}^{p_{z z}^{ \pm}}$have been introduced. In fact, the virtual work of surface loadings can be rewritten in a compact vectorial notation in terms of the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ :

$$
\begin{equation*}
\delta L_{s}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{s} \tag{3.66}
\end{equation*}
$$

where $\mathbf{F}_{\tau i}^{s}$ is the fundamental nucleus of the vector of nodal forces equivalent to the surface loadings of the 1D CUF FE model. This nucleus is a vector with dimensions $3 \times 1$ and, in some sense, it can be seen as the vector of surface loadings energetically consistent to the FE discretization. According to Eq. 3.65, the nucleus is:

$$
\mathbf{F}_{\tau i}^{s}=\left\{\begin{array}{c}
F_{x \tau i}^{s}=F_{x \tau i}^{p_{x x}^{ \pm}}+F_{x \tau i}^{p_{z x}^{ \pm}}  \tag{3.67}\\
F_{y \tau i}^{s}=F_{y \tau i}^{p_{x y}^{ \pm}}+F_{y \tau i}^{p_{z y}^{ \pm}} \\
F_{z \tau i}^{s}=F_{z \tau i}^{p_{x z}^{ \pm}}+F_{z \tau i}^{p_{z z}^{ \pm}}
\end{array}\right\}
$$

For the sake of completeness, the case of pressures constant along the single finite element lenght $L_{\text {EL }}$ is here considered. In this case, it is possible to split the integral along the element length and the integral along the perpendicular direction ( $x$ or $z$ ) into two different contributions to be multiplied. Hence:

$$
\begin{align*}
\delta L_{s}^{p_{x x}^{ \pm}} & =\delta q_{u_{x} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{b^{ \pm}} F_{\tau} p_{x x}^{ \pm} \mathrm{d} z=\delta q_{u_{x} \tau i} E^{i} I_{\tau}^{p_{x x}^{ \pm}}=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{x x}^{ \pm}} \\
\delta L_{s}^{p_{x y}^{ \pm}} & =\delta q_{u_{y} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{b^{ \pm}} F_{\tau} p_{x y}^{ \pm} \mathrm{d} z=\delta q_{u_{y} \tau i} E^{i} I_{\tau}^{p_{x y}^{ \pm}}=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{x y}^{ \pm}} \\
\delta L_{s}^{p_{x z}^{ \pm}} & =\delta q_{u_{z} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{b^{ \pm}} F_{\tau} p_{x z}^{ \pm} \mathrm{d} z=\delta q_{u_{z} \tau i} E^{i} I_{\tau}^{p_{x z}^{ \pm}}=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{x z}^{ \pm}}  \tag{3.68}\\
\delta L_{s}^{p_{z x}^{ \pm}} & =\delta q_{u_{x} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{a^{ \pm}} F_{\tau} p_{z x}^{ \pm} \mathrm{d} x=\delta q_{u_{x} \tau i} E^{i} I_{\tau}^{p_{z x}^{ \pm}}=\delta q_{u_{x} \tau i} F_{x \tau i}^{p_{z x}^{ \pm}} \\
\delta L_{s}^{p_{z y}^{ \pm}} & =\delta q_{u_{y} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{a^{ \pm}} F_{\tau} p_{z y}^{ \pm} \mathrm{d} x=\delta q_{u_{y} \tau i} E^{i} I_{\tau}^{p_{z y}^{ \pm}}=\delta q_{u_{y} \tau i} F_{y \tau i}^{p_{z y}^{ \pm}} \\
\delta L_{s}^{p_{z z}^{ \pm}} & =\delta q_{u_{z} \tau i} \int_{l} N_{i} \mathrm{~d} y \int_{a^{ \pm}} F_{\tau} p_{z z}^{ \pm} \mathrm{d} x=\delta q_{u_{z} \tau i} E^{i} I_{\tau}^{p_{z z}^{ \pm}}=\delta q_{u_{z} \tau i} F_{z \tau i}^{p_{z z}^{ \pm}}
\end{align*}
$$

where the term $E^{i}$ has already been defined in Eq. 3.63::

$$
\begin{equation*}
E^{i}=\int_{l} N_{i} \mathrm{~d} y \tag{3.69}
\end{equation*}
$$

and the terms $I_{\tau}^{p_{x h}^{ \pm}}$and $I_{\tau}^{p_{z h}^{ \pm}}$are defined as:

$$
\begin{array}{rlr}
I_{\tau}^{p_{x h}^{ \pm}}=\int_{b^{ \pm}} F_{\tau} p_{x h}^{ \pm} \mathrm{d} y & h=x, y, z \\
I_{\tau^{z h}}^{p_{z h}^{ \pm}}=\int_{a^{ \pm}} F_{\tau} p_{z h}^{ \pm} \mathrm{d} y & h=x, y, z \tag{3.71}
\end{array}
$$

## Volume Loadings

The work of volume loadings is obvioulsy due to loadings applied either on part of the three-dimensional domain of the structure or on its whole domain. For the sake of brevity, a volume loading $\mathbf{p}^{V}=\left\{\begin{array}{lll}p_{x}^{V} & p_{y}^{V} & p_{z}^{V}\end{array}\right\}^{T}$ applied to the whole domain is here considered.

Using Eq. 3.20, the expression of the virtual work of volume loadings is:

$$
\begin{equation*}
\delta L_{V}=\int_{V} \delta \mathbf{u}^{T} \mathbf{p}^{V} \mathrm{~d} V=\int_{l} \int_{\Omega} \delta \mathbf{u}^{T} \mathbf{p}^{V} \mathrm{~d} \Omega \mathrm{~d} y=\delta \mathbf{q}_{\tau i}^{T} \int_{l} N_{i}\left[\int_{\Omega} F_{\tau} \mathbf{p}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y \tag{3.72}
\end{equation*}
$$

where it is reminded that the shape functions $N_{i}$ of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$ and then the term $N_{i}$ can be taken out of the integral over $\Omega$.

The virtual work of volume loadings can be rewritten in a compact vectorial notation in terms of the transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$ :

$$
\begin{equation*}
\delta L_{V}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{V} \tag{3.73}
\end{equation*}
$$

where $\mathbf{F}_{\tau i}^{V}$ is the fundamental nucleus of the vector of nodal forces equivalent to the volume loadings of the 1D CUF FE model. This nucleus is a vector with dimensions $3 \times 1$ and, in some sense, it can be seen as the vector of volume loadings energetically consistent to the FE discretization. According to Eq. 3.72, the nucleus is:

$$
\mathbf{F}_{\tau i}^{V}=\left\{\begin{array}{l}
F_{x \tau i}^{V}=\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{x}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y  \tag{3.74}\\
F_{y \tau i}^{V}=\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{y}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y \\
F_{z \tau i}^{V}=\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{z}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y
\end{array}\right\}
$$

For the sake of completeness, the case of volume loadings constant along the single finite element lenght $L_{\text {EL }}$ is here considered. In this case, it is possible to split the integral along the element length and the integral over the cross-section $\Omega$ into two different contributions to be multiplied. Hence:

$$
\begin{equation*}
\delta L_{V}=\delta \mathbf{q}_{\tau i}^{T} \int_{l} N_{i} \mathrm{~d} y \int_{\Omega} F_{\tau} \mathbf{p}^{V} \mathrm{~d} \Omega=\delta \mathbf{q}_{\tau i}^{T} E^{i} I_{\tau}^{\mathbf{p}^{V}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i}^{V} \tag{3.75}
\end{equation*}
$$

where the term $E^{i}$ has already been defined in Eq. 3.63:

$$
\begin{equation*}
E^{i}=\int_{l} N_{i} \mathrm{~d} y \tag{3.76}
\end{equation*}
$$

and the term $I_{\tau}^{\mathrm{p}^{V}}$ is defined as:

$$
\begin{equation*}
I_{\tau}^{\mathbf{p}^{V}}=\int_{\Omega} F_{\tau} \mathbf{p}^{V} \mathrm{~d} \Omega \tag{3.77}
\end{equation*}
$$

## Applied Loadings: point, line, surface, and volume loadings

Fundamental nuclei of the vector of nodal forces equivalent to point loadings $\mathbf{F}_{\tau i}^{P}$, line loadings $\mathbf{F}_{\tau i}^{l}$, surface loadings $\mathbf{F}_{\tau i}^{s}$, and volume loadings $\mathbf{F}_{\tau i}^{V}$ on the basis of the 1D CUF FE model have been introduced in Eqs. 3.56, 3.60, 3.66, and 3.73, respectively. In these equations each contribution to the virtual work of applied loadings is expressed in terms of the generic transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$. According to Eq. 3.30, the virtual work of applied loadings can be written in terms of this vector as follows:

$$
\begin{equation*}
\delta L_{\mathrm{al}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i} \tag{3.78}
\end{equation*}
$$

where $\mathbf{F}_{\tau i}$ is the fundamental nucleus of the Vector of Equivalent Nodal Forces of the 1D CUF FE model. This nucleus is a vector with dimensions $3 \times 1$ and, in some sense, it can be seen as the vector of all applied loadings energetically consistent to the FE discretization. Substituting Eqs. 3.56, 3.60, 3.66, and 3.73 into Eq. 3.30, the fundamental nucleus is the sum of the four fundamental nuclei:

$$
\begin{equation*}
\mathbf{F}_{\tau i}=\mathbf{F}_{\tau i}^{P}+\mathbf{F}_{\tau i}^{l}+\mathbf{F}_{\tau i}^{s}+\mathbf{F}_{\tau i}^{V} \tag{3.79}
\end{equation*}
$$

### 3.3 Governing equations

As discussed in section 3.2.1, the Principle of Virtual Displacements (PVD) is the variational statement used for the present formulation. Referring to Eqs. 3.29 and 3.30, the expression of PVD is here proposed again:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta L_{\mathrm{al}}-\delta L_{\mathrm{ine}} \tag{3.80}
\end{equation*}
$$

Sections 3.2.2, 3.2.3, and 3.2.4 have addressed each of the virtual work terms in Eq. 3.80 introducing the fundamental nuclei of the Structural Stiffness Matrix, of the Mass Matrix and of the Vector of Equivalent Nodal forces. Substituting Eqs. 3.42, 3.51, and 3.78 into Eq. 3.80 (i.e. Eq. 3.30), it is obtained:

$$
\delta \mathbf{q}_{\tau i}^{T} \mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\delta \mathbf{q}_{\tau i}^{T} \mathbf{K}^{\tau s i j} \mathbf{q}_{s j}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i} \quad \begin{align*}
& \tau, s=1, \ldots, N_{u}  \tag{3.81}\\
& i, j=1, \ldots, N_{N}
\end{align*}
$$

This is the PVD in terms of the generic transposed virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}^{T}$, the generic nodal displacement vector $\mathbf{q}_{s j}$ and the generic nodal acceleration vector $\ddot{\mathbf{q}}_{s j}$. For the sake of completeness, it is reminded that the formulation so far described is based on Einstein's notation for indices $\tau, s, i$, and $j$, see Eq. 3.20. In fact, in Eq. 3.81 it is reminded that the terms have to be summed up as the values of these indices increase. However, it is important to note that so far the formulation has been written for the single finite element. Hence, the values of indices $i$ and $j$ have been considered only from 1 to $N_{N}$. Following the standard finite element method, the structural domain is discretized by a mesh, where the total number of nodes is equal to $N_{N \text { тот }}$, see section 3.1.4. Consistently with the finite element mesh, Eq. 3.81 has to be expanded over the $N_{N \text { tot }}$ nodes:

$$
\delta \mathbf{q}_{\tau i}^{T} \mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\delta \mathbf{q}_{\tau i}^{T} \mathbf{K}^{\tau s i j} \mathbf{q}_{s j}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i} \quad \begin{align*}
& \tau, s=1, \ldots, N_{u}  \tag{3.82}\\
& i, j=1, \ldots, N_{N \text { тот }}
\end{align*}
$$

Equation 3.81 can be rewritten in the following form:

$$
\delta \mathbf{q}_{\tau i}^{T}\left[\mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\mathbf{K}^{\tau s i j} \mathbf{q}_{s j}-\mathbf{F}_{\tau i}\right]=\delta \mathbf{q}_{\tau i}^{T} \mathbf{0} \quad \begin{align*}
& \tau, s=1, \ldots, N_{u}  \tag{3.83}\\
& i, j=1, \ldots, N_{N \text { TOT }}
\end{align*}
$$

where the term $\mathbf{0}$ is a $3 \times 1$ vector of zeros. Hence, Eq. 3.83 can be seen as:

$$
\delta \mathbf{q}_{\tau i}^{T}\left\{\mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\mathbf{K}^{\tau s i j} \mathbf{q}_{s j}-\mathbf{F}_{\tau i}=\mathbf{0}\right\} \quad \begin{align*}
& \tau, s=1, \ldots, N_{u}  \tag{3.84}\\
& i, j=1, \ldots, N_{N \text { тот }}
\end{align*}
$$

Equation 3.84 states that the equilibrium configuration (given by $\mathbf{q}_{s j}$ and $\ddot{\mathbf{q}}_{s j}$ ) satisfies the equation in curly brackets for any admissible virtual nodal displacement vector $\delta \mathbf{q}_{\tau i}$. Thus, the Principle of Virtual Displacements leads to the following governing equation written in a compact vectorial notation:

$$
\begin{align*}
\mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\mathbf{K}^{\tau s i j} \mathbf{q}_{s j}=\mathbf{F}_{\tau i} \quad s=1, \ldots, N_{u} \\
\forall \tau=1, \ldots, N_{u} \quad \forall i=1, \ldots, N_{N \text { TOT }} \tag{3.85}
\end{align*}
$$

where repeated subscripts $s$ and $j$ indicate summation based on Einstein's notation. The governing equation in Eq. 3.85 has therefore to be expanded with respect to the indices $\tau$, $s, i$, and $j$ in order to obtain the governing equations of the elasticity problem:

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}+\mathbf{K q}=\mathbf{F} \tag{3.86}
\end{equation*}
$$

where the Structural Stiffness Matrix K, the Mass Matrix M, the Vector of Equivalent Nodal Forces $\mathbf{F}$, the nodal displacement vector $\mathbf{q}$, and the nodal acceleration vector $\ddot{\mathbf{q}}$ are involved. As mentioned in section 3.1.4 the nodal displacement vector $\mathbf{q}$ contains all the nodal degrees of freedom of the present structural model, which are commonly referred as DOFs. In particular, $\mathbf{q}$ contains all the $N_{u}$ generic nodal displacement vectors $\mathbf{q}_{\tau i}$ for each of the $N_{N \text { tot }}$ nodes (for all values of $\tau\left(\tau=1, \ldots, N_{u}\right)$ and $i\left(i=1, \ldots, N_{N}\right)$ of each element). The expansion of $\mathbf{q}$ on indices $\tau$ and $i$ is consistent with the assembly procedure followed in section 3.4 to build the finite element matrices $\mathbf{K}, \mathbf{M}$ and vector $\mathbf{F}$.

It should be noted that no assumptions on the expansion order have been made so far. Therefore, it is possible to obtain refined variable kinematic 1D models without changing the formal expression of the nuclei components. Thanks to the CUF, the present model is invariant with respect to the order of the beam theory and the type of element used in the finite element axial discretization. CUF is therefore the ideal tool to easily compare different higher-order theories. In fact, the expansion order of the model, i.e. its accuracy, is a free parameter of the analysis by exploiting a systematic procedure that leads to governing FE matrices whose form does not depend on the order of expansion used for the displacement unknowns over the cross-section.

The governing equations of the elasticity problem expressed in Eq. 3.86 are now separately considered for static, free vibration and dynamic response analysis.

### 3.3.1 Static analysis

When the structural static analysis is performed, the contribution of the virtual work of inertial loadings $\delta L_{\text {ine }}$ is neglected in the Principle of Virtual Displacements (Eq. 3.30):

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta L_{\mathrm{al}} \tag{3.87}
\end{equation*}
$$

As a result, for the static analysis the equation of motion in a compact vectorial form is:

$$
\begin{array}{cc}
\mathbf{K}^{\tau s i j} \mathbf{q}_{s j}=\mathbf{F}_{\tau i} & s=1, \ldots, N_{u}  \tag{3.88}\\
& j=1, \ldots, N_{N \text { тОт }} \\
\forall \tau=1, \ldots, N_{u} & \forall i=1, \ldots, N_{N \text { тот }}
\end{array}
$$

By expanding Eq. 3.88 on indices $\tau, s, i$, and $j$, the governing equations for the static analysis become:

$$
\begin{equation*}
\mathbf{K q}=\mathbf{F} \tag{3.89}
\end{equation*}
$$

which are the equations obtained from Eq. 3.86 by neglecting the contribution of the Mass Matrix derived from the work of inertial loadings.

## Solution

As far as the numerical method used in the present doctoral research to solve the system in Eq. 3.89 is concerned, a direct solution is performed using an algorithm based on Gauss elimination. The basic procedure of the Gauss elimination solution is to reduce the system to an upper triangular coefficient matrix from which the unknown nodal displacement vector $\mathbf{q}$ can be calculated by a back-substitution. Since the Structural Stiffness Matrix K is symmetric (by definition, see section 3.4.1), a factorization of $\mathbf{K}$ is performed:

$$
\begin{equation*}
\mathbf{K}=\mathbf{L} \mathbf{D} \mathbf{L}^{T} \tag{3.90}
\end{equation*}
$$

where $\mathbf{L}$ is a lower unit triangular matrix (product of permutation and unit lower triangular matrices) and $\mathbf{D}$ is a diagonal matrix. In general $\mathbf{L} \mathbf{D} \mathbf{L}^{T}$ is a symmetric indefinite factorization. For the sake of completeness, the routine used in this work to perform this factorization is the routine DSYTRF contained in the freely available software package LAPACK (Linear Algebra PACKage) [110]. The algorithm in DSYTRF uses the BunchKaufman diagonal pivoting method. More details can be found in [111]. After imposition of constraints (i.e. boundary conditions, see appendix D) on the Structural Stiffness Matrix $\mathbf{K}$, it becomes a non-singular matrix. In particular, $\mathbf{K}$ is a positive definite matrix. Hence, the terms of the diagonal matrix $\mathbf{D}$ are positive: $d_{i i}>0 ; i=1, \ldots$, DOFs.

The $\mathbf{L} \mathbf{D} \mathbf{L}^{T}$ factorization can be used effectively to obtain the solution of Eq. 3.89 in the following two steps by means of the auxiliary vector $\mathbf{v}$ :

$$
\begin{gather*}
\mathbf{L} \mathbf{v}=\mathbf{P}  \tag{3.91}\\
\mathbf{D} \mathbf{L}^{T} \mathbf{q}=\mathbf{v} \tag{3.92}
\end{gather*}
$$

The solution of the system is performed by means of the routine DSYTRS of package LAPACK [110]. More details are not reported here, but can be found in [42].

### 3.3.2 Free vibration analysis

When the free vibration analysis is performed, the contribution of the virtual work of applied loadings $\delta L_{\mathrm{al}}$ is neglected in the Principle of Virtual Displacements (Eq. 3.30):

$$
\begin{equation*}
\delta L_{\mathrm{int}}=-\delta L_{\mathrm{ine}} \tag{3.93}
\end{equation*}
$$

As a result, the governing equations in a compact vectorial form are:

$$
\begin{align*}
\mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j}+\mathbf{K}^{\tau s i j} \mathbf{q}_{s j}=\mathbf{0} \quad s=1, \ldots, N_{u} \\
\forall \tau=1, \ldots, N_{u} \quad \forall i=1, \ldots, N_{N \text { тот }} \tag{3.94}
\end{align*}
$$

By expanding Eq. 3.94 on indices $\tau, s, i$, and $j$, the governing equations for the free vibration analysis become the homogeneous case of Eq. 3.86 by neglecting the contribution of $\mathbf{F}$ :

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}+\mathbf{K} \mathbf{q}=\mathbf{0} \tag{3.95}
\end{equation*}
$$

Introducing harmonic solutions, it is possible to compute the natural angular frequencies $\omega_{h}$ and the natural frequencies $f_{h}$ by solving an eigenvalue problem:

$$
\begin{equation*}
\left[-\omega_{h}^{2} \mathbf{M}+\mathbf{K}\right] \mathbf{q}_{h}=\mathbf{0} \tag{3.96}
\end{equation*}
$$

where $\mathbf{q}_{h}$ is the $h^{\text {th }}$ eigenvector.

## Solution

In the present doctoral research, the numerical solution of the eigenvalue problem in Eq. 3.96 has been provided by a mathematical tool based upon an algorithmic variant of the Arnoldi process called the Implicitly Restarted Arnoldi Method (IRAM) [112]. Since for the present case matrices $\mathbf{K}$ and $\mathbf{M}$ are symmetric, the mathematical method IRAM reduces to a variant of the Lanczos process called the Implicitly Restarted Lanczos Method (IRLM) [113]. For the sake of completeness, these variants may be viewed as a synthesis of the Arnoldi/Lanczos process with the Implicitly Shifted QR technique that is suitable for large scale problems. The routines of the freely available software package ARPACK have been used in the present doctoral work. More details are not reported here, but can be found in [114].

### 3.3.3 Dynamic response analysis

The dynamic response analysis takes into account all the terms of the governing equations of the elasticity problem in Eq. 3.86. Hence, the equation of motion to be solved for the dynamic response analysis is precisely:

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}(t)+\mathbf{K} \mathbf{q}(t)=\mathbf{F}(t) \tag{3.97}
\end{equation*}
$$

where the dependence of nodal displacements, nodal accelerations and equivalent nodal forces on time $t$ has been highlighted for the sake of completeness. The quantities $\ddot{\mathbf{q}}$ and $\mathbf{q}$ describe the time-dependent response of the system to time-dependent applied loadings. In general, any kind of dynamic loadings such as point, line, surface, volume (traveling and nontraveling) loads with arbitrary time-dependence can be taken into account by the present formulation.

It is important to note that so far the damping contribution to the governing equations has not been considered in the formulation. Nonetheless, the formulation easily permits the introduction of a damping matrix $\mathbf{C}$ in Eq. 3.97. For example a Rayleigh damping can be assumed:

$$
\begin{equation*}
\mathbf{C}=\gamma \mathbf{M}+\beta \mathbf{K} \tag{3.98}
\end{equation*}
$$

where $\gamma$ and $\beta$ are Rayleigh coefficients. If the damping contribution is considered in the equation of motion, Eq. 3.97 for the dynamic response analysis becomes:

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}(t)+\mathbf{C} \dot{\mathbf{q}}(t)+\mathbf{K} \mathbf{q}(t)=\mathbf{F}(t) \tag{3.99}
\end{equation*}
$$

where $\dot{\mathbf{q}}$ is the nodal velocity vector.

## Solution: Newmark time integration method

In the present work, the solution of the equation of motion is carried out through a direct time integration method. In particular, the Newmark method, widely used in structural dynamics, is implemented. In direct integration the governing equations are integrated using a numerical step-by-step procedure. The term "direct" means that prior to the numerical integration, no transformation of the equations into a different form is carried out. Instead of trying to satisfy Eq. 3.97 (or Eq. 3.99) at any time $t$, direct numerical integration is aimed to satisfy Eq. 3.97 (or Eq. 3.99) only at discrete time intervals $\Delta t$ apart. This means that, basically, the equilibrium, which includes the effect of inertia (and damping) forces, is sought at discrete time points within the interval solution. The time interval $T$ over which the response of the system is evaluated is therefore subdivided into $N_{T}$ equal time steps $\Delta t$ (i.e. $\Delta t=T / N_{T}$ ). Referring to Eq. 3.97, the undamped equation of motion at time $t+\Delta t$ is then:

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}_{t+\Delta t}+\mathbf{K} \mathbf{q}_{t+\Delta t}=\mathbf{F}_{t+\Delta t} \tag{3.100}
\end{equation*}
$$

For the sake of generality, the damping is now taken into account. Referring to Eq. 3.99, the equation of motion at time $t+\Delta t$ is therefore:

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{q}}_{t+\Delta t}+\mathbf{C} \dot{\mathbf{q}}_{t+\Delta t}+\mathbf{K} \mathbf{q}_{t+\Delta t}=\mathbf{F}_{t+\Delta t} \tag{3.101}
\end{equation*}
$$

In time integration methods a variation of displacements, velocities, and accelerations within each time step $\Delta t$ is assumed. The form of the assumption on this variation determines the accuracy, stability, and cost of the solution procedure. More details can be found in [42].

The Newmark method [115] is an implicit time integration scheme. The term "implicit" means that the displacement computed at time instant $t+\Delta t$ depends on displacements, velocities, accelerations computed at time instants previous to time instant $t+\Delta t$ (known quantities) and on velocities, accelerations computed at time instant $t+\Delta t$ (unknown quantities). The Newmark time integration method can be seen as an extension of the linear acceleration method. The following assumptions for displacements and displacements within the time step $\Delta t$ are used:

$$
\begin{gather*}
\dot{\mathbf{q}}_{t+\Delta t}=\dot{\mathbf{q}}_{t}+\left[(1-\delta) \ddot{\mathbf{q}}_{t}+\delta \ddot{\mathbf{q}}_{t+\Delta t}\right] \Delta t  \tag{3.102}\\
\mathbf{q}_{t+\Delta t}=\mathbf{q}_{t}+\dot{\mathbf{q}}_{t} \Delta t+\left[(1 / 2-\alpha) \ddot{\mathbf{q}}_{t}+\alpha \ddot{\mathbf{q}}_{t+\Delta t}\right] \Delta t^{2} \tag{3.103}
\end{gather*}
$$

where the values $\delta=0.5$ and $\alpha=0.25$ are employed for the related constants. Equation 3.103 proves to be an implicit time integration method. From Eq. 3.103:

$$
\begin{equation*}
\ddot{\mathbf{q}}_{t+\Delta t}=\frac{1}{\alpha \Delta t^{2}} \mathbf{q}_{t+\Delta t}-\frac{1}{\alpha \Delta t^{2}} \mathbf{q}_{t}-\frac{1}{\alpha \Delta t} \dot{\mathbf{q}}_{t}-\frac{(1 / 2-\alpha)}{\alpha} \ddot{\mathbf{q}}_{t} \tag{3.104}
\end{equation*}
$$

Equation 3.104 shows the dependence of the nodal accelerations at time instant $t+\Delta t$ on the other terms:

$$
\begin{equation*}
\ddot{\mathbf{q}}_{t+\Delta t}=\ddot{\mathbf{q}}_{t+\Delta t}\left(\ddot{\mathbf{q}}_{t}, \dot{\mathbf{q}}_{t}, \mathbf{q}_{t}, \mathbf{q}_{t+\Delta t}\right) \tag{3.105}
\end{equation*}
$$

Substituting Eq. 3.104 into Eq. 3.102, the following expression can be derived:

$$
\begin{equation*}
\dot{\mathbf{q}}_{t+\Delta t}=\left(1-\frac{\delta}{\alpha}\right) \dot{\mathbf{q}}_{t}+\left(1-\frac{\delta}{2 \alpha}\right) \Delta t \ddot{\mathbf{q}}_{t}+\frac{\delta}{\alpha \Delta t} \mathbf{q}_{t+\Delta t}-\frac{\delta}{\alpha \Delta t} \mathbf{q}_{t} \tag{3.106}
\end{equation*}
$$

Equation 3.106 shows dependence of the nodal velocities at time instant $t+\Delta t$ on the other terms:

$$
\begin{equation*}
\dot{\mathbf{q}}_{t+\Delta t}=\dot{\mathbf{q}}_{t+\Delta t}\left(\ddot{\mathbf{q}}_{t}, \dot{\mathbf{q}}_{t}, \mathbf{q}_{t}, \mathbf{q}_{t+\Delta t}\right) \tag{3.107}
\end{equation*}
$$

It is important to note that the only unknowns at time instant $t+\Delta t$ the nodal accelerations $\ddot{\mathbf{q}}_{t+\Delta t}$ and velocities $\dot{\mathbf{q}}_{t+\Delta t}$ depend on in Eqs. 3.104 and 3.106 are displacements $\mathbf{q}_{t+\Delta t}$. As a consequence, substituting Eqs. 3.104 and 3.106 into the equation of motion at time $t+\Delta t$ (Eq. 3.101) it is possible to obtain the following equation:

$$
\begin{align*}
& \frac{1}{\alpha \Delta t^{2}} \mathbf{M} \mathbf{q}_{t+\Delta t}-\frac{1}{\alpha \Delta t^{2}} \mathbf{M} \mathbf{q}_{t}-\frac{1}{\alpha \Delta t} \mathbf{M} \dot{\mathbf{q}}_{t}+\left(1-\frac{1}{2 \alpha}\right) \mathbf{M} \ddot{\mathbf{q}}_{t}+ \\
& +\left(1-\frac{\delta}{\alpha}\right) \mathbf{C} \dot{\mathbf{q}}_{t}+\left(1-\frac{\delta}{2 \alpha}\right) \Delta t \mathbf{C} \ddot{\mathbf{q}} t+\frac{\delta}{\alpha \Delta t} \mathbf{C} \mathbf{q}_{t+\Delta t}-\frac{\delta}{\alpha \Delta t} \mathbf{C} \mathbf{q}_{t}+  \tag{3.108}\\
& +\mathbf{K} \mathbf{q}_{t+\Delta t}=\mathbf{F}_{t+\Delta t}
\end{align*}
$$

Collecting the terms multipling the unknown nodal displacement vector $\mathbf{q}_{t+\Delta t}$ at time $t+\Delta t$, Eq. 3.108 becomes:

$$
\begin{align*}
& {\left[\mathbf{K}+\frac{1}{\alpha \Delta t^{2}} \mathbf{M}+\frac{\delta}{\alpha \Delta t} \mathbf{C}\right] \mathbf{q}_{t+\Delta t}=\left[\mathbf{F}_{t+\Delta t}+\frac{1}{\alpha \Delta t^{2}} \mathbf{M} \mathbf{q}_{t}+\frac{1}{\alpha \Delta t} \mathbf{M} \dot{\mathbf{q}}_{t}+\right.} \\
& \left.+\left(\frac{1}{2 \alpha}-1\right) \mathbf{M} \ddot{\mathbf{q}}_{t}+\left(\frac{\delta}{\alpha}-1\right) \mathbf{C} \dot{\mathbf{q}}_{t}+\left(\frac{\delta}{2 \alpha}-1\right) \Delta t \mathbf{C} \ddot{\mathbf{q}}_{t}+\frac{\delta}{\alpha \Delta t} \mathbf{C} \mathbf{q}_{t}\right] \tag{3.109}
\end{align*}
$$

Exploiting a compact vectorial notation, the following system has to be solved to compute the dynamic response $\mathbf{q}_{t+\Delta t}$ at time instant $t+\Delta t$ :

$$
\begin{equation*}
\overline{\mathbf{K}} \mathbf{q}_{t+\Delta t}=\overline{\mathbf{F}}_{t+\Delta t} \tag{3.110}
\end{equation*}
$$

where $\overline{\mathbf{K}}$ is the effective Structural Stiffness Matrix and $\overline{\mathbf{F}}_{t+\Delta t}$ is the effective Vector of Equivalent Nodal Forces at time instant $t+\Delta t$ :

$$
\begin{gather*}
\overline{\mathbf{K}}=\mathbf{K}+\frac{1}{\alpha \Delta t^{2}} \mathbf{M}+\frac{\delta}{\alpha \Delta t} \mathbf{C}  \tag{3.111}\\
\overline{\mathbf{F}}_{t+\Delta t}=\mathbf{F}_{t+\Delta t}+\frac{1}{\alpha \Delta t^{2}} \mathbf{M} \mathbf{q}_{t}+\frac{1}{\alpha \Delta t} \mathbf{M} \dot{\mathbf{q}}_{t}+\left(\frac{1}{2 \alpha}-1\right) \mathbf{M} \ddot{\mathbf{q}}_{t}+ \\
+\left(\frac{\delta}{\alpha}-1\right) \mathbf{C} \dot{\mathbf{q}}_{t}+\left(\frac{\delta}{2 \alpha}-1\right) \Delta t \mathbf{C} \ddot{\mathbf{q}}_{t}+\frac{\delta}{\alpha \Delta t} \mathbf{C} \mathbf{q}_{t} \tag{3.112}
\end{gather*}
$$

The system to be solved in Eq. 3.110 is similar to the system to be solved for the static analysis (see Eq. 3.89). The step-by-step solution of Eq. 3.110 using the Newmark integration method is now faced. Firstly, it is required to initialize the values of nodal displacement, velocity and acceleration vectors at time instant $t=0$ which are referred as $\mathbf{q}_{0}, \dot{\mathbf{q}}_{0}$ and $\ddot{\mathbf{q}}_{0}$, respectively. Once the value of the time step $\Delta t$ is selected, the effective Structural Stiffness Matrix $\overline{\mathbf{K}}$ is computed via Eq. 3.111. At each time instant $t+\Delta t$, the effective Vector of Equivalent Nodal Forces is computed via Eq. 3.112. The numerical solution algorithm employed in static analysis based on $\mathbf{L} \mathbf{D} \mathbf{L}^{T}$ factorization can basically be used in direct time integration to compute the step-by-step solution of Eq. 3.110, obtaining:

$$
\begin{equation*}
\overline{\mathbf{K}}=\mathbf{L} \mathbf{D} \mathbf{L}^{T} \tag{3.113}
\end{equation*}
$$

Once the nodal displacement vector $\mathbf{q}_{t+\Delta t}$ is obtained from Eq. 3.111 at each time instant $t+\Delta t$ the nodal acceleration and velocity vectors are computed by means of Eqs. 3.104 and 3.106 previously derived from the assumptions of the Newmark method. In case of undamped system (Eq. 3.100), the contribution of the damping matrix $\mathbf{C}$ in Eqs. 3.111 and 3.112 is neglected.

### 3.4 Finite Element Matrices and Vectors

The previous sections 3.2.2, 3.2.3, and 3.2.4 have introduced the fundamental nucleus of the Structural Stiffness Matrix $\mathbf{K}^{\tau s i j}$, the fundamental nucleus of the Mass Matrix $\mathbf{M}^{\tau s i j}$, and the fundamental nuclei of the vector of nodal forces equivalent to point loadings $\mathbf{F}_{\tau i}^{P}$, line loadings $\mathbf{F}_{\tau i}^{l}$, surface loadings $\mathbf{F}_{\tau i}^{s}$, and volume loadings $\mathbf{F}_{\tau i}^{V}$ on the basis of the Principle of Virtual Displacements and the finite element method. The expressions of the components of these nuclei are here explicitly computed for orthotropic and isotropic materials and then the procedure to build the finite element matrices and vectors is addressed.

### 3.4.1 Structural Stiffness Matrix

Equation 3.43 presents the expression of the fundamental nucleus of the Structural Stiffness Matrix $\mathbf{K}^{\tau s i j}$. As previously mentioned, the material properties are assumed to be constant along the single finite element axis. This fact has allowed the contributions of the material stiffness matrices to be taken out of the integral along the element length, see Eq. 3.41. As far as the material properties over the cross-section are concerned, two kinds of lamination can appear. Firstly, the cross-section might be homogeneous, i.e it presents a unique material lamination. Otherwise, the cross-section might be nonhomogeneous, i.e it can be subdivided into different subsections with their own different laminations. More details about the latter case are given later.

## Homogeneous cross-section case

When the material over the cross-section is homogeneous, the material stiffness matrices can be taken out of the integral over $\Omega$ in Eq. 3.43. The nine integrals of the products of cross-section functions and their derivatives over $\Omega$ are introduced:

$$
\begin{array}{lll}
J_{s}^{\tau}=\int_{\Omega} F_{\tau} F_{s} \mathrm{~d} \Omega & J_{s}^{\tau, x}=\int_{\Omega} F_{\tau, x} F_{s} \mathrm{~d} \Omega & J_{s, x}^{\tau}=\int_{\Omega} F_{\tau} F_{s, x} \mathrm{~d} \Omega \\
J_{s}^{\tau, z}=\int_{\Omega} F_{\tau, z} F_{s} \mathrm{~d} \Omega & J_{s, z}^{\tau}=\int_{\Omega} F_{\tau} F_{s, z} \mathrm{~d} \Omega & J_{s, z}^{\tau, x}=\int_{\Omega} F_{\tau, x} F_{s, z} \mathrm{~d} \Omega  \tag{3.114}\\
J_{s, x}^{\tau, z}=\int_{\Omega} F_{\tau, z} F_{s, x} \mathrm{~d} \Omega & J_{s, x}^{\tau, x}=\int_{\Omega} F_{\tau, x} F_{s, x} \mathrm{~d} \Omega & J_{s, z}^{\tau, z}=\int_{\Omega} F_{\tau, z} F_{s, z} \mathrm{~d} \Omega
\end{array}
$$

It is noteworthy to highlight that the nine terms in Eq. 3.114 are scalar quantities as well as the four terms in Eq. 3.44. As can be seen in Eqs. 3.41 and 3.43, the fundamental nucleus of the Structural Stiffness Matrix is the result of the summation of nine terms. Each of such terms is a $3 \times 3$ matrix and, in general, represents the integration over the single element domain of the product of different quantities such as cross-section functions, shape functions and material stiffness matrices. Of course, the result is again a $3 \times 3$ matrix, whose nine components can be computed by performing explicitly all the multiplications in Eq. 3.43 .

For a cross-section made of homogeneous orthotropic material, the components of the
fundamental nucleus $\mathbf{K}^{\text {qsij }}$ are:

$$
\begin{align*}
& K_{x x}^{\tau s i j}=\widetilde{C}_{22} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{44} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j, y}^{i} J_{s}^{\tau, x}+\widetilde{C}_{26} E_{j}^{i, y} J_{s, x}^{\tau}+\widetilde{C}_{66} E_{j, y}^{i, y} J_{s}^{\tau} \\
& K_{x y}^{\tau s i j}=\widetilde{C}_{23} E_{j, y}^{i} J_{s}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{36} E_{j, y}^{i, y} J_{s}^{\tau}+\widetilde{C}_{66} E_{j}^{i, y} J_{s, x}^{\tau} \\
& K_{x z}^{\tau s i j}=\widetilde{C}_{12} E_{j}^{i} J_{s, z}^{\tau, x}+\widetilde{C}_{44} E_{j}^{i} J_{s, x}^{\tau, z}+\widetilde{C}_{45} E_{j, y}^{i} J_{s}^{\tau, z}+\widetilde{C}_{16} E_{j}^{i, y} J_{s, z}^{\tau} \\
& K_{y x}^{\tau s i j}=\widetilde{C}_{23} E_{j}^{i, y} J_{s, x}^{\tau}+\widetilde{C}_{45} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{36} E_{j, y}^{i, y} J_{s}^{\tau}+\widetilde{C}_{66} E_{j, y}^{i} J_{s}^{\tau, x} \\
& K_{y y}^{\tau s i j}=\widetilde{C}_{33} E_{j, y}^{i, y} J_{s}^{\tau}+\widetilde{C}_{55} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{36} E_{j, y}^{i} J_{s}^{\tau, x}+\widetilde{C}_{36} E_{j}^{i, y} J_{s, x}^{\tau}+\widetilde{C}_{66} E_{j}^{i} J_{s, x}^{\tau, x} \\
& K_{y z}^{\tau s i j}=\widetilde{C}_{13} E_{j}^{i, y} J_{s, z}^{\tau}+\widetilde{C}_{55} E_{j, y}^{i} J_{s}^{\tau, z}+\widetilde{C}_{45} E_{j}^{i} J_{s, x}^{\tau, z}+\widetilde{C}_{16} E_{j}^{i} J_{s, z}^{\tau, x} \\
& K_{z x}^{\tau s i j}=\widetilde{C}_{12} E_{j}^{i} J_{s, x}^{\tau, z}+\widetilde{C}_{44} E_{j}^{i} J_{s, z}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i, y} J_{s, z}^{\tau}+\widetilde{C}_{16} E_{j, y}^{i} J_{s}^{\tau, z} \\
& K_{z y}^{\tau s i j}=\widetilde{C}_{13} E_{j, y}^{i} J_{s}^{\tau, z}+\widetilde{C}_{55} E_{j}^{i, y} J_{s, z}^{\tau}+\widetilde{C}_{45} E_{j}^{i} J_{s, z}^{\tau, x}+\widetilde{C}_{16} E_{j}^{i} J_{s, x}^{\tau, z} \\
& K_{z z}^{\tau s i j}=\widetilde{C}_{11} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{44} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{55} E_{j, y}^{i, y} J_{s}^{\tau}+\widetilde{C}_{45} E_{j, y}^{i} J_{s}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i, y} J_{s, x}^{\tau} \tag{3.115}
\end{align*}
$$

For a cross-section made of homogeneous isotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ present less terms than the ortotropic case:

$$
\begin{align*}
& K_{x x}^{\tau s i j}=C_{22} E_{j}^{i} J_{s, x}^{\tau, x}+C_{44} E_{j}^{i} J_{s, z}^{\tau, z}+C_{66} E_{j, y}^{i, y} J_{s}^{\tau} \\
& K_{x y}^{\tau s i j}=C_{23} E_{j, y}^{i} J_{s}^{\tau, x}+C_{66} E_{j}^{i, y} J_{s, x}^{\tau} \\
& K_{x z}^{\tau s i j}=C_{12} E_{j}^{i} J_{s, z}^{\tau, x}+C_{44} E_{j}^{i} J_{s, x}^{\tau, z} \\
& K_{y x}^{\tau s i j}=C_{23} E_{j}^{i, y} J_{s, x}^{\tau}+C_{66} E_{j, y}^{i} J_{s}^{\tau, x} \\
& K_{y y}^{\tau s i j}=C_{33} E_{j, y}^{i, y} J_{s}^{\tau}+C_{55} E_{j}^{i} J_{s, z}^{\tau, z}+C_{66} E_{j}^{i} J_{s, x}^{\tau, x}  \tag{3.116}\\
& K_{y z}^{\tau s i j}=C_{13} E_{j}^{i, y} J_{s, z}^{\tau}+C_{55} E_{j, y}^{i} J_{s}^{\tau, z} \\
& K_{z x}^{\tau s i j}=C_{12} E_{j}^{i} J_{s, x}^{\tau, z}+C_{44} E_{j}^{i} J_{s, z}^{\tau, x} \\
& K_{z y}^{\tau s i j}=C_{13} E_{j, y}^{i} J_{s}^{\tau, z}+C_{55} E_{j}^{i, y} J_{s, z}^{\tau} \\
& K_{z z}^{\tau s i j}=C_{11} E_{j}^{i} J_{s, z}^{\tau, z}+C_{44} E_{j}^{i} J_{s, x}^{\tau, x}+C_{55} E_{j, y}^{i, y} J_{s}^{\tau}
\end{align*}
$$

It is important to note the symmetry of the fundamental nucleus, which is preserved also for the nonhomogeneous cross-section case. In fact:

$$
\begin{equation*}
\mathbf{K}^{\tau s i j}=\mathbf{K}^{s \tau j i T} \tag{3.117}
\end{equation*}
$$

## Nonhomogeneous cross-section case

When the lamination of the material over the cross-section of the structure is nonhomogeneous, the cross-section can be subdivided into different subsections with their own different laminations. The shape of the cross-section is arbitrary as well as the shape of each subsection the cross-section is made of. The sample case of three subsections is depicted in Fig. 3.11. The number of subsections is referred as $N_{S}$ and the index $k$ is employed to refer to the $k^{t h}$ subsection $\Omega_{k}$.


Figure 3.11: Example of a nonhomogeneous cross-section with three subsections $\left(N_{S}=3\right)$.

In general, the material properties can be different for each subsection. It means that different Young's moduli, Poisson's ratios, lamination angles or material densities can be taken into account over the cross-section $\Omega$. As addressed in section 2.3, orthotropic and isotropic materials are here considered for the $k^{t h}$ subsection. More details on the material properties of orthotropic and isotropic materials can be found in sections 2.3.2 and 2.3.1, respectively.

If the material of the $k^{t h}$ subsection of the nonhomogeneous cross-section is orthotropic, the corresponding $k^{\text {th }}$ Hooke's law refers to Eq. 2.27:

$$
\left\{\begin{align*}
\boldsymbol{\sigma}_{p}^{k} & =\widetilde{\mathbf{C}}_{p p}^{k} \boldsymbol{\varepsilon}_{p}+\widetilde{\mathbf{C}}_{p n}^{k} \boldsymbol{\varepsilon}_{n}  \tag{3.118}\\
\boldsymbol{\sigma}_{n}^{k} & =\widetilde{\mathbf{C}}_{n p}^{k} \boldsymbol{\varepsilon}_{p}+\widetilde{\mathbf{C}}_{n n}^{k} \boldsymbol{\varepsilon}_{n}
\end{align*}\right.
$$

where matrices $\widetilde{\mathbf{C}}_{p p}^{k}, \widetilde{\mathbf{C}}_{p n}^{k}, \widetilde{\mathbf{C}}_{n p}^{k}$, and $\widetilde{\mathbf{C}}_{n n}^{k}$ derive from the matrices in Eq. 2.28:

$$
\begin{gather*}
\widetilde{\mathbf{C}}_{p p}^{k}=\left[\begin{array}{ccc}
\widetilde{C}_{11}^{k} & \widetilde{C}_{12}^{k} & 0 \\
\widetilde{C}_{12}^{k} & \widetilde{C}_{22}^{k} & 0 \\
0 & 0 & \widetilde{C}_{44}^{k}
\end{array}\right] ; \quad \widetilde{\mathbf{C}}_{p n}^{k}=\left[\begin{array}{ccc}
0 & \widetilde{C}_{11}^{k} & \widetilde{C}_{13}^{k} \\
0 & \widetilde{C}_{26}^{k} & \widetilde{C}_{23}^{k} \\
\widetilde{C}_{45}^{k} & 0 & 0
\end{array}\right] ; \\
\widetilde{\mathbf{C}}_{n p}^{k}=\widetilde{\mathbf{C}}_{p n}^{k T}=\left[\begin{array}{ccc}
0 & 0 & \widetilde{C}_{45}^{k} \\
\widetilde{C}_{16}^{k} & \widetilde{C}_{26}^{k} & 0 \\
\widetilde{C}_{13}^{k} & \widetilde{C}_{23}^{k} & 0
\end{array}\right] ; \quad \widetilde{\mathbf{C}}_{n n}^{k}=\left[\begin{array}{ccc}
\widetilde{C}_{55}^{k} & 0 & 0 \\
0 & \widetilde{C}_{66}^{k} & \widetilde{C}_{36}^{k} \\
0 & \widetilde{C}_{36}^{k} & \widetilde{C}_{33}^{k}
\end{array}\right] \tag{3.119}
\end{gather*}
$$

Hooke's law for the $k^{t h}$ subsection made of isotropic material refers instead to Eq. 2.32:

$$
\left\{\begin{align*}
\boldsymbol{\sigma}_{p}^{k} & =\mathbf{C}_{p p}^{k} \varepsilon_{p}+\mathbf{C}_{p n}^{k} \varepsilon_{n}  \tag{3.120}\\
\boldsymbol{\sigma}_{n}^{k} & =\mathbf{C}_{n p}^{k} \boldsymbol{\varepsilon}_{p}+\mathbf{C}_{n n}^{k} \varepsilon_{n}
\end{align*}\right.
$$

where matrices $\mathbf{C}_{p p}^{k}, \mathbf{C}_{p n}^{k}, \mathbf{C}_{n p}^{k}$, and $\mathbf{C}_{n n}^{k}$ derive from the matrices in Eq. 2.33:

$$
\begin{array}{cc}
\mathbf{C}_{p p}^{k}=\left[\begin{array}{ccc}
C_{11}^{k} & C_{12}^{k} & 0 \\
C_{12}^{k} & C_{22}^{k} & 0 \\
0 & 0 & C_{44}^{k}
\end{array}\right] ; & \mathbf{C}_{p n}^{k}=\left[\begin{array}{ccc}
0 & 0 & C_{13}^{k} \\
0 & 0 & C_{23}^{k} \\
0 & 0 & 0
\end{array}\right] ; \\
\mathbf{C}_{n p}^{k}=\mathbf{C}_{p n}^{k}=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
C_{13}^{k} & C_{23}^{k} & 0
\end{array}\right] ; & \mathbf{C}_{n n}^{k}=\left[\begin{array}{ccc}
C_{55}^{k} & 0 & 0 \\
0 & C_{66}^{k} & 0 \\
0 & 0 & C_{33}^{k}
\end{array}\right] \tag{3.121}
\end{array}
$$

The formula of the fundamental nucleus of the Structural Stiffness Matrix in Eq. 3.43 is still valid for the nonhomogeneous cross-section case, since this formula is general. However,
the choice of a nonhomogeneous cross-section influences the integration over $\Omega$ indicated by the symbol $\triangleleft \ldots \triangleright_{\Omega}$ in Eq. 3.45. In fact, for nonhomogeneous sections the integral over $\Omega$ includes the contributions corresponding to each of the $N_{S}$ subsections as expressed in Eq. 3.122:

$$
\begin{equation*}
\triangleleft \ldots \triangleright_{\Omega}=\int_{\Omega} \ldots \mathrm{d} \Omega=\sum_{k=1}^{N_{S}} \int_{\Omega_{k}} \ldots \mathrm{~d} \Omega_{k} \tag{3.122}
\end{equation*}
$$

where $\Omega_{k}$ is the $k^{\text {th }}$ subsection. This method is consistent with the equivalent single-layer approach widely used for layered structures, where a homogenization of the material properties is conducted by summing the contributions of each layer in the stiffness matrix. Each term in Eq. 3.43 has thus to be integrated following the formula in Eq. 3.122. For the sake of completeness, a sample case is here presented. Let the sample cross-section with three subsections depicted in Fig. 3.11 to be considered. These subsections are denoted with indices 1, 2 and 3. For instance, the last integral term of Eq. 3.43 is computed via three contributions as follows:

$$
\begin{equation*}
\triangleleft F_{\tau} \widetilde{\mathbf{C}}_{n n} F_{s} \triangleright \Omega_{\Omega}=\int_{\Omega_{1}} F_{\tau} \widetilde{\mathbf{C}}_{n n}^{1} F_{s} \mathrm{~d} \Omega_{1}+\int_{\Omega_{2}} F_{\tau} \widetilde{\mathbf{C}}_{n n}^{2} F_{s} \mathrm{~d} \Omega_{2}+\int_{\Omega_{3}} F_{\tau} \widetilde{\mathbf{C}}_{n n}^{3} F_{s} \mathrm{~d} \Omega_{3} \tag{3.123}
\end{equation*}
$$

where $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ are the areas of subsections 1,2 and 3 . In other words, for a nonhomogeneous cross-section the material stiffness matrices cannot be taken out of the integral over $\Omega$, unlike the homogeneous case (Eq. 3.114).

For a cross-section made of nonhomogeneous orthotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ are:

$$
\begin{aligned}
K_{x x}^{\tau s i j}= & E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{22} F_{s, x} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{44} F_{s, z} \triangleright \Omega+E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s} \triangleright \Omega+ \\
& E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{26} F_{s, x} \triangleright \Omega+E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{66} F_{s} \triangleright \Omega \\
K_{x y}^{\tau s i j}= & E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{23} F_{s} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s, x} \triangleright \Omega+ \\
& E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s} \triangleright \Omega+E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{66} F_{s, x} \triangleright \Omega \\
K_{x z}^{\tau s i j}= & E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{12} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{44} F_{s, x} \triangleright \Omega+E_{j, y}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s} \triangleright \Omega+ \\
& E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{16} F_{s, z} \triangleright \Omega \\
K_{y x}^{\tau s i j}= & E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{23} F_{s, x} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s, x} \triangleright \Omega+ \\
& E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s} \triangleright \Omega+E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{66} F_{s} \triangleright \Omega \\
K_{y y}^{\tau s i j}= & E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{33} F_{s} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{55} F_{s, z} \triangleright \Omega+E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{36} F_{s} \triangleright \Omega+ \\
& E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s, x} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{66} F_{s, x} \triangleright \Omega
\end{aligned}
$$

$$
\begin{align*}
K_{y z}^{\tau s i j}= & E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{13} F_{s, z} \triangleright{ }_{\Omega}+E_{j, y}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{55} F_{s} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, x} \triangleright{ }_{\Omega}+ \\
& E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{16} F_{s, z} \triangleright{ }^{\prime} \\
K_{z x}^{\tau s i j}= & E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{12} F_{s, x} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{44} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{45} F_{s, z} \triangleright{ }_{\Omega}+ \\
& E_{j, y}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{16} F_{s} \triangleright \Omega \\
K_{z y}^{\tau s i j}= & E_{j, y}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{13} F_{s} \triangleright \Omega+E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{55} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{45} F_{s, z} \triangleright \Omega+ \\
& E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{16} F_{s, x} \triangleright{ }_{\Omega} \\
K_{z z}^{\tau s i j}= & E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{11} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{44} F_{s, x} \triangleright \Omega+E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{55} F_{s} \triangleright{ }_{\Omega}+ \\
& E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{45} F_{s} \triangleright \Omega+E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{45} F_{s, x} \triangleright \Omega \tag{3.124}
\end{align*}
$$

For a cross-section made of nonhomogeneous isotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ present less terms than the ortotropic case:

$$
\begin{align*}
& K_{x x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} C_{22} F_{s, x} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} C_{44} F_{s, z} \triangleright{ }_{\Omega}+E_{j, y}^{i, y} \triangleleft F_{\tau} C_{66} F_{s} \triangleright_{\Omega} \\
& K_{x y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, x} C_{23} F_{s} \triangleright_{\Omega}+E_{j}^{i, y} \triangleleft F_{\tau} C_{66} F_{s, x} \triangleright_{\Omega} \\
& K_{x z}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} C_{12} F_{s, z} \triangleright_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} C_{44} F_{s, x} \triangleright{ }_{\Omega} \\
& K_{y x}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} C_{23} F_{s, x} \triangleright{ }_{\Omega}+E_{j, y}^{i} \triangleleft F_{\tau, x} C_{66} F_{s} \triangleright{ }_{\Omega} \\
& K_{y y}^{\tau s i j}=E_{j, y}^{i, y} \triangleleft F_{\tau} C_{33} F_{s} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} C_{55} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} C_{66} F_{s, x} \triangleright{ }_{\Omega} \\
& K_{y z}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} C_{13} F_{s, z} \triangleright_{\Omega}+E_{j, y}^{i} \triangleleft F_{\tau, z} C_{55} F_{s} \triangleright{ }_{\Omega} \\
& K_{z x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, z} C_{12} F_{s, x} \triangleright \Omega_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} C_{44} F_{s, z} \triangleright_{\Omega} \\
& K_{z y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, z} C_{13} F_{s} \triangleright_{\Omega}+E_{j}^{i, y} \triangleleft F_{\tau} C_{55} F_{s, z} \triangleright_{\Omega} \\
& K_{z z}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, z} C_{11} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} C_{44} F_{s, x} \triangleright{ }_{\Omega}+E_{j, y}^{i, y} \triangleleft F_{\tau} C_{55} F_{s} \triangleright{ }_{\Omega} \tag{3.125}
\end{align*}
$$

## Construction of the Element Structural Stiffness Matrix

The nine components of the $3 \times 3$ fundamental nucleus of the Structural Stiffness Matrix have been explicitly computed in the previous section for a homogeneous cross-section (Eqs. 3.115 and 3.116 for orthotropic and isotropic materials) and a nonhomogeneneous cross-section (Eqs. 3.124 and 3.125 for orthotropic and isotropic materials).

As can be seen in virtual strain energy equation (Eq. 3.42), the fundamental nucleus $\mathbf{K}^{\tau s i j}$ refers to the generic virtual nodal displacement vector $\mathbf{q}_{\tau i}$, which contains the degrees of freedom of the generic $\tau^{t h}$ expansion term corresponding to the $i^{t h}$ element node, and to the generic nodal displacement vector $\mathbf{q}_{s j}$, which contains the degrees of freedom of the generic $s^{\text {th }}$ expansion term corresponding to the $j^{t h}$ element node:

$$
\begin{equation*}
\delta L_{\mathrm{int}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{K}^{\tau s i j} \mathbf{q}_{s j} \tag{3.126}
\end{equation*}
$$

where it is important to remind that repeated subscripts indicate summation based on Einstein's notation. Hence, the fundamental nucleus $\mathbf{K}^{\tau s i j}$ is now expanded with respect to the indices $\tau, s, i$, and $j$ in order to build the Element Structural Stiffness Matrix $\mathbf{K}_{\mathrm{EL}}$,
that is the stiffness matrix of the single finite element with $N_{N}$ nodes based on the CUF formulation with expansion order $N$. The procedure to build this matrix is illustrated in Fig. 3.12, where the sample case of a $B 3$ finite element (defined in section 3.1.2) and a second-order Taylor expansion $(N=2)$ is considered.

( $N=2$ ), the number of expansion terms $N_{u}$ is equal to 6 , according to Eq. 2.35. As depicted in Fig. 3.12, 36 fundamental nuclei for the generic combination of $i^{t h}$ and $j^{t h}$ nodes are collected. The notation used here employs index $\tau$ for rows and index $s$ for columns. The order used to sort these 36 nuclei is of course arbitrary. For the sake of simplicity, indices $\tau$ and $s$ are considered in ascending order. In this case, according to Eq. 3.25, the dimensions of the block of these 36 nuclei are equal to $\mathrm{DOFs}_{N} \times \mathrm{DOFs}_{N}=18 \times 18$.

The second expansion to be carried out is on the nodes of the single element considered. Different one-dimensional finite elements have been defined in sections 3.1.1, 3.1.2 and 3.1.3. The fundamental nuclei previously expanded on indices $\tau$ and $s$ for the generic combination of $i$ and $j$ values are now computed for all the combinations of $i$ and $j$ values of the finite element, i.e. $N_{N} \times N_{N}$ combinations. The sample case here proposed refers to the generic $B 3$ element $\left(N_{N}=3\right)$ depicted in Fig. 3.4. As depicted in Fig. 3.12, 9 combinations $(i, j)$ are possible and then 9 blocks are collected. The notation used here employs index $i$ for rows and index $j$ for columns. The order used to sort these 9 blocks nuclei is of course arbitrary, but a particular sorting choice is convenient and the reason will be clear in the assembly procedure, see section 3.4.1. This choice does not sort $i$ and $j$ indices in ascending order, but following a physical order. In other words, looking at Fig. 3.4, the order of $i$ and $j$ values follows axis $y$, i.e. Nodes $1^{\mathrm{EL}}, 3^{\mathrm{EL}}, 2^{\mathrm{EL}}$. This sorting choice is followed in the construction of the Element Structural Stiffness Matrix, as shown in Fig. 3.12. In this case, according to Eq. 3.21, the dimensions of this square matrix $\mathbf{K}_{\text {EL }}$ are equal to $\mathrm{DOFs}_{E L} \times \mathrm{DOFs}_{E L}=54 \times 54$.

## Assembly procedure

In the previous section, the procedure to build the Element Stiffness Matrix $\mathbf{K}_{\text {EL }}$ of the single finite element with $N_{N}$ nodes has been described. The numerical procedure that allows to build the Structural Stiffness Matrix K starting from the Element Stiffness Matrices associated to all the $N_{\text {EL }}$ elements of the mesh is called assembly procedure and is a standard process of the finite element method.

As introduced in sections 3.1.4 and 3.3, the structural domain is discretized by a mesh, i.e. a series of connected (non-overlapping) one-dimensional finite elements. The sample case of two connected $B 3$ elements previously depicted in Fig. 3.8 is now slightly modified in Fig. 3.13. Given a mesh of one-dimensional finite elements, an identifier $\mathrm{ID}_{N}$ is assigned


Figure 3.13: Example of a mesh with two $B 3$ elements assembled and notation used to enumerate nodes.
to each node of the mesh. The assembly procedure is considerably simplified if there are no "gaps" in the node numbering sequence. For the sake of simplicity, here it is thus assumed that there are no gaps. For instance, the five nodes of the sample mesh in Fig. 3.13 are numbered from 1 to 5 . It is noteworthy that the identifier $\mathrm{ID}_{N}$ is unique for each node of the mesh. The notation here employed includes the identifier of the node of the mesh into a circle, as depicted in Fig. 3.13, and is used in order to not confuse the identifier $\mathrm{ID}_{N}$ with the local identification number of the node on the element, i.e. $1^{\mathrm{EL}}, 2^{\mathrm{EL}}$ for a $B 2$ element, $1^{\mathrm{EL}}, 2^{\mathrm{EL}}, 3^{\mathrm{EL}}$ for a $B 3$ element, $1^{\mathrm{EL}}, 2^{\mathrm{EL}}, 3^{\mathrm{EL}}, 4^{\mathrm{EL}}$ for a $B 4$ element.

In finite element modeling, a standard procedure consists in building a connectivity matix, which relates the local identification numbers of the nodes on each element to the identifiers $\mathrm{ID}_{N}$ of each node of the mesh. In other words, the connectivity matrix summarizes the numbers associated to the nodes of each element and highlights the node that two adjacent elements share. An example of connectivity matrix is reported in Table 3.2 for the case depicted in Fig. 3.13. As far as DOFs are concerned, the connectivity matrix relates to each element degree of freedom the corresponding assemblage degree of freedom.

Table 3.2: Connectivity matrix for the sample case of a mesh of two $B 3$ elements.

| ID-EL | Node $1^{\mathrm{EL}}$ | Node $3^{\mathrm{EL}}$ | Node $2^{\mathrm{EL}}$ |
| :---: | :---: | :---: | :---: |
| 1 | (1) | (2) | (3) |
| 2 | (3) | (4) | (5) |

The assembly procedure realizes numerically the connection of different finite elements and satisfies the congruence and equilibrium of the structure. The assembly procedure consists in collecting the Element Structural Stiffness Matrices computed for each element in order to build the Structural Stiffness Matrix K. The position of the Element Structural Stiffness Matrices over $\mathbf{K}$ follows the order of the node identifiers in the connectivity matrix. For the sample case proposed, the two $B 3$ elements share the node with $\mathrm{ID}_{N}$ equal to (3). This node is the node referred as $2^{\mathrm{EL} 1}$ for element 1 and is the node referred as $1^{\mathrm{EL}} 2$ for element 2, according to Table 3.2. This means that the Element Structural Stiffness Matrices $\mathbf{K}_{\mathrm{EL} 1}$ and $\mathbf{K}_{\mathrm{EL} 2}$ are collected in matrix $\mathbf{K}$ as illustrated in Fig. 3.14, where the blocks corresponding to the shared node (3) $=2^{\mathrm{EL} 1}=1^{\mathrm{EL} 2}$ are superimposed. The same order used to build the Structural Stiffness Matrix $\mathbf{K}$ will be coherently followed in the construction of the Mass Matrix $\mathbf{K}$ and the Vector of Equivalent Nodal Forces $\mathbf{F}$ in following sections. Obviously, the same method is used to collect the degrees of freedom in the nodal displacement and acceleration vectors $\mathbf{q}$ and $\ddot{\mathbf{q}}$ introduced in section 3.3 to write the governing equations of the elasticity problem.

### 3.4.2 Mass Matrix

Equation 3.52 presents the expression of the fundamental nucleus of the Mass Matrix $\mathbf{M}^{\tau s i j}$. In section 3.4.1 the difference between a homogeneous or a nonhomogeneous crosssection about the material properties integration has been highlighted fo the Structural Stiffness Matrix. Also for the Mass Matrix computation, the choice of a homogeneous or a nonhomogeneous cross-section influences the integration over $\Omega$, since the integral in Eq. 3.52 contains the density material.

## Homogeneous cross-section case

When the material over the cross-section is homogeneous, the density material is a unique scalar coefficient and then can be taken out of the integral over $\Omega$ in Eq. 3.52. The integral of the product of cross-section functions over $\Omega$ is retrieved from Eq. 3.114:

$$
\begin{equation*}
J_{s}^{\tau}=\int_{\Omega} F_{\tau} F_{s} \mathrm{~d} \Omega \tag{3.127}
\end{equation*}
$$



Figure 3.14: Assembly procedure to build the Structural Stiffness Matrix $\mathbf{K}$ of two $B 3$ elements.

For a cross-section made of homogeneous orthotropic material, the components of the fundamental nucleus $\mathbf{M}^{\tau s i j}$ are:

$$
\begin{gather*}
M_{x x}^{\tau s i j}=M_{y y}^{\tau s i j}=M_{z z}^{\tau s i j}=\rho E_{j}^{i} J_{s}^{\tau}  \tag{3.128}\\
M_{x y}^{\tau s i j}=M_{x z}^{\tau s i j}=M_{y x}^{\tau s i j}=M_{y z}^{\tau s i j}=M_{z x}^{\tau s i j}=M_{z y}^{\tau s i j}=0
\end{gather*}
$$

It is important to note the symmetry of the fundamental nucleus, which is preserved also for the nonhomogeneous cross-section case. In fact:

$$
\begin{equation*}
\mathbf{M}^{\tau s i j}=\mathbf{M}^{s \tau j i T}=\mathbf{M}^{s \tau j i} \tag{3.129}
\end{equation*}
$$

## Nonhomogeneous cross-section case

The formula of the fundamental nucleus of the Mass Matrix in Eq. 3.52 is still valid for the nonhomogeneous cross-section case, since this formula is general. However, for a nonhomogeneous cross-section, the integral over $\Omega$ includes the contributions corresponding to each of the $N_{S}$ subsections, as expressed in Eq. 3.122. For the sake of completeness, let the sample cross-section with three subsections depicted in Fig. 3.11 to be considered. According to Eq. 3.123 for the fundamental nucleus of the Structural Stiffness Matrix, the integral over $\Omega$ for the Mass Matrix nucleus in Eq. 3.52 is computed as follows:

$$
\begin{equation*}
\triangleleft \rho F_{\tau} F_{s} \triangleright \Omega_{\Omega}=\rho_{1} \int_{\Omega_{1}} F_{\tau} F_{s} \mathrm{~d} \Omega_{1}+\rho_{2} \int_{\Omega_{2}} F_{\tau} F_{s} \mathrm{~d} \Omega_{2}+\rho_{3} \int_{\Omega_{3}} F_{\tau} F_{s} \mathrm{~d} \Omega_{3} \tag{3.130}
\end{equation*}
$$

where $\rho_{1}, \rho_{2}$ and $\rho_{3}$ are the density of the materials which subsections 1,2 and 3 are made of. $\Omega_{1}, \Omega_{2}$ and $\Omega_{3}$ are the areas of subsections 1,2 and 3 . In other words, for a
nonhomogeneous cross-section the material density cannot be taken out of the integral over $\Omega$, unlike the homogeneous case (Eq. 3.127).

For a cross-section made of nonhomogeneous orthotropic material, the components of the fundamental nucleus $\mathbf{M ~}^{\tau s i j}$ are:

$$
\begin{gather*}
M_{x x}^{\tau s i j}=M_{y y}^{\tau s i j}=M_{z z}^{\tau s i j}=E_{j}^{i} \triangleleft \rho F_{\tau} F_{s} \triangleright \Omega \\
M_{x y}^{\tau s i j}=M_{x z}^{\tau s i j}=M_{y x}^{\tau s i j}=M_{y z}^{\tau s i j}=M_{z x}^{\tau s i j}=M_{z y}^{\tau s i j}=0 \tag{3.131}
\end{gather*}
$$

## Construction of the Element Mass Matrix

The nine components of the $3 \times 3$ fundamental nucleus of the Mass Matrix have been explicitly computed in the previous section for a homogeneous cross-section (Eq. 3.128) and a nonhomogeneneous cross-section (Eq. 3.131).

As can be seen in virtual work of inertial loadings (Eq. 3.51), the fundamental nucleus $\mathbf{M}^{\tau s i j}$ refers to the generic virtual nodal displacement vector $\mathbf{q}_{\tau i}$, which contains the degrees of freedom of the generic $\tau^{\text {th }}$ expansion term corresponding to the $i^{\text {th }}$ element node, and to the generic nodal displacement vector $\mathbf{q}_{s j}$, which contains the degrees of freedom of the generic $s^{t h}$ expansion term corresponding to the $j^{t h}$ element node:

$$
\begin{equation*}
\delta L_{\mathrm{ine}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{M}^{\tau s i j} \ddot{\mathbf{q}}_{s j} \tag{3.132}
\end{equation*}
$$

where it is important to remind that repeated subscripts indicate summation based on Einstein's notation. Hence, the fundamental nucleus $\mathbf{M}^{\tau s i j}$ is now expanded with respect to the indices $\tau, s, i$, and $j$ in order to build the Element Mass Matrix $\mathbf{M}_{\mathrm{EL}}$, that is the mass matrix of the single finite element with $N_{N}$ nodes based on the CUF formulation with expansion order $N$. The procedure to build this matrix is exactly the same of that followed for the Element Structural Stiffness Matrix in the previous section. For the sake of completeness, it is illustrated in Fig. 3.15, where the sample case of a $B 3$ finite element (defined in section 3.1.2) and a second-order Taylor expansion $(N=2)$ is considered.

Firstly, as done for the construction of $\mathbf{K}_{\mathrm{EL}}$, the expansion on the CUF indices $\tau$ and $s$ is carried out. This leads to collect $N_{u} \times N_{u}$ fundamental nuclei for the generic combination of $i$ and $j$ values. In the sample case proposed $(N=2)$, the number of expansion terms $N_{u}$ is equal to 6, according to Eq. 2.35. As depicted in Fig. 3.15, 36 fundamental nuclei for the generic combination of $i^{\text {th }}$ and $j^{\text {th }}$ nodes are collected. The expansion on the nodes of the single element is then performed by considering the $N_{N} \times N_{N}$ combinations of $i$ and $j$ values. The sample case here proposed refers to the generic $B 3$ element ( $N_{N}=3$ ) depicted in Fig. 3.4. Hence, 9 combinations $(i, j)$ are possible and then the Element Mass Matrix $\mathbf{M}_{\mathrm{EL}}$ is composed of 9 blocks, as depicted in Fig. 3.15.

## Assembly procedure

In this section the method to build the Mass Matrix M starting from the Element Mass Matrices previously computed is presented. The method to be implemented is exactly the same used in section 3.4.1 to obtain the Structural Stiffness Matrix. Referring to the same sample mesh of two $B 3$ elements depicted in Fig. 3.13 the position of the Element Mass Matrices over $\mathbf{M}$ follows the order of the node identifiers in the connectivity matrix in Table 3.2. The elements share the node with $\mathrm{ID}_{N}$ equal to (3). Since this node is also referred as $2^{\mathrm{EL} 1}$ for element 1 as well as $1^{\mathrm{EL} 2}$ for element 2 , the block with dimensions $\mathrm{DOFs}_{N} \times \mathrm{DOFs}_{N}$ corresponding to the combination $(i=2, j=2)$ of $\mathrm{M}_{\mathrm{EL} 1}$ is superimposed to the block with dimensions $\mathrm{DOFs}_{N} \times \mathrm{DOFs}_{N}$ corresponding to the


Figure 3.15: Procedure to build the Element Mass Matrix $\mathbf{M}_{\text {EL }}$ of a $B 3$ element with $N=2$.
combination $(i=1, j=1)$ of $\mathbf{M}_{\mathrm{EL} 2}$. For the sake of completeness, the procedure here described to obtain the Mass Matrix $\mathbf{M}$ to be used in governing equations of section 3.3 is sketched in Fig. 3.16.

### 3.4.3 Vector of Equivalent Nodal Forces

Equation 3.79 presents the expression of the fundamental nucleus of the Vector of Equivalent Nodal Forces $\mathbf{F}_{\tau i}$. In this section the components of this fundamental nucleus are computed


Figure 3.16: Assembly procedure to build the Mass Matrix $\mathbf{M}$ of two $B 3$ elements.
and the procedure to build the Vector of Equivalent Nodal Forces starting from the nucleus is addressed.

The components of $\mathbf{F}_{\tau i}$ can be expressed following Eq. 3.79:

$$
\begin{align*}
F_{x \tau i} & =F_{x \tau i}^{P}+F_{x \tau i}^{l}+F_{x \tau i}^{s}+F_{x \tau i}^{V} \\
F_{y \tau i} & =F_{y \tau i}^{P}+F_{y \tau i}^{l}+F_{y \tau i}^{s}+F_{y \tau i}^{V}  \tag{3.133}\\
F_{z \tau i} & =F_{z \tau i}^{P}+F_{z \tau i}^{l}+F_{z \tau i}^{s}+F_{z \tau i}^{V}
\end{align*}
$$

Using Eqs. 3.61 and 3.67, Eq. 3.133 becomes:

$$
\begin{align*}
& F_{x \tau i}=F_{x \tau i}^{P}+F_{x \tau i}^{p_{x x}^{l \pm}}+F_{x \tau i}^{p_{z x}^{l \pm}}+F_{x \tau i}^{p_{x x}^{ \pm}}+F_{x \tau i}^{p_{z x}^{ \pm}}+F_{x \tau i}^{V} \\
& F_{y \tau i}=F_{y \tau i}^{P}+F_{y \tau i}^{p_{x y}^{l \pm}}+F_{y \tau i}^{p_{z y}^{l \pm}}+F_{y \tau i}^{p_{x y}^{ \pm}}+F_{y \tau i}^{p_{z y}^{ \pm}}+F_{y \tau i}^{V}  \tag{3.134}\\
& F_{z \tau i}=F_{z \tau i}^{P}+F_{z \tau i}^{p_{x z}^{l \pm}}+F_{z \tau i}^{p_{z}^{l \pm}}+F_{z \tau i}^{p_{x z}^{ \pm}}+F_{z \tau i}^{p_{z z}^{ \pm}}+F_{z \tau i}^{V}
\end{align*}
$$

The components of the fundamental nucleus $\mathbf{F}_{\tau i}$ can be explicitly computed following Eq. 3.134 and retrieving the expressions of the components of the fundamental nuclei $\mathbf{F}_{\tau i}^{P}$, $\mathbf{F}_{\tau i}^{l}, \mathbf{F}_{\tau i}^{s}, \mathbf{F}_{\tau i}^{V}$ in Eqs. 3.57, 3.59, 3.65, 3.74, respectively:

$$
\begin{aligned}
F_{x \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{x}}+\int_{L_{x}^{ \pm}} N_{i} p_{x x}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)+ \\
& \int_{L_{z}^{ \pm}} N_{i} p_{z x}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)+\int_{l} N_{i}\left[\int_{b^{ \pm}} F_{\tau} p_{x x}^{ \pm} \mathrm{d} z\right] \mathrm{d} y+ \\
& \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z x}^{ \pm} \mathrm{d} x\right] \mathrm{d} y+\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{x}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y
\end{aligned}
$$

$$
\begin{align*}
F_{y \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{y}}+\int_{L_{x}^{ \pm}} N_{i} p_{x y}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)+ \\
& \int_{L_{z}^{ \pm}} N_{i} p_{z y}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)+\int_{l} N_{i}\left[\int_{b \pm} F_{\tau} p_{x y}^{ \pm} \mathrm{d} z\right] \mathrm{d} y+  \tag{3.135}\\
& \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z y}^{ \pm} \mathrm{d} x\right] \mathrm{d} y+\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{y}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y \\
F_{z \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{z}}+\int_{L_{x}^{ \pm}} N_{i} p_{x z}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right)+ \\
& \int_{L_{z}^{ \pm}} N_{i} p_{z z}^{l \pm} \mathrm{d} y F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right)+\int_{l} N_{i}\left[\int_{b^{ \pm}} F_{\tau} p_{x z}^{ \pm} \mathrm{d} z\right] \mathrm{d} y+ \\
& \int_{l} N_{i}\left[\int_{a^{ \pm}} F_{\tau} p_{z z}^{ \pm} \mathrm{d} x\right] \mathrm{d} y+\int_{l} N_{i}\left[\int_{\Omega} F_{\tau} p_{z}^{V} \mathrm{~d} \Omega\right] \mathrm{d} y
\end{align*}
$$

For the sake of completeness, the case of line, surface and volume loadings constant along the single finite element lenght $L_{\text {EL }}$ is here considered. In this case, it is possible to simplify Eq. 3.4.3 by taking into account Eqs. 3.62, 3.68 and 3.75. Hence:

$$
\begin{align*}
F_{x \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{x}}+E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x x}^{l \pm}+ \\
& E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z x}^{l \pm}+E^{i} I_{\tau}^{p_{x x}^{ \pm}}+E^{i} I_{\tau}^{p_{z x}^{ \pm}}+E^{i} I_{\tau}^{p_{x}^{V}} \\
F_{y \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{y}}+E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x y}^{l \pm}+ \\
& E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z y}^{l \pm}+E^{i} I_{\tau}^{p_{x y}^{ \pm}}+E^{i} I_{\tau}^{p_{z y}^{ \pm}}+E^{i} I_{\tau}^{p_{y}^{V}}  \tag{3.136}\\
F_{z \tau i}= & N_{i}\left(y_{P}\right) F_{\tau}\left(x_{P}, z_{P}\right) P_{u_{z}}+E^{i} F_{\tau}\left(x_{x}^{ \pm}, z_{x}^{ \pm}\right) p_{x z}^{l \pm}+ \\
& E^{i} F_{\tau}\left(x_{z}^{ \pm}, z_{z}^{ \pm}\right) p_{z z}^{l \pm}+E^{i} I_{\tau}^{p_{z z}^{ \pm}}+E^{i} I_{\tau}^{p_{z y}^{ \pm}}+E^{i} I_{\tau}^{p_{z}^{V}}
\end{align*}
$$

where terms $E^{i}, I_{\tau}^{p^{ \pm}}, I_{\tau}^{p^{ \pm}}$, and $I_{\tau}^{\mathbf{p}^{V}}$ have been defined in Eqs. 3.63, 3.70, 3.71, and 3.77, respectively.

## Construction of the Element Vector of Equivalent Nodal Forces

The three components of the $3 \times 1$ fundamental nucleus of the Vector of Equivalent Nodal Forces have been explicitly computed in the previous section by taking into account point, line, surface, and volume loadings (Eqs. 3.4.3 and 3.136).

As can be seen in virtual work of applied loadings (Eq. 3.78), the fundamental nucleus $\mathbf{F}_{\tau i}$ refers to the generic virtual nodal displacement vector $\mathbf{q}_{\tau i}$, which contains the degrees of freedom of the generic $\tau^{\text {th }}$ expansion term corresponding to the $i^{\text {th }}$ element node:

$$
\begin{equation*}
\delta L_{\mathrm{al}}=\delta \mathbf{q}_{\tau i}^{T} \mathbf{F}_{\tau i} \tag{3.137}
\end{equation*}
$$

where it is important to remind that repeated subscripts indicate summation based on Einstein's notation. Hence, the fundamental nucleus $\mathbf{F}_{\tau i}$ is now expanded with respect to the indices $\tau$ and $i$ in order to build the Element Vector of Equivalent Nodal Forces $\mathbf{F}_{\text {EL }}$, that is the vector of nodal forces of the single finite element with $N_{N}$ nodes based on the CUF formulation with expansion order $N$. The procedure to build this vector is similar to that followed for the Element Structural Stiffness Matrix and the Element Mass Matrix in previous sections. For the sake of completeness, it is illustrated in Fig. 3.17, where the sample case of a $B 3$ finite element (defined in section 3.1.2) and a second-order Taylor


Figure 3.17: Procedure to build the element Vector of Equivalent Nodal Forces $\mathbf{F}_{\text {EL }}$ of a $B 3$ element with $N=2$.
expansion $(N=2)$ is considered. The first expansion to be implemented is on the CUF index $\tau$. In other words, for the generic $i^{t h}$ node of the element, the fundamental nuclei for all the $\tau$ values ( $N_{u}$ combinations) have to be computed. In the sample case proposed in Fig. 3.17, $N$ is equal to 2 and the number of expansion terms $N_{u}$ is therefore equal to 6, according to Eq. 2.35. For the generic $i^{t h}$ node, $N_{u}$ nuclei are collected and index $\tau$ is employed for rows in ascending order as done for the construction of $\mathbf{K}_{\text {EL }}$ and $\mathbf{M}_{\mathrm{EL}}$. For $N=2$, the dimensions of the block of these 6 nuclei are equal to $\operatorname{DOFs}_{N} \times 1=18 \times 1$, according to Eq. 3.25.

The second expansion to be carried out is on the nodes of the single element. The fundamental nuclei previously expanded on index $\tau$ for the generic $i$ value are now computed for all the $N_{N}$ values of $i$ for the finite element nodes. For a $B 3$ element, 3 blocks are considered as can be shown in Fig. 3.17. The order to be followed to sort these block is consistent to the choice made for $\mathbf{K}_{\mathrm{EL}}$ and $\mathbf{M}_{\mathrm{EL}}$, where $i$ follows axis $y$, i.e. Nodes $1^{\mathrm{EL}}$, $3^{\mathrm{EL}}, 2^{\mathrm{EL}}$. In the case depicted in Fig. 3.17, the dimensions of the vector $\mathbf{F}_{\mathrm{EL}}$ are equal to DOFseL $\times 1=54 \times 1$, according to Eq. 3.21.

## Assembly procedure

In this section the method to build the Vector of Equivalent Nodal Forces $\mathbf{F}$ starting from the Element Vector of Equivalent Nodal Forces previously computed is presented. The method to be implemented is exactly the same used in sections 3.4.1 and 3.4.2 to obtain the Structural Stiffness Matrix and the Mass Matrix. Referring to the same sample mesh of two $B 3$ elements depicted in Fig. 3.13 the position of the Element Vectors of Equivalent Nodal Forces over $\mathbf{F}$ follows the order of the node identifiers in the connectivity matrix
in Table 3.2. The elements share the node with $\mathrm{ID}_{N}$ equal to (3). Since this node is also referred as $2^{\mathrm{EL} 1}$ for element 1 as well as $1^{\mathrm{EL} 2}$ for element 2, the block with dimensions $\mathrm{DOFs}_{N} \times 1$ corresponding to the combination $(i=2, j=2)$ of $\mathbf{F}_{\mathrm{EL} 1}$ is superimposed to the block with dimensions $\mathrm{DOFs}_{N} \times 1$ corresponding to the combination $(i=1, j=1)$ of $\mathbf{F}_{\text {EL 2 }}$. For the sake of completeness, the procedure here described to obtain the Vectors of Equivalent Nodal Forces $\mathbf{F}$ to be used in governing equations of section 3.3 is sketched in Fig. 3.18.


Figure 3.18: Assembly procedure to build the Vector of Equivalent Nodal Forces $\mathbf{F}$ of two $B 3$ elements.

## Chapter 4

## Results: free vibration analysis of thin and thick shell structures

### 4.1 Assessment with 3D elasticity solutions

First of all, the proposed 1D structural model is compared to free vibration results based on three-dimensional analysis in [88] and exact analysis of reference [85]. In [88] the governing equations of three-dimensional linear elasticity were solved by using an iterative approach based on the introduction of fictitious layers along the shell thickness. Armenàkas et al. [85] provided exact natural frequencies of harmonic elastic waves propagating in an infinitely long isotropic hollow cylinder. Nonetheless, this work may be used directly to obtain the frequency of standing waves propagating in simply supported shells of finite length. The analyses in [85] were instead based on closed form solutions of the governing three-dimensional equations, which were obtained in terms of Bessel functions.

A circular cylindrical shell with middle surface radius $R$ equal to 0.5 m and axial length $L$ equal to 0.5 m is introduced, see Fig. 4.1(a). The values here considered for the cylinder thickness $t$ are 0.05 m and 0.1 m . The simply supported boundary conditions $u_{x}=u_{z}=0$ are imposed on the free edges (at $y=0$ and $y=L$ ). They correspond to the three-dimensional constraints used in [88] and analogues of what are classified, in two-dimensional shell theories, as S2 simply supported edge boundary conditions according to Almroth's classification [116]. The isotropic material considered is aluminium: Young's modulus $E=73 \mathrm{GPa}$, Poisson's ratio $\nu=0.3$, and density $\rho=2700 \mathrm{~kg} \mathrm{~m}^{-3}$.


Figure 4.1: Cross-sections geometry for the one-layer and three-layer cylinders.

This geometrical layout has been chosen since it represents a very severe test case for the present one-dimensional model. In fact, both the configurations are very short cylinders $(L / R=1)$ with a thin $(R / t=10)$ or a thicker $(R / t=5)$ cross-section. Classical beam theories are thus completely ineffective for studying this kind of structure due to their kinematic hypotheses on the cross-section and shear deformation. The cylinder is analyzed by means of higher-order CUF models and modeled through a 1D mesh of $10 B 4$ finite elements along the $y$ axis. This choice of mesh ensues from the conclusions made in previous CUF works on the dynamics of thin-walled structures [117, 118]. A convergence study on the mesh is not repeated here for the sake of brevity. Different values of the circumferential half-wave number $n$ ( $2 n$ in [85]) are investigated, whereas the axial half-wave number $m$ is set to 1 .

In Table 4.1 the three first frequency parameters $\bar{\omega}$ based on the present 1D CUF model are compared with corresponding 3D results obtained in references [85, 88] according to Eq. 4.1:

$$
\begin{equation*}
\bar{\omega}=\frac{\Omega \pi L}{t \sqrt{2}}=\omega L \sqrt{\frac{\rho(1+\nu)}{E}} \tag{4.1}
\end{equation*}
$$

where $\Omega$ is the frequency parameter used in [85] and $\omega$ is the natural angular frequency of vibration. Values of 4,7 , and 9 are employed for the expansion order $N$. Table 4.1 shows that it is necessary to enhance the displacement field with higher-order terms to correctly describe the dynamic behavior of a thin-walled cylinder. This statement is true especially for vibrating modes with a high half-wave number. For instance, even a fourth-order model provides good results for $n=2$, whereas the frequency parameters computed for $n=6$ are clearly wrong. The results of the present 1D CUF model (with $N=9$ ) are in excellent agreement with the results based on three-dimensional exact and quasi-exact elasticity solutions [85, 88].

It is interesting to note a particular behavior of $\bar{\omega}$. Considering the ninth-order model, for a fixed combination of $t / R$ ratio and $n$, the first frequency parameter (I) is always affected by the higher error compared to exact 3D results. On the contrary, the second value (II) shows the best agreement with even fifth-digit precision. From the results in Table 4.1, it seems that an increasing value of $N$ might be necessary for the correct study of thicker cylinders. The ninth-order model is accurate also for $t / R=0.2$, but it seems more powerful for $t / R=0.1$. For a $t / R$ ratio higher than 0.2 the $N=9$ model might be not refined enough, especially for the frequency detection of the first vibrational mode $n=6$. However, accuracy in at least three significant digits is achieved for all the vibrational modes of both the shell structures involved. This validates the correctness of the proposed 1D CUF analysis even for short structures, according to the previous CUF work [119].

### 4.2 Assessment with 3D finite element solutions

The assessment procedure regarding homogeneous shells is completed. A thin-walled cylinder with a nonhomogeneous cross-section is now introduced. As depicted in Fig. 4.1(b), the cross-section is composed of three thin circular layers denoted as layers 1,2 and 3 . The layers of the cylinder are made of three different isotropic materials. The material and geometrical properties of the layers are summarized in Table 4.2. The thickness $t=$ 1 mm is constant for each layer and is small enough to consider overall the cylinder as a thin-walled structure, since the external and internal diameters are equal to $d_{e}=100$ mm and $d_{i}=94 \mathrm{~mm}$, respectively. The length $L$ of the cylinder is equal to 500 mm . A clamped boundary condition is taken into account for both the free edges of the cylinder (at $y=0$ and $y=L$ ).

Table 4.1: Comparison of frequency parameters $\bar{\omega}$ based on the present 1D models with 3D elasticity solutions. $m=1$.

| $t / R$ | $n$ |  | $N=4$ | $N=7$ | $N=9$ | Exact $3 \mathrm{D}[85]$ | $3 \mathrm{D}[88]$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0.1 | 2 | I | 1.0804 | 1.0620 | 1.0620 | 1.0623 | 1.0624 |
|  |  | II | 2.3758 | 2.3745 | 2.3744 | 2.3744 | 2.3745 |
|  |  | III | 3.9659 | 3.9634 | 3.9632 | 3.9634 | 3.9634 |
|  | 4 | I | 0.9937 | 0.8838 | 0.8819 | 0.8823 | 0.8826 |
|  |  | II | 2.9118 | 2.7160 | 2.7159 | 2.7159 | 2.7159 |
|  |  | III | 4.8497 | 4.4877 | 4.4875 | 4.4874 | 4.4876 |
|  | 6 | I | 1.7500 | 0.8388 | 0.8112 | 0.8093 | 0.8096 |
|  |  | II | 4.1441 | 3.1562 | 3.1534 | 3.1533 | 3.1533 |
|  |  | III | 5.9431 | 5.2485 | 5.2367 | 5.2365 | 5.2367 |
| 0.2 | 2 | I | 1.2430 | 1.1891 | 1.1887 | 1.1889 | 1.1891 |
|  |  | II | 2.3806 | 2.3757 | 2.3757 | 2.3757 | 2.3758 |
|  |  | III | 3.9620 | 3.9531 | 3.9529 | 3.9527 | 3.9528 |
|  | 4 | I | 1.2823 | 1.1184 | 1.1015 | 1.1009 | 1.1012 |
|  |  | II | 2.9260 | 2.7192 | 2.7182 | 2.7182 | 2.7184 |
|  | 6 III | 4.7903 | 4.4667 | 4.4661 | 4.4659 | 4.4661 |  |
|  | I | 1.8907 | 1.2647 | 1.2181 | 1.1975 | 1.1979 |  |
|  | II | 4.0979 | 3.1688 | 3.1569 | 3.1566 | 3.1569 |  |
|  | III | 5.8511 | 5.2247 | 5.1965 | 5.1949 | 5.1952 |  |

Table 4.2: Material and geometrical properties of the cylinder layers.

| Property | Layer 1 | Layer 2 | Layer 3 |
| :--- | ---: | ---: | ---: |
| $\mathrm{t}[\mathrm{mm}]$ | 1 | 1 | 1 |
| $\mathrm{E}[\mathrm{GPa}]$ | 69 | 30 | 15 |
| $\nu$ | 0.33 | 0.33 | 0.33 |
| $\rho\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ | 2700 | 2000 | 1800 |

One-dimensional theories are usually employed to study slender beams because of their limiting kinematic hypotheses. Instead, the cylinder here considered is relatively short since the span-to-external diameter ratio $L / d_{e}$ is equal to 5 . Nevertheless, the free vibration analysis of this shell structure is performed by solving Eq. 3.95. The 1D CUF model, which has been previously assessed for homogeneous shells, is employed with a variable expansion order up to $N=9$ as well as a 1D mesh of $10 B 4$ finite elements. A solid finite element analysis is also carried out via the commercial code NASTRAN and taken as reference in order to assess the present refined 1D model for a nonhomogeneous shell case. Due to the small layer thickness and the well-known aspect ratio restrictions typical of solid finite elements, the model in NASTRAN consists of 86880 nodes and 64800 HEX8 elements. The number of degrees of freedom (DOFs) is thus equal to 257760 . All the vibrational modes obtained can be divided into four categories: bending, axial, radial, and lobe-type modes. In axial modes, the cylinder vibrates along its longitudinal axis $y$ and the cross-sections remain annular-type. In radial modes, the cross-sections vibrate radially remaining annular-type and rotate along the circumferential direction. In lobe-type modes,
the cross-sections do not remain annular-type since two, theee, or four lobes and so on compare along the circumferential direction in the deformed configuration, see Fig. 4.2.


Figure 4.2: Cross-section deformation for different lobe-type modes.
Table 4.3 summarizes the three first natural frequencies of the bending modes computed through different models. Each frequency value refers to two numbers, put as superscripts and denoted as mode indices, whose meaning is important to be explained. Let the vibrational modes computed by EBBM to be considered for instance. The first and second modes correspond to the same natural frequency ( 1963.0 Hz ) and represent the same way of vibrating, that is a single-half-wave bending mode. In fact, the cylinder is axisymmteric and thus can vibrate along a single-half-wave either in the $x-y$ plane or in the $y-z$ plane. Hence, the mode indices related to EBBM single-half-wave bending mode ( 1963.0 Hz ) are 1 and 2. Instead, the single-half-wave bending mode computed by $N=4$ model corresponds to the third and fourth overall modes of the cylinder. Although EBBM is basically a bending beam theory, for this thin-walled short structure it is not able to properly detect even bending frequencies with respect to the reference 3D solution. The first-order shear deformation theories (TBM and $N=1$ ) are also not accurate enough and only a theory order higher than 2 provides good results compared to NASTRAN solid bending frequencies. The increase of the expansion order $N$ improves the results approaching the reference data with a convergent trend.

Let a dimensionless frequency parameter $f / f_{\text {REF }}$ be defined as the ratio between the frequency computed and the reference value obtained by the solid FE model. The trend of this parameter is depicted in Fig. 4.3 for different FE models and bending modes. It is noteworthy that the error obtained by the first-order theories is significant even for bending frequencies. As can be seen, $f / f_{\text {REF }}$ seems to rise as the number of bending half-waves increases and the error is likely to propagate dramatically for higher mode numbers even for $N=3$.

An opposite behavior is instead visible for $N>3$. In fact, for this bending case, the introduction of fourth-order terms in the displacement field expansion makes the present 1D model accurate enough to achieve an excellent agreement with the solid model. As shown in Table 4.3, for $N>3$ the percent difference with 3D results is almost negligible and slightly decreases as the mode number increases. As far as the mode index is concerned, an increasing expansion order is required to achieve a perfect agreement for increasing

Table 4.3: Comparison of natural frequencies $[\mathrm{Hz}]$ based on the present 1D models with 3D FE solution. Bending modes.

|  | Mode number |  |  |  |
| :--- | ---: | :--- | ---: | ---: |
| Model | DOFs |  |  |  |
| EBBM | 1 | 2 | 3 |  |
| TBM | $1963.0^{1,2}$ | $5044.9^{4,5}$ | $9014.9^{7,8}$ | 93 |
| $N=1$ | $1572.6^{1,2}$ | $3530.3^{3,4}$ | $5823.0^{6,7}$ | 155 |
| $N=2$ | $1572.6^{1,2}$ | $3530.3^{6,7}$ | $5823.0^{12,13}$ | 279 |
| $N=3$ | $1597.7^{1,2}$ | $3562.1^{4,5}$ | $5852.6^{8,9}$ | 558 |
| $N=4$ | $1368.0^{1,2}$ | $2925.0^{6,7}$ | $4725.5^{11,12}$ | 930 |
| $N=5$ | $1364.3^{3,4}$ | $2912.6^{10,11}$ | $4679.5^{19,20}$ | 1395 |
| $N=6$ | $1364.3^{3,4}$ | $2909.7^{10,11}$ | $4674.0^{25,26}$ | 1953 |
| $N=7$ | $1364.2^{3,4}$ | $2909.6^{16,17}$ | $4673.9^{25,26}$ | 2604 |
| $N=8$ | $1364.2^{3,4}$ | $2909.5^{16,17}$ | $4673.8^{29,30}$ | 3348 |
| $N=9$ | $1364.0^{3,4}$ | $2908.6^{16,17}$ | $4673.8^{33,34}$ | 4185 |
| Solid FEM | $1360.9^{3,4}$ | $2906.3^{16,17}$ | $4671.0^{37,38}$ | 5115 |



Figure 4.3: Dimensionless frequency parameter for different 1D models. Bending modes.
mode numbers. With $N=9$, both the bending frequency values and mode indices are in excellent agreement with the reference solution for all the three first mode numbers. It should be foreseen that a correct computation of the mode indeces here is related to a correct analysis of all the four kinds of vibrational modes of the cylinder, not only the bending modes. The last column of Table 4.3 reports the DOFs required by the models. It is worth pointing out that the bending dynamic behavior of the cylinder is well described by the proposed 1D CUF model with a considerably smaller computational cost with respect to the solid FE model.

The natural frequencies related to the radial and axial modes are presented in Table 4.4 for several structural models. As will be shown afterwards, when the cylinder vibrates radially the thin cross-sections remain circular-type but they are subjected to a torsion
about the longitudinal $y$ axis. It means that the radial mode is also a torsional mode of vibration. Hence, Euler-Bernoulli and Timoshenko beam theories are unable to detect any radial mode due to the kinematic hypotheses they are based on. On the contrary, the radial natural frequencies are well-computed even by the first-order model $(N=1)$, which takes into account the torsional deformation of the cross-section. It is noteworthy that in this particular case the introduction of higher-order terms in the a priori displacement field does not improve the radial frequencies computation. For instance, the seventh-order model provides the same values as those computed by $N=1$. For the sake of brevity, the results corresponding to $N=8$ and $N=9$ are not reported in Table 4.4 even for axial modes, since the five digits do not change when an expansion order higher than the seventh is considered.

Table 4.4: Comparison of natural frequencies [ Hz ] based on the present 1D models with 3D FE solution. Radial and axial modes.

| Model | Radial |  | Axial |  |
| :--- | :---: | :---: | :---: | ---: |
|  | DOFs |  |  |  |
|  | Mode 1 | Mode 2 | Mode 1 |  |
| EBBM | - | - | $4173.2^{3}$ | 93 |
| TBM | - | - | $4173.2^{5}$ | 155 |
| $N=1$ | $2540.9^{4}$ | $5081.7^{10}$ | $4173.3^{8}$ | 279 |
| $N=2$ | $2540.9^{3}$ | $5081.7^{7}$ | $4191.3^{6}$ | 558 |
| $N=3$ | $2540.9^{5}$ | $5081.7^{15}$ | $4182.5^{10}$ | 930 |
| $N=4$ | $2540.9^{7}$ | $5081.7^{23}$ | $4182.4^{18}$ | 1395 |
| $N=5$ | $2540.9^{7}$ | $5081.7^{29}$ | $4182.1^{22}$ | 1953 |
| $N=6$ | $2540.9^{11}$ | $5081.7^{31}$ | $4182.0^{24}$ | 2604 |
| $N=7$ | $2540.9^{11}$ | $5081.7^{39}$ | $4182.0^{24}$ | 3348 |
| Solid FEM | $2540.2^{13}$ | $5080.1^{43}$ | $4172.2^{32}$ | 257760 |

Unlike bending and radial modes, the natural frequency of the first axial mode is accurately computed by classical beam theories, even better than higher-order models. In axial modes, the cylinder vibrates along its longitudinal axis $y$ and the cross-sections remain annular-type. This kind of deformation is consistent with the kinematic hypotheses which classical beam models are based on. Nonetheless, it is foreseen that classical models are not so accurate in the evaluation of the axial modal shape. In fact, the thin-walled surface of the cylinder induces some in-plane deformations which are not detectable by classical beam theories.

As occurred for bending modes, an increase of the expansion order $N$ corresponds to a decrease of the numerical value of the axial frequency, see Table 4.4. The main reason of this behavior stands in the fact that higher-order models reduce the overall structural stiffness since the enrichment of the displacement field enables the structure to deform in a more realistic way. This general consideration is consistent with those made in previous works on higher-order models [101]. Nonetheless, it is interesting to note that the value of the axial natural frequency decreases when the theory changes from a first-order to a second-order form. The same behavior occurred for bending modes as can be seen in Table 4.3. This turnaround is mainly due to the required correction of a phenomenon known in literature as Poisson's locking, which is explained in detail in [120, 121]. Poisson's locking correction is here correctly adopted only for classical and first-order theories (EBBM, TBM, $N=1$ ), according to the works of Carrera and Giunta [44] on beams and Carrera and Brischetto
$[120,121]$ on plates and shells. It means that if this correction was disabled, first-order models would provide a higher value for the axial frequency corresponding to an increase of the cylinder stiffness.

The analysis now addresses the investigation of the lobe-type modes, which are typical of shell structures. This kind of vibrating involves a lobe-type deformation over the cross-section (see Fig. 4.2) and cannot be consistent with the kinematic hypotheses which classical beam models are based on. Classical models are therefore not expected to yield accurate results. By enriching the displacement field, the first-order model provide a linear displacement distribution in all the three directions. Nevertheless, it is not refined enough for the investigation of lobe-type modes as well as the second-order expansion. This statement is confirmed by the fact that none of the lobe-type modes is detected for $N<3$. Numerical results in Table 4.5 present the natural frequencies of two-lobe modes computed by the present hierarchical models. It is necessary to enhance the displacement field with higher-order terms to correctly describe the dynamic behavior of the thin-walled cylinder. In fact, an expansion order lower than the sixth provides a remarkable error in computing the two-lobe frequencies as illustrated in Fig. 4.4. This error is maximum when the axial half-wave number $m$ is equal to 1 and decreases as the mode number increases. However, the proposed 1D FEs provide a convergent solution by approaching the NASTRAN 3D results as the refinement of the expansion increases, until a well agreement for $N=9$ is achieved for every $m$ considered.

Table 4.5: Comparison of natural frequencies [Hz] based on the present 1D models with 3D FE solution. Two-lobe modes.

| Model | Mode number |  |  |  |  |  |  |  |  |  |
| :--- | ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 |  |  |  |  | 3 |  | 4 | 5 |
| $N=3$ | $1617.9^{3,4}$ | $2960.7^{8,9}$ | $4755.5^{13,14}$ | $6832.6^{18,19}$ | $9094.5^{24,25}$ |  |  |  |  |  |
| $N=4$ | $1209.1^{1,2}$ | $1795.0^{5,6}$ | $2732.8^{8,9}$ | $3850.7^{14,15}$ | $5049.5^{21,22}$ |  |  |  |  |  |
| $N=5$ | $1005.4^{1,2}$ | $1666.3^{5,6}$ | $2643.3^{8,9}$ | $3765.4^{18,19}$ | $4937.7^{27,28}$ |  |  |  |  |  |
| $N=6$ | $995.8^{1,2}$ | $1640.9^{5,6}$ | $2602.6^{12,13}$ | $3707.4^{20,21}$ | $4859.5^{27,28}$ |  |  |  |  |  |
| $N=7$ | $862.7^{1,2}$ | $1561.9^{5,6}$ | $2551.7^{12,13}$ | $3670.0^{20,21}$ | $4829.1^{35,36}$ |  |  |  |  |  |
| $N=8$ | $862.6^{1,2}$ | $1561.6^{5,6}$ | $2551.3^{14,15}$ | $3669.4^{20,21}$ | $4828.0^{35,36}$ |  |  |  |  |  |
| $N=9$ | $860.4^{1,2}$ | $1560.3^{5,6}$ | $2550.6^{14,15}$ | $3669.3^{24,25}$ | $4827.4^{39,40}$ |  |  |  |  |  |
| Solid FEM | $859.9^{1,2}$ | $1557.0^{5,6}$ | $2545.4^{14,15}$ | $3662.9^{24,25}$ | $4820.9^{41,42}$ |  |  |  |  |  |

Table 4.6 reports the natural frequencies of three-lobe modes obtained by the CUF models. None of the three-lobe modes is computed by the third-order model. A further increase of $N$ is required and this means that the introduction of higher-order terms is even more important here than in the previous two-lobe case. The higher the theory order employed the more the results approach the solid FEM frequencies. However, the convergence obtained by increasing $N$ shows that the proposed hierarchical models do not introduce additional numerical problems in the free vibration analysis. This trend is consistent with the considerations made in previous CUF works for static [122], aeroelastic [123] and dynamic response [118] analyses. The choice of $N=8$ seems to be accurate enough for three-lobe modes, even if the ninth-order theory is more powerful especially for high values of $m$. Moreover, the choice of $N=9$ provides the exact evaluation of the mode indices even for $m=6$. In regards to the DOFs required, it is worth pointing out that an accurate evaluation even of the lobe-type dynamic behavior is provided by the proposed 1D CUF model with a sizeable reduction in computational cost with respect to


Figure 4.4: Dimensionless frequency parameter for different 1D models. Two-lobe modes.
the solid FE model. For the sake of brevity, the results for models with an expansion order higher than the ninth are not reported here, since an excellent agreement is achieved in comparison with three-dimensional FE results with a convergent trend on $N$.

Table 4.6: Comparison of natural frequencies $[\mathrm{Hz}]$ based on the present 1D models with 3D FE solution. Three-lobe modes.

| Model | Mode number |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 | 3 | 4 | 5 | 6 |
| $N=4$ | $3281.2^{12,13}$ | $4099.2^{16,17}$ | $5354.2{ }^{24,25}$ | $6946.7^{30,31}$ | $8795.1{ }^{40,41}$ | $10837.6^{53,54}$ |
| $N=5$ | $3027.4^{12,13}$ | $3160.3^{14,15}$ | $3453.3^{16,17}$ | $3930.7{ }^{20,21}$ | $4572.9{ }^{23,24}$ | $5339.8^{30,31}$ |
| $N=6$ | $2319.2^{7,8}$ | $2504.7^{9,10}$ | $2882.6^{14,15}$ | $3454.6{ }^{18,19}$ | $4178.1^{22,23}$ | $5001.9^{29,30}$ |
| $N=7$ | $2306.9^{7,8}$ | $2458.7^{9,10}$ | $2789.4^{14,15}$ | $3308.7^{18,19}$ | $3978.0{ }^{22,23}$ | $4746.3^{33,34}$ |
| $N=8$ | $1826.3^{7,8}$ | $2006.7^{9,10}$ | $2390.3^{11,12}$ | $2968.6^{18,19}$ | $3688.1^{22,23}$ | $4493.8{ }^{29,30}$ |
| $N=9$ | $1826.2^{7,8}$ | $2006.2^{9,10}$ | $2389.1^{11,12}$ | $2966.7^{18,19}$ | $3685.0{ }^{26,27}$ | $4490.2^{35,36}$ |
| Solid FEM | $1818.0{ }^{7,8}$ | $1997.1^{9,10}$ | $2378.6^{11,12}$ | $2954.9{ }^{18,19}$ | $3672.2{ }^{26,27}$ | $4475.1^{35,36}$ |

The accuracy of the CUF approach as $N$ increases is highlighted in Fig. 4.5, where the trend of the frequency parameter is depicted for a raising mode number. The choice of $N=5$ provides frequencies which are clearly wrong with respect to the reference solution. The $f / f_{\text {REF }}$ ratio, i.e. the error with respect to the reference solution, decreases when a sixth- or a seventh-order model is employed. Nonetheless, only with a higher-order model than the seventh the curve trend changes and becomes practically the same as the reference one. The convergent trend which has been mentioned above is clearly shown in Fig. 4.5 confirming the numerical consistency of Carrera Unified Formulation. As in the case of two-lobe vibrations, the error computed decreases as the axial half-wave number $m$ increases. The frequency parameter is maximum for $m=1$. It is noteworthy that this behavior is typical of lobe-modes. For instance, the situation is opposite as far as the bending vibrations are concerned, see Fig. 4.3.

As can be seen, the lobe-type modes have two mode indexes for each frequency, due to the cylinder axisymmetry. In general, the higher the mode number the more is the expansion order required to compute the correct mode index. In order to understand this


Figure 4.5: Dimensionless frequency parameter for different 1D models. Three-lobe modes.
statement, the following consideration is crucial. It is important to note that the increase of $N$ usually corresponds to a detection of new lobe-type modes thanks to the displacement field enrichment. As previously reported, the second-order model is not able to detect any two-lobe mode. Instead, the third-order model is able to compute such modes, though the corresponding frequencies are sizeable far from the correct vales. As a consequence, it has been necessary to increase the expansion order to 7 to match the NASTRAN reference results, see Table 4.5. Nevertheless, the seventh-order model is not refined enough for the frequency computation of three-lobe modes, which appear only with $N \geq 4$, see Table 4.6. In the same way, the four-lobe modes are not detectable for an expansion order lower than 5. This is the reason why the increase of the accuracy of the 1D model improves not only the computation of the frequency values, but also the corresponding mode indices.

Taking the solid FE model as reference, the results summarized in Table 4.7 improve again as the expansion order $N$ increases. In particular, for the present ninth-order model the four-lobe natural frequencies are accurately computed and the agreement with the 3D solution is remarkable as well as the sizeable reduction in computational cost. On the contrary, the eighth-order expansion is no longer as refined as it was in the analysis of the other vibrational modes. Numerical results are depicted in Fig. 4.6 in terms of the dimensionless frequency parameter $f / f_{\text {REF }}$. The trend of the curves is very similar to that illustrated in Fig. 4.5 corresponding to three-lobe modes. Given a theory, the error decreases as $m$ increases. A remarkable difference between these figures is that $N=8$ is not an appropriate choice to compute the correct four-lobe frequencies.

As far as lobe-type modes are concerned, it is interesting to note a noteworhty behavior that occurs when the half-wave number $m$ is equal to 1 . Sometimes an increment by one of the expansion order $N$ seems to improve the numerical results very slightly in terms of frequencies computed by CUF models. The particularity stands in the fact that it occurs even when the convergence on $N$ is not achieved. This particular behavior is clearly shown in Figs. 4.4-4.6. For instance, for four-lobe modes the first frequencies computed by the seventh- and eighth-order models are approximately the same. Regarding three-lobe modes, the same goes for $N=6$ and $N=7$. Figure 4.4 illustrates a similar behavior for the fifthand sixth-order models.

Looking at the numerical results presented so far, the ninth-order model seems to


Figure 4.6: Dimensionless frequency parameter for different 1D models. Four-lobe modes.

Table 4.7: Comparison of natural frequencies $[\mathrm{Hz}]$ based on the present 1D models with 3D FE solution. Four-lobe modes.

| Model | Mode number |  |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | ---: |
|  | 1 | 2 | 3 | 4 | 5 | 6 |
| $N=5$ | $5961.4^{32,33}$ | $6489.5^{40,41}$ | $7350.1^{46,47}$ | $8512.2^{56,57}$ | $9937.2^{66,67}$ | $11585.3^{86,87}$ |
| $N=6$ | $5805.4^{32,33}$ | $5877.9^{34,35}$ | $6017.7^{40,41}$ | $6243.3^{42,43}$ | $6527.3^{46,47}$ | $6993.4^{50,51}$ |
| $N=7$ | $4380.8^{25,26}$ | $4494.4^{27,28}$ | $4702.8^{31,32}$ | $5020.9^{37,38}$ | $5453.5^{40,41}$ | $5994.1^{46,47}$ |
| $N=8$ | $4368.4^{25,26}$ | $4445.6^{27,28}$ | $4595.8^{31,32}$ | $4837.5^{37,38}$ | $5180.6^{40,41}$ | $5623.1^{44,45}$ |
| $N=9$ | $3441.1^{20,21}$ | $3528.8^{22,23}$ | $3700.4^{28,29}$ | $3976.8^{30,31}$ | $4364.8^{33,34}$ | $4856.7^{41,42}$ |
| Solid FEM | $3417.8^{20,21}$ | $3503.3^{22,23}$ | $3672.4^{28,29}$ | $3945.3^{30,31}$ | $4329.1^{33,34}$ | $4816.1^{39,40}$ |

be refined enough to study the dynamic behavior of the layered cylinder considered. This statement is confirmed by Table 4.8, where the first thirty-eight natural frequencies computed by the present 1D model $(N=9)$ and the NASTRAN solid model. Each mode presents a superscript composed of two terms. The first term indicates the kind of mode whereas the second term indicates the value of $m$. The results involve bending, radial, axial, two-, three- and four-lobe vibrational modes with an axial half-wave number up to 6 . Despite the one-dimensional approach of the proposed higher-order model, it provides an error lower than 0.8 percent for all these modes, with a remarkably lower computational effort than that required by the reference solid model. In fact, about a $98 \%$ saving of the degrees of freedom involved in the free vibration analysis is obtained. The maximum error is computed for three- and four-lobe modes, for which an expansion with an even higher order, $N=10$ for instance, would further increase the computational accuracy.

Further results regarding five-lobe modes are not reported here for the sake of brevity. In fact, the same conclusions made for two-, three- and four-lobe modes would be valid also for more complex lobe-type vibrational modes. However, for the considered cylinder a ninth-order expansion theory detects vibrational modes with a five-lobe deformation of the cross-section. Nonetheless, a further increase of $N$ might be required to achieve a good accuracy regarding the numerical frequencies of these even more complex vibrational modes. This behavior is consistent with the considerations previously mentioned about the

Table 4.8: Natural frequencies $[\mathrm{Hz}]$ of the first thirty-eight vibrational modes of the cylinder.

| Mode | $N=9$ | Solid FEM | \% Difference |
| :---: | ---: | :---: | :---: |
| $1,2^{2.1}$ | 860.4 | 859.9 | 0.058 |
| $3,4^{b .1}$ | 1364.0 | 1360.9 | 0.228 |
| $5,6^{2.2}$ | 1560.3 | 1557.0 | 0.212 |
| $7,8^{3.1}$ | 1826.2 | 1818.0 | 0.451 |
| $9,10^{3.2}$ | 2006.2 | 1997.1 | 0.456 |
| $11,12^{3.3}$ | 2389.1 | 2378.6 | 0.441 |
| $13^{r .1}$ | 2540.9 | 2540.2 | 0.028 |
| $14,15^{2.3}$ | 2550.6 | 2545.4 | 0.204 |
| $16,17^{b .2}$ | 2908.6 | 2906.3 | 0.079 |
| $18,19^{3.4}$ | 2966.7 | 2954.9 | 0.399 |
| $20,21^{4.1}$ | 3441.1 | 3417.8 | 0.682 |
| $22,23^{4.2}$ | 3528.8 | 3503.3 | 0.728 |
| $24,25^{2.4}$ | 3669.3 | 3662.9 | 0.175 |
| $26,27^{3.5}$ | 3685.0 | 3672.2 | 0.349 |
| $28,29^{4.3}$ | 3700.4 | 3672.4 | 0.762 |
| $30,31^{4.4}$ | 3976.8 | 3945.3 | 0.798 |
| $32^{a .1}$ | 4190.6 | 4172.2 | 0.441 |
| $33,34^{4.5}$ | 4364.8 | 4329.1 | 0.825 |
| $35,36^{3.6}$ | 4490.2 | 4475.1 | 0.337 |
| $37,38^{b .3}$ | 4671.0 | 4670.9 | 0.002 |
| DOFs | 5115 | 257760 | -98.02 |
| $2.1:$ two-lobe mode, $m=1$ | $b .1:$ bending mode, $m=1$ |  |  |
| $3.1:$ three-lobe mode, $m=1$ | $r .1:$ radial mode, $m=1$ |  |  |
| $4.1:$ four-lobe mode, $m=1$ | $a .1:$ axial mode, $m=1$ |  |  |
| $4.5:$ four-lobe mode, $m=5$ |  |  |  |

expansion enrichment.
A summary of the first thirty-eight vibrational modes detected by the present onedimensional models is reported in Table 4.9, where the accuracy in computing natural frequencies with respect to the solid FE analysis is shown by varying the expansion order $N$. It should be pointed out that a different higher-order expansion is required depending on the kind of vibrational mode investigated. Some structural models are not even able to detect all the kinds of vibrational modes. For instance, classical beam theories do not consider radial modes. Third-, fourth- and fifth-order models are necessary to compute two-, three- and four-lobe modes, respectively. As far as the accuracy is concerned, a further higher expansion order is required. However, the results show that the introduction of higher-order terms is fundamental for the free vibration analysis of a thin-walled layered structure, according to previous dynamic response computations through 1D CUF models [118].

The assessment procedure of 1D CUF models on natural frequencies is now completed. A graphical comparison between the modal shapes based on the present 1D CUF model and those computed by the solid FEM is now carried out. Some interesting modal shapes have been chosen and compared in Figs. 4.7-4.12. The three-dimensional deformation as well as the front-view of each modal shape are depicted. It is noteworthy that the modal shapes perfectly match for all the kinds of vibrational modes considered, including the

Table 4.9: Summary of the vibrational modes detected with an increasing expansion order $N$ by 1 D models and comparison of the corresponding natural frequencies with 3D FE solution.

| Model | Vibrational mode |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 2.1 | b. 1 | 2.2 | 3.1 | 3.2 | 3.3 | r. 1 | 2.3 | b. 2 | 3.4 | 4.1 | 4.2 | 2.4 | 3.5 | 4.3 | 4.4 | a. 1 | 4.5 | 3.6 | b. 3 |
| EBBM | - | $\diamond$ | - | - | - | - | - | - | $\diamond$ | - | - | - | - | - | - | - | - | - | - | $\diamond$ |
| TBM | - | > | - | - | - | - | - | - | $\diamond$ | - | - | - | - | - | - | - | - | - | - | $\diamond$ |
| $N=1$ | - | < | - | - | - | - | - | - | $\diamond$ | - | - | - | - | - | - | - | - | - | - | $\diamond$ |
| $N=2$ | - | > | - | - | - | - | - | - | $\diamond$ | - | - | - | - | - | - | - | - | - | - | $\diamond$ |
| $N=3$ | $\diamond$ | - | $\diamond$ | - | - | - | - | $\diamond$ | - | - | - | - | $\diamond$ | - | - | - | - | - | - | - |
| $N=4$ | $\diamond$ | - | > | $\diamond$ | $\diamond$ | $\diamond$ | - | - | - | $\diamond$ | - | - | - | $\diamond$ | - | - | - | - | $\diamond$ | - |
| $N=5$ | > | - | - | $\diamond$ | $\diamond$ | $\diamond$ | - | - | - | $\diamond$ | $\diamond$ | $\diamond$ | * | $\diamond$ | $\diamond$ | $\diamond$ | - | $\diamond$ | > | - |
| $N=6$ | > | - | - | $\diamond$ | $\diamond$ | $\diamond$ | - | * | - | > | $\diamond$ | $\diamond$ | * | > | $\diamond$ | $\diamond$ | - | $\diamond$ | > | - |
| $N=7$ | $\checkmark$ | - | - | $\diamond$ | $\diamond$ | > | - | - | - | > | $\diamond$ | $\diamond$ | - | - | $\diamond$ | $\diamond$ | - | $\diamond$ | - | - |
| $N=8$ | - | - | - | - | - | $\checkmark$ | - |  | - | $\checkmark$ | $\diamond$ | $\diamond$ | - | - |  | $\diamond$ | - |  | - |  |
| $N=9$ | - | - |  | - | - | - |  |  | - |  |  |  |  |  |  | - | - |  |  |  |

- Mode not detected.
$\diamond$ Mode detected. Error $>20 \%$.
> Mode detected. $10 \% \leq$ Error $\leq 20 \%$.
lobe-type modes which are typical of shell structures.
The lobe-type shapes are correctly described by one-dimensional higher-order models even when these models do not provide accurate frequencies in comparison with the reference three-dimensional results. For example, the shape of two-lobe modes is correctly described by using a third-order displacement expansion, although this choice does not provide accurate two-lobe frequencies in comparison with higher-order models and solid FE solution, see Table 4.5. As a consequence, when the accuracy in computing the numerical frequency of a certain lobe-type mode is low, nonetheless the corresponding modal shape is usually well-described.

Figure 4.10 highlights the rotation of the cross-sections about the longitudinal $y$ axis that occurs when the cylinder vibrates radially. Instead, when the cylinder vibrates along its longitudinal $y$ axis, the cross-sections which are close to the clamped edges are subjected to streching or dilation effects. These effects are emphasized in Fig. 4.9. Euler-Bernoulli is very effective in the computation of the axial frequencies of the cylinder, see Table 4.4. Nonetheless, although the axial mode computed by Euler-Bernoulli beam theory is not represented here, this classical theory is not able to describe the in-plane deformation of the compressed and dilated cross-sections. On the contrary, these in-plane deformations are well-described by the present higher-order 1D models and a good agreement with the three-dimensional solution is achieved not only in the frequency computation but also in the vibrational shape description, see Figs. 4.7-4.12.

The results in Figs. 4.7-4.12 clearly show the accuracy of the present refined model in detecting the three-dimensional deformation despite its one-dimensional approach, according to previous dynamic computations through 1D CUF models [118]. The present method shows features not present in standard one-dimensional theories such as the thickness changing of the thin-walled laminated surface and the in-plane and out-of-plane cross-section deformations.
(a) Present 1D model
(b) NASTRAN solid model


Figure 4.7: Third bending modal shape (b.3).
(a) Present 1D model
(b) NASTRAN solid model


Figure 4.8: Second radial modal shape (r.2).
(a) Present 1D model
(b) NASTRAN solid model


Figure 4.9: First axial modal shape (a.1).
(a) Present 1D model
(b) NASTRAN solid model


Figure 4.10: Fifth two-lobe modal shape (2.5).
(a) Present 1D model
(b) NASTRAN solid model


Figure 4.11: Fourth three-lobe modal shape (3.4).

(a) Present 1D model



Figure 4.12: Second four-lobe modal shape (4.2).

## Chapter 5

## Results: free vibration analysis of conventional and joined wings

A free vibration analysis of different beam models is conducted in this chapter. The adopted coordinate frame is shown in 5.1. An isotropic material is used. Young's modulus, $E$, is equal to 75 [GPa]. The Poisson ratio, $\nu$, is equal to 0.33 . The density of the material, $\rho$, is equal to $2700\left[\mathrm{Kg} / \mathrm{m}^{3}\right]$. Beams geometries are described in the following sections.


Figure 5.1: Coordinate frame of the beam model.

### 5.1 Solid rectangular cross-section beam

A rectangular compact beam is considered as a first example to assess the proposed FE model. The coordinate frame and the cross-section geometry are shown in Fig. 5.2. The span-to-height ratio, $L / h$, is equal to 100 . The straight and the swept beams have square cross-sections with $b$ equal to 0.2 m . Fig. 5.3 shows the notation used to deal with the swept configuration. The case considered has $h_{\Lambda}$ equal to 5 m , this choice makes the sweep angle, $\Lambda$, equal to $14.3^{\circ}$. The swept tapered beam keeps this angle while $b$ varies linearly along the span-wise direction, $y$. The clamped section is square with $b=0.2 \mathrm{~m}$ and the free tip section is rectangular with $b$ equal to 0.1 m , and $h$ equal to 0.2 m .

A static analysis of a cantilever is first carried out to evaluate the convergence properties of the finite element mesh. Four-node elements are used (B4). A vertical force is applied at the center of the free-tip cross-section, $[b / 2, L, h / 2]$. The vertical displacement, $u_{z}$,


Figure 5.2: Rectangular cross-section.


Figure 5.3: Graphical definition of the sweep angle, $\Lambda$.
is evaluated at $[0, L, h / 2]$. An MSC Nastran model made of solid elements is used as a reference solution. The computed results are presented in Table 5.1.

Table 5.1: Vertical displacement, $u_{z} \times 10^{-2} \mathrm{~m}$, for different beam models and meshes. Rectangular cross-section cantilever beam.

| Number of Elements | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| :--- | :---: | :--- | :---: | :--- | :---: | :---: |
| Straight beam | -1.333 | -1.333 | -1.333 | -1.324 | -1.324 | -1.324 |
| 10 | -1.333 | -1.333 | -1.333 | -1.328 | -1.329 | -1.329 |
| 20 | -1.333 | -1.333 | -1.333 | -1.330 | -1.330 | -1.330 |
| 30 |  |  | $-1.507^{*}$ |  |  |  |
| Swept beam | -1.515 | -1.515 | -1.515 | -1.504 | -1.504 | -1.504 |
| 10 | -1.515 | -1.515 | -1.515 | -1.510 | -1.510 | -1.510 |
| 20 | -1.515 | -1.515 | -1.515 | -1.510 | -1.510 | -1.510 |
| 30 |  |  | $-1.753^{*}$ |  |  |  |
| Swept tapered beam | -1.755 | -1.755 | -1.755 | -1.744 | -1.744 | -1.744 |
| 10 | -1.753 | -1.753 | -1.753 | -1.748 | -1.748 | -1.748 |
| 20 | -1.753 | -1.753 | -1.753 | -1.749 | -1.749 | -1.749 |
| 30 |  |  |  |  |  |  |

(*): computed with MSC Nastran, solid elements

The free-vibration analysis is also considered. The first five bending modes are analyzed. The natural frequencies, $f_{i}$, are compared with those obtained by the Euler-Bernoulli beam
model (EBBM):

$$
\begin{equation*}
f_{i}=\frac{1}{2 \pi}\left(\frac{\left(\lambda_{i} L\right)^{2}}{L^{2}}\left(\frac{E I}{\rho A}\right)^{\frac{1}{2}}\right. \tag{5.1}
\end{equation*}
$$

Table 5.2: First four bending frequencies, Hz , for different beam models and meshes. Cantilever straight beam with rectangular cross-section.

| Number of Elements | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{1}$ | 0.426* |  |  |  |  |  |
| 10 | 0.428 | 0.426 | 0.426 | 0.428 | 0.428 | 0.428 |
| 20 | 0.426 | 0.426 | 0.426 | 0.427 | 0.427 | 0.427 |
| 30 | 0.426 | 0.426 | 0.426 | 0.426 | 0.426 | 0.426 |
| $f_{2}$ | 2.668* |  |  |  |  |  |
| 10 | 2.668 | 2.667 | 2.667 | 2.680 | 2.679 | 2.679 |
| 20 | 2.668 | 2.667 | 2.667 | 2.673 | 2.673 | 2.673 |
| 30 | 2.668 | 2.667 | 2.667 | 2.671 | 2.671 | 2.671 |
| $f_{3}$ | 7.470* |  |  |  |  |  |
| 10 | 7.468 | 7.461 | 7.461 | 7.498 | 7.497 | 7.497 |
| 20 | 7.468 | 7.461 | 7.461 | 7.480 | 7.479 | 7.479 |
| 30 | 7.468 | 7.461 | 7.461 | 7.474 | 7.473 | 7.473 |
| $f_{4}$ | 14.639* |  |  |  |  |  |
| 10 | 14.630 | 14.607 | 14.607 | 14.680 | 14.676 | 14.676 |
| 20 | 14.630 | 14.607 | 14.607 | 14.644 | 14.639 | 14.639 |
| 30 | 14.630 | 14.607 | 14.607 | 14.632 | 14.628 | 14.628 |

(*): reference value computed by means of Eq. (5.1)
For the sake of brevity, the values of $\lambda_{i} L$ are not reported here. They can be found in [124] or [125]. Table 5.2 shows the first four bending frequencies in the case of a cantilever straight beam for different models and meshes. Figs. 5.4 present the fourth bending frequency values obtained with and without the Poisson locking correction for the EBBM, TBM, and full linear models. Table 5.3 shows the first four bending frequencies of a fixed-pinned straight beam, values from Eq. 5.1 are used as references. The cantilever swept and swept-tapered beam models are addressed in Tables 5.4 and 5.5 , respectively. The first three bending frequencies along the $x-$ and $z$ - directions are reported. An MSC Nastran solid model is used as a benchmark. Solid elements with an almost unitary aspect ratio have been used. The first row reports the total number of degrees of freedom (DOFs) of the considered models, that is, the computational cost of each analysis.

The static analysis of the compact rectangular slender beam suggests the following conclusions.

1. In all the considered beam configurations, the linear models, $N<2$, furnishes the largest displacement values. This result is due to the Poisson locking correction that is just activated for the linear cases, and it is coherent with [117].
2. No significant differences have been observed amongst the classical models, EBBM and TBM, and the full linear case, that is, the shear effect and the linear terms of the cross-section displacements haven't remarkable effects in this case. This is due to the fact that the beam is slender, $L / h=100$, the section is compact, and a bending load is applied.


Figure 5.4: Effect of the Poisson locking correction on the fourth bending frequency of the straight beam with rectangular cross-section.

Table 5.3: First four bending frequencies, Hz , for different beam models and meshes. Fixed-Pinned straight beam with rectangular cross-section, 30 B4 mesh.

| EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | $\begin{aligned} & \text { NASTRAN } \\ & \text { (solid) } \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f_{1}$ |  |  |  | .866* |  |  |
| 1.867 | 1.866 | 1.866 | 1.869 | 1.869 | 1.869 | 1.868 |
| $f_{2}$ |  |  |  | 6.050* |  |  |
| 6.048 | 6.044 | 6.044 | 6.054 | 6.053 | 6.053 | 6.049 |
| $f_{3}$ |  |  |  | 2.621* |  |  |
| 12.617 | 12.598 | 12.598 | 12.620 | 12.617 | 12.617 | 12.609 |
| $f_{4}$ |  |  |  | $1.584^{*}$ |  |  |
| 21.570 | 21.518 | 21.518 | 21.556 | 21.546 | 21.546 | 21.535 |

(*): reference value computed by means of Eq. (5.1)

Table 5.4: First three bending frequencies, Hz , in the $x$ - and $z$ - directions. Cantilever swept rectangular beam. 30 B 4 mesh.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (solid) |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of DOFs | 455 | 455 | 819 | 1638 | 2730 | 4095 | 5565 |
| $x-$ direction |  |  |  |  |  |  |  |
| $f_{1}$ | 0.388 | 0.387 | 0.387 | 0.388 | 0.388 | 0.388 | 0.389 |
| $f_{2}$ | 2.427 | 2.426 | 2.426 | 2.430 | 2.430 | 2.430 | 2.437 |
| $f_{3}$ | 6.795 | 6.789 | 6.789 | 6.801 | 6.800 | 6.800 | 6.819 |
|  | $\quad z-$ direction |  |  |  |  |  |  |
| $f_{1}$ | 0.400 | 0.400 | 0.400 | 0.400 | 0.400 | 0.400 | 0.400 |
| $f_{2}$ | 2.505 | 2.504 | 2.504 | 2.508 | 2.508 | 2.508 | 2.507 |
| $f_{3}$ | 7.012 | 7.006 | 7.006 | 7.018 | 7.017 | 7.017 | 7.014 |

Table 5.5: First three bending frequencies, Hz , in the $x-$ and $z$ - directions. Cantilever swept-tapered rectangular beam. 30 B 4 mesh.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (solid) |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of DOFs | 455 | 455 | 819 | 1638 | 2730 | 4095 | 5565 |
| $x-$ direction |  |  |  |  |  |  |  |
| $f_{1}$ | 0.430 | 0.422 | 0.422 | 0.422 | 0.422 | 0.422 | 0.427 |
| $f_{2}$ | 2.020 | 2.018 | 2.018 | 2.021 | 2.021 | 2.021 | 2.046 |
| $f_{3}$ | 5.207 | 5.205 | 5.205 | 5.212 | 5.212 | 5.212 | 5.279 |
| $f_{1}$ | 0.499 | 0.491 | 0.491 | 0.491 | 0.491 | 0.491 | 0.490 |
| $f_{2}$ | 2.675 | 2.672 | 2.672 | 2.677 | 2.676 | 2.676 | 2.668 |
| $f_{3}$ | 7.181 | 7.175 | 7.175 | 7.187 | 7.186 | 7.186 | 7.165 |

3. The results are in good agreement with those furnished by the solid elements.
4. As far as the mesh refinement is concerned, the use of thirty four-node elements offers appreciable convergent capabilities.
5. The good match between the results given by the present beam formulation and MSC Nastran is confirmed, even in the cases of swept and swept-tapered beam configurations. This aspect offers validation of the adopted formulation.

As far as the free vibration analysis is concerned, the following conclusions hold.

1. The results match those obtained with the analytical model. As seen for the static case, the Poisson locking correction greatly enhances the flexibility of the finite element model for EBBM, TBM, and full linear models. It has to be highlighted that this correction can lead to lower frequencies than those from higher-order models. This result agrees with [117] and [126].
2. The free vibration analysis highlights the difference between the EBBM solution and the TBM one. The higher the mode number, the larger the differences. No very significant differences were observed between TBM and the full linear case.
3. A ten element mesh furnishes the first natural frequency with an appreciable accuracy. However, the higher the mode number, the larger the influence of a finer mesh. Since the thirty element mesh offers good convergent behavior, it will be used for all the subsequent analyses.
4. The results show a good match with those obtained with the solid model. This confirms the validity of the adopted formulation in dealing with arbitrary oriented structures (swept) with varying cross-section geometries along the longitudinal axis (tapered).
5. The higher the order of the beam model, the larger the total number of DOFs. A fourth-order model requires a similar total number of DOFs to that of the solid model because $b / h$ is close to unity and the cross-section is compact.

### 5.2 Airfoil-Shaped Beam

A cantilever arbitrary shaped thin-walled beam is considered in this section. The crosssection contour is defined by the NACA 2415 airfoil profile. The cross-section geometry is shown in Fig. 5.5. The chord length, $b$, is assumed equal to 1 m . A three-cell section is evaluated. The cells are obtained by inserting two beams along the span-wise direction at $25 \%$ and $75 \%$ of the chord. The span-to-chord ratio, $L / b$, is assumed to be equal to 5 , that is, a moderately short structure is considered. The graphical definitions of the sweep and dihedral angles are shown in Figs. 5.3 and 5.6. Two different configurations are considered: a straight wing, i.e. $h_{\Lambda}, h_{\Gamma}=0$, and a wing with both sweep and dihedral angle. In this latter case $h_{\Lambda}$ and $h_{\Gamma}$ are equal to 2 m , i.e. $\Lambda=21.8^{\circ}$ and $\Gamma=21.8^{\circ}$. An Ansys solid model is adopted for result comparison purposes.


Figure 5.5: Wing cross-section.


Figure 5.6: Graphical definition of the dihedral angle, $\Gamma$.

Table 5.6: Bending natural frequencies, Hz , of the cantilever wing models for different theories, 10 B4 mesh.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (solid) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of DOFs | 155 | 155 | 279 | 558 | 930 | 1395 | $>600000$ |
| Straight |  |  |  |  |  |  |  |
| $f_{1}$, bending $z$-dir | 5.872 | 5.866 | 5.866 | 5.984 | 5.934 | 5.925 | 5.864 |
| $f_{2}$, bending $x$-dir | 33.340 | 32.709 | 32.709 | 32.891 | 32.712 | 32.696 | 32.335 |
| $f_{3}$, bending $z-\operatorname{dir}$ | 36.735 | 36.581 | 36.581 | 37.206 | 36.447 | 36.288 | 34.844 |
| $f_{4}$, bending $z$-dir | 102.634 | 101.617 | 101.617 | 103.430 | 99.100 | 98.318 | 81.976 |
| Sweep + Dihedral |  |  |  |  |  |  |  |
| $f_{1}$, bending $z$-dir | 4.126 | 4.128 | 4.128 | 4.200 | 4.171 | 4.166 | 4.114 |
| $f_{2}$, bending $x$-dir | 23.736 | 23.432 | 23.432 | 23.560 | 23.472 | 23.465 | 23.778 |
| $f_{3}$, bending $z$-dir | 25.854 | 25.782 | 25.783 | 26.189 | 25.798 | 25.722 | 24.991 |
| $f_{4}$, bending $z$-dir | 72.288 | 71.815 | 71.815 | 73.035 | 70.837 | 70.459 | 64.731 |

Tables 5.6 and 5.7 show the first four bending frequencies for the two considered wing configurations, 10 B 4 and 30 B 4 meshes are used, respectively. The first torsional frequency is shown in Table 5.8 . Figure 5.7 shows the cross-section modal shape of the free tip

Table 5.7: Bending natural frequencies, Hz , of the cantilever wing models for different theories, 30 B4 mesh.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (solid) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of DOFs | 455 | 455 | 819 | 1638 | 2730 | 4095 | > 600000 |
| Straight |  |  |  |  |  |  |  |
| $f_{1}$, bending $z$-dir | 5.872 | 5.866 | 5.866 | 5.972 | 5.922 | 5.913 | 5.864 |
| $f_{2}$, bending $x$-dir | 33.340 | 32.709 | 32.709 | 32.834 | 32.656 | 32.625 | 32.335 |
| $f_{3}$, bending $z$-dir | 36.735 | 36.581 | 36.581 | 37.127 | 36.376 | 36.216 | 34.844 |
| $f_{4}$, bending $z$-dir | 102.634 | 101.617 | 101.617 | 103.210 | 98.918 | 98.133 | 81.976 |
| Sweep + Dihedral |  |  |  |  |  |  |  |
| $f_{1}$, bending $z$-dir | 4.126 | 4.128 | 4.128 | 4.191 | 4.162 | 4.157 | 4.114 |
| $f_{2}$, bending $x$-dir | 23.736 | 23.432 | 23.432 | 23.513 | 23.426 | 23.411 | 23.778 |
| $f_{3}$, bending $z$-dir | 25.854 | 25.782 | 25.783 | 26.132 | 25.746 | 25.668 | 24.991 |
| $f_{4}$, bending $z-\operatorname{dir}$ | 72.288 | 71.815 | 71.815 | 72.866 | 70.696 | 70.317 | 64.731 |

obtained via a fourth-order model with a significant distortion of the airfoil contour. The following considerations can be made.

1. A good match is found with the reference solutions.
2. The bending modes can be detected using classical models. However, the importance of higher-order terms increases for higher vibration modes.
3. The investigation of the torsional modes underlines the importance of the refined models. At least a second-order theory is needed to obtain a reliable estimation of the torsional frequency. In the presence of sweep and dihedral angles, the ineffectiveness of linear models is even more evident. It has to be highlighted that, while EBBM and TBM cannot provide torsional modes, the $N=1$ model is able to detect the torsion of the cross-section, however higher-order terms are needed to compute more accurate torsional frequencies. This aspect is consistent to what presented in [117] for the static case.
4. The influence of the higher-order terms is more relevant on the torsional modes than on the bending ones.
5. Refined models are able to detect the distortion of the cross-section. It is noteworthy that this aspect is particular relevant when thin-walled structures are considered.
6. The TBM and the full linear models provide the same results for bending modes.
7. The combined presence of the sweep and dihedral angles does not corrupt the accuracy of the solution.
8. The difference in computational cost between the beam and the solid models is more evident in the case of a thin-walled structure than for the compact one of the previous section.
9. A fourth-order model cannot be enough to detect the exact torsional frequency or higher bending modes. This issue is typical of thin-walled structures as clearly shown in [127] where $N=11$ models were used to detect the right distortion of a thin-walled cylinder loaded by a point force.

Table 5.8: First torsional natural frequency, Hz , of the cantilever wing models for different theories, 30 B 4 mesh.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (solid) |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Number of DOFs | 455 | 455 | 819 | 1638 | 2730 | 4095 | $>600000^{a}$ |
| Straight | $-^{*}$ | $-^{*}$ | 161.581 | 56.857 | 54.457 | 53.979 | 44.481 |
| Sweep + Dihedral | $-^{*}$ | $-^{*}$ | $-^{* *}$ | 48.689 | 46.692 | 46.304 | 39.443 |

(*): No torsional modes are provided by this model
$(* *)$ : No torsional modes have been found for frequencies up to 200 Hz

$$
\begin{array}{r}
\mathrm{N}=4 \\
\text { NACA } 2415
\end{array}
$$



Figure 5.7: Wing cross-section $17^{\text {th }}$ natural modal shape, $\mathrm{f}=604 \mathrm{~Hz}$, straight wing model.

### 5.3 Joined Wing

A joined wing model is considered as the last assessment of the present beam formulation. The geometry of the wing is shown in Fig. 5.8. The structure is composed of three segments: two horizontal and one vertical. As far as the boundary conditions are considered, the horizontal segments are both clamped at $y=0$. Two cross-section geometries are considered: rectangular and airfoil-shaped.


Figure 5.8: Joined-wing scheme, the horizontal segments are both clamped at $y=0$.
A compact rectangular cross-section is first considered, and is shown in Fig. 5.2. The height, $h$, is equal to 0.1 m and the width, $b$, is equal to 1 m . The horizontal segments have $L_{1} / h$ and $L_{3} / h$ equal to 100 , while for the vertical segment, $L_{2} / h$ is equal to 30.45 four-node beam elements are used as the mesh. An MSC Nastran shell model is used to compare the results. The different types of natural frequencies are reported in Table 5.9.

A torsional mode shape is shown in Fig. 5.9. The last row in Table 5.9 is related to a particular mode shape characterized by a differential bending of the horizontal segments which induces a torsion of the vertical part. This mode is shown in Fig. 5.10.

Table 5.9: Natural frequencies, Hz , of the joined rectangular wing for different beam models and comparison with those obtained via shell elements in NASTRAN.

| Model Type | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN <br> (shell) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Number of DOFs | 680 | 680 | 1224 | 2448 | 4080 | 6120 | 4242 |
| Bending $z$-direction |  |  |  |  |  |  |  |
| $f_{1}$ | 1.203 | 1.152 | 1.148 | 1.175 | 1.164 | 1.162 | 1.116 |
| $f_{2}$ | 5.417 | 5.414 | 5.373 | 5.532 | 5.466 | 5.454 | 4.620 |
| $f_{3}$ | 6.620 | 6.613 | 6.575 | 6.760 | 6.680 | 6.667 | 6.287 |
| $f_{4}$ | 14.929 | 14.911 | 14.798 | 15.260 | 15.052 | 15.022 | 13.110 |
| $f_{5}$ | 16.571 | 16.553 | 16.450 | 16.961 | 16.725 | 16.693 | 15.738 |
| Bending $x$-direction |  |  |  |  |  |  |  |
| $f_{1}$ | 11.510 | 11.361 | 7.913 | 7.933 | 7.882 | 7.875 | 6.682 |
| $f_{2}$ | 65.600 | 62.511 | 46.150 | 46.296 | 45.965 | 45.941 | 43.112 |
| Torsional |  |  |  |  |  |  |  |
| $f_{1}$ | -* | -* | -** | 33.774 | 33.548 | 33.177 | 31.417 |
| $f_{2}$ | -* | -* | -** | 34.034 | 33.803 | 33.442 | 31.658 |
|  |  |  | Others |  |  |  |  |
| $f_{1}$ | 119.290 | 107.005 | 56.101 | 47.803 | 47.553 | 47.512 | 47.118 |

(*): No torsional modes are provided by this model
$(* *)$ : No torsional modes have been found for frequencies up to 200 Hz

(a) Present Beam, $\mathrm{f}=33.216 \mathrm{~Hz}$.

(b) NASTRAN, $\mathrm{f}=31.417 \mathrm{~Hz}$.

Figure 5.9: Comparison between the present beam and the NASTRAN solution of a torsional mode of the joined-wing.

The second joined wing model has the airfoil shaped cross-section shown in Fig. 5.5. The horizontal segments have $L_{1} / b$ and $L_{3} / b$ equal to 10 , while for the vertical segment,


Figure 5.10: Differential bending mode of the rectangular joined wing. $\mathrm{f}=47.512 \mathrm{~Hz}$.
$L_{2} / b$ is equal to 3 . Some natural frequencies are reported in Table 5.10. Some modal shapes are reported in Fig. 5.11.

Table 5.10: Natural frequencies, Hz , of the joined wing with the airfoil-shaped cross-section.

| Model Type <br> Number of DOFs | $\begin{aligned} & \text { EBBM } \\ & 680 \end{aligned}$ | $\begin{aligned} & \text { TBM } \\ & 680 \end{aligned}$ | $\begin{aligned} & N=1 \\ & 1224 \end{aligned}$ | $\begin{aligned} & N=2 \\ & 2448 \end{aligned}$ | $\begin{aligned} & N=3 \\ & 4080 \end{aligned}$ | $\begin{aligned} & N=4 \\ & 6120 \end{aligned}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Bending z-direction |  |  |  |  |  |  |
| $f_{1}$ | 1.735 | 1.728 | 1.717 | 1.747 | 1.739 | 1.736 |
| $f_{2}$ | 8.125 | 8.115 | 8.010 | 8.193 | 8.135 | 8.118 |
| $f_{3}$ | 9.923 | 9.912 | 9.817 | 10.024 | 9.957 | 9.939 |
| $f_{4}$ | 22.384 | 22.324 | 22.043 | 22.566 | 22.384 | 22.337 |
| $f_{5}$ | 24.850 | 24.786 | 24.530 | 25.100 | 24.898 | 24.850 |
| Bending $x$-direction |  |  |  |  |  |  |
| $f_{1}$ | 11.511 | 11.361 | 8.148 | 8.158 | 8.089 | 8.080 |
| $f_{2}$ | 65.600 | 62.511 | 46.827 | 46.918 | 46.528 | 46.476 |
| Torsional |  |  |  |  |  |  |
| $f_{1}$ | -* | - | 152.689 | 49.721 | 49.409 | 48.375 |
| $f_{2}$ | - | - | 162.443 | 52.803 | 52.476 | 52.722 |
| Others |  |  |  |  |  |  |
| $f_{1}$ | 119.291 | 107.005 | 56.488 | 45.982 | 45.722 | 45.338 |

$(*)$ : No torsional modes are provided by this model
The following remarks can be made.

1. In the case of a rectangular cross-section, the total number of degrees of freedom is lower than that of the shell model in the cases of linear and parabolic beam models. In the case of cubic expansion, the number of DOFs is almost the same, while the fourth-order model requires a larger number of displacement variables than the shell model. This is due to the fact that a compact rectangular cross-section is considered. It should be pointed out that the analysis of the airfoil-shaped cross-section requires the same amount of DOFs as the compact rectangular one, while modeling via shell


Figure 5.11: Various modal shapes of the joined-wing with airfoil-shaped cross-section.
elements would probably need a considerably larger effort.
2. The results are in good agreement with those furnished by the shell model.
3. It has been confirmed that the torsional modes are detected by higher models than the linear ones.
4. The proper detection of the torsional modes as well as of the differential bending ones shows that the present beam model is able to detect complex modal deformed configurations that are usually furnished by shell models. This kind of result is therefore referred to shell-like.
5. The x-direction bending frequencies by TBM and EBBM present larger differences with respect to the refined models than the z-direction ones. This is most likely due to the fact that such a bending mode makes the vertical portion of the joined wing rotate along its local longitudinal axis, this rotation cannot be properly detected by TBM and EBBM because no in-plane distortions are foreseen by these models.

## Chapter 6

## Results: static and dynamic response of homogeneous structures

The present chapter report the results concerning the implementation of one-dimensional CUF models in the dynamic response analysis of slender structures. A preliminary assessment to test the accuracy of finite elements based on the CUF in a Newmark direct integration process is therefore required. Some reference cases retrieved from the structural dynamics literature [128] are taken as a benchmark. Moreover, a number of thin-walled structures under dynamc loads are afterwards analyzed through refined 1D theories in order to highlight the shell-type capabilities of the formulation.

### 6.1 Compact square section

A square simply supported cross shaped beam is considered. The sides of the cross-section are equal to 0.1 m , whereas the span-to-height ratio $L / h$ is equal to 100 . This slender structure is modeled through a one-dimensional mesh of 10 B4 finite elements along the $y$ axis, as done for all the following analyses. In this chapter, the isotropic material considered is aluminium: Young's modulus $E=69 \mathrm{GPa}$, Poisson's ratio $\nu=0.33$, and density $\rho=2700$ $\mathrm{kg} \mathrm{m}^{-3}$.
The first assessment case is a single harmonic force applied at the mid-span section of the beam:

$$
\begin{equation*}
P_{z}(t)=P_{z 0} \sin (\omega t) \quad y_{L}=L / 2 \tag{6.1}
\end{equation*}
$$

where $P_{z 0}=-1000 \mathrm{~N}$ is the amplitude of the sinusoidal load with angular frequency $\omega=7$ $\mathrm{rad} \mathrm{s}^{-1}$. The analytical undamped dynamic response of an Euler-Bernoulli beam made of isotropic material and loaded by this kind of force is well-known [128]. Let $\omega_{1}$ be the fundamental angular frequency of the beam corresponding to a bending modal shape. For the sake of brevity, when $\omega<\omega_{1}$ some reference values for the maximum transverse dynamic and static deflections occurring at the load application point are reported here:

$$
\begin{gather*}
u_{z \text { max }, \mathrm{DYN}}^{\text {anal }} \cong \frac{2 P_{z 0} L^{3}}{\pi^{4} E I} \frac{1}{1-\omega / \omega_{1}} \quad u_{z \text { max }, \mathrm{ST}}^{\text {anal }}=\frac{P_{z 0} L^{3}}{48 E I} \cong \frac{2 P_{z 0} L^{3}}{\pi^{4} E I}  \tag{6.2}\\
\frac{u_{z \text { max }, \mathrm{DYN}}^{\text {ana }}}{u_{z \text { max }, \mathrm{ST}}^{\text {ana }}} \cong \frac{1}{1-\omega / \omega_{1}} \tag{6.3}
\end{gather*}
$$



Figure 6.1: Effect of the time step $\Delta t$ of the transverse displacement at the mid-span section. Sinusoidal force. EBBM.
where $I$ is the moment of inertia of the beam cross-section.
The numerical dynamic response of the system is investigated through direct time integration via the Newmark method over the interval $[0,8] \mathrm{s}$. Since the analytical solution refers to an Euler-Bernoulli beam, for a proper comparison the analysis involves that classical beam theory, which is obtained as a particular case of the first-order CUF model. Here, the Newmark method is unconditionally stable. However, the numerical solution approaches the reference trend as the time step $\Delta t$ decreases. A convergence study is carried out to evaluate the dependence of the results on the time step chosen. The time-history of the transverse displacement $u_{z}$ at the mid-span section is depicted in Fig. 6.1. It is noteworthy that the choice of $\Delta t=0.08 \mathrm{~s}$ represents a coarse time discretization for this problem, whereas a good agreement with the analytical deflection is achieved for $\Delta t=$ 0.004 s . The dynamic response is approximately the sum of two sinusoidal functions with angular frequencies equal to $\omega$ and $\omega_{1}$. The maximum dynamic displacement computed through FEs based on EBBM differs in about $0.03 \%$ from the analytical value, as reported in Table 6.1. It occurs for $t=3.816 \mathrm{~s}$ at the mid-span beam section. The static solution of the system is also evaluated by disabling the inertial contribution of the mass matrix. As expected, it is a time-dependent sinusoid with the same frequency as that of the point force. Unlike the dynamic case, the amplitude is constant and equal to $u_{z \max , \mathrm{ST}}^{\mathrm{anal}}$.

Table 6.1: Maximum dynamic and static displacements for different beam models. Sinusoidal load. $\Delta t=0.004 \mathrm{~s}$ for the Newmark method.

| Theory | $u_{z \max , \mathrm{DYN}}$ | $u_{z \max , \mathrm{ST}}$ | $\omega_{1}$ | $\frac{u_{z \max , \mathrm{DYN}}}{u_{z \max , \mathrm{ST}}}$ | $\frac{1}{1-\omega / \omega_{1}}$ |
| :--- | :--- | :---: | :---: | :---: | :---: |
| Analytical | -70.0116 | -36.2319 | 14.4030 | 1.9323 | 1.9456 |
| EBBM | -69.9886 | -36.2318 | 14.4024 | 1.9317 | 1.9456 |
| $N=3$ | -70.0232 | -36.2427 | 14.4006 | 1.9321 | 1.9459 |
| $N=7$ | -70.0233 | -36.2428 | 14.4006 | 1.9321 | 1.9459 |

The time-response analysis is also conducted through refined beam models. Table 6.1
summarizes the maximum dynamic and static displacements for the third- and seventhorder models. However, for this case the increase of the theory expansion order $N$ does not reveal any remarkable difference in comparison with Euler-Bernoulli beam theory. In fact, the use of a compact square section for a slender beam subjected to a bending load restricts the local effects of the beam cross-section, which are eventually detectable by the higher-order terms of the beam displacement field. In confirmation of this fact, the value of the fundamental bending frequency of the beam $\omega_{1}$ is substantially the same for all the theories involved, as evident from Table 6.1. EBBM is therefore an effective theory for this case.
The second assessment consists in a single sinusoidal force starting from the left support and traveling along the beam axis with constant velocity $v_{y}=1.25 \mathrm{~m} \mathrm{~s}^{-1}$ :

$$
\begin{equation*}
P_{z}(t)=P_{z 0} \sin (\omega t) \quad y_{L}(t)=v_{y} t \tag{6.4}
\end{equation*}
$$

The analytical dynamic response to this load can be found in [128]. Unlike the non-traveling load case, in this second assessment the maximum displacement of the beam is not placed at the mid-span section for both the static and dynamic analyses. Nevertheless, as in the previous case, the time-dependent displacement of the mid-span section is investigated through EBBM and plotted in Fig. 6.2. The influence of the time step exploited in the Newmark method is less evident with respect to the non-traveling load case. However, the choice of $\Delta t=0.004 \mathrm{~s}$ is again an appropriate time discretization. In fact, the amplitude of the curve obtained with $\Delta t=0.08$ s noticeably differs from the analytical benchmark even over $t=2 \mathrm{~s}$ and this error is likely to propagate dramatically in even longer simulations. The inertial effect due to the mass matrix emphasizes the transverse deflection of the beam with respect to the static response. For the sake of brevity, the results for refined theories are not reported here because there is no noticeable difference with Euler-Bernoulli beam theory.


Figure 6.2: Effect of the time step $\Delta t$ of the transverse displacement at the mid-span section. Sinusoidal traveling force. EBBM.

A concentrated force $P_{z 0}$ advancing along the beam axis with the same velocity is considered as a third assessment case:

$$
\begin{equation*}
P_{z}(t)=P_{z 0} \quad y_{L}(t)=v_{y} t \tag{6.5}
\end{equation*}
$$

At first, the inertial effect is neglected in order to evaluate the static behavior of the beam, in terms of the displacement $u_{z}(y=L / 2, t)$ of the mid-span section. This time-dependent displacement is described by the continuous cubic function illustrated by the square symbols in Fig. 6.3. The static curve is defined on two subsets and is symmetrical with respect to $t=4 \mathrm{~s}$. In fact, for Euler-Bernoulli beam theory:
$u_{z}(y=L / 2, t)= \begin{cases}\frac{P_{z 0}}{48 E I}\left(3 L^{2}-4\left(v_{y} t\right)^{2}\right)\left(v_{y} t\right) & 0 \leq v_{y} t \leq L / 2 \\ \frac{P_{z 0}}{48 E I}\left(-L^{2}+8 L\left(v_{y} t\right)-4\left(v_{y} t\right)^{2}\right)\left(L-\left(v_{y} t\right)\right) & L / 2 \leq v_{y} t \leq L\end{cases}$


Figure 6.3: Effect of the time step $\Delta t$ of the transverse displacement at the mid-span section. Constant traveling force. EBBM.

For the static analysis, it is possible to demonstrate that the maximum deflection of the beam, $u_{z \max }=P_{z 0} L^{3} / 48 E I$, is placed at $y=L / 2$ for $t_{u_{z \max }}=4 \mathrm{~s}$, that is for $y_{L}=v_{y} t=5 \mathrm{~m}$. On the contrary, the dynamic response curve oscillates over the static cubic function and is no longer symmetrical. As a result, the maximum value of $u_{z}$ during the considered time interval is no longer reached at the mid-span section. The Newmark method fails to compute the frequency of this oscillating trend through a coarse time discretization ( $\Delta t=0.08 \mathrm{~s}$ ). Instead, the direct time integration with $\Delta t=0.004 \mathrm{~s}$ ensures a convergent solution by approaching the analytical results accross the entire time interval of study. Furthermore, the error in computing the maximum dynamic displacement at $y=L / 2$ decreases as $\Delta t$ decreases. As shown in Table 6.2, the same occurs for the corresponding time instant $t_{u_{z \max }}$. In fact, the position of the load application point $y_{L}\left(t_{u_{z \max }}\right)$ at the time instant $t_{u_{z \max }}$ is moderately different from the analytical value, except for the refined value $\Delta t=0.004 \mathrm{~s}$.

### 6.2 Thin-walled rectangular section

The assessment procedure is completed. A clamped beam with a thin-walled rectangular cross-section is now introduced. As illustrated in Fig. 6.4, the width $a$ of the section is equal

Table 6.2: Maximum displacement of the mid-span section and corresponding time instant for different time steps through the Newmark method. Constant traveling load. EBBM.

| Analysis | $u_{z \max }$ | $t_{u_{z \max }}$ | $y_{L}\left(t_{\left.u_{z \max }\right)}\right) / L$ |
| :--- | :---: | :---: | :---: |
| Newmark $\Delta t=0.08 \mathrm{~s}$ | -36.9617 | 4.160 | 0.52 |
| Newmark $\Delta t=0.016 \mathrm{~s}$ | -37.1458 | 3.840 | 0.48 |
| Newmark $\Delta t=0.004 \mathrm{~s}$ | -37.1292 | 3.824 | 0.47 |
| Analytical solution | -37.1185 | 3.824 | 0.47 |
| Static solution | -36.2319 | 4.000 | 0.50 |

to 1 m and the width-to-height ratio is equal to 0.1 . The thickness $t$ of the skin is constant and equal to 0.005 m . The beam is relatively short since the span-to-width ratio $L / a$ is now equal to 10 . The section at $y=2.5 \mathrm{~m}$ is loaded by a concentrated force $P_{z}$. Four points are chosen as characteristic positions. The application point is denoted as point 1 in Fig. 6.4. The effect of the point load on this thin-walled section is evaluated through a preliminary static analysis with $P_{z}=-10000 \mathrm{~N}$. This case cannot be consistent with the kinematic hypotheses which classical beam models are based on. Classical models are therefore not expected to yield accurate results. In fact, while EBBM assumes an undeformed section, the increase of the expansion order $N$ provides a remarkable shell-like deformation of the loaded cross-section. Figure 6.5 illustrates that the upper skin is particularly deflected due to the position of $P_{z}$ at point 1. However, at least a tenth-order model ( 6138 DOFs ) is necessary to detect such an effect and simulate the FE NASTRAN solution well, obtained through a 396000 DOFs analysis.


Figure 6.4: Thin-walled rectangular cross-section.


Figure 6.5: Static deformation of the loaded beam cross-section. $t=0 \mathrm{~s}$.
A time-dependent sinusoidal load with the same amplitude $P_{z 0}=-10000 \mathrm{~N}$ and angular frequency $\omega=30 \mathrm{rad} \mathrm{s}^{-1}$ is applied at point 1 . Given the static results, a tenth-order structural model is considered in the dynamic analysis over the interval $[0,1.5]$ s. Figure 6.6 plots the convergence of the solution as the time step used in the Newmark method decreases. For instance, point 2 is taken as a control point. Again, $\Delta t$ affects the correct evaluation of the amplitude and frequency of the beam dynamic response. It can be demonstrated that this convergence is also required for classical and lower-order theories. Once an appropriate
$\Delta t$ is chosen, the transverse displacement is evaluated for points $1,2,3$ and 4 and depicted in Fig. 6.7. As expected, $u_{z}$ reaches the maximum value at loading point 1, whose trend is described mainly by an oscillation with angular frequency equal to $\omega$. The other points seem to be affected in a different way by some local oscillations due to inertial effects, see in particular point 3. Such effects can be observed thanks to higher-order terms. In fact, for EBBM the four curves coincide perfectly, given the undeformed section hypothesis of classical models.


Figure 6.6: Effect of the time step $\Delta t$ of the transverse displacement of Point 2. $N=10$.


Figure 6.7: Dynamic response of different points on the loaded cross-section. $\Delta t=7.510^{-4} \mathrm{~s}$. $N=10$.

Figure 6.8 displays the response of point 1 computed for different theories. EBBM and the fouth-order model provide very similar results. For $N=7$ the maximum displacement increases, but it is via the tenth-order model that $u_{z}$ dramatically rises, even over 250 \% with respect to EBBM and $N=4$. The difference consists not only in the amplitude of the oscillation, but also in the trend shape. Unlike lower-order models, for $N=10$ the displacement of point 1 vs. time is an oscillating curve dominated by the angular
frequency of the load. The same behavior does not occur at point 3, which lies on the lower skin of the section. The refinement of models does not reveal a striking difference in maximum displacement whereas their trend shapes are considerably dissimilar, see Fig. 6.9. Since higher-order terms are powerful in evaluating the section deformation, refined models are able to detect local shell-like oscillations related to different inertial accelerations of points over the loaded section. Table 6.3 summarizes the results in terms of maximum displacement of points $1,2,3$ and 4 obtained over the interval. Especially for $N=10$ these values are very different within the section and correspond to different time instants.

Table 6.3: Maximum displacement [mm] of points 1, 2, 3 and 4 and corresponding time instant for different theories.

| $u_{z \max }$ | EBBM | $N=4$ | $N=7$ | $N=10$ |
| :---: | :---: | :---: | :---: | :---: |
| Point 1 | -24.660 | 24.812 | 39.359 | 57.838 |
|  | $(1.08225 \mathrm{~s})$ | $(1.38975 \mathrm{~s})$ | $(1.41225 \mathrm{~s})$ | $(0.88125 \mathrm{~s})$ |
| Point 2 | -24.660 | 23.964 | 32.944 | 32.071 |
|  | $(1.08225 \mathrm{~s})$ | $(1.38900 \mathrm{~s})$ | $(1.41300 \mathrm{~s})$ | $(1.42425 \mathrm{~s})$ |
| Point 3 | -24.660 | 23.964 | 31.055 | 26.193 |
|  | $(1.08225 \mathrm{~s})$ | $(1.38900 \mathrm{~s})$ | $(1.41225 \mathrm{~s})$ | $(1.43100 \mathrm{~s})$ |
| Point 4 | -24.660 | 24.810 | 37.419 | -39.633 |
|  | $(1.08225 \mathrm{~s})$ | $(1.38975 \mathrm{~s})$ | $(1.41225 \mathrm{~s})$ | $(0.87900 \mathrm{~s})$ |



Figure 6.8: Time-dependent transverse displacement of Point 1 for different theories. $\Delta t=7.510^{-4}$ s.

### 6.3 Thin-walled annular section

A clamped-clamped beam with a thin-walled annular cross-section is considered in the last analysis case of the present chapter. The outer diameter of the cylinder $d$ is equal to 0.1 m whereas the thickness is 0.001 m . The span-to-diameter ratio $L / d$ is equal to 10. As displayed in Fig. 6.10, four particular points are considered over the mid-span


Figure 6.9: Time-dependent transverse displacement of Point 3 for different theories. $\Delta t=7.510^{-4}$ s.
cross-section. Four concentrated forces are applied at points $A, B, C$ and $D$ within the thin-walled cross-section, in an outward direction. They are time-dependent sinusoids with the same amplitude $P_{z 0}=10000 \mathrm{~N}$ and a phase shift:

$$
\begin{array}{ll}
P_{z A}(t)=P_{z 0} \sin \left(\omega t+\phi_{A}\right) & \phi_{A}=0^{\circ} \\
P_{x B}(t)=P_{z 0} \sin \left(\omega t+\phi_{B}\right) & \phi_{B}=30^{\circ} \\
P_{z C}(t)=-P_{z 0} \sin \left(\omega t+\phi_{C}\right) & \phi_{C}=60^{\circ}  \tag{6.7}\\
P_{x D}(t)=-P_{z 0} \sin \left(\omega t+\phi_{D}\right) & \phi_{D}=90^{\circ}
\end{array}
$$

where the angular frequency $\omega=100 \mathrm{rad} \mathrm{s}^{-1}$. The dynamic response of the structure is evaluated over the time interval $[0,0.025]$ s by involving classical as well as refined 1 D models. The deformed configuration of the mid-span section at $t=0 \mathrm{~s}$ is presented in Fig. 6.11 and compared with NASTRAN shell-FE solution. EBBM and TBM cannot evaluate any displacement at any point of the section. The fourth-order theory shows a global deformation but cannot detect any local effect due to the concentrated loads. With $N=7$ the refined elements are able to detect the shell-like displacement field of all the cross-section points except the loading ones. An expansion order $N$ at least equal to 10 is necessary to obtain a detailed description of the loading points displacement field. This conclusion is consistent with the results obtained through CUF models by Carrera et al.[129].

Table 6.4 summarizes the transverse displacements of loading points $A$ and $D$ at $t=0$ s. The third and fifth columns present the percentage error computed with respect to the NASTRAN solution, taken as a reference, for $u_{x D}$ and $u_{z A}$, respectively. As expected, the computation of local displacements on these loading points is not trivial for lower-order models and impossible for classical models. A slight improvement is noticed for $N=7$, but the error decreases remarkably for an expansion order higher than 10. The last column shows the total number of degrees of freedom, DOFs, for each model. A good convergent trend is observed as $N$ increases with a considerably smaller computational effort than that required by the reference shell model. In Fig. 6.12 the static three-dimensional deformation of the cylinder computed via $N=10$ is graphically compared with NASTRAN shell solution. The spectrum used on the surface corresponds to the resultant displacement. As expected, the mid-span section is subjected to the overall maximum deflection due to the


Figure 6.10: Thin-walled annular cross-section.


Figure 6.11: Static response of the mid-span cross-section. $t=0 \mathrm{~s}$.
concentrated loads. Figure 6.12 emphasizes the capabilities of the proposed refined model in describing typical shell-like lobs over the thin-walled structure with a sizeable reduction in computational cost in terms of DOFs ( 6138 vs. 250000).

Table 6.4: Displacements [mm] of the loading points $A$ and $D$ for different FE models. $t=0 \mathrm{~s}$.

| Theory | $u_{x D}$ | Error $u_{x D}$ | $u_{z A}$ | Error $u_{z A}$ | DOFs |
| :--- | :---: | :---: | :---: | :---: | ---: |
| EBBM | 0. | - | 0. | - | 93 |
| TBM | 0. | - | 0. | - | 155 |
| $N=1$ | -2.0937 | $-91.08 \%$ | -1.4362 | $-85.47 \%$ | 271 |
| $N=3$ | -2.9313 | $-87.51 \%$ | -3.5311 | $-64.27 \%$ | 930 |
| $N=4$ | -5.9690 | $-74.56 \%$ | -6.8900 | $-30.29 \%$ | 1395 |
| $N=7$ | -15.7213 | $-32.99 \%$ | -9.3591 | $-5.31 \%$ | 3348 |
| $N=10$ | -19.7523 | $-15.81 \%$ | -9.7314 | $-1.54 \%$ | 6138 |
| $N=14$ | -21.1939 | $-9.67 \%$ | -9.8418 | $-0.43 \%$ | 11160 |
| NASTRAN | -23.4628 | - | -9.8840 | - | 250000 |



Figure 6.12: Static three-dimensional resultant displacement of the thin-walled cylinder. $t=0 \mathrm{~s}$.

As far as the dynamic response is concerned, the analysis involves variable kinematic models with an expansion order $N$ up to 10 because of the considerations exposed above for the static case. The mid-span cross-section remains the most stressed section and the high sensitivity of its shape to the point loads is detectable for higher-order models. For instance, the configuration at the final time instant $t=0.025 \mathrm{~s}$ is depicted in Fig. 6.13. According to Fig. 6.11, $N=4$ is again unable to detect any local effect near the loading points; it only detects a global deflection of the annular section. On the contrary, with $N=7$ and $N=10$ the proposed 1D model makes it possible to take into account local deformations typical of a shell-like behavior.


Figure 6.13: Deformation of the mid-span cross-section for different beam models. $t=0.025 \mathrm{~s}$.

## Chapter 7

## Results: static and dynamic response of nonhomogeneous structures


#### Abstract

In this chapter, applications of 1D CUF models to the static and dynamic response of structures with arbitrary cross-sections are presented. The capabilities of the CUF one-dimensional (1D) model in describing the structural static and dynamic behavior of arbitrary nonhomogeneous structures are investigated. The simulation of biomechanical systems requires nowadays a multifield approach involving computational fluid mechanics of haematic flows, structural modeling of biological tissues and high-fidelity fluid-structure techniques [46]. In order to gain more insight into the complex biomedical processes during therapeutical interventions and for the optimization of treatment methods and disease preventions, constitutive modeling of biological tissues and related computer simulations are active subjects of current research [3]. As well as in other physical applications such as aerospace and civil engineering, nonhomogeneous structures are widely used in biomechanical field. For example, healthy arteries consist of three layers with different mechanical properties: the intima, the media and the adventitia. In last decades, a large use of three-dimensional models for the structural analysis of biological tissues has been necessary with the main disadvantage of huge computational cost simulations.


### 7.1 Static analysis of a layered cylinder

The structure studied in the present section is a thin-walled three-layer cylinder. As depicted in Fig. 7.1, the cross-section is composed of three thin circular layers denoted as layers 1,2 and 3 . The layers of the cylinder are made of three different isotropic materials. The material and geometrical properties of the layers are summarized in Table 7.1. The thickness $t=1 \mathrm{~mm}$ is constant for each layer and is small enough to consider overall the cylinder as a thin-walled structure, since the external and internal diameters are equal to $d_{e}=100 \mathrm{~mm}$ and $d_{i}=94 \mathrm{~mm}$, respectively. The length $L$ of the cylinder is equal to 500 mm . A clamped boundary condition is taken into account for the edges of the cylinder at $y=0$ and $y=L$.

In order to easily present the deformation of the cylinder, a cylindrical coordinate system $r-\theta-y$ is now introduced. The plane $r-\theta$ is the cross-section plane. The $r$ coordinate goes along the radial direction, whereas the $\theta$ coordinate is an angle measured counterclockwise from the axis $-z$, see Fig. 7.1. For the sake of simplicity, the origin of


Figure 7.1: Pressure applied to the cylinder with three different layers for the static analysis.
Table 7.1: Material and geometrical properties of the cylinder layers.

| Property | Layer 1 | Layer 2 | Layer 3 |
| :--- | ---: | ---: | ---: |
| $\mathrm{t}[\mathrm{mm}]$ | 1 | 1 | 1 |
| $\mathrm{E}[\mathrm{GPa}]$ | 69 | 30 | 15 |
| $\nu$ | 0.33 | 0.33 | 0.33 |
| $\rho\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ | 2700 | 2000 | 1800 |

the cylindrical system overlaps the cross-section center of mass as well as the origin of the cartesian coordinate system.

A uniform pressure $p=14.8 \mathrm{MPa}$ is applied on the internal surface of the cylinder as shown in Fig. 7.1. In particular, the loading is applied along all the length of the structure and only on the upper side of layer $1\left(r=d_{i} / 2,90^{\circ} \leq \theta \leq 270^{\circ}, 0 \leq y \leq L\right)$. This sample case is retrieved from the work [130]. Although the structure here analyzed is axisymmetric, its deformed configuration is not expected to be axisymmetric due to this particular loading distribution. Nevertheless, the solution will be symmetrical with respect to both $x=0$ and $y=L / 2$ planes. Given the solution's simmetry with respect to $y=L / 2$ plane, the maximum deformation is placed on the section lying on this plane. This section is denoted as mid-span section from this point forward.

One-dimensional theories are usually employed to study slender beams because of their limiting kinematic hypotheses. Instead, the cylinder here considered is relatively short since the span-to-external diameter ratio $L / d_{e}$ is equal to 5 . Nevertheless, the static response of the structure is computed through the 1D CUF model with a variable expansion order up to $N=8$ and a 1D mesh of $10 B 4$ finite elements by solving Eq. 7.1, i.e. Eq. 3.89. A solid finite element analysis is also carried out via the commercial code NASTRAN and taken as reference in order to assess the present refined 1D model for a nonhomogeneous shell case. Due to the small layer thickness and the well-known aspect ratio restrictions typical of solid finite elements, the model in NASTRAN consists of 64800 HEX8 elements and 86880 nodes. The number of degrees of freedom (DOFs) is thus equal to 257760 .

$$
\begin{equation*}
\mathbf{K q}=\mathbf{F} \tag{7.1}
\end{equation*}
$$

The deformation of the cylinder mid-span section, which is expressed in terms of the magnitude of displacement vector $\boldsymbol{u}$, is depicted in Fig. 7.2 for different 1D models and
compared with the solid NASTRAN solution. Instead, the magnitude of displacement vector $\mathbf{u}$ computed on the external edge of the annular mid-span section $\left(r=d_{e} / 2, y=L / 2\right)$ along the angular $\theta$ coordinate is depicted in Fig. 7.3. The solution confirms to be symmetrical with respect to the $x=0$ plane regardless the model used. In addition to a bending behavior, the particular distribution of pressure loadings is supposed to deform the beam cross-section. This type of deformation cannot be consistent with the kinematic hypotheses of undeformed cross-section shape which classical beam models are based on. Classical models are therefore not expected to yield accurate results and this statement is confirmed by the constant displacement trend computed by Euler-Bernoulli beam theory.

By enriching the displacement field, the first-order model provides a linear displacement distribution along the $x$ and $z$ directions, but it results not to be realistic. Taking the solid FE model as reference, the results obtained by the present formulation improve as the expansion order $N$ increases. In particular, for an eighth-order model the deformation of the mid-span section is accurately described and its agreement with the 3D solution is remarkable. Even though it is not reported here, it is noteworthy that for an expansion order higher than 8 the analysis would provide the same results as those obtained by $N=8$, thus confirming a convergent trend on $N$. However, for $N<8$ the displacement is remarkably overestimated at point $\theta=180^{\circ}$ and dangerously underestimated at points where the displacement is actually maximum.

The maximum displacement on the external edge of the mid-span section is reported in Table 7.2. The error in computing the maximum deflection is significant for classical and low-order models, except for $N=1$. Nonetheless, the solution for the first-order model is completely unrealistic, see Fig. 7.3. In general, $u_{\max }$ increases as $N$ increases approaching the 3D value. Also the position of the maximum displacement along the external edge in terms of $\theta$ noticeably changes for different models. $\theta_{u_{\max }}$ for 1D models with an expansion order higher than 4 coincides with the reference solution, whereas lower-order models show their low accuracy even in the position evaluation. This aspect has not to be underestimate because it turns out to be fundamental for a failure investigation, for instance. In conclusion, the eighth-order model proves its capability in detecting exactly the three-dimensional deformation of the layered thin-walled cylinder with a sizeable reduction in computational cost in terms of DOFs (4185 vs. 257760).

Table 7.2: Maximum displacement [mm] on the external edge of the mid-span section. Layered cylinder case.

| Model | $u_{\max }$ | \% Difference | $\theta_{u_{\max }}$ | DOFs |
| :--- | ---: | :---: | ---: | ---: |
| EBBM | 5.639 | -60.663 | - | 93 |
| $\mathrm{~N}=1$ | 14.272 | -0.439 | 180 | 279 |
| $\mathrm{~N}=2$ | 8.780 | -38.751 | 93 | 558 |
| $\mathrm{~N}=3$ | 12.011 | -16.212 | 0 | 930 |
| $\mathrm{~N}=4$ | 12.625 | -11.929 | 120 | 1395 |
| $\mathrm{~N}=5$ | 12.685 | -11.510 | 117 | 1953 |
| $\mathrm{~N}=6$ | 13.281 | -7.353 | 117 | 2604 |
| $\mathrm{~N}=7$ | 13.300 | -7.220 | 117 | 3348 |
| $\mathrm{~N}=8$ | 14.236 | -0.691 | 117 | 4185 |
| Solid FEM | 14.335 | - | 117 | 257760 |



Figure 7.2: Deformation of the mid-span section for higher-order 1D models.

### 7.2 Static analysis of a nonhomogeneous arterial atherosclerotic plaque

The static structural analysis of a clinic artery case retreived from the biomechanical literature $[2,3]$ is now addressed. This example represents a preliminary application of the 1D CUF model to the study of a biomechanical case with arbitrary cross-section and nonhomogeneous materials. In the work [3] only a two-dimensional structural simulation of the cross-section under a time-dependent internal blood pressure is carried out in order to keep the computational effort relatively low. This simplified approach totally neglects


Figure 7.3: Displacement of the external edge of the mid-span section. Layered cylinder case.
the important effects due to the third out-of-plane dimension. In general, these effects are fundamental especially in a biomechanical case where the haematic flow field and the non-standard structural behavior of biological tissues need a complete three-dimensional description. Obviously, the introduction of the third direction would typically need the use of solid (3D) elements instead of 2D plate or shell FEs and, consequently, a much higher computational effort. In order to take into account the out-of-plane direction and analyze a complete solid structure, 1D CUF models are thus here proposed since they require a low computational cost though showing remarkable three-dimensional performance.

Once the capabilities of higher-order 1D CUF models have been assessed for the previous nonhomogeneous cylinder case with a classic annular section, a human external iliac artery with a pronounced atherosclerotic plaque is now considered as an important application in biomechanics of an arbitrary cross-section structure. In particular, a portion of the athesclerotic artery with a severe stenoses (lumen reduction) presented in [2] is introduced. The components of the artery are identified by hrMRI (high resolution magnetic resonance imaging) and histological analysis [2], see Fig. 7.4. These approaches consider eight different tissue types: fibrous cap (FC), i.e. the fibrotic part at the luminal border, calcification (C), lipid pool (LP), adventitia (A), non-diseased media (M), non-diseased intima, fibrotic intima at the medial border and diseased fibrotic media. As done in [3], for the following numerical investigations the non-diseased intima is neglected. Furthermore, the fibrotic intima at the medial border and the diseased fibrotic media are treated as one component, the fibrotic media (FM). According to these assumptions, in [3] the same cross-section including the above mentioned components is discretized with 6048 triangular elements with quadratic Ansatz functions, as depicted in Fig. 7.5.

The section width and height of the cross-section are approximately the same and equal to 20 mm . For the sake of simplicity, the arterial cross-section is extruded along the out-of-plane direction ( $y$ axis) for 40 mm and a clamped boundary condition is taken into account for the edges at $y=0$ and $y=L$. The structure is here modeled with a one-dimensional mesh of 10 B4 finite elements (31 nodes), as shown in Fig. 7.6(c), and analyzed through the CUF formulation. Furthermore, a FE model is built in NASTRAN


Figure 7.4: Arterial cross-section with a pronounced atherosclerotic plaque in a human external iliac artery. This figure is retrieved from [2].


Figure 7.5: Two-dimensional model for the discretization of atherosclerotic plaque section used in [3].
and discretized with a mesh of 244320 HEX8 solid elements ( 260172 nodes) with a total number of DOFs equal to 761244 . Figures 7.6 (a) and 7.6 (b) show the solid model of the atherosclerotic plaque, obtained by extruding the same cross-section shape as that used in [3] (see Fig. 7.4(c)). The linear static analysis is performed with a uniform pressure load of $180 \mathrm{mmHg}(\cong 24 \mathrm{kPa})$ applied on the surface bounding the lumen, i.e. the inside space of the artery, as illustrated in Fig. 7.6(a). This pressure level may be seen as an upper bound for the hypertensive internal blood pressure.

Higher-order CUF models with a variable expansion order $N$ are employed and the results are compared to the solid FEM solution. Both the cases of homogeneous and nonhomogeneous materials are considered and afterwards described. The clinic application above described is a very preliminary application of the 1D CUF model to the study of a biomechanical case. For the sake of completeness, it is emphasized that the following results provide a numerical example of limited validity with respect to quantitative results. However, this section has the goal to show that the proposed one-dimensional CUF models provide an excellent agreement with a three-dimensional solution in the context of finite element simulations, with a remarkable reduction in computational cost (in terms of DOFs).


Figure 7.6: Simplified solid model of the arterial atherosclerotic plaque discretized with 244320 HEX8 solid finite elements in NASTRAN (a),(b). Mesh of the one-dimensional CUF model discretized with $10 B 4$ finite elements (c). Homogeneous material case.

## Homogeneous material case

A simplified test case is here addressed by assuming the cross-section made of homogeneous isotropic material. Averaging grossly the Young's moduli of the six arterial tissue types reported in [3], the Young's modulus considered is $E=2.4 \mathrm{MPa}$ and Poisson's ratio is $\nu=0.33$. Figure 7.7 shows the deformation of the mid-span arterial cross-section $(y=L / 2)$, where the maximum displacement is located, for different one-dimensional models (up to $N=20$ ) compared to the solid FE solution. In particular, the coloured map of each subfigure represents the magnitude of the displacement vector $\mathbf{u}$ computed over the section by each one-dimensional theory. On the contrary, the two coloured curves shows the internal and external plaque contours in the deformed configuration computed through the solid FE method, which is taken as reference. The remaining blue lines are the edges of the six arterial tissues.

As expected, classical beam theories (EBBM and TBM) are completely not able to study this case due to their kinematic hypotheses about the cross-section deformation. In fact, they show a uniform quasi-null displacement over the cross-section. The first-order model $(N=1)$ enables the in-plane deformation of the cross-section but the result is again completely wrong with respect to the 3 D solution. In this case, even low-order theories are not accurate enough to catch an acceptable solution compared to the 3 D simulation. According to the reference solid solution, the reason is that the plaque deforms locally around the lumen, i.e. the load application region, whereas a quasi-null displacement is observed far from the lumen. This particular deformation requires a high expansion order for the present formulation to reach an acceptable accuracy. In particular, the tissues most interested by the internal blood pressure are the media and the part of adventitia in contact with the media. Though surrounding part of the lumen and being thus directly loaded by the pressure, the fibrous cap is barely deformed and its maximum displacement is placed close to the media. As will be seen in the nonhomogeneous material case, this fact is directly related to the assumption of homogeneous material for all the tissue types. In fact, when a homogeneous material is taken into account the volume occupied by the lipid pool and calcification remarkably stiffens the side of the section on the right of the lumen with respect to the left one.

Despite its one-dimensional approach, the proposed higher-order model is able to accurately detect the in-plane deformation of this kind of cross-section with arbitrary


Figure 7.7: Displacement $|\mathbf{u}|[\mathrm{mm}]$ over the mid-span cross-section of the atherosclerotic plaque for different one-dimensional models compared to the solid FE solution. Homogeneous material case.
geometry. In fact, the proposed 1D FEs provide a convergent solution by approaching the NASTRAN 3D results as the refinement of the expansion increases, until a well agreement is achieved for $N=20$. As can be seen in Fig. 7.7, the region subjected to the maximum displacement lies on the central part of media (M) and adventitia (A). Thus, the maximum displacements on media and adventitia computed by the different models are reported in Table 7.3 and indicated as $u_{\text {max }}^{\mathrm{M}}$ and $u_{\text {max }}^{\mathrm{A}}$, respectively. For the sake of completeness, also the maximum displacement on fibrous cap $u_{\text {max }}^{\mathrm{FC}}$ is reported, even though it is much lower as previously mentioned. As expected, classical beam theories are completely ineffective in
studying this kind of structure, giving a constant quasi-null displacement over the section. For all the displacements summarized, the increase of the expansion order $N$ improves the results approaching the reference data with a convergent trend. In fact, the introduction of higher-order terms enables the structure to deform in a more realistic way and results to be fundamental in order to catch properly the artery deformation. It is important to note the remarkably lower computational effort required by 1D CUF model. In fact, the $N=20$ model provides an acceptable maximum error but with a number of degrees of freedom equal to 21483 , about 35 times lower than the DOFs required by the solid FE model.

Table 7.3: Maximum displacements [mm] on media, adventitia and fibrous cap of the atherosclerotic plaque over the mid-span cross-section for different models. Homogeneous material case.

| Model | $u_{\max }^{\mathrm{M}}$ | $(\%$ Error $)$ | $u_{\max }^{\mathrm{A}}$ | $(\%$ Error $)$ | $u_{\max }^{\mathrm{FC}}$ | $(\%$ Error $)$ | DOFs |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| EBBM | 0.0000 | $(-100.00 \%)$ | 0.0000 | $(-100.00 \%)$ | 0.0000 | $(-100.00 \%)$ | 93 |
| TBM | 0.0001 | $(-99.99 \%)$ | 0.0001 | $(-99.99 \%)$ | 0.0001 | $(-99.95 \%)$ | 155 |
| $N=1$ | 0.4212 | $(-58.88 \%)$ | 0.4632 | $(-53.96 \%)$ | 0.3656 | $(+93.44 \%)$ | 279 |
| $N=4$ | 0.0468 | $(-95.43 \%)$ | 0.0471 | $(-95.32 \%)$ | 0.0263 | $(-86.08 \%)$ | 1395 |
| $N=7$ | 0.1631 | $(-84.08 \%)$ | 0.1609 | $(-84.01 \%)$ | 0.0402 | $(-78.73 \%)$ | 3348 |
| $N=9$ | 0.2774 | $(-72.92 \%)$ | 0.2674 | $(-73.42 \%)$ | 0.0569 | $(-69.89 \%)$ | 5115 |
| $N=10$ | 0.3999 | $(-60.96 \%)$ | 0.3839 | $(-61.84 \%)$ | 0.0770 | $(-59.26 \%)$ | 6138 |
| $N=12$ | 0.5696 | $(-44.39 \%)$ | 0.5469 | $(-45.64 \%)$ | 0.0912 | $(-51.75 \%)$ | 8463 |
| $N=14$ | 0.7628 | $(-25.53 \%)$ | 0.7468 | $(-25.77 \%)$ | 0.1322 | $(-30.05 \%)$ | 11160 |
| $N=17$ | 0.9001 | $(-12.13 \%)$ | 0.8884 | $(-11.70 \%)$ | 0.1581 | $(-16.35 \%)$ | 15903 |
| $N=19$ | 0.9441 | $(-7.83 \%)$ | 0.9320 | $(-7.36 \%)$ | 0.1661 | $(-12.12 \%)$ | 19530 |
| $N=20$ | 0.9645 | $(-5.84 \%)$ | 0.9537 | $(-5.21 \%)$ | 0.1714 | $(-9.31 \%)$ | 21483 |
| NASTRAN | 1.0243 | - | 1.0061 | - | 0.1890 | - | 761244 |
| solid |  |  |  |  |  |  |  |

Numerical results are presented also for strain and stress quantities in Table 7.4. In particular, the maximum value of the transverse normal strain $\varepsilon_{z z}^{\max }$, the minimum (negative) value of the longitudinal normal strain $\varepsilon_{y y}^{\min }$, and the maximum value of the transverse normal stress $\sigma_{z z}^{\max }$ (all lying in the adventitia) are reported for different one-dimensional models in comparison with the three-dimensional solution. The reason of this choice is to compare the 1D higher-order formulation with classical beam theories, which neglect $\varepsilon_{z z}$ and $\sigma_{z z}$ by definition. For the arterial case studied, higher-order models highlight that these transverse quantities are not negligible in agreement with 3D results. Moreover, although EBBM and TBM take into account the longitudinal normal strain $\varepsilon_{y y}$, Table 7.4 shows that even this quantity is completely wrongly computed by classical beam theories, which in this case are able to catch only a uniform quasi-null displacement, see Fig. 7.7 and Table 7.3. In general, the convergent trend obtained for displacements as $N$ increases occurs also for strain and stress computation, approaching the reference 3 D results with a remarkably lower number of DOFs.

In addition to the evaluation of the maximum value of $\sigma_{z z}$, a comparison of this transverse normal stress between the model $N=20$ and the 3D reference solution is presented in Fig. 7.8. The figure clearly shows that the $\sigma_{z z}^{\max }$ value is obtained in the adventitia of the mid-span atherosclerotic plaque and that the internal blood pressure causes a stress field locally influenced on the right of the lumen. It is important to remark the accuracy achieved by the 1D twentieth-order model in detecting the stress field all over the cross-section, reaching with a much lower computational cost ( 21438 vs. 761244 DOFs) an approximation comparable to the three-dimensional model. It is noteworthy that higher-order terms are necessary for the proper evaluation of all transverse normal

Table 7.4: Some maximum and minimum strain and stress [ MPa ] values on the atherosclerotic plaque over the mid-span cross-section for different models. Homogeneous material case.

| Model | $10^{2} \varepsilon_{z z}^{\max }$ | (\% Error) | $10^{2} \varepsilon_{y y}^{\min }$ |  | $(\%$ Error) | $\sigma_{z z}^{\max }$ | $(\%$ Error) |
| :--- | ---: | :---: | ---: | ---: | ---: | ---: | ---: |
| EBBM | 0.0000 | $(-100.00 \%)$ | -0.0004 | $(-99.86 \%)$ | 0.00 | $(-100.00 \%)$ | 93 |
| TBM | 0.0000 | $(-100.00 \%)$ | -0.0004 | $(-99.86 \%)$ | 0.00 | $(-100.00 \%)$ | 155 |
| $N=1$ | 3.7706 | $(-52.97 \%)$ | -0.0004 | $(-99.86 \%)$ | 0.00 | $(-100.00 \%)$ | 279 |
| $N=4$ | 0.6539 | $(-91.84 \%)$ | -0.0421 | $(-85.39 \%)$ | 31.40 | $(-85.68 \%)$ | 1395 |
| $N=7$ | 1.4728 | $(-81.63 \%)$ | -0.0505 | $(-82.47 \%)$ | 59.53 | $(-72.86 \%)$ | 3348 |
| $N=9$ | 2.3757 | $(-70.37 \%)$ | -0.0736 | $(-74.45 \%)$ | 110.60 | $(-49.57 \%)$ | 5115 |
| $N=10$ | 3.3719 | $(-57.94 \%)$ | -0.1045 | $(-63.73 \%)$ | 112.31 | $(-48.79 \%)$ | 6138 |
| $N=12$ | 4.7096 | $(-41.25 \%)$ | -0.1453 | $(-49.57 \%)$ | 134.44 | $(-38.70 \%)$ | 8463 |
| $N=14$ | 6.2810 | $(-21.65 \%)$ | -0.1997 | $(-30.68 \%)$ | 184.00 | $(-16.10 \%)$ | 11160 |
| $N=17$ | 7.1202 | $(-11.18 \%)$ | -0.2382 | $(-17.32 \%)$ | 200.49 | $(-8.58 \%)$ | 15903 |
| $N=19$ | 7.5109 | $(-6.31 \%)$ | -0.2514 | $(-12.74 \%)$ | 194.66 | $(-11.24 \%)$ | 19530 |
| $N=20$ | 7.7067 | $(-3.87 \%)$ | -0.2590 | $(-10.10 \%)$ | 204.56 | $(-6.73 \%)$ | 21483 |
| NASTRAN | 8.0169 | - | -0.2881 | - | 219.31 | - | 761244 |
| solid |  |  |  |  |  |  |  |

and shear strains and stresses, which is a feature not present in standard beam models. A more accurate comparison of strain and stress terms over the structure will be carried out in the following nonhomogeneous material case.


Figure 7.8: Comparison of stress $\sigma_{z z}[\mathrm{MPa}]$ over the mid-span cross-section of the atherosclerotic plaque between the present $N=20$ model ( 21483 DOFs) and NASTRAN solid model (761244 DOFs). Homogeneous material case.

## Nonhomogeneous material case

The assessment procedure on the simplified case of homogeneous atherosclerotic plaque is completed. The same static analysis is carried out now considering a different material for each of the six tissues constituting the atherosclerotic plaque. As illustrated in Fig. 7.9, the mesh of the solid finite element model is the same as that employed for the analysis of the homogeneous material case (in Fig. 7.6). In [3] the calcification is assumed to be isotropic and the lipid pool is a neo-Hookean material. The adventitia, media, fibrous cap, and fibrotic are instead modeled as hyperelastic materials defined via several hyperelastic and damage parameters. For the sake of simplicity, all the six tissues of Fig. 7.9(a) are here assumed to be made of linear isotropic materials. The isotropic material properties of each
tissue are introduced extrapolating grossly the Young's moduli used in [3] and reported in Table 7.5 with the corresponding acronyms. It is important to note that this approximation is not very relevant for the purpose of this work, i.e. the assessment of the 1D formulation with respect to a solid FE model in presence of significant nonhomogeneities. In fact, it is emphasized that the following results provide a numerical example of limited validity with respect to quantitative results.


Figure 7.9: Simplified solid model of the arterial atherosclerotic plaque discretized with 64800 solid HEX8 finite elements in NASTRAN. Nonhomogeneous material case.

Table 7.5: Material properties of the tissue types used for the analysis the arterial atherosclerotic plaque. Nonhomogeneous material case.

| Tissue type | $E[\mathrm{MPa}]$ | $\nu$ |
| :--- | :---: | :---: |
| Calcification (C) | 12 | 0.33 |
| Lipid pool (LP) | 0.1 | 0.33 |
| Fibrous cap (FC) | 2.4 | 0.33 |
| Media (M) | 1 | 0.33 |
| Fibrotic media (FM) | 5 | 0.33 |
| Adventitia (A) | 2.5 | 0.33 |

Also for the nonhomogeneous case, the displacements computed by 1D CUF models over the mid-span cross-section are depicted in Fig. 7.10 and compared with a commercial solid finite element solution (NASTRAN). The equivalent results of the homogeneous case have been presented in Fig. 7.7. Comparing Figs. 7.7 and 7.10, it points out that now the deformation of the cross-section, under the same internal blood pressure, is higher than the deformation obtained in the homogeneous case. In fact, when each tissue is modeled through a different material, the large volume occupied by the lipid pool is very deformable and "relaxes" the in-plane cross-section rigidity. The lipid pool deforms significantly as well as the fibrous cap, unlike for the homogeneous case. As a consequence, also the adventitia and the media present displacement values higher than the previous case. Nonetheless, the deformation of the calcification and fibrotic media remains quasi-null due to their high material stiffness. In general, this larger in-plane corss-section deformation requires an expansion order higher than the homogeneous case to obtain a good agreement with the solid finite element solution, which is achieved with $N=22$. It is noteworthy that even for this very complex structure made of nonhomogeneous material with arbitrary cross-section
geometry, the more the expansion order $N$ is, the more the results obtained through the $1 D$ formulation are accurate, approaching the NASTRAN results.


Figure 7.10: Displacement $|\mathbf{u}|[\mathrm{mm}]$ over the mid-span cross-section of the atherosclerotic plaque for different one-dimensional models compared to the solid FE solution. Nonhomogeneous material case.

The same maximum results over the cross-section as those presented in Table 7.3 are now reported for the nonhomogeneous case in Table 7.6. The analysis involves models with $N$ up to 22 and shows the larger deformation due to the nonhomogeneous material. While about a 60 percent increase is observed in $u_{\max }^{\mathrm{M}}$ and $u_{\max }^{\mathrm{A}}$, the maximum displacement in the fibrous cap $u_{\max }^{\mathrm{FC}}$ shows about a 450 percent increase, mainly due to the butter-like
behavior of the lipid pool. The considerations about the inefficiency of classical beam theories made for the homogeneous case are still valid here. On the contrary, the 1D CUF FEs provide again a convergent solution by approaching the NASTRAN 3D results and a good agreement is obtained for $N=22$. Although the error computed with respect to the 3 D solution is about $8-11 \%$, it is important to note that this error is on the overall maximum displacement values and that this approximation is achieved via about a $96 \%$ reduction in degrees of freedom (see Table 7.6).

Table 7.6: Maximum displacements [mm] on media, adventitia and fibrous cap of the atherosclerotic plaque over the mid-span cross-section for different models. Nonhomogeneous material case.

| Model | $u_{\max }^{\mathrm{M}}$ | $(\%$ Error $)$ | $u_{\max }^{\mathrm{A}}$ |  | $(\%$ Error $)$ | $u_{\max }^{\mathrm{FC}}$ | $(\%$ Error) |
| :--- | ---: | :---: | :---: | :---: | :---: | :---: | ---: |$)$ DOFs

Despite its one-dimensional approach, the proposed higher-order model is able to accurately detect the in-plane deformation of the cross-section even for this kind of crosssection made of nonhomogeneous material. A thorough assessment on strain and stress fields over the mid-span cross-section is now carried out. The two transverse normal strains $\varepsilon_{x x}$ and $\varepsilon_{z z}$ and the shear stress $\varepsilon_{x z}$ are evaluated for the six arterial tissues and the relative maps are depicted on the deformed configuration in Figs. 7.11, 7.12, and 7.13, respectively. These strain quantities are related to the beam cross-section, i.e. the components of vector $\varepsilon_{p}$, and are neglected by classical beam theories. Considering the mid-span cross-section, the maximum values of these strains are achieved in the adventitia and media, whereas the minimum ones occur in the lipid pool and media. This fact clearly highlights the complexity of the case studied, given the markedly material nonhomogeneity. In fact, as can be seen also in Fig. 7.10, unlike the homogeneous material case, the whole section is affected by the internal blood pressure. Nonetheless, for all the strains mentioned, an expansion order equal to 21 provides a solution in excellent agreement with the reference 3 D results. In general, a further increase of $N$ might be required to achieve an even better accuracy, consistent with the considerations previously mentioned about the expansion enrichment.

Numerical results for the transverse normal strain $\varepsilon_{z z}$ are summarized in Table 7.7. In particular, the maximum value, lying in the adventitia, and the minimum value, lying in the lipid pool, are reported for different one-dimensional models up to $N=21$ and compared to the reference solution. As can be seen, classical and low-order models provide a unrealistic behavior of the arterial plaque. In fact, EBBM and TBM neglect $\varepsilon_{z z}$, whereas $N=1$ takes into account an inaccurate non-null constant strain distribution. On the


Figure 7.11: Comparison of strain $\varepsilon_{x x}$ over the mid-span cross-section of the atherosclerotic plaque between the present $N=21$ model ( 23529 DOFs) and NASTRAN solid model ( 761244 DOFs). Nonhomogeneous material case.


Figure 7.12: Comparison of strain $\varepsilon_{z z}$ over the mid-span cross-section of the atherosclerotic plaque between the present $N=21$ model ( 23529 DOFs) and NASTRAN solid model ( 761244 DOFs). Nonhomogeneous material case.
contrary, the proposed 1D FEs provide a convergent solution by approaching the NASTRAN


Figure 7.13: Comparison of strain $\varepsilon_{x z}$ over the mid-span cross-section of the atherosclerotic plaque between the present $N=21$ model ( 23529 DOFs) and NASTRAN solid model ( 761244 DOFs). Nonhomogeneous material case.

3D results as the refinement of the expansion increases. According to Fig. 7.12, a good agreement is achieved via a remarkably lower number of DOFs. The maximum value of the corresponding transverse normal stress $\sigma_{z z}^{\max }$ computed through an increasing expansion order is also presented in Table 7.7. The accuracy obtained demonstrates once again the three-dimensional capabilities of the CUF higher-order approach in computing the displacement and strain fields, since all the different strain terms are involved in the computation of stresses via the constitutive equations (Eq. 3.120). These capabilities are nonstandard for a one-dimensional formulation.

Table 7.7: Some maximum and minimum strain and stress [ MPa ] values on the atherosclerotic plaque over the mid-span cross-section for different models. Nonhomogeneous material case.

| Model | $10^{2} \varepsilon_{z z}^{\max }$ | $(\%$ Error $)$ | $10^{2} \varepsilon_{z z}^{\min }$ | $(\%$ Error $)$ | $\sigma_{z z}^{\max }$ | $(\%$ Error $)$ | DOFs |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| EBBM | 0.000 | $(-100.00 \%)$ | 0.000 | $(-100.00 \%)$ | 0.00 | $(-100.00 \%)$ | 93 |
| TBM | 0.000 | $(-100.00 \%)$ | 0.000 | $(-100.00 \%)$ | 0.00 | $(-100.00 \%)$ | 155 |
| $N=1$ | 3.025 | $(-72.31 \%)$ | 3.025 | $(-129.94 \%)$ | 0.00 | $(-100.00 \%)$ | 279 |
| $N=4$ | 1.050 | $(-90.39 \%)$ | -0.468 | $(-95.37 \%)$ | 42.04 | $(-86.51 \%)$ | 1395 |
| $N=7$ | 2.097 | $(-80.80 \%)$ | -0.472 | $(-95.33 \%)$ | 74.70 | $(-76.03 \%)$ | 3348 |
| $N=9$ | 3.629 | $(-66.78 \%)$ | -1.259 | $(-87.54 \%)$ | 162.98 | $(-47.70 \%)$ | 5115 |
| $N=10$ | 4.953 | $(-54.66 \%)$ | -2.567 | $(-74.59 \%)$ | 171.68 | $(-44.91 \%)$ | 6138 |
| $N=12$ | 6.206 | $(-43.19 \%)$ | -4.889 | $(-51.61 \%)$ | 179.47 | $(-42.41 \%)$ | 8463 |
| $N=14$ | 8.032 | $(-26.47 \%)$ | -6.587 | $(-35.54 \%)$ | 228.49 | $(-26.68 \%)$ | 11160 |
| $N=17$ | 9.287 | $(-14.99 \%)$ | -8.173 | $(-19.11 \%)$ | 274.38 | $(-11.95 \%)$ | 15903 |
| $N=20$ | 10.301 | $(-5.70 \%)$ | -9.014 | $(-10.79 \%)$ | 287.15 | $(-7.86 \%)$ | 21483 |
| $N=21$ | 10.569 | $(-3.25 \%)$ | -9.114 | $(-9.80 \%)$ | 311.23 | $(-0.13 \%)$ | 23529 |
| NASTRAN | 10.924 | - | -10.104 | - | 311.63 | - | 761244 |
| solid |  |  |  |  |  |  |  |

Figures 7.14 and 7.15 compare the longitudinal stress $\sigma_{y y}$ and the transverse normal stress $\sigma_{z z}$ computed by the CUF model to the three-dimensional solution, respectively. The stress maps are depicted on the deformed configuration of the media, which is particularly stressed and subjected to both traction and compression. Even though the EBBM and TBM results are not reported here, it can be demonstrated that these classical beam theories are again completely ineffective for this case even in evaluating the axial stress, which is not neglected by kinematic hypotheses. Obviously, Figs. 7.14 and 7.15 show that the introduction of higher-order terms is fundamental not only for the accurate evaluation of the deformation, but also of the stress field. In fact, the higher the theory order employed the more the results approach the solid FEM solution with a convergent trend.

The present model allows the computation of strain and stress fields in every point of the structure analyzed. The shear stress $\sigma_{x z}$ distribution over the cross-section is investigated with an expansion order equal to 21 and compared with the solid FE solution, bearing in mind that EBBM and TBM completely neglect it. The prediction of the shear stress $\sigma_{x z}$ is slightly underestimated by the $N=21$ model as can be noted by the different scales used in Fig. 7.16, but its distribution is well-detected all over the cross-section with an acceptable approximation with respect to the 3D solution, which involves a number of DOFs about 32 times higher.

In conclusion, the one-dimensional CUF formulation provides not only a correct evaluation of the displacements of the structure, but also a proper computation of the strain and stress fields. The atherosclerotic plaque studied here represents a very severe test case for the present one-dimensional model, from different points of view. First of all, this configuration is very short, given that the ratio between the length $L$ and the characteristic


Figure 7.14: Comparison of stress $\sigma_{y y}[\mathrm{MPa}]$ over the media of the atherosclerotic plaque (mid-span cross-section) for different models. Nonhomogeneous material case.


Figure 7.15: Comparison of stress $\sigma_{z z}[\mathrm{MPa}]$ over the media of the atherosclerotic plaque (mid-span cross-section) for different models. Nonhomogeneous material case.
cross-section dimension is about equal to 2 . Furthermore, the cross-section has an arbitrary nonconventional geometrical layout. The material employed is markedly nonhomogeneous and, finally, the internal pressure load is applied on a nonplanar surface again of arbitrary


Figure 7.16: Comparison of stress $\sigma_{x z}[\mathrm{MPa}]$ over the mid-span cross-section of the atherosclerotic plaque between the present $N=21$ model ( 23529 DOFs) and NASTRAN solid model (761244 DOFs). Nonhomogeneous material case.
geometry.

### 7.3 Dynamic analysis of nonhomogeneous structures with arbitrary cross-sections

In this section, some structural cases with arbitrary cross-sections are subjected to timedependent loads of different kind. The analyses aim at proving the capabilities of the higher-order 1D CUF model in the dynamic response analysis of nonhomogeneous structures.

The first study focuses on the propagation of a wave in a three-layer cylinder with a thin circular cross-section. Let the cylinder introduced in section 4.2 for the free vibration analysis to be employed again here. As a resume, more details are given in Table 7.8 and Fig. 7.17(a), which are equal to Table 4.2 and Fig. 4.1(b), respectively. The three layers of the cylinder are made of different isotropic materials. The external and internal diameter are respectively equal to $d_{e}=100 \mathrm{~mm}$ and $d_{i}=94 \mathrm{~mm}$. The length $L$ of the cylinder is equal to 500 mm . The simply supported cylinder is modeled with a 1D finite element mesh of 10 B 4 elements.

Table 7.8: Material and geometrical properties of the cylinder and half-cylinder layers.

| Property | Layer 1 | Layer 2 | Layer 3 |
| :--- | ---: | ---: | ---: |
| $\mathrm{t}[\mathrm{mm}]$ | 1 | 1 | 1 |
| $\mathrm{E}[\mathrm{GPa}]$ | 69 | 30 | 15 |
| $\nu$ | 0.33 | 0.33 | 0.33 |
| $\rho\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ | 2700 | 2000 | 1800 |



Figure 7.17: Cross-sections geometry of the three-layer cylinder and five points chosen on the circular cross-section at $y=L / 2$ to evaluate the deformation as the time increases.

The wave is here described as a step profile of pressure which travels along the longitudinal axis with constant velocity $v_{y}=90 \mathrm{~m} \mathrm{~s}^{-1}$ and its extension is described by the length $a=L / 10=50 \mathrm{~mm}$, as can be seen in Fig. 7.18. In particular, the cylinder is subjected to a uniform pressure $p_{1}=14.8 \mathrm{MPa}$ applied to the internal surface of the cylinder $\left(r=d_{i} / 2\right.$,
$90^{\circ} \leq \theta \leq 270^{\circ}, 0 \leq y \leq L$, see Fig. 7.1), to which a step of length $a$ and pressure value $p_{2}=-172.7 \mathrm{MPa}$ (the negative sign means that this pressure $p_{2}$ is along the opposite versus with respect to $p_{1}$ ).


Figure 7.18: Step profile pressure distribution which travels along the $y$ axis with constant velocity $v_{y}$.

The dynamic response of the structure is computed over the interval [0, 0.005] s via the Newmark's method. Figure 7.17 (b) shows five points chosen at the mid-span cross-section $(y=L / 2)$ as characteristic positions where to evaluate the deformation as the time increases. The corresponding curves are depicted in Fig. 7.19 for $N=8$. It is worth pointing out that these five curves would be equal and coincident in the case of classical beam theories, such as Euler-Bernoulli (EBBM) and Timoshenko (TBM), due to their kinematic limitations on the cross-section deformation.


Figure 7.19: Dynamic response of different points at $y=L / 2$. Circular cross-section case. $\Delta t=$ $1.67 \cdot 10^{-5}$ s. $N=8$.

The three-dimensional time-dependent deformation of the cylinder is presented in Fig. 7.20. At the starting time instant, the wave is close to the constrained section at $y=0$ and so it slightly affects the cylinder which basically bends as though the load were a uniform pressure on the upper half of the cylinder. As the wave moves in the cylinder, the deformed configuration of the cross-sections changes along the longitudinal axis assuming a typical triangle-like shape, similar to the deformation already described for the static analysis in section 7.1 (see Fig. 7.2). Local effects due to the wave loading and inertial effects
are computed by an eight-order model $(N=8)$ as highlighted in Fig. 7.20 for different time instants. The results clearly show the accuracy of the present refined model in detecting the three-dimensional deformation despite its one-dimensional approach, according to previous dynamic computations through 1D CUF models [118]. The present method shows features not present in standard one-dimensional theories such as the thickness changing of the thin-walled laminated surface and the in-plane and out-of-plane cross-section deformations.


Figure 7.20: Three-dimensional deformation of the cylinder as time increases. $t=[0: 0.005] \mathrm{s}$.

The same wave propagation as that described in the previous case is now applied on a half-cylinder, i.e. a structure whose cross-section is shown in Fig. 7.21(a). This structure results to be exactly the upper half-part of the previous cylinder; in fact, the external and internal radius are respectively equal to $r_{e}=50 \mathrm{~mm}$ and $r_{i}=47 \mathrm{~mm}$. The material and the geometrical propoerties are summarized in Table 7.8.


Figure 7.21: Cross-section geometry of the three-layer half-cylinder and five points chosen to evaluate the deformation as the time increases.

The dynamic response of the structure is computed over the interval [0, 0.005] s as in the previous case. Five points are chosen at the mid-span cross-section ( $y=L / 2$ ) as characteristic positions where to evaluate the deformation as the time increases, see Fig. 7.21(b). The corresponding curves are depicted in Fig. 7.22 for $N=8$. Moreover, the three-dimensional time-dependent deformation of the cylinder is presented in Fig. 7.23. It is possible to compare the completely different dynamic response of the cylinder and the half-cylinder, mainly due to the contrasting choice of section (close section for cylinder vs. open section for half-cylinder).


Figure 7.22: Dynamic response of different points at $y=L / 2$. Half-circular cross-section case. $\Delta t=1.67 \cdot 10^{-4}$ s. $N=8$.


Figure 7.23: Three-dimensional deformation of the half-cylinder as time increases. $t=[0: 0.05]$ s.

The third and last analysis of this section addresses the dynamic response of a opensection structure over the time interval $[0,0.2] \mathrm{s}$ to an impulse load in presence of damping. The cross-section chosen for this assessment is different from the previous cases in order to show the versatility of the proposed 1D CUF approach independently of the cross-section geometry employed. In particular, the triangle-like configuration is illustrated in Fig. 7.24(a) and composed of three layers with material and geometrical properties listed in Table 7.9. The length of the structure is $L=5 \mathrm{~m}$.

The impulse load is a uniform pressure load $p=90 \mathrm{kPa}$ applied at $t=0 \mathrm{~s}$ over the region $1.8 \mathrm{~m} \leq x \leq 2.0 \mathrm{~m}, 0.9 \mathrm{~m} \leq y \leq 1.1 \mathrm{~m}$ of the upper-side region of the structure. Rayleigh damping with coefficients $\gamma=0.005$ and $\beta=0.0075$ is assumed to compute the damping matrix $\mathbf{C}$, see Eq. 3.98. Five points are chosen at the one-third-of-the-span cross-section ( $y=L / 3$ ) as characteristic positions where to evaluate the deformation as the time increases, see Fig. 7.24(b). The corresponding curves are depicted in Fig. 7.25 for $N=8$. Moreover, the three-dimensional time-dependent deformation of the cylinder is presented in Fig. 7.26. The same conclusion about the limitations of classical beam theories and the ability of higher-order theories to detect a three-dimensional response as


Figure 7.24: Triangle-like three-layer cross-section geometry and five points chosen to evaluate the deformation as the time increases.

Table 7.9: Material and geometrical properties of the three layers for the triangle-like cross-section.

| Property | Layer 1 | Layer 2 | Layer 3 |
| :--- | ---: | ---: | ---: |
| $\mathrm{t}[\mathrm{mm}]$ | 2 | 2 | 2 |
| $\mathrm{E}[\mathrm{GPa}]$ | 69 | 30 | 15 |
| $\nu\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$ | 0.33 | 0.33 | 0.33 |
| $\rho\left[\begin{array}{l}\text { l }\end{array}\right.$ | 2700 | 2000 | 1800 |

those mentioned above can be made for the present case.


Figure 7.25: Dynamic response of different points at $y=L / 3$. Triangle-like cross-section case. $\Delta t=1 \cdot 10^{-3} \mathrm{~s} . N=8$.


Figure 7.26: Three-dimensional deformation of the structure with triangle-like cross-section as time increases. $t=[0: 0.02] \mathrm{s}$.

## Part II

## Aeroelastic Formulation

## Chapter 8

## Aeroelastic Preliminaries

### 8.1 Introduction

This part of the work is dedicated to address the multidisciplinary topic of the aeroelastic analysis. So far the aerodynamic problem has not been taken into account yet. It will be faced in the following discussion and then interfaced to the structural analysis. The result will be a unified formulation for the aeroelastic beam-like structures.
This first chapter included in such a part plays an introductory role to chapter 9. It presents the general concepts at the base of the model, whose aeroelastic formulation will be faced in detail and will refer to the particular notation exposed here.

### 8.2 Aeroelastic Notation

First of all, the choice of the global coordinate system is faced. It plays an important role because it permits to define univocally the geometrical configuration and its coordinates. Furthermore, it is the system taken as reference in writing some vectorial quantities corresponding to the wing system considered as a whole. For that reason, the choice of such a reference must be as useful as possible for the sake of ease.

Therefore, the system is so that its global $x$ axis is parallel to the free stream velocity $\boldsymbol{V}_{\infty}$ and directed toward the trailing edge of the wing system. Instead, the global $y$ axis goes along the spanwise direction. In particular its direction is from the root wing section to the tip of right half-wing. The third global axis $z$ is located so that the reference coordinate system is orthogonal according to the right-hand rule. Thus, it is oriented from the lower to the upper surface of the wing section airfoil. The origin of the system is placed on the intersection point between the leading edge of right half-wing and the fusolage longitudinal axis of the aircraft. In other words, it is the leading edge point of the root wing section profile. Assuming the yaw angle of the aircraft corresponding to the considered wing system to be zero, then the global coordinate system is parallel to aircraft wind axes. The difference consists in the choice of $x$ and $z$ axes directions, that for wind axes are opposite

The proposed method is able to analyze wing systems generally oriented in the 3D space, so including the conventional and unconventional wing configurations. Hence, it is aimed at the study the variety of innovative wing configurations that could replace the traditional aircraft shape. For this purpose, the model allows to divide the wing system into a set of large trapezoidal Wing Segments, according to the same logic used in other previous aeroelastic works (see Demasi and Livne [12]). The number of these trapezi is denoted as $N_{W S}$. As it will be seen later, the Wing Segments will be subdivided into aerodynamic panels. In the present formulation they are located on the aerodynamic reference surfaces
of the wing system with initial angle of attack equal to zero. Then, the Wing Segments are identified with these reference surfaces. Hence it is assumed that all the Wing Segments have two opposite segment parallel to the wind direction, i.e. parallel to the global $x$ axis.

Thanks to the possibility of studying nonplanar configuration, each Wing Segment can have dihedral or sweep angle. That is more clear by looking at Fig. 8.1, which shows a typical C-Wing configuration. The presented model allows to divide that unconventional wing system in a set of Wing Segments. In such a figure, each differently coloured surface represents a Wing Segment. It has to be noticed also the orientation of the global coordinate system according to what described above.


Figure 8.1: An example of subdivision in Wing Segments for a C-Wing configuration.

The previous paragraph has introduced a global reference system, but it is not the only one utilized for the configuration. In fact, each Wing Segment contains a local coordinate system. Let superscript ${ }^{S}$ be the index for the generic Wing Segment; its local reference system has orthogonal axes denoted as $x^{S}, y^{S}$ and $z^{S}$. As it can be seen in Fig. 8.2, the first two axes are located in a way so that the Wing Segment itself lies in the plane $x^{S}-y^{S}$. In particular, the $x^{S}$ axis has to be always parallel to the free stream $\boldsymbol{V}_{\infty}$. As a consequence, $x^{S}$ is parallel to global $x$ for each Wing Segment. Then, it is important to notice that $y^{S}$ lies on the reference surface of the Wing Segment along the direction perpendicular to $x^{S}$. Instead of $x^{S}, y^{S}$ is not parallel to $y$ only if the Wing Segment has a dihedral different from zero. In that case also the local $z^{S}$ axis would be not parallel to the global $z$. The origin of the system is located on one of the two leading edges of the Wing Segment. The point is chosen so that the other one has a positive value of the local $y^{S}$ coordinate. So, once the direction of $y^{S}$ is decided, then the origin is fixed according to the exposed logic.

Now, it is carried out the procedure to number the four vertices of each Wing Segment. As done for all the following operations, such a procedure has to be general and unique, leading to a computerized implementation. In fact, the numerical code will be able to automatically treat the quantities involved in the formulation. Let $S$ be the index for the generic Wing Segment and $1^{S}, 2^{S}, 3^{S}, 4^{S}$ its four vertices; the utilized scheme is so that the numbering will respect the conditions:


Figure 8.2: Local coordinate system and numbering convention for a Wing Segment.

$$
\begin{align*}
x_{1^{S}}^{S} & >x_{2^{S}}^{S} \\
x_{4^{S}}^{S} & >x_{3^{S}}^{S} \\
y_{3^{S}}^{S} & >y_{2^{S}}  \tag{8.1}\\
y_{4^{S}}^{S} & >y_{1}^{S}
\end{align*}
$$

where the involved coordinates are local. According to the previous reasoning, among the four vertices the point $2^{S}$ is the right candidate to be the origin of the local coordinate system. Hence:

$$
\begin{align*}
x_{2^{S}}^{S} & =0 \\
y_{2^{S}}^{S} & =0  \tag{8.2}\\
z_{2^{S}}^{S} & =0
\end{align*}
$$

As it will be seen in chapter 9 , the local reference system plays an important role because a lot of quantities will be written with respect to it. For instance, the application of the splining will be performed by considering such a system.

As remembered, the present notation must follow a scheme implementation oriented. As a consequence, also the numbering of the Wing Segments is performed in such a way. Being the structure subdivided into $N_{W S}$ Wing Segments, it is assumed that they have identification numbers ( $=\mathrm{IDs}$ ) from 1 to $N_{W S}$. In other words, the IDs numerating has to be consecutive. This constraint is very useful in view of a more easy implementation. No other assumptions have been taken into account about the Wing Segments. For example they have not to be necessarily sorted following the physical order, but randomly.

Once introduced the division of the whole structure in Wing Segments, now the discussion focuses on the generic Wing Segment $S$. As written above, the aeroelastic analysis carries out not only the structural analysis, but combines such an analysis with the aerodynamic problem. The aerodynamic method here chosen has been the Vortex Lattice Method (VLM), which will be described in the following sections. Just to introduce its logic, the whole structure is divided into a lattice of quadrilateral aerodynamic panels. Since the structure has been split into a set of Wing Segments, the aerodynamic mesh is performed on these reference surfaces.

The discretization is automatically carried out by the aeroelastic code and follows a general scheme for each Wing Segment. The four straight lines bounding the generic Wing

Segment $S$ have been divided by means of a mesh seed, whose parameters are chosen by the user. Such parameters are the numbers of subdivisions of each Wing Segment along the spanwise and chord directions. They coincide with the local axes $x^{S}$ and $y^{S}$. As a result, the mesh seeds on two opposite lines bounding the Wing Segment are equal. Because of the parallelism between $x^{S}$ and $x$ (and $V_{\infty}$ ), then the two edges along the chord direction of the aerodynamic panels are parallel. As a consequence, the generic $i^{t h}$ aerodynamic panel has the same geometrical features of the Wing Segment to which the panel corresponds. For that reason, the automatic method of numbering its vertices is assumed equal to the one utilized for the Wing Segments. Thus:

$$
\begin{align*}
x_{1^{i}}^{S} & >x_{2^{i}}^{S} \\
x_{4^{i}}^{S} & >x_{3^{i}}^{S} \\
y_{3^{i}}^{S} & >y_{2^{i}}^{S}  \tag{8.3}\\
y_{4^{i}}^{S} & >y_{1^{i}}^{S}
\end{align*}
$$

where the involved coordinates are written with respect to the local reference system of Wing Segment $S$ in which the panel lies. Considering the $i^{t h}$ panel, the Vortex Lattice Method will perform the aerodynamic analysis via the definition of a couple of points on the panel. They are denoted as Control Point $P_{C}^{i}$ and Load Point $P_{L}^{i}$. They result from the location of a bound vortex at the panel quarter chord line. As a consequence, the Control Point $P_{C}^{i}$ (also called Collocation Point) is placed at the center of the panel's three-quarter chord line. Concerning the Load Point, it is the point at the center of the panel's quarter chord line because there it is supposed to be concentrated the resulting lifting force, equivalent to the pressure distribution on the panel. As it will be resumed, such a choice of the positions is not a theoretical law, simply a placement that works well and has become a rule of thumb. It was discovered by Pistolesi [131]. Mathematical derivations of more precise vortex/control point locations are available (see [132]), but the $1 / 4-3 / 4$ rule is widely used, and has proven to be sufficiently accurate in practice.

However, the two described points for the generic panel $i$ are shown in Fig. 8.3.


Figure 8.3: Conventions for a generic aerodynamic panel.
It should be noted that the aerodynamic mesh is carried out on the Wing Segments; as
a consequence it discretizes only the reference surfaces of the structure, which are assumed to have an angle of attack equal to zero. That does not mean that the incidence of the wing system is not considered in the model, but just that it is not faced in the discetization. In fact, the angle of attack of the structure will be considered in the construction of a term denoted as $\boldsymbol{L}_{R H S}$. The details are addressed in chapter 9, but at the moment it has to be clear that it is allowed to have an angle of attack. However, it must not be very large, in order to have a problem case where the linear aerodynamic analysis remains a valid approximation.

The numbering of aerodynamic panels has been chosen for the sake of ease. At first, the number of aerodynamic panels into which the generic Wing Segment $S$ is divided has been named $N_{A P}^{S}$, where the superscript ${ }^{S}$ refers to the index of the segment. Furthermore, the first $N_{A P}^{1}$ panels are located on the Wing Segment having ID equal to 1 . The second $N_{A P}^{2}$ panels are placed on the Wing Segment 2 and so forth. It has to be noted that the segments do not share any aerodynamic panel; in fact there are no common panels even between two adjacent Wing Segments.

$$
\begin{equation*}
N_{A P}=\sum_{S=1}^{N_{W S}} N_{A P}^{S} \tag{8.4}
\end{equation*}
$$

As a result, during the assembly procedure developed in chapter 9 to obtain the matrix $\boldsymbol{K}_{\text {aero }}$, all the contributions due to panels of different Wing Segments will not be summed up. Differently from the classical FEM assembly procedure, the method of superposition will just collect different terms on the assembled matrix, but without superposing any value.

So far the discussion has not cited yet the structural FEM mesh. The formulation of the refined finite element has already been described in chapters 2 and 3, but no details have been added about the discretization of the structure. The first step of the aeroelastic formulation will be the extension of the finite element formulated so far to the study of generally oriented beam-like structures. Such an extensions leads to write the fundamental nucleus of the Structural Stiffness Matrix in the global coordinate system. Now, it is interesting to discuss where the beam elements are located in the generic wing system. First of all, the subdivision of the structure into $N_{W S}$ Wing Segments is taken into account while discretizing it. In fact, for each Wing Segment it is assigned a straight beam, over which a structural 1D mesh is created. For the sake of completeness, it should be remembered that here the finite elements analyzing the structure are one-dimensional and formulated by the hierarchical 1D CUF approach exposed in chapter 2.

For the generic Wing Segment $S$ a set of $N_{E L}^{S}$ finite elements is assigned. Each beam element must be entirely contained in a plane perpendicular to the global $x$ axis (wind direction). For the Wing Segment attached to the fuselage, in which the local $x^{S}$ axis coincide with the global $x$ one, the finite elements are located along the local $y^{S}$ axis of that Wing Segment. It is important to know that the last element corresponding to such a Wing Segment ends exactly at the same $y^{S}$ of its edge $3^{S}-4^{S}$. Then for the Wing Segment adjacent to the first one the corresponding elements are aligned along a direction parallel to its local $y^{S}$ axis, but starting from the last node of the previous Wing Segment and not from the origin point $2^{S}$. As a matter of fact, the beam elements of each Wing Segment are located on a straight line parallel to the local $y^{S}$ and on the global plane $x-z$. Furthermore, all the elements constituting the mesh are adjacent, i.e. all the elements are connected. In general, only the Wing Segment joined to the fuselage has its finite elements along its local $y^{S}$ axis. Figure 8.4 shows more clearly all the above remarks.


Figure 8.4: One-dimensional structural mesh and bidimensional aerodynamic mesh of Wing Segments.

The conclusion is that the origin of the coordinate system on the section of a generic finite element does not necessarily coincide with the centroid of the cross-section. Furthermore, such an origin in general is not necessarily included in the cross-section area. It can be located outside it. This is very useful to analyze wings with sweep angles or joints. To summarize, the model studies the deformation of a wing system generally oriented in the 3D space by means of refined finite elements in general not lying on the wing surface.

Another obvious notice is that each element of the mesh has to correspond to one and only one Wing Segment. In fact, different Wing Segments do not share common elements. It is resumed by the following formula:

$$
\begin{equation*}
N_{E L}=\sum_{S=1}^{N_{W S}} N_{E L}^{S} \tag{8.5}
\end{equation*}
$$

On the contrary, different Wing Segments could share some common Structural Nodes. In fact, as said before, the elements have to be all joined, also at the connection point between two adjacent Wing Segments. Such a remark reflects to the assembly operations performed in chapter 9 to construct the Structural Stiffness Matrix. In fact, the terms referring to same nodes belonging to different beam elements are summed up via the method of superposition, according to the classical assembly procedure. The only particularity is that such a procedure has to be conducted with respect to the global system. The reason is that the terms summed up has to be written in the same coordinate system. For the sake of convenience, such a system has been chosen to be the global one.

As done for the aerodynamic panels, now the numbering issue is faced. Let us start from the beam elements. There are not particular assumptions about their IDs. In fact the first element could have an ID different from 1 and the last element could have ID higher or lower than $N_{E L}$. Furthermore, the sorting of the elements can be not consecutive. For instance, if the first element has ID equal to 3 , the last element could have ID equal to $5+N_{E L}$; in that case three ID number between the starting and the ending points are not
associated to any element. The model allows this situation.
The discussion above is not valid for the grid nodes. In fact, their IDs have to start from 1 and go to $N_{S N}$ ( $\mathrm{SN}=$ Structural Nodes). Thus, the convention for the nodes is the same adopted for the Wing Segments. It is convenient because in this way the vector of nodal degrees of freedom is sorted following the ID numbering, which starts from 1 and has not any discontinuity.

$$
\begin{equation*}
N_{S N} \neq \sum_{S=1}^{N_{W S}} N_{S N}^{S} \tag{8.6}
\end{equation*}
$$

The proposed program code will be able to study the possible symmetry of the wing system. The fundamental method is exposed in section 8.3, where the Vortex Lattice Method will be described in detail. It will be noted that such a feature will be performed without discretizing both the half-wings of the wing system, but just the desired right or left half-wing. This strategy turns out to be advantageous, since the number of degrees of freedom will be smaller, so saving appreciably analysis time.

The last point to be treated is the coupling of structures and aerodynamics. As it will be seen in detail in the following sections, the utilized method to perform that is the splining. Now any mathematical concept is introduced, but it is important to anticipate the logic adopted in the proposed formulation. The spline methods analyzed in Section 8.4 are two: the first is called Beam Spline Method and the second one is denoted as Infinite Spline Method. The Beam Spline Method is usually performed to interpolate a surface when a set of structural values is known only along a line lying on the surface. For that reason such a method is addressed to one-dimensional discretizations of the space, such as it happens here. Instead, the Infinite Plate Method utilizes a set of structural quantities spread on the surface to find its mathematical description. Thus, it is addressed mainly to a FEM bidimensional discretization, since the degrees of freedom at the element nodes could directly represent the set required as input.

In this work the utilize of a spline method is unconventional with respect to what said before. In fact, although the Finite Element Model is 1D, the splining is not performed by means of the Beam Spline method, but indeed of the Infinite Plate Spline method (IPS). The reason is due to the formulated hierarchical element, which shows some good accuracy in the prediction of displacements on a set of desired points on the surface to fit, even if they are not necessarily coincident with the actual FEM nodes and not even located on the element axis. So, for each Wing Segment (the generic one is indicated with $S$ ) a set of $N_{P S}^{S}$ aeroelastic points is chosen on the reference plane of the each Wing Segment and the corresponding displacements are computed by means of the structural formulation. Then, these deflections are utilized as input data in order to compute slopes and displacements on other desired points on the Wing Segment via IPS method.

The points forming the set are denoted as Pseudo-Structural Points, precisely because they have the meaning of structural points (the spline surface is treated as a plate by IPS method). The adjective Pseudo is adopted to not confuse them with the Structural Nodes of the beam elements lying in the Wing Segment. Their meaning is also shown in Fig. 8.5, where it is clear how they can be spread randomly on the surface to fit. Obviously, a quite uniform distribution can be lead to a more correct interpolation of the surface.

In conclusion, it is anticipated that the following formulation will take the structural degrees of freedom as the independent variables of the problem. It means that the quantities involved in the model are all written as functions of these unknwns, in order to obtain a sole final matricial system to solve. This remark is obvious in the classical structural FEM codes, since it is a method based on displacements unknowns. Now also the aerodynamic


Figure 8.5: Meaning of Pseudo-Structural Points.
quantities are evaluated. Thus, the formulation will describe how they can be written as functions of the degrees of freedom. It will appear more clear during the construction of the Aerodynamic Stiffness Matrix. This is the practical meaning of the coupling between structural and aerodynamic fields.

### 8.3 The Vortex Lattice Method

### 8.3.1 Introduction

The method chosen to carry out the aerodynamic analysis of the aeroelastic model is the Vortex Lattice Method (VLM). It was among the earliest numerical methods utilizing computers to actually assist aerodynamicists in estimating aircraft fluid dynamics. The Vortex Lattice Method models the lifting surfaces, such as a wing, of an aircraft as an infinitely thin sheet of discrete vortices to compute lift and induced drag. The influence of the thickness and viscosity is neglected. In particular, the VLM is very easy to use and capable of providing remarkable insight into wing aerodynamics and component interaction. It is based on solutions to Laplace's Equation, and is subject to the same basic theoretical restrictions that apply to panel methods.
As a summary, the main features of such a method are listed in the following scheme:

- singularities are placed on a surface;
- the non-penetration condition is satisfied at a number of Control Points;
- a system of linear algebraic equations is solved to determine singularity strengths;
- method oriented toward thin lifting effects, and classical formulations ignoring thickness;
- boundary conditions (BCs) are applied on a mean surface, not the actual surface.

Vortex Lattice Methods were first formulated in the late '30s, and the method was first called "Vortex Lattice" in 1943 by Falkner. The main concept is simple, but because of its purely numerical approach practical applications awaited sufficient development of computers. The early '60s saw widespread adoption of the method.

### 8.3.2 Preliminaries

Nowadays, there are many different vortex lattice schemes. The first implementation procedure follows the classical way to face the aerodynamic problem via the Vortex Lattice Method. Instead, the second part of this section will be dedicated to the variation by Katz and Plotkin [133] and a further adaptation to the present aeroelastic model. As briefly explained in the previous section, the thin wing reference surface is divided into a lattice of quadrilateral panels, as shown in Fig. 8.6. In order to perform the aerodynamic analysis of the wing-system, a horseshoe element is placed on each panel. This element consists of a straight bound vortex segment $B C$ that models the lifting properties, and of two semi-infinite trailing vortex lines that models the wake, $A B$ and $C D$. Here, the bound vortex is placed at the panel quarter chord line and the Control Point (also called Collocation Point) is at the center of the panel's three-quarter chord line.


Figure 8.6: Lattice of trapezoidal aerodynamic panels for the VLM.
The segment $B C$ does not necessarily have to be parallel to $y$ or $y^{S}$ axes, but at the element tips the vortex is shed into the flow where it must be parallel to the streamlines so that no force will act on the trailing vortices. In order to not violate the Helmholtz condition, these vortex elements are viewed as the near portions of vortex rings whose starting vortices extend far back, so that the effect of this segment $A D$ is negligible. On the practical point of view, it means that such edges are located at least 20 wing spans behind the wing.

The requirement that the "far wake" must be parallel to the free stream poses some modeling difficulties in the case the angle of attack of the structure is not as small as considered among the assumptions of the lifting-line model. However, some solutions could be adopted in that situation. For instance, one adeguate solution has the trailing wake bent near the trailing edge in order to meet the "free wake" condition. Another possibility takes into account the horseshoe vortex, but modified because the trailing segments are not shed at the trailing edge. Thus, in general the method can be easily modified to treat the desired general case. However, the following discussion will assume small angles of attack.

The strength of the vortex is assumed to be constant for the horseshoe element and a positive circulation is defined as shown in Fig. 8.7. Furthermore, it is assumed that the introduced three-dimensional model accounts for (in an approximate way) the Kutta condition:

$$
\begin{equation*}
\gamma_{\text {T.E. }}=0 \tag{8.7}
\end{equation*}
$$

where the subscript T.E. stands for trailing edge.
The procedure carried out to compute the velocity $\left(u^{\prime}, v^{\prime}, w^{\prime}\right)$ induced at a point $P(x, y, z)$ by a straight vortex line segment with circulation $\Gamma^{\prime}$ is based on the Biot-Savart


Figure 8.7: The horseshoe convention followed for the VLM.

Law. Here, the details are not explained, but it is assumed that all the operations can be joined in a function called VORTEXL (stands for Vortex Line):

$$
\begin{equation*}
\left(u^{\prime}, v^{\prime}, w^{\prime}\right)=\operatorname{VORTEXL}\left(x, y, z, x_{1}, y_{1}, z_{1}, x_{2}, y_{2}, z_{2}, \Gamma^{\prime}\right) \tag{8.8}
\end{equation*}
$$

The superscript ' indicates that the quantities are not normalized, i.e. with physical units; it is utilized because the discussion below is going to involve dimensionless quantities. The coordinates having subscripts 1 and 2 are the coordinates of the considered vortex line segment.

Concerning the horseshoe element, the velocity induced by such an element at an arbitrary point $P(x, y, z)$ of the 3 D space can be computed by applying three times the function VORTEXL, that is for each vortex line segment. As a result:

$$
\begin{align*}
& \left(u_{1}^{\prime}, v_{1}^{\prime}, w_{1}^{\prime}\right)=\operatorname{VORTEXL}\left(x, y, z, x_{A}, y_{A}, z_{A}, x_{B}, y_{B}, z_{B}, \Gamma^{\prime}\right)  \tag{8.9}\\
& \left(u_{2}^{\prime}, v_{2}^{\prime}, w_{2}^{\prime}\right)=\operatorname{VORTEXL}\left(x, y, z, x_{B}, y_{B}, z_{B}, x_{C}, y_{C}, z_{C}, \Gamma^{\prime}\right)  \tag{8.10}\\
& \left(u_{3}^{\prime}, v_{3}^{\prime}, w_{3}^{\prime}\right)=\operatorname{VORTEXL}\left(x, y, z, x_{C}, y_{C}, z_{C}, x_{D}, y_{D}, z_{D}, \Gamma^{\prime}\right) \tag{8.11}
\end{align*}
$$

Now let these three computations be included in a sole function, which is addressed as $H S H O E$. Hence, the velocity induced by the generic horseshoe vortex element $j$ with vortex line strength $\Gamma_{j}$ at the generic Control Point of panel $i$ is given by:

$$
\begin{align*}
\left(u^{\prime}, v^{\prime}, w^{\prime}\right)_{i j}=\operatorname{HSHOE}( & x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j} \\
& \left.x_{B_{j}}, y_{B_{j}}, z_{B_{j}}, x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \Gamma_{j}^{\prime}\right) \tag{8.12}
\end{align*}
$$

where $i$ and $j$ are counters for all the aerodynamic panels of the structure. In fact, they go from 1 to $N_{A P}$, which is the total number of aerodynamic panels located in the domain of the whole structure. Such a number is the sum of the different $N_{A P}^{S}$, i.e. the number of aerodynamic panels of each Wing Segment S , where $S=1,2, \ldots, N_{W S}$. The term $\Gamma_{j}^{\prime}$ is the circulation, that is the strength of the vorteces of panel $j$. Besides, $\left(x_{i}, y_{i}, z_{i}\right)$ are the coordinates of the generic point in which the induced velocity has to be evaluated. The other coordinates indicate the position of the horseshoe's edges $A, B, C, D$.

In general, there is not a particular reference system in which the coordinates have to be written. They can be expressed in a general reference system, provided that all of them are in the same system. The only restriction is the "free wake" condition. According to it, at each element tip the two trailing vortex segments must be parallel to the streamlines. This condition has to be satisfied and its consequence in terms of coordinates depends on the choice of the reference system. The result of the function $H S H O E$ will be the components of induced velocity with respect to the same reference system of the input coordinates.

In this work the discussion of Vortex Lattice Method follows the aeroelastic notation described in section 8.2. According to such a notation, the global $x$ and local $x^{S}$ axes are parallel to the free stream $\boldsymbol{V}_{\infty}$. Then, the aerodynamic mesh of the generic Wing Segment $S$ lies on its reference plane surface parallel to the free stream. The conclusion is that each element corresponding to the Wing Segment $S$ lies on the plane $x^{S}-y^{S}$ parallel to $\boldsymbol{V}_{\infty}$ and without any angle of attack. This does not have to confuse the reader about the inclusion of an eventual angle of attack for the structure. In fact, the aeroelastic model is formulated so that the angles of attack of the configuration are treated in the construction of $L_{R H S}$, as it will be seen afterwards.

However, it is preferred to write Eq. 8.12 in the global reference system; the reason is related to the ability of the presented aeroleastic model to analyze wing systems composed of different Wing Segments, generally oriented in the 3D space. This choice will be more clear in the following.

Thus, in this particular case, the restriction "free wake" condition becomes:

$$
\begin{align*}
y_{A_{j}} & =y_{B_{j}} \\
y_{D_{j}} & =y_{C_{j}}  \tag{8.13}\\
z_{A_{j}} & =z_{B_{j}} \\
z_{D_{j}} & =z_{C_{j}}
\end{align*}
$$

At this point, the relation between the vortex line strength of a horseshoe and the velocity which it induces at a Control Point has been shown. The following Eq. 8.14 is able to provide the value of the lift force, in modulus, of the bound vortex segment once known its circulation on the panel $j$. It is the Kutta-Joukowsky theorem and is given by:

$$
\begin{equation*}
L_{j}=\rho_{\infty} V_{\infty} 2 e_{j} \Gamma_{j}^{\prime} \tag{8.14}
\end{equation*}
$$

where $2 e_{j}$ is the panel bound vortex projection normal to the free stream. In other words, $e_{j}$ refers to half length of panel $j$ along the wing span, i.e. along the $y^{S}$ local direction of the corresponding Wing Segment. It is convenient now to introduce the dimensionless vortex line strengths, denoted without adding the superscript ':

$$
\begin{equation*}
\Gamma_{j}=\frac{\Gamma_{j}^{\prime}}{\Gamma_{r e f}} \tag{8.15}
\end{equation*}
$$

The choice of the reference value $\left[\Gamma_{r e f}\right]=\frac{m m^{2}}{s}$ is arbitrary and has been chosen for the sake of convenience. In fact, the millimeter will be the unit of measurement for the distances. However, this is not an important aspect. Notice that such a reference value will be repeated in the following formulae just for the sake of completeness. Thus, Eq. 8.16 replaces the previous Eq. 8.14 as follows:

$$
\begin{equation*}
L_{j}=\rho_{\infty} V_{\infty} 2 e_{j} \Gamma_{r e f} \Gamma_{j} \tag{8.16}
\end{equation*}
$$

The formula 8.16 can be extended to all the $N_{A P}$ aerodynamic panels of the whole configuration. It allows to write the Lift Forces as functions of the circulations on all the panels. As a result, Eq. 8.17 assumes a matricial notation given by:

$$
\begin{equation*}
\boldsymbol{L}=\rho_{\infty} V_{\infty} \cdot \boldsymbol{\Delta} \boldsymbol{y} \cdot \Gamma_{r e f} \boldsymbol{\Gamma} \tag{8.17}
\end{equation*}
$$

where the vector $\boldsymbol{L}$ contains the $N_{A P}$ Lift Forces applied on all the Load Points. The same procedure is followed for the vortex strengths. Instead, the matrix $\boldsymbol{\Delta} \boldsymbol{y}$ collects the panel bound vortex projections normal to the free stream in its main diagonal. Then, their definitions are given by the following schemes:

$$
\left.\begin{array}{c}
\boldsymbol{\Gamma}=\left\{\begin{array}{c}
\Gamma_{1} \\
\Gamma_{2} \\
\Gamma_{3} \\
\vdots \\
\Gamma_{N_{A P}}
\end{array}\right\} \\
\boldsymbol{\Delta} \boldsymbol{y}=\left[\begin{array}{ccccc}
2 e_{1} & 0 & 0 & \ldots & 0 \\
0 & 2 e_{2} & 0 & \ldots & 0 \\
0 & 0 & 2 e_{3} & \ldots & 0 \\
\vdots \\
L_{N_{A P}}
\end{array}\right\} \\
\vdots \\
0 \\
L_{3} \\
L_{3} \\
0
\end{array}\right\}
$$

### 8.3.3 The Aerodynamic Influence Coefficient Matrix

As it will be seen in Section 9.4, the vortex circulations do not appear among the aerodynamic quantities involved. In fact, pressures acting on the aerodynamic panels are in place of these circulations. On the countrary, the discussion presented so far takes into account the vortex line strengths rather than pressures. Thus, it is useful to find an expression relating the two quantities in order to adapt the Vortex Lattice Method to the aeroelastic model.

The connection between pressures and circulations is carried out by the definition of Lift Force provided by Eq. 8.18. As known, the modulus of the Aerodynamic Force applied at the Load Point of generic $j^{\text {th }}$ panel is obtained by multiplying the pressure load by the panel surface:

$$
\begin{equation*}
L_{j}=\Delta x_{j} 2 e_{j} \Delta p_{j}^{\prime} \tag{8.18}
\end{equation*}
$$

where $\Delta x_{j}$ is the average chord of $j^{t h}$ panel. The equation indicates how the Aerodynamic Loads acting on the surface are transferred as Lift Forces located on the Loads Points of the aerodynamic panels of the whole structure. It means that the pressure distribution charging the wing system becomes a set of concentrated loads.

As done about the vortex strengths, it is better to treat a dimensionless pressure, which will be addressed as $\Delta p_{j}$, lacking of the superscript ' used for the real quantities. Thus,
they are normalized with respect to the dynamic pressure and defined in Eq. 8.19:

$$
\begin{equation*}
\Delta p_{j}=\frac{\Delta p_{j}^{\prime}}{\Delta p_{\text {ref }}}=\frac{\Delta p_{j}^{\prime}}{\left(\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\right)} \tag{8.19}
\end{equation*}
$$

Since the dimensionless quantities have been introduced, for a generic panel $j$ the modulus of the Aerodynamic Force, applied at its Load Point, is obtained by multiplying the dynamic pressure by the panel surface and by the dimensionless pressure load:

$$
\begin{equation*}
L_{j}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{j} 2 e_{j} \Delta p_{j} \tag{8.20}
\end{equation*}
$$

Now, let $\Delta \boldsymbol{p}$ be a vector which contains all the dimensionless pressure loads on all aerodynamic panels of the structure. It represents the ratio of the real pressures $\Delta p^{\prime}$ to the dynamic pressure $\frac{1}{2} \rho_{\infty} V_{\infty}$, taken as reference value. The same matrix notation utilized in Eq. 8.17 leads to:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \cdot \boldsymbol{\Delta} \boldsymbol{x} \cdot \boldsymbol{\Delta} \boldsymbol{y} \cdot \Delta \boldsymbol{p} \tag{8.21}
\end{equation*}
$$

where $\boldsymbol{L}$ and $\boldsymbol{\Delta} \boldsymbol{y}$ have been defined previously. On the contrary, the other two entities are written as:

$$
\begin{gathered}
\boldsymbol{\Delta} \boldsymbol{p}=\left\{\begin{array}{c}
\Delta p_{1} \\
\Delta p_{2} \\
\Delta p_{3} \\
\vdots \\
\Delta p_{N_{A P}}
\end{array}\right\} \\
\boldsymbol{\Delta} \boldsymbol{x}=\left[\begin{array}{ccccc}
\Delta x_{1} & 0 & 0 & \ldots & 0 \\
0 & \Delta x_{2} & 0 & \ldots & 0 \\
0 & 0 & \Delta x_{3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \Delta x_{N_{A P}}
\end{array}\right]
\end{gathered}
$$

It is useful to join the terms of matrices $\boldsymbol{\Delta x}$ and $\boldsymbol{\Delta x}$ in a single matrix, addressed as $\boldsymbol{I}_{D}$. By means of the new matrix, Eq. 8.21 becomes more compact as shown in Eq. 8.22.

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \cdot \boldsymbol{I}_{D} \cdot \Delta \boldsymbol{p} \tag{8.22}
\end{equation*}
$$

where:

$$
\boldsymbol{I}_{D}=\left[\begin{array}{ccccc}
\Delta x_{1} 2 e_{1} & 0 & 0 & \cdots & 0 \\
0 & \Delta x_{2} 2 e_{2} & 0 & \cdots & 0 \\
0 & 0 & \Delta x_{3} 2 e_{3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \Delta x_{N_{A P}} 2 e_{N_{A P}}
\end{array}\right]
$$

Notice that $\boldsymbol{I}_{D}$ is a square matrix and its dimension is $N_{A P} \times N_{A P}$. An easy practical example of construction of these matrices and vectors will be provided in Section 9.4.

Now, the procedure to connect pressures and circulations on the aerodynamic panels is faced. It is carried out according to the matrix forms written in Eqs. 8.17 and 8.21. Their coupling, i.e. their equality, conducts to:

$$
\rho_{\varnothing} V_{\infty} \cdot \Delta y \cdot \Gamma_{r e f} \Gamma=\frac{1}{2} \rho_{\infty} V_{\infty}^{\not 2} \cdot \Delta x \cdot \Delta y \cdot \Delta p
$$

By multiplying both the right and left sides by matrix $[\boldsymbol{\Delta} \boldsymbol{y}]^{-1}$, finally Eq. 8.23 expresses the relationship between dimensionless circolations and dimensionless pressures.

$$
\begin{equation*}
\Gamma_{r e f} \boldsymbol{\Gamma}=\frac{1}{2} V_{\infty} \cdot \boldsymbol{\Delta} \boldsymbol{x} \cdot \Delta \boldsymbol{p} \tag{8.23}
\end{equation*}
$$

By isolating the contribution of the generic $j^{t h}$ panel in the previous equation, it is possible to relate its single dimensionless circulation to its single dimensionless pressure:

$$
\begin{equation*}
\Gamma_{r e f} \Gamma_{j}=\frac{1}{2} V_{\infty} \cdot \Delta x_{j} \cdot \Delta p_{j} \tag{8.24}
\end{equation*}
$$

Then Eq. 8.12 is resumed; considering the $j^{\text {th }}$ panel, the induced velocity at the Control Point of $i^{\text {th }}$ panel is written as function of real vortex line circulation $\Gamma_{j}^{\prime}$. The transition to dimensionless circulation is performed by means of the following expedient:

$$
\begin{align*}
\left(u^{\prime}, v^{\prime}, w^{\prime}\right)_{i j}=\operatorname{HSHOE}( & x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}, x_{B_{j}}, y_{B_{j}}, z_{B_{j}} \\
& \left.x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \Gamma=\Gamma_{r e f}\right) \cdot \Gamma_{j} \tag{8.25}
\end{align*}
$$

As a matter of fact, the input value for the function is the reference value of circulation; it's important to notice that such an input has physical units. The purpose of the previous derivation of Eqs. 8.23 and 8.24 was to relate the vortex line strength to the pressure of the generic panel $j$. Thus, let us utilize the latter equation in order to build a correspondence between the induced velocity at Control Point $i$ and such a pressure. Furthermore, according to the aeroelastic notation, it is better to find the dimensionless components $(u, v, w)_{i j}$ of such an induced velocity. Obviously, the normalization is performed considering the modulus of free stream velocity as the reference value:

$$
\begin{equation*}
(u, v, w)_{i j}=\frac{\left(u^{\prime}, v^{\prime}, w^{\prime}\right)_{i j}}{V_{\infty}} \tag{8.26}
\end{equation*}
$$

The combination of Eqs. 8.25 and 8.26 leads to:

$$
\begin{aligned}
(u, v, w)_{i j}=\operatorname{HSHOE}( & \left(x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}, x_{B_{j}}, y_{B_{j}}, z_{B_{j}}\right. \\
& \left.x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \Gamma=1\right) \cdot \frac{1}{2} \Delta x_{j} \cdot \Delta p_{j}
\end{aligned}
$$

The value $\Gamma=1$ could be changed, but particular attention has to be payed. It is remembered that the function $H S H O E$ is based on the Biot-Savart Law and then it involves the quantity $\Gamma$. This is the reason why the input of $H S H O E$ must remains a quantity playing the role of vortex line strength. Besides, the Biot-Savart Law is directly proportional to
the circulation. Thus, the term $\frac{1}{2} \Delta x_{j}$ can be included into the input data of function HSHOE . Such a term has the meaning of the circulation of vorteces on $j^{\text {th }}$ panel so that its dimensionless pressure is equal to the unit and so that the free stream has velocity equal to $\left(1 \frac{m m}{s}\right)$.

At the end, the formula giving the dimensionless components of the velocity induced by the horseshoe on aerodynamic panel $j$ to the Control Point of aerodynamic panel $i$ as a function of the dimensionless pressure on the panel $j$ is Eq. 8.27:

$$
\begin{align*}
& (u, v, w)_{i j}=\operatorname{HSHOE}\left(x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}, x_{B_{j}}, y_{B_{j}}, z_{B_{j}}\right. \\
& \left.x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \frac{1}{2} \Delta x_{j}\right) \cdot \Delta p_{j} \tag{8.27}
\end{align*}
$$

Considering the Control Point of the generic panel $i$, its final induced velocity resulting from the aerodynamic analysis will be the summation of the contributes corresponding to all the aerodynamic panels $j$ of the whole structure. In fact the method of superposition is applied to sum up all the values deriving from Eq. 8.27. It appears more clear looking at Eq. 8.28:

$$
\begin{align*}
& (u, v, w)_{i}=\sum_{j=1}^{N_{A P}} \operatorname{HSHOE}\left(x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}, x_{B_{j}}, y_{B_{j}}, z_{B_{j}},\right.  \tag{8.28}\\
& \\
& \left.x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \frac{1}{2} \Delta x_{j}\right) \cdot \Delta p_{j}
\end{align*}
$$

As said above, in general there is not any constraint about the choice of the reference system in which to write Eq. 8.27. However, since the exposed aeroelastic model will be able to study wing systems composed of different Wing Segments generally oriented in the 3D space, then the global reference results the most useful choice. The meaning is explained as follows.

Even if the final result has to be written in the local reference system of the panel $i$, i.e. the local reference system of the Wing Segment where it lies, it is advantageous that such values are all written in the same reference system, so as to avoid any further coordinate tranformation. For instance, in the case the panel $i$ lies in a Wing Segment different from panel $j$, the input coordinates could not refer to different reference systems. So, a further transformation of either the $i^{t h}$ panel or the $j^{\text {th }}$ panel would be required. By the way, that transformation from a local reference to another local reference would require two operations, through the global system. Finally, the adoption of global reference system appears to be the most convenient choice.

As a consequence, both the input and the output coordinates of Eq. 8.27 will be written in the global reference system. According to the aeroelastic notation, the transformation of the output induced velocity at Control Point $i$ of such an equation from global to local components is required. The local reference system to be chosen is the one corresponding to the Wing Segment $S$ where the panel $i$ lies. It follows from the fact that the following boundary condition will involve the slopes of the panels and such quantities are expressed in local coordinates.

Besides, in the cited boundary condition only the dimensionless normalwash is considered. Looking at a generic panel $i$, the normalwash $w_{i}^{\prime}$ is the component of the velocity normal to the reference undeformed surface of the Wing Segment where the panel lies, i.e. normal to the considered aerodynamic panel. Being this surface parallel to the free stream
direction $\boldsymbol{V}_{\infty}$, then the normalwash results to be perpendicular to the free stream too. As a consequence, the dimensionless normalwash $w_{i}$ at Control Point of $i^{\text {th }}$ panel is the projection of the computed vector $(u, v, w)_{i}$ (written in the global reference) toward the vector $\boldsymbol{n}_{i}$ perpendicular to the panel. Hence:

$$
\begin{gather*}
w_{i}=\sum_{j=1}^{N_{A P}}\left[\operatorname { H S H O E } \left(x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}, x_{B_{j}}, y_{B_{j}}, z_{B_{j}}, x_{C_{j}}, y_{C_{j}}, z_{C_{j}},\right.\right.  \tag{8.29}\\
\left.\left.x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \frac{1}{2} \Delta x_{j}\right)\left.\right|_{g l o b} \bullet \boldsymbol{n}_{i}\right] \cdot \Delta p_{j}
\end{gather*}
$$

The $N_{A P}$ terms appearing between the box brackets are addressed as $A_{i j}^{D}$. They can be computed by only knowing the geometrical quantities. In fact, the input data consist of the vertices' coordinates of the horseshoes, the coordinates of the Control Points and finally the chords and the normal vectors of all the panels of the structure. Thus, the dimensionless normalwash for each panel of the structure can be easily evaluated. Via the extension of Eq. 8.29 to all the aerodynamic panels of the structure, the problem can be summarized by the following matricial form:

$$
\begin{equation*}
\boldsymbol{w}=\boldsymbol{A}^{D} \cdot \Delta \boldsymbol{p} \tag{8.30}
\end{equation*}
$$

where $\boldsymbol{A}^{D}$ is the Aerodynamic Influence Coefficient Matrix for the aerodynamic panels. This matrix is calculated once by using the geometry of the aerodynamic reference configuration. In fact it depends only on the aerodynamic discretization. Equation 8.30 will be resumed in Section 9.3 to relate the dimensionless normalwash to the dimensionless pressures.

### 8.3.4 The Boundary Condition

Finally, the boundary condition for the Vortex Lattice Method has to be written. The VLM is based on the theory of the vortex line, which is a solution of the Laplace's equation. As a consequence, the only boundary condition that needs to be satisfied is the zero normal flow accross the thin wing's solid surface. According to the aeroelastic notation and what obtained above, the boundary condition here adopted is given by Eq. 8.31:

$$
\begin{equation*}
\boldsymbol{w}=\frac{\partial \mathcal{Z}_{\text {loc }}}{\mathrm{d} x} \tag{8.31}
\end{equation*}
$$

Concerning the equation, the normalization of the normalwash with respect to $V_{\infty}$ means that the dimensionless normalwash is equal to the tangent of the angle from the velocity of the stream in proximity of the panel, in the deformed configuration, to the free stream direction $\boldsymbol{V}_{\infty}$. As a consequence, considering small angles of deflection because of the model's linearity, the dimensionless normalwash has to equal the slope at the aerodynamic Control Point.

### 8.3.5 Modelling of Wing Symmetry

In situations when symmetry exists between the left and right halves of the body's surface a rather simple method can be used to include this feature in the numerical scheme. In terms of programming simplicity, these modifications will affect only the aerodynamic influence coefficient calculation.

Usally, when a wing is symmetric it could be better to model only its right-hand half. The influence of the generic panel $j$ on point P (indicated as the generic $i^{\text {th }}$ point) can be
obtained by the function $H S H O E$ so far introduced. According to Eq. 8.12, the velocity induced (indicated with the subscript 1 ) at point P by the $j^{\text {th }}$ element is:

$$
\begin{align*}
& \left(u_{1}^{\prime}, v_{1}^{\prime}, w_{1}^{\prime}\right)_{i j}=\operatorname{HSHOE}\left(x_{i}, y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}\right. \\
&  \tag{8.32}\\
& \left.x_{B_{j}}, y_{B_{j}}, z_{B_{j}}, x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \Gamma_{j}^{\prime}\right)
\end{align*}
$$

Now let us evaluate the effect of the wing left/right symmetry. Let the panel symmetrical to panel $j$ be called image panel $j^{\prime}$. Obviously it lies in the left half-wing and has the same strenght of the panel $j$, because of the symmetry. The scope is to perform the new aerodynamic analysis without meshing the left symmetrical half-wing, reducing the number of unknowns.

It has been seen how the induced velocity on a point depends only on the circulation of the vortices and the distances from the considered point to the vorteces' edges. Then, in order to measure the influence of the image panel $j^{\prime}$ on the point $P$, it is possible to evaluate the influence of $j^{t h}$ panel on point $P^{\prime}$ (indicated as the generic point $i^{\prime}$ ), which is symmetrical to $P$ with respect to the longitudinal aircraft plane. The choice of that image panel is appropriate since in this way the two strengths of the panels and the two distances to the points are the same; as a consequence, the modulus of the velocity induced at point $P$ by the image panel $j^{\prime}$ is equal to the modulus of the velocity induced (indicated with the subscript 2) at point $P^{\prime}$ by panel $j$ (Fig. 8.8).


Figure 8.8: Influence of panel $j$ and image panel $j^{\prime}$
However, not only the modulus has to be considered. In fact, also the direction of the induced velocity has to be taken into account. It is easy to understand that the two resulting induced velocity are symmetrical with respect to the longitudinal plane. Hence, they differs only for the component along the spanwise direction, coincident with the $y$ axis. It means that in evaluating the velocity induced at point $P$ it is necessary to involve the velocity induced at point $P^{\prime}$, but inverting the sign of its $y$ component.

$$
\begin{align*}
& \left(u_{2}^{\prime}, v_{2}^{\prime}, w_{2}^{\prime}\right)_{i^{\prime} j}=\operatorname{HSHOE}\left(x_{i},-y_{i}, z_{i}, x_{A j}, y_{A j}, z_{A j}\right. \\
&  \tag{8.33}\\
& \left.\quad x_{B_{j}}, y_{B_{j}}, z_{B_{j}}, x_{C_{j}}, y_{C_{j}}, z_{C_{j}}, x_{D_{j}}, y_{D_{j}}, z_{D_{j}}, \Gamma_{j}^{\prime}\right)
\end{align*}
$$

By utilizing the method of superposition between contributes of Eqs. 8.32 and 8.33, the final result for the induced velocity at the generic $i^{\text {th }}$ point in the case of symmetrical wing system is given by:

$$
\begin{equation*}
\left(u^{\prime}, v^{\prime}, w^{\prime}\right)_{i j}=\left(u_{1}^{\prime}+u_{2}^{\prime}, v_{1}^{\prime}-v_{2}^{\prime}, w_{1}^{\prime}+w_{2}^{\prime}\right) \tag{8.34}
\end{equation*}
$$

This procedure can reduce the number of unknowns by half, and only the vortices of the right half-wing need to be modeled. Therefore, when scanning the elements of the half-span in the "influence coefficient" computation the coefficients have to be modified according to Eq. 8.34.

### 8.4 Spline Methods

Aeroelastic analysis, as an interdisciplinary problem, requires the coupling of the aerodynamics and structural responses. In practice, the requirements to generate the discretized models of these disciplines are subject to different engineering considerations. For instance, the grid of the discretized aerodynamic model is usually placed on the external surface, whereas that of the Structural model is located on the internal load-carrying component. This gives rise to the data-transfer problem between two computational grid systems. This would amount to the following problems:

- the proper transfer of the displacements computed in the Structural grid to those located on the aerodynamic grid, where the incidences with respect to the free stream have to be evaluated too;
- the transformation of Aerodynamic Loads from aerodynamic grid points into Equivalent Nodal Loads by which the Structure is charged.

The development of a suitable methodology for solving this type of data transfer problem is by no means a trivial task. In fact, such a methodology should be further developed as the Aerodynamic and Structural methods advance. The interpolation method used is called "splining". The spline theory involves the mathematical analysis of beams and plates. Briefly, a Beam spline is a generalization of the simple beam, which allows torsional as well as bending behavior. Instead, a Surface spline is a solution for an infinite uniform plate.

There are several ways to analyze splines. These include the three-moment method, the stiffness method and the influence function methods. Here, the influence function methods have been chosen. The advantages include a uniform formulation for beams and plates, the ease of interpolation for the Aerodynamic Points, and the ease of eventually putting springs at the point attachments, even if not described here.

This chapter will describe the spline theories adopted for Aeroelastic problems nowadays. The purpose of the following is to provide a transformation matrix which gives structural displacements at a set of interpolated (or extrapolated) locations in terms of deflections at structural points. As said, the matrix coefficients are determined by using Beam and Surface splines, which give "structural-like" deformation patterns since they are beams and plates.

However, according to the most popular aeroelastic convention (as well as commercial software MSC.Nastran or ZAERO), the Structural degrees of freedom have been chosen as the independent degrees of freedom, whereas the aerodynamic degrees of freedom are dependent. It is important to note that this mathematical tool is essential for both the
two transformations required in aeroelastic problems. At first, the interpolation from the structural deflections to the aerodynamic deflections is necessary. In the second place also the relationship between the Aerodynamic Forces and the "structurally equivalent"forces acting on the Structural Nodes is equally required.

### 8.4.1 The Infinite Plate Spline Method (IPS)

The IPS method was first proposed by Harder and Desmarais, which was a significant improvement over two-dimensional (2D) interpolation method of Rodden, McGrew and Kalman. This development was motivated by the advent of lifting surface methods in aerodynamics at that time, which required a 2D interpolation method such as IPS. The 2D surface is defined as the plane of the lifting surface. Therefore, IPS is ideally suited for displacements and forces transfer of wing-like components. Today, IPS is one of the most popular methods of interpolation used in aerospace industry.

As a matter of fact, the Surface Spline is a mathematical tool used to find a surface function $w(x, y)$ for all points $(x, y)$ when $w$ is known for a discrete set of $N$ Structural Points (the $i^{\text {th }}$ displacement is $w_{i}\left(x_{i}, y_{i}\right)$, for $\left.i=1,2, \ldots, N\right)$, lying within a 2D domain with Cartesian coordinates $x$ and $y$; i.e. it is the problem of a plate with multiple deflecting supports. In fact the theory introduces an infinite plate with uniform thickness and solves its partial differential equation of equilibrium. The resulting deformation satisfies the given deflections $w_{i}\left(x_{i}, y_{i}\right)$ at the $N$ Structural Points. Moreover, the deflection at other points, for istance the Aerodynamic Points, can be determinated on the the plate.

To solve the surface spline problem, at first the governing Differential Equation 8.35 of an infinite plate with bending stiffness is taken into account:

$$
\begin{equation*}
D \nabla^{4} w=q \tag{8.35}
\end{equation*}
$$

where $w$ is the plate deflection, $D$ is the plate bending stiffness and $q$ is the distributed load on the plate. Introducing polar coordinates, $x=r \cos \vartheta$ and $y=r \sin \vartheta$, then the differential operator $\nabla^{4}$ is given by:

$$
\begin{equation*}
\nabla^{4}=\frac{1}{r} \frac{d}{d r}\left\{r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d}{d r}\right)\right]\right\} \tag{8.36}
\end{equation*}
$$

Let us start deducting the solution to the previous Differential Equation, by means of a series of integrations:

$$
\begin{aligned}
D \nabla^{4} w=q \quad \Rightarrow \quad \frac{1}{r} \frac{d a}{d r}=\frac{q}{D} \quad \text { where } a=\left\{r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right]\right\} \\
\frac{d a}{d r}=r \frac{q}{D} \quad \Rightarrow \quad a=\int d a=\int r \frac{q}{D} d r \quad \Rightarrow \quad a=\frac{r^{2}}{2} \frac{q}{D}+K_{1}
\end{aligned}
$$

where $q$ is considered constant and then independent of radius $r$.

$$
\begin{gathered}
\frac{r^{2}}{2} \frac{q}{D}+K_{1}=r \frac{d b}{d r} \quad \text { where } b=\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right] \\
\frac{d b}{d r}=\frac{r}{2} \frac{q}{D}+\frac{K_{1}}{r} \quad \Rightarrow \quad b=\int d b=\int \frac{r}{2} \frac{q}{D} d r+\int \frac{K_{1}}{r} d r \\
\Rightarrow \quad b=\frac{r^{2}}{4} \frac{q}{D}+K_{1} \ln r+K_{2}
\end{gathered}
$$

$$
\begin{gathered}
\frac{r^{2}}{4} \frac{q}{D}+K_{1} \ln r+K_{2}=\frac{1}{r} \frac{d c}{d r} \quad \text { where } c=r \frac{d w}{d r} \\
c=\int d c=\int \frac{r^{3}}{4} \frac{q}{D} d r+\int r K_{1} \ln r d r+\int r K_{2} d r \\
\Rightarrow \quad c=\frac{r^{4}}{16} \frac{q}{D}+\int r K_{1} \ln r d r+\frac{K_{2}}{2} r^{2}+K_{3} \\
\text { where } \int r \ln r d r=\frac{r^{2}}{2} \ln r-\int \frac{r}{2} d r=\frac{r^{2}}{2} \ln r-\frac{r^{2}}{4} \\
\Rightarrow \quad c=\frac{r^{4}}{16} \frac{q}{D}+K_{1} \frac{r^{2}}{2} \ln r-K_{1} \frac{r^{2}}{4}+\frac{K_{2}}{2} r^{2}+K_{3} \\
w=\int d w=\int \frac{r^{4}}{16} \frac{q}{D}+K_{1} \frac{r^{2}}{2} \ln r-K_{1} \frac{r^{2}}{4}+\frac{K_{2}}{2} r^{2}+K_{3}=r \frac{d w}{d r} \\
w+\int K_{1}\left[\frac{r}{2} \ln r-\frac{r}{4}\right] d r+\int K_{2} \frac{r}{2} d r+\int \frac{K_{3}}{r} d r
\end{gathered}
$$

In conclusion, Eq. 8.37 represents the solution of the Differential Equation, i.e. the deformation of an infinite plate charged by the distributed load $q$ in the origin.

$$
\begin{equation*}
w(r)=K_{4}+K_{3} \ln r+K_{2} \frac{r^{2}}{4}-K_{1} \frac{r^{2}}{4}+K_{1} \frac{r^{2}}{4} \ln r+\frac{q}{D} \frac{r^{4}}{64} \tag{8.37}
\end{equation*}
$$

Since, from now on, the discussion will focus on a concentrated load rather than a distributed one charging the plate, the integral of the homogeneous form of Eq. 8.35 is written. To do that, let us assume the distributed load $q$ equal to zero, obtaining:

$$
\begin{equation*}
\left.w\right|_{q=0}=K_{4}+K_{3} \ln r+K_{2} \frac{r^{2}}{4}-K_{1} \frac{r^{2}}{4}+K_{1} \frac{r^{2}}{4} \ln r+\frac{q}{\not D} \frac{y^{4}}{64} \tag{8.38}
\end{equation*}
$$

By isolating some terms and recalling them, it becomes:

$$
\begin{equation*}
\left.w\right|_{q=0}=C_{0}+C_{1} r^{2}+C_{2} \ln r+C_{3} r^{2} \ln r \tag{8.39}
\end{equation*}
$$

where :

$$
C_{0}=K_{4} \quad C_{1}=\frac{K_{2}}{4}-\frac{K_{1}}{4} \quad C_{2}=K_{3} \quad C_{3}=\frac{K_{1}}{4}
$$

The transverse displacement $w$ must assume finite values when $r=0$. That is why it is useful to look at the limit of Eq. 8.39 as follows:

$$
\begin{aligned}
\lim _{r \rightarrow 0} w & =\lim _{r \rightarrow 0}\left[C_{0}+C_{1} r^{2}+C_{2} \ln r+C_{3} r^{2} \ln r\right] \\
& =\underbrace{\lim _{r \rightarrow 0} C_{0}}_{=C_{0}}+\underbrace{\lim _{r \rightarrow 0}\left(C_{1} r^{2}\right)}_{=0}+\underbrace{\lim _{r \rightarrow 0}\left(C_{2} \ln r\right)}_{=C_{2}(-\infty)}+\underbrace{\lim _{r \rightarrow 0}\left(C_{3} r^{2} \ln r\right)}_{\text {to be evaluated }}
\end{aligned}
$$

The last term of previous equation is studied below by means of L'Hôpital's rule:

$$
\lim _{r \rightarrow 0}\left(C_{3} r^{2} \ln r\right)=\lim _{r \rightarrow 0} C_{3} \frac{\ln r}{\frac{1}{r^{2}}}=\lim _{r \rightarrow 0} C_{3} \frac{\frac{1}{r}}{-\frac{2}{r^{3}}}=\lim _{r \rightarrow 0}\left(-\frac{C_{3}}{2}\right) r^{2}=0
$$

Therefore, to have finite values for $\lim _{r \rightarrow 0} w$ only $C_{2}$ must be set to zero: $C_{2}=0$. Thus:

$$
\begin{equation*}
w(r)=C_{0}+C_{1} r^{2}+C_{3} r^{2} \ln r \tag{8.40}
\end{equation*}
$$

The concentrated load $P$ is applied on the origin of the plane ( $r=0$ ) instead of $q$, multiplied by $2 \pi r$ and integrated from 0 to $r$ :

$$
\begin{equation*}
P=\lim _{r \rightarrow 0}\left[\int_{0}^{r} 2 \pi r q d r\right] \tag{8.41}
\end{equation*}
$$

Remembering that:

$$
q=D \frac{1}{r} \frac{d}{d r}\left\{r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right]\right\}
$$

the new Differential Equation 8.42 valid for concentrated load $P$ acting the plate results to be by integration:

$$
\begin{align*}
P=\lim _{r \rightarrow 0}\left[\int_{0}^{r} 2 \pi r q d r\right] & =\lim _{r \rightarrow 0}\left[\int_{0}^{r} 2 \pi \gamma^{\prime} D \frac{1}{r} \frac{d}{d r}\left\{r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right]\right\} d r\right] \\
& =\lim _{r \rightarrow 0}\left[2 \pi D r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(r \frac{d w}{d r}\right)\right]\right] \tag{8.42}
\end{align*}
$$

Here Eq. 8.42 assumes the role of a boundary condition useful to derive one of the coefficients listed in the fundamental solution 8.40. In fact, combining that equation to the previous just obtained, $C_{3}$ becomes dependent on the load's intensity and the bending stiffness $D$. The procedure consists in easily substituting the formula for $w$ and computing the limit value when $r \rightarrow 0$.

$$
\begin{array}{r}
P=\lim _{r \rightarrow 0}\left[2 \pi D r \frac{d}{d r}\left[\frac{1}{r} \frac{d}{d r}\left(2 C_{1} r^{2}+r C_{3}(2 r \ln r+r)\right)\right]\right] \\
\quad=\lim _{r \rightarrow 0}\left[2 \pi D r \frac{d}{d r}\left[4 C_{1}+2 C_{3}+4 \ln r+2 C_{3}\right]\right] \\
=\lim _{r \rightarrow 0}\left[2 \pi D\left(4 C_{3}\right)\right]=8 \pi D C_{3} \quad \Rightarrow \quad C_{3}=\frac{P}{8 \pi D}
\end{array}
$$

Thanks to the property of the logarithm: $\ln r=\frac{1}{2} 2 \ln r=\frac{1}{2} \ln r^{2}$, hence:

$$
w(r)=C_{0}+C_{1} r^{2}+\frac{P}{16 \pi D} r^{2} \ln r^{2}
$$

which, according to the traditional spline notation, represents the following fundamental solution:

$$
\begin{equation*}
w(r)=A+B r^{2}+F r^{2} \ln r^{2} \tag{8.43}
\end{equation*}
$$

where:

$$
A=C_{0} \quad B=C_{1} \quad F=\frac{P}{16 \pi D}
$$

The deflection due to a single concentrated load is the fundamental solution and has polar symmetry. When the concentrated load is not located at the origin of the coordinate system $(x, y), r_{i}$ has to be put in place of $r$ in Eq. 8.43. The quantity $r_{i}$ is the distance between the generic point $(x, y)$, where $w$ is computed, and the load's application point.

Looking at the former statement, the application of several loads on different points of the domain doesn't involve any problem. In fact, in general the infinite plate could be charged not only by a sole concentrated load, but by $N$ forces acting on $N$ different Load Points. In that case it's allowed to apply the method of superposition as follows, where $\left(x_{i}, y_{i}\right)$ represent the coordinates of the $i^{\text {th }}$ Load Point. In other words the deflection of the plate is synthesized as the sum of deflections due to a set of point loads on the infinite plate. Figure 8.9 shows the physical meaning of the introduced quantities.


Figure 8.9: Coordinate system and Generic Load's Application Point.

As said before, the quantity $r_{i}$ represents the distance between the generic point $(x, y)$, where $w$ is computed, and the $i^{\text {th }}$ load's application point. It's advantageous to introduce $r_{i}$, according to the polar coordinates used so far. As a resume:

$$
\left\{\begin{array}{l}
x=r \cos \vartheta \\
y=r \sin \vartheta \\
r^{2}=x^{2}+y^{2} \\
r_{i}^{2}=\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}
\end{array}\right.
$$

The method of superposition leads to the expression given by:

$$
\begin{equation*}
w(x, y)=\sum_{i=1}^{N}\left[A_{i}+B_{i} r_{i}^{2}+F_{i} r_{i}^{2} \ln r_{i}^{2}\right] \tag{8.44}
\end{equation*}
$$

The aim is to write a formula dependent on $x_{i}, y_{i}, r$ and $\vartheta$, instead of $x, y$ and $r_{i}$, quantities appearing in Eq. 8.44. Therefore, the substitution of the relation among $r_{i}, x_{i}$ and $y_{i}$ leads
to Eq. 8.45.

$$
\begin{align*}
w(x, y) & =\underbrace{\sum_{i=1}^{N} A_{i}}_{\text {TERM } 1}+\underbrace{\sum_{i=1}^{N} B_{i}\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right]}_{\text {TERM } 2}+ \\
& +\underbrace{\sum_{i=1}^{N} F_{i}\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right] \ln \left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right]}_{\text {TERM } 3} \tag{8.45}
\end{align*}
$$

The analysis of TERM 2 is performed in order to eliminate the dependence on $r_{i}$, by splitting it into minimum elements.

$$
\begin{gathered}
\sum_{i=1}^{N} B_{i}\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right]= \\
\left.=\sum_{i=1}^{N} B_{i}\left[\left(x^{2}+y^{2}\right)+x_{i}^{2}+y_{i}^{2}-2 x x_{i}-2 y y_{i}\right)^{2}\right]= \\
=r^{2} \sum_{i=1}^{N} B_{i}+\sum_{i=1}^{N} B_{i} x_{i}^{2}+\sum_{i=1}^{N} B_{i} y_{i}^{2}-2 x \sum_{i=1}^{N} B_{i} x_{i}-2 y \sum_{i=1}^{N} B_{i} y_{i} \\
-2 x \sum_{i=1}^{N} B_{i} x_{i}=-2 r \sum_{i=1}^{N} B_{i} x_{i} \cos \vartheta \\
-2 y \sum_{i=1}^{N} B_{i} y_{i}=-2 r \sum_{i=1}^{N} B_{i} y_{i} \sin \vartheta
\end{gathered}
$$

Finally, TERM 2 becomes:

$$
\begin{gathered}
\sum_{i=1}^{N} B_{i}\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right]= \\
=r^{2} \sum_{i=1}^{N} B_{i}+\sum_{i=1}^{N} B_{i} x_{i}^{2}+\sum_{i=1}^{N} B_{i} y_{i}^{2}-2 r \sum_{i=1}^{N} B_{i} x_{i} \cos \vartheta-2 r \sum_{i=1}^{N} B_{i} y_{i} \sin \vartheta
\end{gathered}
$$

In the same way, also TERM 3 is given by:

$$
\begin{aligned}
& \sum_{i=1}^{N} F_{i}\left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right] \ln \left[\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}\right]= \\
& \sum_{i=1}^{N} F_{i}\left[r^{2}+x_{i}^{2}+y_{i}^{2}-2 r \cos \vartheta x_{i}-2 r \sin \vartheta y_{i}\right] \times \\
& \quad \times \ln \left[r^{2}+x_{i}^{2}+y_{i}^{2}-2 r \cos \vartheta x_{i}-2 r \sin \vartheta y_{i}\right]
\end{aligned}
$$

For the purpose of determining the undetermined spline coefficients $A_{i}, B_{i}, F_{i}$ certain information about the solution need to be used. The considered plate is infinite, therefore the displacement progress has to be at maximum linear at infinity, when $r \rightarrow \infty$. In fact,
the Surface Spline is a smooth continuous function that will become nearly linear in $x$ and $y$ at large distances from Load Points $\left(x_{i}, y_{i}\right)$. That statement describes a boundary condition; radial lines emanating from load's application points (which all may be regarded as at the origin relative to infinity) appear to be straight lines.

At first the expansion of $w(x, y)$ for large values of $r$ is performed. But to do that, TERM 3 must be written in a different manner, since it's present a natural logarithmic part. Since it should be expanded, the trick consists in referring to Taylor series of natural logarithm $\ln (1+z)$ when $z \rightarrow 0$, i.e. $r \rightarrow \infty$.

$$
\begin{aligned}
& \ln \left[r^{2}+x_{i}^{2}+y_{i}^{2}-2 r \cos \vartheta x_{i}-2 r \sin \vartheta y_{i}\right]= \\
= & \ln \left[r^{2}\left(1+\frac{x_{i}^{2}+y_{i}^{2}-2 r \cos \vartheta x_{i}-2 r \sin \vartheta y_{i}}{r^{2}}\right)\right]= \\
= & \ln r^{2}+\underbrace{\ln [1+\underbrace{\frac{x_{i}^{2}+y_{i}^{2}-2 r\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}{r^{2}}}_{=z}]}_{\text {TERM } 3.1}
\end{aligned}
$$

By using the big O notation, the Taylor series of natural logarithm $\ln (1+z)$ is introduced in Eq. 8.46 and utilized to rewrite TERM 3.1:

$$
\begin{align*}
\ln (1+z) \approx & \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^{n}=z-\frac{1}{2} z^{2}+\frac{1}{3} z^{3}-\frac{1}{4} z^{4}+O\left(z^{5}\right)  \tag{8.46}\\
\ln (1+z) \approx & \frac{x_{i}^{2}}{r^{2}}+\frac{y_{i}^{2}}{r^{2}}-\frac{2\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}{r}-\frac{1}{2} \frac{x_{i}^{4}}{r^{4}}-\frac{1}{2} \frac{y_{i}^{4}}{r^{4}}+ \\
& -\frac{2}{r^{2}} x_{i}^{2} \cos ^{2} \vartheta-\frac{2}{r^{2}} y_{i}^{2} \sin ^{2} \vartheta-\frac{4 x_{i} y_{i} \cos \vartheta \sin \vartheta}{r^{2}}-\frac{x_{i}^{2} y_{i}^{2}}{r^{4}}+ \\
& +\frac{2}{r^{3}}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right) x_{i}^{2}+\frac{2}{r^{3}}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right) y_{i}^{2}+\ldots \\
\ln (1+z) \approx- & \frac{2\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}{r}+\frac{x_{i}^{2}+y_{i}^{2}-2\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)^{2}}{r^{3}}+O\left(\frac{1}{r^{2}}\right)
\end{align*}
$$

Hence, when $r \rightarrow \infty$ TERM 3 has the form:

$$
\begin{gathered}
\sum_{i=1}^{N} F_{i}\left[r^{2}+x_{i}^{2}+y_{i}^{2}-2 r \cos \vartheta x_{i}-2 r \sin \vartheta y_{i}\right] \times \\
\times\left[\ln r^{2}-\frac{2\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}{r}+\frac{x_{i}^{2}+y_{i}^{2}-2\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)^{2}}{r^{3}}+O\left(\frac{1}{r^{2}}\right)\right]
\end{gathered}
$$

In conclusion, when $r \rightarrow \infty$ the deflection $w(r, \vartheta)$ is given by:

$$
\begin{gathered}
\left.w(r, \vartheta)\right|_{r \rightarrow \infty}=\sum_{i=1}^{N} A_{i}+\underbrace{r^{2} \sum_{i=1}^{N} B_{i}}_{\text {TERM }}+\sum_{i=1}^{N} B_{i} x_{i}^{2}+\sum_{i=1}^{N} B_{i} y_{i}^{2}+ \\
\underbrace{-2 r \sum_{i=1}^{N} B_{i}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}_{\text {TERM } 7.1}+\underbrace{r^{2} \ln r^{2} \sum_{i=1}^{N} F_{i}}_{\text {TERM } 4} \underbrace{-2 r \sum_{i=1}^{N} F_{i}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}_{T E R M 7.2} \\
+\sum_{i=1}^{N} F_{i}\left(x_{i}^{2}+y_{i}^{2}\right)-2 \sum_{i=1}^{N} F_{i}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)^{2}+\underbrace{\ln r^{2} \sum_{i=1}^{N} F_{i}\left(x_{i}^{2}+y_{i}^{2}\right)}_{T E R M 8}+ \\
-2 r \ln r^{2} \sum_{i=1}^{N} F_{i}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right) \\
\underbrace{}_{T E R M 6} \\
\end{gathered}
$$

By neglecting all the terms $O\left(\ln r^{2}\right)$, which are negligible with respect to the other terms, the expansion for $w$ is a list of the indicated TERMS 4-8:

$$
\begin{gathered}
\left.w(r, \vartheta)\right|_{r \rightarrow \infty}=\underbrace{r^{2} \ln r^{2} \sum_{i=1}^{N} F_{i}}_{\text {TERM }}+\underbrace{r^{2} \sum_{i=1}^{N} B_{i}}_{\text {TERM }} \underbrace{-2 r \ln r^{2} \sum_{i=1}^{N} F_{i}\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}_{\text {TERM } 6}+ \\
\underbrace{-2 r \sum_{i=1}^{N}\left(B_{i}+F_{i}\right)\left(x_{i} \cos \vartheta+y_{i} \sin \vartheta\right)}_{\text {TERM } 7.1+7.2}+\underbrace{\ln r^{2} \sum_{i=1}^{N} F_{i}\left(x_{i}^{2}+y_{i}^{2}\right)+O\left(\ln r^{2}\right)}_{\text {TERM } 8}
\end{gathered}
$$

Remembering what said before, both first derivatives approching $\pm \infty$ and irregularities in the $w$ trend are forbidden far from the load points. Therefore, looking for monotonic and limited trend when $r \rightarrow \infty$, the conditions to be satisfied are:

$$
\begin{align*}
\lim _{r \rightarrow \infty} D^{1}(w(r, \vartheta)) & \neq \pm \infty  \tag{8.47}\\
\lim _{r \rightarrow \infty} D^{2}(w(r, \vartheta)) & =0^{ \pm} \tag{8.48}
\end{align*}
$$

where $D^{1}$ and $D^{2}$ are differential operators of first and second derivatives respectively. Since the summations appearing in TERMS 4-8 are constants independent of $r$, let us evaluate the functions multiplying the summations to find the boundary conditions at infinity :

TERM 4:

$$
\begin{aligned}
\lim _{r \rightarrow \infty} D^{1}\left(r^{2} \ln r^{2}\right) & =\lim _{r \rightarrow \infty}\left(2 r \ln r^{2}+2 r\right)=\infty \\
\lim _{r \rightarrow \infty} D^{2}\left(r^{2} \ln r^{2}\right) & =\lim _{r \rightarrow \infty}\left(2 \ln r^{2}+8\right)=\infty \neq 0^{ \pm}
\end{aligned}
$$

TERM 5:

$$
\begin{aligned}
\lim _{r \rightarrow \infty} D^{1}\left(r^{2}\right) & =\lim _{r \rightarrow \infty}(2 r)=\infty \\
\lim _{r \rightarrow \infty} D^{2}\left(r^{2}\right) & =\lim _{r \rightarrow \infty}(2)=2 \neq 0^{ \pm}
\end{aligned}
$$

TO BE ELIMINATED

TERM 6:
$\lim _{r \rightarrow \infty} D^{1}\left(-2 r \ln r^{2}\right)=\lim _{r \rightarrow \infty}\left(-2 \ln r^{2}-4\right)=-\infty$
$\lim _{r \rightarrow \infty} D^{2}\left(-2 r \ln r^{2}\right)=\lim _{r \rightarrow \infty}\left(-\frac{4}{r}\right)=0^{-}$
TO BE ELIMINATED

TERM 7:
$\lim _{r \rightarrow \infty} D^{1}(-2 r)=\lim _{r \rightarrow \infty}(-2)=-2 \neq \pm \infty$
$\lim _{r \rightarrow \infty} D^{2}(-2 r)=0$
COMPATIBLE TERM

TERM 8 :

$$
\begin{aligned}
\lim _{r \rightarrow \infty} D^{1}\left(\ln r^{2}\right) & =\lim _{r \rightarrow \infty}\left(\frac{2}{r}\right)=0^{+} \\
\lim _{r \rightarrow \infty} D^{2}\left(\ln r^{2}\right) & =\lim _{r \rightarrow \infty}\left(-\frac{2}{r^{2}}\right)=0^{-}
\end{aligned}
$$

Finally, the boundary conditions at infinity, necessary to compute later the spline coefficients, are given by:

$$
\begin{array}{ll}
\text { To eliminate TERM 4: } & \sum_{i=1}^{N} F_{i}=0 \\
\text { To eliminate TERM 5: } & \sum_{i=1}^{N} B_{i}=0 \\
\text { To eliminate TERM } 6: & \left\{\begin{array}{l}
\sum_{i=1}^{N} F_{i} x_{i}=0 \\
\sum_{i=1}^{N} F_{i} y_{i}=0
\end{array}\right.
\end{array}
$$

Here Eq. 8.49 can be recognized as the discrete force equilibrium equation, whereas Eq. 8.51 assumes the role of discrete moment equilibrium equation. As reported by Harder and Desmarais, the physical significance of Eq. 8.50 is not clear, but that is necessary to satisfy the desired behavior for $w$ at infinity.

After involving the boundary conditions, the transverse deflection of a generic point $(r, \vartheta)$ expressed in polar coordinates is given by:

$$
\begin{align*}
w(r, \vartheta)= & \sum_{i=1}^{N} A_{i}+\sum_{i=1}^{N} B_{i} x_{i}^{2}+\sum_{i=1}^{N} B_{i} y_{i}^{2}+ \\
& -2 r \cos \vartheta \sum_{i=1}^{N} B_{i} x_{i}-2 r \sin \vartheta \sum_{i=1}^{N} B_{i} y_{i}+\sum_{i=1}^{N} F_{i} r_{i}^{2} \ln r_{i}^{2} \tag{8.52}
\end{align*}
$$

The first three terms appearing in the previous equation are constants. Thus, it is advantageous to gather them in a sole constant $a_{0}$. Being more useful coming back to a function dependent on Coordinates $x$ and $y$, also coefficients $a_{1}$ and $a_{2}$ have been introduced:

$$
\begin{gather*}
\sum_{i=1}^{N} A_{i}+\sum_{i=1}^{N} B_{i} x_{i}^{2}+\sum_{i=1}^{N} B_{i} y_{i}^{2}=a_{0} \\
-2 r \cos \vartheta \sum_{i=1}^{N} B_{i} x_{i}=x\left[-2 \sum_{i=1}^{N} B_{i} x_{i}\right]=a_{1} x  \tag{8.53}\\
-2 r \sin \vartheta \sum_{i=1}^{N} B_{i} y_{i}=y\left[-2 \sum_{i=1}^{N} B_{i} y_{i}\right]=a_{2} y
\end{gather*}
$$

In conclusion, Eq. 8.54 describes the final transverse deflection at a generic point $(x, y)$, expressed in scalar coordinates, and lying on the infinite plate, charged by $N$ concentrated loads located in $N$ Load Points as follows:

$$
\begin{equation*}
w(x, y)=a_{0}+a_{1} x+a_{2} y+\sum_{i=1}^{N} F_{i} \underbrace{r_{i}^{2} \ln r_{i}^{2}}_{=K_{i}(x, y)} \tag{8.54}
\end{equation*}
$$

where the definition of $r_{i}$ is retrived by using the coordinates $\left(x_{i}, y_{i}\right)$ of the generic $i^{\text {th }}$ Load Point:

$$
r_{i}^{2}=\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}
$$

At this point the unknowns of the spline are the following:

$$
a_{0}, a_{1}, a_{2}, F_{i} \quad N+3 \text { UnKNOWNS }
$$

Given $N$ Load Points, it is necessary to find $N+3$ equations in order to obtain all the above unknowns, essential for the complete description of surface spline. The first three equations to be considered are the boundary conditions at infinity involving the unknowns $F_{i}$, i.e. Eqs. 8.49 and 8.51. $N$ equations could be added to the set if the displacements at the $N$ load points are known. For that reason, $N$ Structural Points playing the role of $N$ Load Points are considered. Thus, the load's application points will be the Structural Points, for which it is assumed to know all the displacements $w_{j}$, for $j=1,2, \ldots, N$.

By the way, it is noteworthy that these Structural Points will be denoted as Pseudo Points according to the aeroelastic formulation adopted in this work. The reason is just that this particular work discusses about Beams with Structural Nodes on their axis. The displacements on the surface necessary for the spline will be computed by means of the structural formulation and only not to confuse them with the Structural Nodes, it is preferred to address them as Pseudo Points, according to the aeroelastic formulation.

Thus, other $N$ equations are added to the set by writing Eq. 8.55 for all the $N$ Structural Points and assuming their displacements $w$ known.

$$
\begin{equation*}
w_{j}\left(x_{j}, y_{j}\right)=a_{0}+a_{1} x_{j}+a_{2} y_{j}+\sum_{i=1}^{N} F_{i} K_{i j}(x, y) \tag{8.55}
\end{equation*}
$$

where:

$$
\begin{gather*}
r_{i j}^{2}=\left(x_{i}-x_{j}\right)^{2}+\left(y_{i}-y_{j}\right)^{2}  \tag{8.56}\\
K_{i j}=r_{i j}^{2} \ln r_{i j}^{2}
\end{gather*}\left\{\begin{array}{l}
K_{i j}=K_{j i} \\
K_{i j}=0 \quad \text { when } i=j
\end{array}\right.
$$

It could be interesting to demonstrate why $K_{i j}$ is null when $i=j$. In that situation $r_{i j}=0$ and $\lim _{r_{i j} \rightarrow 0} r_{i j}^{2} \ln r_{i j}^{2}=0$. In fact:

$$
\lim _{r_{i j} \rightarrow 0}\left(r_{i j}^{2} \ln r_{i j}^{2}\right)=\lim _{r_{i j} \rightarrow 0} \frac{\ln r_{i j}^{2}}{\frac{1}{r_{i j}^{2}}}=\lim _{r_{i j} \rightarrow 0}\left(-r_{i j}^{2}\right)=0
$$

Now the following column vectors and matrices are introduced in order to build later, in matrix notation, the final system to solve. Its resolution will provide the surface spline unknows:

$$
\begin{gathered}
\{w\}=\left\{\begin{array}{c}
w_{1} \\
w_{2} \\
\vdots \\
w_{N}
\end{array}\right\} \quad\{a\}=\left\{\begin{array}{c}
a_{0} \\
a_{1} \\
a_{2}
\end{array}\right\} \quad\{F\}=\left\{\begin{array}{c}
F_{1} \\
F_{2} \\
\vdots \\
F_{N}
\end{array}\right\} \\
{[R]=\left[\begin{array}{ccc}
1 & x_{1} & y_{1} \\
1 & x_{2} & y_{2} \\
\vdots & \vdots & \vdots \\
1 & x_{N} & y_{N}
\end{array}\right] \quad[K]=\left[\begin{array}{cccc}
K_{11} & K_{12} & \ldots & K_{1 N} \\
K_{21} & K_{22} & \ldots & K_{2 N} \\
\vdots & \vdots & \ddots & \vdots \\
K_{N 1} & K_{N 2} & \ldots & K_{N N}
\end{array}\right]}
\end{gathered}
$$

The entire set of $N+3$ equations can be listed in the system 8.57 ; it contains the boundary conditions 8.49 and 8.51 and Eq. 8.55 written for all $N$ Structural Points.

$$
\left\{\begin{array}{l}
{[0]=[0]\{a\}+[R]^{T}\{F\}}  \tag{8.57}\\
{[w]=[R]\{a\}+[K]\{F\}}
\end{array}\right.
$$

In a matrix notation, it becomes:

$$
\left\{\begin{array}{l}
\{0\}  \tag{8.58}\\
\{w\}
\end{array}\right\}=\left[\begin{array}{cc}
{[0]} & {[R]^{T}} \\
{[R]} & {[K]}
\end{array}\right]\left\{\begin{array}{l}
\{a\} \\
\{F\}
\end{array}\right\}=[G]\left\{\begin{array}{l}
\{a\} \\
\{F\}
\end{array}\right\}
$$

Inverting the relation expressed by Eq. 8.58, it is possible to find the $N+3$ unknowns represented by the components of vectors $\{a\}$ and $\{F\}$.

So far only the Structural Points on the surface have been considered. Once obtained the coefficients necessary to describe the spline, then the Aerodynamic Points on the surface can be taken into account; the scope consists in computing their displacements $w^{\star}$ and slopes, called $w^{\star^{\prime}}$.

The first step concerns one point with coordinates $\left(x_{k}, y_{k}\right)$. That lies on the plate's surface and represents the $k^{t h}$ Aerodynamic Point on which the unknowns are computed. Its interpolated displacement $w_{k}^{\star}\left(x_{k}, y_{k}\right)$ can be obtained by evaluating $w(x, y)$ of Eq. 8.54
at the desired location:

$$
\begin{aligned}
w_{k}^{\star}\left(x_{k}, y_{k}\right) & =a_{0}+a_{1} x_{k}+a_{2} y_{k}+\sum_{i=1}^{N} F_{i} r_{k i}^{2} \ln r_{k i}^{2} \\
& =a_{0}+a_{1} x_{k}+a_{2} y_{k}+\sum_{i=1}^{N} F_{i} \mathcal{K}_{k i}
\end{aligned}
$$

where:

$$
\begin{gather*}
r_{k i}^{2}=\left(x_{k}-x_{i}\right)^{2}+\left(y_{k}-y_{i}\right)^{2} \\
\mathcal{K}_{k i}=r_{k i}^{2} \ln r_{k i}^{2} \tag{8.59}
\end{gather*}
$$

In a matrix notation, it becomes:

$$
\begin{aligned}
w_{k}^{\star}\left(x_{k}, y_{k}\right) & =\left[1, x_{k}, y_{k}, \mathcal{K}_{k 1}, \mathcal{K}_{k 2}, \ldots, \mathcal{K}_{k N}\right]\left\{\begin{array}{c}
\{a\} \\
\{F\}
\end{array}\right\} \\
& =\left[1, x_{k}, y_{k}, \mathcal{K}_{k 1}, \mathcal{K}_{k 2}, \ldots, \mathcal{K}_{k N}\right][G]^{-1}\left\{\begin{array}{c}
\{0\} \\
\{w\}
\end{array}\right\}
\end{aligned}
$$

After having discussed about the deformation of $k^{t h}$ Aerodynamic Point, the formulation focuses on its slope $w^{\star^{\prime}}$, which is defined in Eq. 8.60:

$$
\begin{equation*}
w_{k}^{\star^{\prime}}\left(x_{k}, y_{k}\right)=\left.\frac{\partial w_{k}^{\star}}{\partial x}\right|_{\left(x_{k}, y_{k}\right)} \tag{8.60}
\end{equation*}
$$

According to the definition written above:

$$
\begin{aligned}
w_{k}^{\star^{\prime}}\left(x_{k}, y_{k}\right) & =a_{1}+\left.\sum_{i=1}^{N} F_{i} \frac{\partial\left(r_{i}^{2} \ln r_{i}^{2}\right)}{\partial x}\right|_{\left(x_{k}, y_{k}\right)} \\
& =a_{1}+\sum_{i=1}^{N} F_{i} \mathcal{D} \mathcal{K}_{k i}
\end{aligned}
$$

where:

$$
r_{i}^{2}=\left(x-x_{i}\right)^{2}+\left(y-y_{i}\right)^{2}
$$

The term $\mathcal{D} \mathcal{K}_{k i}$ has to be investigated in order to find a formula similar to $\mathcal{K}_{k i}$ :

$$
\begin{align*}
\mathcal{D K}_{k i}= & \left.\frac{\partial \mathcal{K}_{i}}{\partial x}\right|_{\left(x_{k}, y_{k}\right)}=\left.\frac{\partial\left(r_{i}^{2} \ln r_{i}^{2}\right)}{\partial x}\right|_{\left(x_{k}, y_{k}\right)}= \\
= & \left.\left.\frac{\partial r_{i}^{2}}{\partial x}\right|_{\left(x_{k}, y_{k}\right)} \ln r_{i}^{2}\right|_{\left(x_{k}, y_{k}\right)}+\left.\left.\frac{\partial \ln r_{i}^{2}}{\partial x}\right|_{\left(x_{k}, y_{k}\right)} r_{i}^{2}\right|_{\left(x_{k}, y_{k}\right)}=  \tag{8.61}\\
= & 2\left(x_{k}-x_{i}\right) \ln \left[\left(x_{k}-x_{i}\right)^{2}+\left(y_{k}-y_{i}\right)^{2}\right]+ \\
& +\frac{2}{r_{k i}^{2}}\left(x_{k}-x_{i}\right)\left(r_{k i}^{2}\right)=2\left(x_{k}-x_{i}\right)\left(1+\ln r_{k i}^{2}\right)
\end{align*}
$$

In a matrix notation, the $k^{t h}$ slope becomes:

$$
\begin{aligned}
w_{k}^{\star^{\prime}}\left(x_{k}, y_{k}\right) & \left.=\left[0,1,0, \mathcal{D} \mathcal{K}_{k 1}, \mathcal{D} \mathcal{K}_{k 2}, \ldots, \mathcal{D} \mathcal{K}_{k N}\right] \begin{array}{l}
\{a\} \\
\{F\}
\end{array}\right\} \\
& =\left[0,1,0, \mathcal{D} \mathcal{K}_{k 1}, \mathcal{D} \mathcal{K}_{k 2}, \ldots, \mathcal{D} \mathcal{K}_{k N}\right][G]^{-1}\left\{\begin{array}{c}
\{0\} \\
\{w\}
\end{array}\right\}
\end{aligned}
$$

Following the exposed procedure for all the $N_{A P}$ ( $=$ Number of Aerodynamic Points) locations on the surface, it is possible to create a system in a matrix notation; in particular two new matrices $[\mathcal{D}]$ and $\left[\mathcal{D}^{\star}\right]$ have to be built as done for matrix $[R]$.

$$
\begin{gather*}
\left\{w^{\star}\right\}=\left\{\begin{array}{c}
w_{1 A}^{\star} \\
w_{2 A}^{\star} \\
\vdots \\
w_{N A}^{\star}
\end{array}\right\}=\underbrace{\left[\begin{array}{ccccccc}
1 & x_{1 A} & y_{1 A} & \mathcal{K}_{1 A, 1} & \mathcal{K}_{1 A, 2} & \ldots & \mathcal{K}_{1 A, N} \\
1 & x_{2 A} & y_{2 A} & \mathcal{K}_{2 A, 1} & \mathcal{K}_{2 A, 2} & \ldots & \mathcal{K}_{2 A, N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{N A} & y_{N A} & \mathcal{K}_{N A, 1} & \mathcal{K}_{N A, 2} & \ldots & \mathcal{K}_{N A, N}
\end{array}\right]}_{=\left[D{ }^{\star}\right]}[G]^{-1}\left\{\left\{\begin{array}{l}
\{0\} \\
\{w\}
\end{array}\right\}\right.  \tag{8.62}\\
\left\{w^{\star^{\prime}}\right\}=\left\{\begin{array}{c}
w_{1 A}^{\star^{\prime}} \\
w_{2 A}^{\star^{\prime}} \\
\vdots \\
w_{N A}^{\star^{\prime}}
\end{array}\right\}=\underbrace{\left[\begin{array}{ccccccc}
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{1 A, 1} & \mathcal{D} \mathcal{K}_{1 A, 2} & \ldots & \mathcal{D} \mathcal{K}_{1 A, N} \\
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{2 A, 1} & \mathcal{D} \mathcal{K}_{2 A, 2} & \ldots & \mathcal{D} \mathcal{K}_{2 A, N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{N A, 1} & \mathcal{D} \mathcal{K}_{N A, 2} & \ldots & \mathcal{D} \mathcal{K}_{N A, N}
\end{array}\right]}[G]^{-1}\left\{\left\{\begin{array}{l}
\{0\} \\
\{w\}
\end{array}\right\}\right. \tag{8.63}
\end{gather*}
$$

Note that the matrices $[\mathcal{D}]$ and $\left[\mathcal{D}^{\star}\right]$ have dimension $N_{A P} \times(N+3)$. Observing that the first three rows of the spline coefficients vector are zeros, it is useful to eliminate the first three columns of the matrix $[G]^{-1}$ without changing the result. Defining $\left[G^{\star}\right]^{-1}$ (the matrix $[G]^{-1}$ with the first three columns eliminated) and being $(N+3) \times N$ its new dimension, Eqs. 8.62 and 8.63 can be rewritten as functions of input data:

$$
\begin{align*}
\left\{w^{\star}\right\}= & \left\{\begin{array}{c}
w_{1 A}^{\star} \\
w_{2 A}^{\star} \\
\vdots \\
w_{N A}^{\star}
\end{array}\right\}=\left[\mathcal{D}^{\star}\right]\left[G^{\star}\right]^{-1}\{w\}=\left[T^{\star}\right]\{w\}  \tag{8.64}\\
\left\{w^{\star^{\prime}}\right\} & =\left\{\begin{array}{c}
w_{1 A}^{\star^{\prime}} \\
w_{2 A}^{\star^{\prime}} \\
\vdots \\
w_{N A}^{\star^{\prime}}
\end{array}\right\}=[\mathcal{D}]\left[G^{\star}\right]^{-1}\{w\}=[T]\{w\} \tag{8.65}
\end{align*}
$$

where $\left[T^{\star}\right]$ and $[T]$ result from the matrix multiplications $\left[\mathcal{D}^{\star}\right]\left[G^{\star}\right]^{-1}$ and $[\mathcal{D}]\left[G^{\star}\right]^{-1}$ respectively.

Now, the Surface Spline' examination is completed. As a summary, it results very useful to list all the operations performing the computation of the desired output and to note some important remarks about the IPS method.

## a. INPUT

- $x_{i}, y_{i}$ : Coordinates along the beam axis of $N$ Structural Points, called later Pseudo Points ;
- $w_{i} \quad$ : Transverse Displacements of $N$ Structural Points ;
- $x_{k}, y_{k}$ : Coordinates along the beam axis of $N_{A P}$ Aerodynamic Points.


## b. COMPUTATION

- Computation of $r_{i j}$ and $K_{i j}$ by means of Eq. 8.56:

$$
\begin{gathered}
r_{i j}^{2}=\left(x_{i}-x_{j}\right)^{2}+\left(y_{i}-y_{j}\right)^{2} \\
K_{i j}=r_{i j}^{2} \ln r_{i j}^{2} \quad \forall i, j \text { on Structural Points }
\end{gathered}
$$

- Construction of matrices $[R],[R]^{T}$ and $[K]$, necessary to matrices $[G]$ and $[G]^{-1}$ ;
- Optional Resolution of linear system 8.58 , with the purpose of obtaining the spline coefficients vectors $\{a\}$ and $\{F\}$;
- Computation of $r_{k i}, \mathcal{K}_{k i}$ and $\mathcal{D} \mathcal{K}_{k i}$ by means of Eqs. 8.59 and 8.61 :

$$
\begin{array}{cc}
r_{k i}^{2}=\left(x_{k}-x_{i}\right)^{2}+\left(y_{k}-y_{i}\right)^{2} \\
\mathcal{K}_{k i}=r_{k i}^{2} \ln r_{k i}^{2} & \forall k \text { on Aerodynamic Points } \\
\mathcal{D} \mathcal{K}_{k i}=2\left(x_{k}-x_{i}\right)\left(1+\ln r_{k i}^{2}\right)
\end{array}
$$

- Construction of matrices $\left[\mathcal{D}^{\star}\right]$ and $[\mathcal{D}]$;
- Construction of matrices $\left[T^{\star}\right]$ and $[T]$.


## c. OUTPUT

- Computation of $\left\{w^{\star}\right\}$ by performing Eq. 8.64 ;
- Computation of $\left\{w^{\star^{\prime}}\right\}$ by performing Eq. 8.65.

The solutions of $\left\{w^{\star}\right\}$ and $\left\{w^{\star^{\prime}}\right\}$ exist only if the matrix $[G]$ is non-singular. The singularity in matrix $[G]$ occurs when:

- all Structural Points are aligned along a line. This is obvious since a line fails to define a plane.
- two or more than two Structural Points share the same $x$ and $y$ location on the spline plane.

To perform the IPS method, it is required that all Structural Points and Aerodynamic Points are located on the same plane, called "spline plane". Normally, the plane of the lifting surface (or the mean one of the wing-like component) is selected as the spline plane. However, for Structural Points located in 3D space, they may not be necessarily located on the plane. In this case, it is required to project the Structural Points onto the plane along the normal direction of the spline plane. This can be done by transforming the Structural Point locations to a local coordinate system whose $x-y$ plane coincide with the spline plane.

According to the boundary conditions imposed at infinity, linear extrapolation occurs only if the Aerodynamic Points are far from the domain of the Structural Points. Otherwise, distorsions or oscillations may appear in the extrapolated regions.

The last conclusion is that the IPS method is a scalar operator. For a given set of displacements along one direction, the IPS method does not recover displacements along other directions. However, for this purpose the IPS can be applied in the same way as that of the normal displacements. The result would be an IPS generalized spline method. Here this solution is not necessary, since both the structural and aerodynamic analyses are linear. That means that the eventually displacements along directions different from the normal one of the spline plane are negligible.

## Chapter 9

## Aeroelastic Formulation

### 9.1 Introduction

This is the main chapter of the Aeroelastic Part II. In fact, it reports the detailed description of the aeroelastic beam model. At this point, the fundamental notation and concepts are well known about both the hierarchical structural element and the general logic introduced in chapter 8. The following formulation will make large use of matrices and vector since it represents an easier way to face problems as complicated as the aeroelastic ones. Moreover, this approach has already been used successfully in this field. In addition, it will result well oriented to the implementation into a numerical code, such as this case.

### 9.2 Assumptions

As a summary, it is very useful to list all the assumptions taken into account for the following document. They resume both the structural model and the aeroelastic notation:

- Assumption \# 1

The wing system is divided into $N_{W S}$ trapezoidal Wing Segments, according to the same logic used in other previous aeroelastic works (see Demasi and Livne AIAA J 2009)

- Assumption \# 2

Each Wing Segment can have only one beam (which is divided into elements according to the finite element method). For example, if a planar wing is considered then only a straight beam will be used.

- Assumption \# 3

Each beam element must be entirely contained in a plane perpendicular to the $x$ axis (wind direction). This means that each structural element is always perpendicular to the $x$ axis

- Assumption \# 4

The origin of the coordinate system in one section does not necessarily coincide with the centroid of the cross-section

- Assumption \# 5

The origin of the coordinate system in one section is not necessarily included in the cross-section area. It can be located outside it. This is very useful to analyze wings with sweep angles or joints.

### 9.3 Displacement Model

As said in chapter 3, the structural model analyzed so far cannot yet analyze structures generally oriented in 3D space. In fact the nuclei are obtained in the local coordinate system of the element. For the general purpose of this work, it is necessary to extend the formulation to the global coordinate system, which is defined in chapter 2, according to the aeroelastic notation introduced and used from now on. The problem consists in a typical transformation of coordinates by means of orthogonal matrices.

Let us consider the Wing Segment $S$ and resume the Carrera Unified Formulation (CUF). According to Assumption \# 3, the axis of each beam will always be perpendicular to the $x$ direction. This implies, according to Fig. 8.1, that the local beam axis is parallel to $y^{S}$. The cross-section coordinates are $x^{S}$ and $z^{S}$. Adding the superscript ${ }^{S}$ and the subscript loc respect to the previous notation, the displacement model (in the Wing Segment local coordinate system) is the following (static case):

$$
\begin{equation*}
\boldsymbol{u}_{\mathrm{loc}}^{S}\left(x^{S}, y^{S}, z^{S}\right)=F_{\tau}\left(x^{S}, z^{S}\right) \boldsymbol{u}_{\tau \operatorname{loc}}^{S}\left(y^{S}\right) \tag{9.1}
\end{equation*}
$$

where

$$
\begin{aligned}
\boldsymbol{u}_{\text {loc }}^{S}\left(x^{S}, y^{S}, z^{S}\right) & =\left\{\begin{array}{l}
u_{x \text { loc }}\left(x^{S}, y^{S}, z^{S}\right) \\
u_{y \text { loc }}\left(x^{S}, y^{S}, z^{S}\right) \\
u_{z \text { loc }}\left(x^{S}, y^{S}, z^{S}\right)
\end{array}\right\} \\
\boldsymbol{u}_{\tau \operatorname{loc}}^{S}\left(y^{S}\right) & =\left\{\begin{array}{l}
u_{\tau x \text { loc }}^{S}\left(y^{S}\right) \\
u_{\tau \text { loc }}^{S}\left(y^{S}\right) \\
u_{\tau z \text { loc }}^{S}\left(y^{S}\right)
\end{array}\right\}
\end{aligned}
$$

are the vectors of displacements and generalized displacements respectively, evaluated at the generic point of the Wing Segment $S$ with local coordinates $\left(x^{S}, y^{S}, z^{S}\right)$. The FEM approximation, by means of the introduction of Shape Functions, is:

$$
\begin{equation*}
\boldsymbol{u}_{\tau \mathrm{loc}}^{S}\left(y^{S}\right)=N_{i}\left(y^{S}\right) \boldsymbol{q}_{\tau i \mathrm{loc}}^{S} \tag{9.2}
\end{equation*}
$$

where the vector

$$
\boldsymbol{q}_{\tau i}^{S} \operatorname{loc}=\left\{\begin{array}{l}
q_{u_{x} \tau i \operatorname{loc}} \\
q_{u_{y}} \tau i \operatorname{loc} \\
q_{u_{z} \tau i \operatorname{loc}}
\end{array}\right\}
$$

is the vector of nodal degrees of freedom of the finite element corresponding to the local coordinate $y^{S}$. The superscript ${ }^{S}$ is used to denote that these elements have axis parallel to the local $y^{S}$ axis. By adopting the Principle of Virtual Displacements (PVD), the strain energy is defined as:

$$
\begin{equation*}
\delta L_{i}=\int_{l} \int_{\Omega}\left(\delta \varepsilon_{n}^{T} \boldsymbol{\sigma}_{n}+\delta \varepsilon_{p}^{T} \boldsymbol{\sigma}_{p}\right) \mathrm{d} x^{S} \mathrm{~d} z^{S} \mathrm{~d} y^{S} \tag{9.3}
\end{equation*}
$$

where $T$ has the meaning of transposition. Using the procedure presented in chapter 3 such a strain energy can be written as function of the fundamental nucleus, the nodal degrees of freedom and their virtual variations:

$$
\begin{equation*}
\delta L_{i}=\delta \boldsymbol{q}_{\tau i}^{S} \frac{T \mathrm{loc}}{} \boldsymbol{K}_{\mathrm{loc}}^{\tau \operatorname{sij} S} \boldsymbol{q}_{s j \mathrm{loc}}^{S} \tag{9.4}
\end{equation*}
$$

This presentation is in general valid in local coordinate system, whose $y^{S}$ axis is parallel to the element beam axis. It is convenient to use the same coordinate system valid for the $s$-th trapezoidal Wing Segment described in Fig. 8.1. However, the aim of the model proposed is to analyze beam-like structures generally oriented in the 3D space; to do that, before the assembly, the FE matrices must be rotated to impose the compatibility of the displacements expressed in global coordinates. Let us consider known the orientations of both the Wing Segment's local and global coordinate systems. The unit vectors of the former system are expressed as $\boldsymbol{i}^{S}, \boldsymbol{j}^{S}$ and $\boldsymbol{k}^{S}$, while $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ refer to the global coordinates (see Fig. 8.2). The former vectors are expressed in the global reference system by intoducing 9 coefficients:

$$
\begin{align*}
\boldsymbol{i}^{S} & =e_{11}^{S} \boldsymbol{i}+e_{12}^{S} \boldsymbol{j}+e_{13}^{S} \boldsymbol{k} \\
\boldsymbol{j}^{S} & =e_{21}^{S} \boldsymbol{i}+e_{22}^{S} \boldsymbol{j}+e_{23}^{S} \boldsymbol{k}  \tag{9.5}\\
\boldsymbol{k}^{S} & =e_{31}^{S} \boldsymbol{i}+e_{32}^{S} \boldsymbol{j}+e_{33}^{S} \boldsymbol{k}
\end{align*}
$$

Looking at Eq. 9.5 the coefficients $e_{11}^{S}, e_{12}^{S}$ and $e_{13}^{S}$ are the global coordinates of the local unit vector $\boldsymbol{i}^{S}$. Instead, $e_{21}^{S}, e_{22}^{S}$ and $e_{23}^{S}$ are the global coordinates of the local unit vector $\boldsymbol{j}^{S}$. In the same way, $e_{31}^{S}, e_{32}^{S}$ and $e_{33}^{S}$ are the global coordinates of the local unit vector $\boldsymbol{k}^{S}$.

The inverse of relation 9.5 relates the global unit vectors to the local unit vectors by means of other 9 coefficients:

$$
\begin{align*}
\boldsymbol{i} & =\bar{e}_{11}^{S} \boldsymbol{i}^{S}+\bar{e}_{12}^{S} \boldsymbol{j}^{S}+\bar{e}_{13}^{S} \boldsymbol{k}^{S} \\
\boldsymbol{j} & =\bar{e}_{21}^{S} \boldsymbol{i}^{S}+\bar{e}_{22}^{S} \boldsymbol{j}^{S}+\bar{e}_{23}^{S} \boldsymbol{k}^{S}  \tag{9.6}\\
\boldsymbol{k} & =\bar{e}_{31}^{S} \boldsymbol{i}^{S}+\bar{e}_{32}^{S} \boldsymbol{j}^{S}+\bar{e}_{33}^{S} \boldsymbol{k}^{S}
\end{align*}
$$

According to the previous logic, the coefficients $\bar{e}_{11}^{S}, \bar{e}_{12}^{S}$ and $\bar{e}_{13}^{S}$ are the local coordinates of the global unit vector $\boldsymbol{i}$, and so on. Now, the following operations are listed in order to obtain the final transformation equations relating the systems. Let us write $\boldsymbol{q}_{s j \text { loc }}^{S}$ from local coordinates to global coordinates, by using Eq. 9.5:

$$
\begin{aligned}
\boldsymbol{q}_{s j \operatorname{loc}}^{S} & =q_{s j \operatorname{loc} x}^{S} \boldsymbol{i}^{S}+q_{s j \operatorname{loc} y}^{S} \boldsymbol{j}^{S}+q_{s j \operatorname{loc} z}^{S} \boldsymbol{k}^{S} \\
& =q_{s j \operatorname{loc} x}^{S}\left(e_{11}^{S} \boldsymbol{i}+e_{12}^{S} \boldsymbol{j}+e_{13}^{S} \boldsymbol{k}\right)+ \\
& +q_{s j \operatorname{loc} y}^{S}\left(e_{21}^{S} \boldsymbol{i}+e_{22}^{S} \boldsymbol{j}+e_{23}^{S} \boldsymbol{k}\right)+ \\
& +q_{s j \operatorname{loc} z}^{S}\left(e_{31}^{S} \boldsymbol{i}+e_{32}^{S} \boldsymbol{j}+e_{33}^{S} \boldsymbol{k}\right)
\end{aligned}
$$

The same vector can be written in the global coordinates; in this way, the subscript loc is eliminated and the vector is called again as $\boldsymbol{q}_{s j}^{S}$ :

$$
\begin{aligned}
\boldsymbol{q}_{s j}^{S} & =\left(q_{s j \operatorname{loc} x}^{S} e_{11}^{S}+q_{s j \operatorname{loc} y}^{S} e_{21}^{S}+q_{s j \operatorname{loc} z}^{S} e_{31}^{S}\right) \boldsymbol{i}+ \\
& +\left(q_{s j \operatorname{loc} x}^{S} e_{12}^{S}+q_{s j \operatorname{loc} y}^{S} e_{22}^{S}+q_{s j \operatorname{loc} z}^{S} e_{32}^{S}\right) \boldsymbol{j}+ \\
& +\left(q_{s j \operatorname{loc} x}^{S} e_{13}^{S}+q_{s j \operatorname{loc} y}^{S} e_{23}^{S}+q_{s j \operatorname{loc} z}^{S} e_{33}^{S}\right) \boldsymbol{k}
\end{aligned}
$$

In matrix notation, it becomes:

$$
\begin{equation*}
\boldsymbol{q}_{s j}^{S}=\boldsymbol{e}^{S T} \cdot \boldsymbol{q}_{s j \text { loc }}^{S} \tag{9.7}
\end{equation*}
$$

where

$$
\boldsymbol{e}^{S}=\left[\begin{array}{ccc}
e_{11}^{S} & e_{12}^{S} & e_{13}^{S} \\
e_{21}^{S} & e_{22}^{S} & e_{23}^{S} \\
e_{31}^{S} & e_{32}^{S} & e_{33}^{S}
\end{array}\right]
$$

In the same way, let us write $\boldsymbol{q}_{s j}^{S}$ from global coordinates to local coordinates, by resuming Eq. 9.6:

$$
\begin{aligned}
\boldsymbol{q}_{s j}^{S} & =q_{s j x}^{S} \boldsymbol{i}+q_{s j y}^{S} \boldsymbol{j}+q_{s j z}^{S} \boldsymbol{k} \\
& =q_{s j x}^{S}\left(\bar{e}_{11}^{S} \boldsymbol{i}^{S}+\bar{e}_{12}^{S} \boldsymbol{j}^{S}+\bar{e}_{13}^{S} \boldsymbol{k}^{S}\right)+ \\
& +q_{s j y}^{S}\left(\bar{e}_{21}^{S} \boldsymbol{i}^{S}+\bar{e}_{22}^{S} \boldsymbol{j}^{S}+\bar{e}_{23}^{S} \boldsymbol{k}^{S}\right)+ \\
& +q_{s j z}^{S}\left(\bar{e}_{31}^{S} \boldsymbol{i}^{S}+\bar{e}_{32}^{S} \boldsymbol{j}^{S}+\bar{e}_{33}^{S} \boldsymbol{k}^{S}\right)
\end{aligned}
$$

As done previously, the obtainment of the same vector in the local coordinates is completed. Then the subscript loc is added and the vector is called again as $\boldsymbol{q}_{s j \text { loc }}^{S}$ :

$$
\begin{aligned}
\boldsymbol{q}_{s j \text { loc }}^{S} & =\left(q_{s j x}^{S} \bar{e}_{11}^{S}+q_{s j y}^{S} \bar{e}_{21}^{S}+q_{s j z}^{S} \bar{e}_{31}^{S}\right) \boldsymbol{i}^{S}+ \\
& +\left(q_{s j x}^{S} \bar{e}_{12}^{S}+q_{s j y}^{S} \bar{e}_{22}^{S}+q_{s j z}^{S} \bar{e}_{32}^{S}\right) \boldsymbol{j}^{S}+ \\
& +\left(q_{s j x}^{S} \bar{e}_{13}^{S}+q_{s j y}^{S} \bar{e}_{23}^{S}+q_{s j z}^{S} \bar{e}_{33}^{S}\right) \boldsymbol{k}^{S}
\end{aligned}
$$

In matrix notation, it becomes:

$$
\begin{equation*}
\boldsymbol{q}_{s j \text { loc }}^{S}=\overline{\boldsymbol{e}}^{S T} \cdot \boldsymbol{q}_{s j}^{S} \tag{9.8}
\end{equation*}
$$

where

$$
\overline{\boldsymbol{e}}^{S}=\left[\begin{array}{ccc}
\bar{e}_{11}^{S} & \bar{e}_{12}^{S} & \bar{e}_{13}^{S} \\
\bar{e}_{21}^{S} & \bar{e}_{22}^{S} & \bar{e}_{23}^{S} \\
\bar{e}_{31}^{S} & \bar{e}_{32}^{S} & \bar{e}_{33}^{S}
\end{array}\right]
$$

By coupling Eqs. 9.7 and 9.8, the result is a first relationship connecting the two transformation matrices:

$$
\begin{equation*}
\boldsymbol{q}_{s j \text { loc }}^{S}=\overline{\boldsymbol{e}}^{S T} \boldsymbol{e}^{S T} \cdot \boldsymbol{q}_{s j}^{S}=\left[\overline{\boldsymbol{e}}^{S} \boldsymbol{e}^{S}\right]^{T} \cdot \boldsymbol{q}_{\mathrm{sj} \text { loc }}^{S} \tag{9.9}
\end{equation*}
$$

As a consequence:

$$
\left[\bar{e}^{S} \boldsymbol{e}^{S}\right]^{T}=\boldsymbol{I}=\boldsymbol{I}^{T}
$$

where $\boldsymbol{I}$ is the identity matrix. Finally:

$$
\begin{equation*}
e^{S}=\left[\bar{e}^{S}\right]^{-1} \quad \text { or } \quad \bar{e}^{S}=\left[e^{S}\right]^{-1} \tag{9.10}
\end{equation*}
$$

The objective is to write a final equation between the inverse transformation matrix and the transposed one. To do that, a mathematic expedient is utilized; let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two unit vectors related by the expression:

$$
\boldsymbol{a}=\overline{\boldsymbol{e}}^{S} \cdot \boldsymbol{b}
$$

Because of the unit module of both the vectors, the multiplication by $\boldsymbol{a}^{T}$ is performed, obtaining:

$$
\boldsymbol{a}^{T} \cdot \boldsymbol{a}=\boldsymbol{b}^{T} \cdot \overline{\boldsymbol{e}}^{S T} \overline{\boldsymbol{e}}^{S} \cdot \boldsymbol{b}
$$

As a consequence:

$$
\bar{e}^{S T} \bar{e}^{S}=\boldsymbol{I}
$$

where $\boldsymbol{I}$ is the identity matrix. Finally:

$$
\bar{e}^{S T}=\left[\bar{e}^{S}\right]^{-1}
$$

Proceeding in the same way for $\boldsymbol{e}^{S}$, then it is demonstrated that the transformation matrices are orthogonal. In conclusion, Eqs. 9.11 and 9.12 permit to express any vector of nodal local degrees of freedom in global coordinates and vice versa:

$$
\begin{align*}
& \boldsymbol{q}_{s j \text { loc }}^{S}=e^{S} \cdot \boldsymbol{q}_{s j}^{S}  \tag{9.11}\\
& \boldsymbol{q}_{s j}^{S}=\bar{e}^{S} \cdot \boldsymbol{q}_{s j \text { loc }}^{S} \tag{9.12}
\end{align*}
$$

Note that all finite elements in one Wing Segment will have the same transformation matrix, since the local coordinate system is distinctive of a the Wing Segment. Equations 9.4 and 9.11 imply that for all finite elements contained in Wing Segment $S$ the fundamental nuclei must be symbolically changed in global reference as follows:

$$
\begin{equation*}
\delta L_{i}=\delta \boldsymbol{q}_{\tau i}^{S T} \frac{\mathrm{loc}}{} \boldsymbol{K}_{\mathrm{loc}}^{\tau s i j S} \boldsymbol{q}_{s j \mathrm{loc}}^{S}=\delta \boldsymbol{q}_{\tau i}^{S T}\left[\boldsymbol{e}^{S T} \cdot \boldsymbol{K}_{\mathrm{loc}}^{\tau s i j S} \cdot \boldsymbol{e}^{S}\right] \boldsymbol{q}_{s j}^{S} \tag{9.13}
\end{equation*}
$$

Therefore, within Wing Segment $S$, the fundamental nuclei must be modified as follows:

$$
\begin{equation*}
\boldsymbol{K}^{\tau s i j S}=\left[e^{S T} \cdot \boldsymbol{K}_{\mathrm{loc}}^{\tau s i j S} \cdot \boldsymbol{e}^{S}\right] \tag{9.14}
\end{equation*}
$$

where the orthogonal transformation matrices are involved. The assembly phase among beam elements of different Wing Segments will be performed in the classical way, summing up the stiffness terms corresponding to the common nodes. Via this generalization, it will be possible to study accurately beam-like structures, generally oriented in the tridimensional space, and unconventional too.

### 9.4 Aeroelastic Model

As explained before, the interaction between structures and aerodynamics represents a delicate issue. The coupling of these fields focuses on relating the displacements and slopes at the aerodynamic Load and Control Points (see Section 8.3), respectively, to the structural degrees on freedom. Section 8.4 indicates the spline methods as the mathematical tools able to resolve this matter.

### 9.4.1 Introduction of Pseudo-Structural Points

In this work, although the Finite Element Model is 1D, it is not performed the Beam Spline method, but indeed the Infinite Plate Spline method (IPS). The reason is due to the hierarchical element, which is quite thorough in the prediction of displacements on a set of desired Points not necessarily coincident with the actual FEM nodes and not even located on the element axis. So, the user defines a series of aeroelastic points on the plane of the Wing Segment, whose displacements are computed by means of the structural formulation. These deflections are utilized as input data in order to compute slopes and displacements on other desired points (aerodynamic Loads Points and Control Points) via IPS method. The points forming the set are denoted as Pseudo-Structural Points, precisely because they have the meaning of structural points (the spline surface is treated as a plate by IPS method). The adjective Pseudo is adopted to not confuse them with the Structural Nodes of the beam elements.

For each Wing Segment the user must specify these Pseudo-Structural Points by giving the coordinates in the global reference system. The vector $\boldsymbol{x}$ contains the global coordinates of these points. Notice that if there are common points between two adjacent Wing Segments they will be not duplicated.
Suppose that all finite nodal degrees of freedom of all beam elements are known. They are necessarily considered in the global coordinate system, according to the explanation of Section 9.3. Let us consider the $i^{t h}$ node. The nodal degrees of freedom depend on the theory expansion order used in CUF. Let's indicate these quantities in global coordinates with $\boldsymbol{q}_{i}$. For example, if the expansion is composed of three terms, $\boldsymbol{q}_{i}$ contains the following global degrees of freedom:

$$
\boldsymbol{q}_{i}=\left[\begin{array}{lllllllll}
q_{u_{x} 1 i} & q_{u_{y} 1 i} & q_{u_{z} 1 i} & q_{u_{x} 2 i} & q_{u_{y} 2 i} & q_{u_{z} 2 i} & q_{u_{x} 3 i} & q_{u_{y} 3 i} & q_{u_{z} 3 i}
\end{array}\right]^{T}
$$

If $\boldsymbol{q}$ indicates the vector of nodal degrees of freedom (in global coordinate system) of all nodes on the beams (not including the "extra" structural points), it is possible to define a matrix which depends on the expansion order and IDs of the nodes contained in a Wing Segment. This matrix, addressed as $\boldsymbol{I}^{S}$, will select the nodal displacements corresponding to the Wing Segment $S$ only, among the entiree set of Structural Nodes. The vector $\boldsymbol{q}^{S}$ indicates the nodal degrees of freedom (in global coordinates) of all the Structural Nodes of the elements (along the beam axis $y^{S}$ ) corresponding to Wing Segment $S$. From the previous definitions it follows:

$$
\begin{equation*}
\boldsymbol{q}^{S}=\boldsymbol{I}^{S} \cdot \boldsymbol{q} \tag{9.15}
\end{equation*}
$$

As said above, the final purpose is to compute the displacements and slopes of Loads Points and Control Points by means of the input local displacements at Pseudo-Structural Points. The latter can be computed by the CUF just knowing the local coordinates of these Pseudo-Structural Points on the Wing Segment and the nodal degrees of freedom of the corresponding finite element. Thus, let us proceed to find these coordinates, starting from the global ones.

It is now possible to convert the vector of degrees of freedom in local coordinates using a formula similar to Eq. 9.11. Let $\boldsymbol{E}_{q}^{S}$ to be defined as the matrix that transforms the degrees of freedom from global to local coordinates. It is a block diagonal matrix containing the transformation matrix $\boldsymbol{e}^{S}$, along the main diagonal, for each degree of freedom of Structural Nodes corresponding to the Wing Segment. By resuming Eq. 9.15, thus:

$$
\begin{equation*}
\boldsymbol{q}_{\mathrm{loc}}^{S}=\boldsymbol{E}_{q}^{S} \cdot \boldsymbol{q}^{S}=\boldsymbol{E}_{q}^{S} \cdot \boldsymbol{I}^{S} \cdot \boldsymbol{q} \tag{9.16}
\end{equation*}
$$

As done for the Structural Nodes, it is possible to extract the coordinates of the PseudoStructural Points located on the Wing Segment $S$ and define the vector $\boldsymbol{x}^{S}$, which contains their coordinates, written in the global reference system:

$$
\begin{equation*}
\boldsymbol{x}^{S}=\boldsymbol{J}^{S} \cdot \boldsymbol{x} \tag{9.17}
\end{equation*}
$$

where the matrix $\boldsymbol{J}^{S}$ needs in input only the IDs of Pseudo-Structural Points positioned on the reference surface of the considered Wing Segment. Now, the local coordinates of these points have to be computed; at first it is necessary to focus on point $2^{S}$, positioned on the reference surface of Wing Segment $S$, according to the notation of numeration described in chapter 8. Its global coordinates are addressed as $x_{2^{S}}, y_{2^{S}}$ and $z_{2^{S}}$. Moreover, the coordinates of each Pseudo-Structural Point on the Wing Segment $S$ expressed in the local reference system are determined by subtracting the global coordinates of the point $2^{S}$ to their global coordinates and multiplying the result by the already defined matrix $\boldsymbol{e}^{S}$. For this purpose, it is introduced the vector $\boldsymbol{x}_{2^{S}}$ (which has dimension $3 N_{P S}^{S} \times 1$, where $N_{P S}^{S}$ is the number of Pseudo-Structural Points of Wing Segment $S$ ):

$$
\boldsymbol{x}_{2^{S}}=\left[\begin{array}{lllllll}
x_{2 S} & y_{2 S} & z_{2 S} & \ldots & x_{2 S} & y_{2 S} & z_{2 S} \tag{9.18}
\end{array}\right]^{T}
$$

and the matrix $\boldsymbol{E}^{S}$ which is a block diagonal matrix, where the transformation matrix $\boldsymbol{e}^{S}$ is repeated for all the Pseudo-Structural Points of each Wing Segment; its construction is similar to matrix $\boldsymbol{E}_{q}^{S}$. Notice that the dimension of $\boldsymbol{E}^{S}$ is $3 N_{P S}^{S} \times 3 N_{P S}^{S}$. The main point is that the global coordinates of a generic point minus the global coordinates of point $2^{S}$ are the coordinates of the generic point with respect to a reference system parallel to the global one and with the point $2^{S}$ as its origin. That concerns the translation; whereas, the rotation between coordinate systems is performed by matrix $\boldsymbol{E}^{S}$. Finally, the coordinates of the Pseudo-Structural Points on Wing Segment $S$ (in the undeformed configuration with zero angle of attack) can be expressed in the local coordinate system thanks to Eq. 9.19:

$$
\begin{equation*}
\boldsymbol{x}_{\mathrm{loc}}^{S}=\boldsymbol{E}^{S} \cdot\left[\boldsymbol{x}^{S}-\boldsymbol{x}_{2^{S}}\right]=\boldsymbol{E}^{S} \cdot\left[\boldsymbol{J}^{S} \cdot \boldsymbol{x}-\boldsymbol{x}_{2^{S}}\right] \tag{9.19}
\end{equation*}
$$

The purpose of writing the local coordinates of Pseudo-Structural Points has been reached. Note that all these points have zero local $z^{S}$ coordinate because of the zero angle of attack condition. However, for the same reason, also all the Structural Nodes have zero local $z^{S}$ coordinate. Although in general Pseudo-Structural and Structural Nodes are different entities, some of them may be equivalent. In fact it is possible that a structural node were a Pseudo-Structural Point.

To utilize the Finite Element formulation, it is mandatory to individualize the corresponding finite element for each Pseudo-Structural Point. The parameter to be analyzed is the local $y^{S}$ coordinate, which is extracted from vector $\boldsymbol{x}_{\mathrm{loc}}^{S}$. In fact, using that value it is possible to "assign" that Pseudo-Structural Point to a particular beam element on Wing Segment $S$. If the local $y^{S}$ coordinate is exactly coincident with the $y^{S}$ coordinate of a node that two adjacent beam elements have in common, both the elements can be used; however only one has to be selected and the choice is arbitrary, since the displacements are continuous functions. Everything is expressed in local coordinates and so the FEM equation 9.20 can be used to calculate the local displacements:

$$
\begin{equation*}
\boldsymbol{u}_{\mathrm{loc}}^{S}\left(x^{S}, y^{S}, z^{S}\right)=F_{\tau}\left(x^{S}, z^{S}\right) \boldsymbol{u}_{\tau \mathrm{loc}}^{S}\left(y^{S}\right)=F_{\tau}\left(x^{S}, z^{S}\right) N_{i}\left(y^{S}\right) \boldsymbol{q}_{\tau i \mathrm{loc}}^{S} \tag{9.20}
\end{equation*}
$$

where the counter $\tau$ goes from 1 to $N_{u}=N_{u}(N)$. The number $N_{u}$ indicates how many terms in the CUF expansion have to be considered and depends only on the order of the theory $N$. Instead, the counter $i$ in the Einstein notation goes from 1 to $N_{N}$, i.e. the total number of nodes of the element ( 2,3 or 4 ), on the nodes of the element corresponding to the Pseudo-Structural Point.

The generic Pseudo-Structural Point has zero angle of attack and so $z^{S}=0$. Thus:

$$
\begin{equation*}
\boldsymbol{u}_{\mathrm{loc}}^{S}\left(x^{S}, y^{S}, 0\right)=F_{\tau}\left(x^{S}, 0\right) \boldsymbol{u}_{\tau \mathrm{loc}}^{S}\left(y^{S}\right)=F_{\tau}\left(x^{S}, z^{S}\right) N_{i}\left(y^{S}\right) \boldsymbol{q}_{\tau i \mathrm{loc}}^{S} \tag{9.21}
\end{equation*}
$$

The same operations can be repeated for all the Pseudo-Structural Points of Wing Segment $S$. Resuming Eq. 9.16, this means that for each Wing Segment it is possible to define a matrix $\boldsymbol{Y}^{S}$ which relates the vector of nodal degrees of freedom in local coordinates of Wing Segment $S$ with the displacements (in local coordinates) of all the Pseudo-Structural Points:

$$
\begin{equation*}
\tilde{\boldsymbol{u}}_{\mathrm{loc}}^{S}=\boldsymbol{Y}^{S} \cdot \boldsymbol{q}_{\mathrm{loc}}^{S}=\boldsymbol{Y}^{S} \cdot \boldsymbol{E}_{q}^{S} \cdot \boldsymbol{I}^{S} \cdot \boldsymbol{q} \tag{9.22}
\end{equation*}
$$

The creation of matrix $\boldsymbol{Y}^{S}$ is an important point of the discussion. As a matter of fact, it can be built in two ways: the first consists in putting the terms corresponding to the Pseudo-Structural Points directly in the right positions. The second manner is to obtain the matrix by multiplication between tho matrices. In that case, the first matrix would relate the displacements of Pseudo-Structural Points to the vector of generalized displacements $\boldsymbol{u}_{\tau \text { loc }}^{S}\left(y^{S}\right)$. This matrix needs the values of functions approximating the section and thus the coordinates $\left(x^{S}, z^{S}\right)$ of Pseudo-Structural Points. So, the second matrix would connect the generalized displacements to the nodal degrees of freedom $\boldsymbol{q}$ by means of shape functions, according to the FE approximation.

Calling $\boldsymbol{I}_{z}^{S}$ the constant matrix which allows extraction of the $z^{S}$ component of the local displacement it is possible to write Eq. 9.23 (see Eq. 9.22):

$$
\begin{equation*}
\boldsymbol{Z}_{\mathrm{loc}}^{S}=\boldsymbol{I}_{z}^{S} \cdot \tilde{\boldsymbol{u}}_{\mathrm{loc}}^{S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{Y}^{S} \cdot \boldsymbol{E}_{q}^{S} \cdot \boldsymbol{I}^{S} \cdot \boldsymbol{q} \tag{9.23}
\end{equation*}
$$

where $\boldsymbol{I}_{z}^{S}$ is sparse matrix, which is full of null terms and has some unit terms conveniently located. Note that the initial local $z^{S}$ coordinates of all Pseudo-Structural Points are zero. This implies that vector $\boldsymbol{Z}_{\text {loc }}^{S}$ in equation 9.23 contains the $z^{S}$ coordinates of the deformed configuration. Using the fitted surface spline shape it is possible to calculate the derivatives of such a shape and the associated local angle of attack. Let the $i^{\text {th }}$ Pseudo-Structural Point on wing segment $S$ be considered. The local $z$ coordinate of the point $i$ will be $Z_{i \text { loc }}^{S}$.

$$
Z_{i \mathrm{loc}}^{S}=Z_{i \mathrm{loc}}^{S}\left(x_{i \mathrm{loc}}^{S}, y_{i \mathrm{loc}}^{S}\right)
$$

Again it has to be clear that $x_{i \text { loc }}^{S}, y_{i \mathrm{loc}}^{S}$ are the local coordinates of the Pseudo-Structural Point $i$ in which the local $z$ coordinate $Z_{i \text { loc }}^{S}$ is considered. The assumption that the displacements are not very large is made. In fact, a linear theory is utilized, then it's appropriate to assume small displacements. So, the aerodynamic linear theory holds. Also, under this assumption, it is reasonable to consider the local in-plane coordinates of nodes, Load Points and Control Points of a generic Wing Segment constant. Only the out-of-plane local displacement will be different from zero. Under this hypothesis, all the splining matrices are constant and they can be calculated once, precisely because the surface is defined once.

### 9.4.2 The Spline Method

The Infinite Plate Spline method (IPS) is involved now. As resumed in Section 8.4, it's necessary as input a discrete set of points, lying within a 2D domain with Cartesian coordinates $x$ and $y$, in which the displacement is known. In this case, the set is composed af all the Pseudo-Structural Points of the Wing Segment $S$.

For each Pseudo-Structural Point $i$ of Wing Segment $S$ the corresponding $Z_{i \text { loc }}^{S}$ is written as:

$$
\begin{equation*}
Z_{i \mathrm{loc}}^{S}=a_{0}^{S}+a_{1}^{S} x_{i \mathrm{loc}}^{S}+a_{2} y_{i \mathrm{loc}}^{S}+\sum_{j=1}^{N_{P S}^{S}} F_{j}\left(r_{i j \mathrm{loc}}^{S}\right)^{2} \ln \left(r_{i j \mathrm{loc}}^{S}\right)^{2} \tag{9.24}
\end{equation*}
$$

where:

$$
\begin{equation*}
\left(r_{i j \mathrm{loc}}^{S}\right)^{2}=\left(x_{i \mathrm{loc}}^{S}-x_{j \mathrm{loc}}^{S}\right)^{2}+\left(y_{i \mathrm{loc}}^{S}-y_{j \mathrm{loc}}^{S}\right)^{2} \tag{9.25}
\end{equation*}
$$

noting that also the counter j refers to the Pseudo-Structural Points. According to the exposed spline notation, Eq. 9.24 can be rewritten introducing the matrix $\boldsymbol{K}^{S}$, whose generic term lying in $i^{\text {th }}$ row and $j^{\text {th }}$ column is defined as

$$
\begin{equation*}
K_{i j}^{S}=\left(r_{i j}^{S} \text { loc }\right)^{2} \ln \left(r_{i j}^{S} \text { loc }\right)^{2} \tag{9.26}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
Z_{i \mathrm{loc}}^{S}\left(x_{i \mathrm{loc}}^{S}, y_{i \mathrm{loc}}^{S}\right)=a_{0}^{S}+a_{1}^{S} x_{i \mathrm{loc}}^{S}+a_{2} y_{i \mathrm{loc}}^{S}+\sum_{j=1}^{N_{P S}^{S}} F_{j} K_{i j}^{S} \tag{9.27}
\end{equation*}
$$

Also the following conditions have to be satisfied:

$$
\begin{align*}
& \sum_{j=1}^{N_{P S}^{S}} F_{j}^{S}=0 \\
& \sum_{j=1}^{N_{P S}^{S}} F_{j}^{S} x_{j \mathrm{loc}}^{S}=0  \tag{9.28}\\
& \sum_{j=1}^{N_{P S}^{S}} F_{j}^{S} y_{j \mathrm{loc}}^{S}=0
\end{align*}
$$

Equations 9.27 written for all the Pseudo-Structural Points and 9.28 can be combined in the following matricial system:

$$
\left\{\begin{array}{c}
0 \\
0 \\
0 \\
Z_{1 \mathrm{loc}}^{S} \\
Z_{2 \mathrm{loc}}^{S} \\
\vdots \\
Z_{N_{P S}^{S} \operatorname{loc}}^{S}
\end{array}\right\}=\left[\begin{array}{cccccc}
0 & 0 & 0 & 1 & 1 & \ldots \\
0 & 0 & 0 & x_{1 \mathrm{loc}}^{S} & x_{2 \mathrm{loc}}^{S} & \ldots \\
x_{N_{P S}^{S} \mathrm{loc}}^{S} \\
0 & 0 & 0 & y_{1 \mathrm{loc}}^{S} & y_{2 \mathrm{loc}}^{S} & \ldots \\
y_{N_{P S}^{S} \mathrm{loc}}^{S} \\
1 & x_{1 \mathrm{loc}}^{S} & y_{1 \mathrm{loc}}^{S} & 0 & K_{12}^{S} & \ldots \\
K_{1 N_{P S}^{S}}^{S} \\
1 & x_{2 \mathrm{loc}}^{S} & y_{2 \mathrm{loc}}^{S} & K_{21}^{S} & 0 & \ldots \\
K_{2 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\vdots \\
1 & x_{N_{P S}^{S}}^{S} & y_{N_{P S}^{S} \operatorname{loc}}^{S} & K_{N_{P S}^{S} 1}^{S} & K_{N_{P S}^{S} 2}^{S} & \ldots \\
a_{1}^{S} \\
a_{2}^{S} \\
F_{1}^{S} \\
F_{2}^{S} \\
\vdots \\
F_{N_{P S}^{S}}^{S}
\end{array}\right\}
$$

Isolating the terms of the matrix and introducing other vectors and matrices, the system can assume the more compact form expressed by Eq. 9.29. Notice that the vector $\boldsymbol{Z}_{\text {loc }}^{S \star}$ is coincident with $\boldsymbol{Z}_{\text {loc }}^{S}$ except for the fact that three rows of zeros have been added at the top.

$$
\begin{align*}
& \boldsymbol{Z}_{\text {loc }}^{S \star}=\left[\begin{array}{lllllll}
0 & 0 & 0 & Z_{1 \text { loc }}^{S} & Z_{2 \text { loc }}^{S} & \cdots & Z_{N_{P S}^{S}}^{S} \text { loc }
\end{array}\right]^{T} \\
& \boldsymbol{P}^{S}=\left[\begin{array}{lllllll}
a_{0}^{S} & a_{1}^{S} & a_{2}^{S} & F_{1}^{S} & F_{2}^{S} & \ldots & F_{N_{P S}}^{S}
\end{array}\right]^{T} \\
& \boldsymbol{R}^{S}=\left[\begin{array}{cccc}
1 & 1 & \ldots & 1 \\
x_{1 \mathrm{loc}}^{S} & x_{2 \mathrm{loc}}^{S} & \ldots & x_{N_{S S}^{S} \operatorname{loc}}^{S} \\
y_{1 \mathrm{loc}}^{S} & y_{2 \mathrm{loc}}^{S} & \ldots & y_{N_{P S}^{S} \operatorname{loc}}^{S}
\end{array}\right] \\
& \boldsymbol{K}^{S}=\left[\begin{array}{cccc}
0 & K_{12}^{S} & \ldots & K_{1 N_{P S}^{S}}^{S} \\
K_{21}^{S} & 0 & \ldots & K_{2 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \ddots & \vdots \\
K_{N_{P S}^{S} 1}^{S} & K_{N_{P S}{ }^{S}}^{S} & \cdots & 0
\end{array}\right] \\
& \boldsymbol{Z}_{\text {loc }}^{S \star}=\left[\begin{array}{cc}
\mathbf{0} & \boldsymbol{R}^{S} \\
{\left[\boldsymbol{R}^{S}\right]^{T}} & \boldsymbol{K}^{S}
\end{array}\right] \cdot \boldsymbol{P}^{S} \tag{9.29}
\end{align*}
$$

Moreover, if the system matrix is addressed as:

$$
\boldsymbol{G}^{S}=\left[\begin{array}{cc}
\mathbf{0} & \boldsymbol{R}^{S}  \tag{9.30}\\
{\left[\boldsymbol{R}^{S}\right]^{T}} & \boldsymbol{K}^{S}
\end{array}\right]
$$

then the above system is easily summarized by Eq. 9.31:

$$
\begin{equation*}
\boldsymbol{Z}_{\text {loc }}^{S \star}=\boldsymbol{G}^{S} \cdot \boldsymbol{P}^{S} \tag{9.31}
\end{equation*}
$$

Notice that $\boldsymbol{R}^{S}$ has dimension $3 \times N_{P S}^{S}, \boldsymbol{K}^{S}$ has dimension $N_{P S}^{S} \times N_{P S}^{S}, \boldsymbol{P}^{S}$ has dimension $\left(3+N_{P S}^{S}\right) \times 1$ and $\boldsymbol{G}^{S}$ has dimension $\left(3+N_{P S}^{S}\right) \times\left(3+N_{P S}^{S}\right)$.

As explained in Section 8.4, the only input data to construct $\boldsymbol{G}^{S}$ are the coordinates of Pseudo-Structural Points (in the local system according to the aeroelastic notation) and their displacements. Actually, the latter will be known only after having computed the degrees of freedom of the structure. But, as it will be seen, the relationship between the two entities will be written in a matrix notation; so the spline coefficients of the spline, and even better the slopes of Control Points and displacements of Load Points, will be linked by some transformation matrices to the vector $\boldsymbol{q}$.

Inverting the relation expressed by Eq. 9.31, it is possible to find the $N_{P S}^{S}+3$ unknowns represented by the components of $\boldsymbol{P}^{S}$ :

$$
\begin{equation*}
\boldsymbol{P}^{S}=\left[\boldsymbol{G}^{S}\right]^{-1} \cdot \boldsymbol{Z}_{\mathrm{loc}}^{S \star} \tag{9.32}
\end{equation*}
$$

Now the coefficients that have to be used for the spline are known.

## Slopes at Control Points

So far only the Pseudo-Structural Points on the reference surface of Wing Segment $S$ have been considered. Once obtained the coefficients necessary to describe the spline, then the Aerodynamic Points of the panels can be taken into account. The beginning of this chapter has shown that the Wall Tangency Condition (WTC) is enforced at the aerodynamic Control Points. Then, let the local coordinates (in the reference plane) of the $k^{\text {th }}$ Control Point be indicated with $\mathcal{X}_{k \text { loc }}^{S}$ and $\mathcal{Y}_{k \text { loc }}^{S}$. The coordinate $\mathcal{Z}_{k \text { loc }}^{S}$ in the direction $z^{S}$ of the $k^{\text {th }}$ Control Point will be calculated by resuming the spline equation 9.27 :

$$
\begin{equation*}
\mathcal{Z}_{k \text { loc }}^{S}\left(\mathcal{X}_{k \mathrm{loc}}^{S}, \mathcal{Y}_{k \mathrm{loc}}^{S}\right)=a_{0}^{S}+a_{1}^{S} \mathcal{X}_{k \mathrm{loc}}^{S}+a_{2} \mathcal{Y}_{k \mathrm{loc}}^{S}+\sum_{j=1}^{N_{P}^{S} S} F_{j} \mathcal{K}_{k j}^{S} \tag{9.33}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{K}_{k j}^{S}=\left(\mathcal{R}_{k j \operatorname{loc}}^{S}\right)^{2} \ln \left(\mathcal{R}_{k j \text { loc }}^{S}\right)^{2} \tag{9.34}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\mathcal{R}_{k j \text { loc }}^{S}\right)^{2}=\left(\mathcal{X}_{k \text { loc }}^{S}-x_{j \text { loc }}^{S}\right)^{2}+\left(\mathcal{Y}_{k \text { loc }}^{S}-y_{j \text { loc }}^{S}\right)^{2} \tag{9.35}
\end{equation*}
$$

The number of Control Points is the same as the number of aerodynamic panels $N_{A P}^{S}$. Equation 9.33 is utilized for the whole set of $N_{A P}$ locations on the surface. Therefore, it is more advantageous to create a system in a matrix form:

$$
\left\{\begin{array}{c}
\mathcal{Z}_{1 \text { loc }}^{S} \\
\mathcal{Z}_{2 \text { loc }}^{S} \\
\mathcal{Z}_{3 \text { loc }}^{S} \\
\vdots \\
\mathcal{Z}_{N_{A P}^{S} \text { loc }}^{S}
\end{array}\right\}=\left[\begin{array}{cccccc}
1 & \mathcal{X}_{1 \mathrm{loc}}^{S} & \mathcal{Y}_{1 \mathrm{loc}}^{S} & \mathcal{K}_{11}^{S} & \ldots & \mathcal{K}_{1 N_{P S}^{S}}^{S} \\
1 & \mathcal{X}_{2 \mathrm{loc}}^{S} & \mathcal{Y}_{2 \mathrm{loc}}^{S} & \mathcal{K}_{21}^{S} & \ldots & \mathcal{K}_{2 N_{P S}^{S}}^{S} \\
1 & \mathcal{X}_{3 \mathrm{loc}}^{S} & \mathcal{Y}_{3 \mathrm{loc}}^{S} & \mathcal{K}_{31}^{S} & \ldots & \mathcal{K}_{3 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \mathcal{X}_{N_{A P}^{S} \text { loc }}^{S} & \mathcal{Y}_{N_{A P}^{S} \text { loc }}^{S} & \mathcal{K}_{N_{A P}^{S} 1}^{S} & \cdots & \mathcal{K}_{N_{A P}^{S} N_{P S}^{S}}^{S}
\end{array}\right]\left\{\begin{array}{c}
a_{0}^{S} \\
a_{1}^{S} \\
a_{2}^{S} \\
F_{1}^{S} \\
\vdots \\
F_{N_{P S}^{S}}^{S}
\end{array}\right\}
$$

Actually, this matricial form will not be used for Control Points, but for Load Points, since the displacement is required to be computed for these latter points. To impose the boundary conditions the derivatives with respect to $x^{S}$ are required. Actually, there is no difference between $x$ and $x^{S}$ because the reference surfaces do not have angle of attack. In fact, the aeroelastic convention for the Wing Segments is so that the local $x^{S}$ axis is always parallel to the global $x$ axis. Hence, the derivations with respect the two direction are equal. However it's kept here the notation $x^{S}$ to designate that the local coordinate system of wing segment $S$ is considered. Therefore, it is necessary to differentiate the spline equation 9.33 with respect to $x^{S}$ and calculate the result in the local coordinates of the Control Points. As a result, the slope of the $k^{\text {th }}$ Control Point is given by:

$$
\begin{align*}
\frac{\partial \mathcal{Z}_{k \text { loc }}^{S}}{\partial x^{S}}\left(\mathcal{X}_{k l \mathrm{loc}}^{S}, \mathcal{Y}_{k \mathrm{loc}}^{S}\right)= & a_{1}+\left.\sum_{j=1}^{N_{P S}^{S}} F_{j} \frac{\partial\left(\left(\mathcal{R}_{k j \operatorname{loc}}^{S}\right)^{2} \ln \left(\mathcal{R}_{k j \operatorname{loc}}^{S}\right)^{2}\right)}{\partial x^{S}}\right|_{\left(\mathcal{X}_{k l \mathrm{loc}}^{S}, \mathcal{X}_{k l \mathrm{loc}}^{S}\right)}  \tag{9.36}\\
& =a_{1}+\sum_{j=1}^{N_{P S}^{S}} F_{j} \mathcal{D} \mathcal{K}_{k j}^{S}
\end{align*}
$$

where:

$$
\begin{equation*}
\mathcal{D} \mathcal{K}_{k j}^{S}=\left.\frac{\partial K_{j}^{S}}{\partial x^{S}}\right|_{\left(\mathcal{X}_{k \text { loc }}^{S}, \mathcal{Y}_{k \text { loc }}^{S}\right)}=2\left(\mathcal{X}_{k \text { loc }}^{S}-x_{j \text { loc }}^{S}\right)\left[1+\ln \left(\mathcal{R}_{k j \text { loc }}^{S}\right)^{2}\right] \tag{9.37}
\end{equation*}
$$

Following the exposed procedure for all the $N_{A P}^{S}$ ( $=$ Number of Aerodynamic Points) locations on the surface, it is possible to create a system in a matrix notation:

$$
\left\{\begin{array}{c}
\frac{\partial \mathcal{Z}_{1 \mathrm{loc}}^{S}}{\partial x^{S}} \\
\frac{\partial \mathcal{Z}_{2 \mathrm{loc}}^{S}}{\partial x^{S}} \\
\vdots \\
\frac{\partial \mathcal{Z}_{N_{A P}^{S} \operatorname{loc}}^{S}}{\partial x^{S}}
\end{array}\right\}=\left[\begin{array}{ccccccc}
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{11}^{S} & \mathcal{D} \mathcal{K}_{12}^{S} & \ldots & \mathcal{D} \mathcal{K}_{1 N_{P S}^{S}}^{S} \\
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{21}^{S} & \mathcal{D} \mathcal{K}_{22}^{S} & \ldots & \mathcal{D} \mathcal{K}_{2 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{N_{A P}^{S} 1}^{S} & \mathcal{D} \mathcal{K}_{N_{A P}^{S} 2}^{S} & \ldots & \mathcal{D} \mathcal{K}_{N_{A P}^{S} N_{P S}^{S}}^{S}
\end{array}\right]\left\{\begin{array}{c}
a_{0}^{S} \\
a_{1}^{S} \\
a_{2}^{S} \\
F_{1}^{S} \\
F_{2}^{S} \\
\vdots \\
F_{N_{P S}^{S}}^{S}
\end{array}\right\}
$$

Let us indicate with $\frac{\partial \mathcal{Z}_{\text {loc }}^{S}}{\partial x^{S}}$ the vector which contains the slopes of the Control Points of Wing Segment $S$. Instead, $\mathcal{D}^{S}$ is the above matrix which multiplied by the vector containing the coefficients of the spline $\boldsymbol{P}^{S}$ gives the vector $\frac{\partial \mathcal{Z}_{\text {loc }}^{S}}{\partial x^{S}}$. The following quantities are so defined:

$$
\begin{gathered}
\frac{\partial \mathcal{Z}_{\text {loc }}^{S}}{\partial x^{S}}=\left[\begin{array}{cccccc}
\frac{\partial \mathcal{Z}_{1 \mathrm{loc}}^{S}}{\partial x^{S}} & \frac{\partial \mathcal{Z}_{2 \mathrm{loc}}^{S}}{\partial x^{S}} & \frac{\partial \mathcal{Z}_{3 \mathrm{loc}}^{S}}{\partial x^{S}} & \ldots & \frac{\partial \mathcal{Z}_{N^{S} \mathrm{loc}}^{S}}{\partial x^{S}}
\end{array}\right]^{T} \\
\mathcal{D}^{S}=\left[\begin{array}{ccccccc}
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{11}^{S} & \mathcal{D} \mathcal{K}_{12}^{S} & \ldots & \mathcal{D} \mathcal{K}_{1 N_{P S}^{S}}^{S} \\
0 & 1 & 0 & \mathcal{D K}_{21}^{S} & \mathcal{D} \mathcal{K}_{22}^{S} & \ldots & \mathcal{D} \mathcal{K}_{2 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 0 & \mathcal{D} \mathcal{K}_{N_{A P}^{S} 1}^{S} & \mathcal{D} \mathcal{K}_{N_{A P}^{S} 2}^{S} & \ldots & \mathcal{D} \mathcal{K}_{N_{A P}^{S} S}^{S} N_{P S}^{S}
\end{array}\right]
\end{gathered}
$$

Using these definitions, the slopes can be written as functions of the coefficients of the spline fit by means of Eq. 9.38:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \cdot \boldsymbol{P}^{S} \tag{9.38}
\end{equation*}
$$

Just to be exhaustive, the vector $\frac{\partial \mathcal{Z}_{\text {loc }}^{S}}{\partial x^{S}}$ has dimension $N_{A P}^{S} \times 1$, whereas $\mathcal{D}^{S}$ has dimension $N_{A P}^{S} \times\left(N_{P S}^{S}+3\right)$. Besides, the only input required to build the latter matrix are geometrical, i.e. the local coordinates of Pseudo-Structural and Control Points on the Wing Segment $S$. Now, it is advantageous to write an expression able to relate directly the output and the input data, corresponding to the $z^{S}$ coordinates of the Pseudo-Structural Points in the deformed configuration. The result avoids computing the spline coefficients $\boldsymbol{P}^{S}$ and thus:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \cdot \boldsymbol{P}^{S}=\mathcal{D}^{S} \cdot\left[\boldsymbol{G}^{S}\right]^{-1} \cdot \boldsymbol{Z}_{\mathrm{loc}}^{S \star} \tag{9.39}
\end{equation*}
$$

Observing that the first three rows of $\boldsymbol{Z}_{\text {loc }}^{S \star}$ are zeros, it is possible to eliminate the first three columns of the matrix $\left[\boldsymbol{G}^{S}\right]^{-1}$ without changing the result. Defining $\boldsymbol{S}^{S}$, the matrix $\left[\boldsymbol{G}^{S}\right]^{-1}$ with the first three columns eliminated, and coming back to $\boldsymbol{Z}_{\text {loc }}^{S}$, the vector $\boldsymbol{Z}_{\text {loc }}^{S}$ 夫 without the first three rows, Eq. 9.39 can be rewritten as:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \cdot \boldsymbol{S}^{S} \cdot \boldsymbol{Z}_{\mathrm{loc}}^{S} \tag{9.40}
\end{equation*}
$$

Notice that $\boldsymbol{S}^{S}$ has dimension $\left(N_{n}^{S}+3\right) \times N_{n}^{S}$ and $\boldsymbol{Z}_{\text {loc }}^{S}$ has dimension $N_{n}^{S} \times 1$.
In the model proposed, the aim is to express all the introduced quantities as functions of the nodal degrees of freedom. In fact, these are the independent variables as well as the unknowns to be computed. Thus, it is very useful to combine Eqs. 9.40 and 9.23:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \cdot \boldsymbol{S}^{S} \cdot \boldsymbol{I}_{z}^{S} \cdot \boldsymbol{Y}^{S} \cdot \boldsymbol{E}_{q}^{S} \cdot \boldsymbol{I}^{S} \cdot \boldsymbol{q} \tag{9.41}
\end{equation*}
$$

Introducing the definition:

$$
\begin{equation*}
\boldsymbol{a}_{3}^{S}=\boldsymbol{S}^{S} \cdot \boldsymbol{I}_{z}^{S} \cdot \boldsymbol{Y}^{S} \cdot \boldsymbol{E}_{q}^{S} \cdot \boldsymbol{I}^{S} \tag{9.42}
\end{equation*}
$$

it is obtained:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \boldsymbol{a}_{3}^{S} \cdot \boldsymbol{q} \tag{9.43}
\end{equation*}
$$

This formula relates the slope of all the Control points of all panels of Wing Segment $S$ to the vector of nodal degrees of freedom of the whole structure. Equation 9.43 can be written for all Wing Segments and so an assembly procedure is required to have all the local slopes of all the panels of the entire wing system as a function of the degrees of freedom of all the structural finite elements.

## Displacements at Load Points

Since the discussion is about splining method, it is useful to face the matter of aerodynamic Load Points, too. As it will be seen, not only the boundary condition require the IPS method. In fact, in the calculation of the generalized aerodynamic matrices, it is required to also transform loads, i.e. Lift Forces, at aerodynamic Load Points to nodes on the structural grid. This transformation will be performed via the Principle of Virtual Displacements and will involve the displacements of Load Points. Thus, the spline method has to evaluate them.

The matrix of transformation between the input displacements at Pseudo-Struc-tural Points and the output deflections at Load Points is built following what done above about the Control Points. This matrix is addressed as $\tilde{\mathcal{D}}^{S \star}$, where appears the superscript * to indicate the calculus of displacements rather than slopes, according to Section 8.4. While the symbol ${ }^{\sim}$ is added from now on in order to refer to Load Points rather than Control Points:

$$
\tilde{\mathcal{D}}^{S \star}=\left[\begin{array}{cccccc}
1 & \tilde{\mathcal{X}}_{1 \mathrm{loc}}^{S} & \tilde{\mathcal{Y}}_{1 \mathrm{loc}}^{S} & \tilde{\mathcal{K}}_{11}^{S} & \ldots & \tilde{\mathcal{K}}_{1 N_{P S}^{S}}^{S}  \tag{9.44}\\
1 & \tilde{\mathcal{X}}_{2 \mathrm{loc}}^{S} & \tilde{\mathcal{Y}}_{2 \mathrm{loc}}^{S} & \tilde{\mathcal{K}}_{21}^{S} & \ldots & \tilde{\mathcal{K}}_{2 N_{P S}^{S}}^{S} \\
1 & \tilde{\mathcal{X}}_{3 \mathrm{loc}}^{S} & \tilde{\mathcal{Y}}_{3 \mathrm{loc}}^{S} & \tilde{\mathcal{K}}_{31}^{S} & \ldots & \tilde{\mathcal{K}}_{3 N_{P S}^{S}}^{S} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \tilde{\mathcal{X}}_{N_{A P}^{S} \text { loc }}^{S} & \tilde{\mathcal{Y}}_{N_{A P}^{S} \text { loc }}^{S} & \tilde{\mathcal{K}}_{N_{A P}^{S}}^{S} & \cdots & \tilde{\mathcal{K}}_{N_{A P}^{S} N_{P S}^{S}}^{S}
\end{array}\right]
$$

where $\tilde{\mathcal{X}}_{\text {1loc }}^{S}$ and $\tilde{\mathcal{Y}}_{\text {1loc }}^{S}$ are the coordinates of Load Points, written with respect to the local coordinate system of Wing Segment $S$. Thus, the displacements at Load Points can be written as functions of the coefficients of the spline fit by means of Eq. 9.45:

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{\text {loc }}^{S}=\tilde{\mathcal{D}}^{S \star} \cdot \boldsymbol{P}^{S} \tag{9.45}
\end{equation*}
$$

Just to be exhaustive, the vector $\tilde{\mathcal{Z}}_{\text {loc }}^{S}$ has dimension $N_{A P}^{S} \times 1$, whereas $\tilde{\mathcal{D}}^{S \star}$ has dimension $N_{A P}^{S} \times\left(N_{P S}^{S}+3\right)$. Again, the only input required to build the latter matrix are geometrical, i.e. the local coordinates of Pseudo-Structural and Load Points on the Wing Segment $S$.

By using a procedure formally identical to the one used to obtain Eq. 9.43, it is possible to obtain:

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{\text {loc }}^{S}=\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S} \cdot \boldsymbol{q} \tag{9.46}
\end{equation*}
$$

The only difference between Eqs. 9.46 and 9.43 consists in the fact that the local coordinates of the Load Points are considered instead of the local coordinates of the Control Points.

## Assembly procedure

The aerodynamic panels are numbered so as to have the first $N_{A P}^{S}$ panels of the first Wing Segment and then the second $N_{A P}^{S}$ panels of the seconds Wing Segment and so forth. The assembly process is carried out by calculating all the products (for all Wing Segments) $\mathcal{D}^{S} \boldsymbol{a}_{3}^{S}$ and $\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S}$ and observing that each aerodynamic panel can be included only in one trapezoidal Wing Segment. This means that two different Wing Segments do not share common aerodynamic panels. Thus, the assembly procedure is different from the classical way followed in structural finite element method. In fact, since two Wing Segments are adjacent, then they necessarily share common nodes. Instead, adjacent Wing Segments don't have common aerodynamic panels, which are bidimensional, thus no terms have to be summed up. After the assembly, at wing system level, Eqs. 9.43 and 9.46 become:

$$
\begin{gather*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}}{\partial x}=\boldsymbol{A}_{3} \cdot \boldsymbol{q}  \tag{9.47}\\
\tilde{\mathcal{Z}}_{\mathrm{loc}}=\tilde{\boldsymbol{A}}_{3}^{\star} \cdot \boldsymbol{q} \tag{9.48}
\end{gather*}
$$

By means of the exposed matrix notation, Eqs. 9.47 and 9.48 allow to directly relate displacements and slopes at Aerodynamic Points of the structure to its nodal degrees of freedom. That was the target of the Surface Spline method.

### 9.4.3 Steady Aerodynamic Forces

Once completed the discussion about splining, the derivation of Aerodynamic Loads is faced. According to the Vortex Lattice Method exposed in Section 8.3, the Aerodynamic Loads acting on the deflecting surface are transferred as Lift Forces located on Loads Points of the aerodynamic panels of the whole structure. It means that the pressure distribution charging the wing system becomes a set of concentrated loads.

As done in the spline section, here the object is to build a formulation in which the aeroelastic quantities are written as functions of nodal degrees of freedom. In fact, they represent the independent variables of the problem. It counts also for the Aerodynamic Forces. Therefore, the goal to be achieved is to write the relationship between these loads and the vector of nodal degrees of freedom.

At first, a connection between the Lift Forces and the pressure loads on all the aerodynamic panels has to be established. Thus, let $\Delta \boldsymbol{p}$ be a vector which contains all the dimensionless pressure loads on all aerodynamic panels of the structure. It represents the ratio of the real pressures $\Delta \boldsymbol{p}^{\prime}$ to the dynamic pressure $\frac{1}{2} \rho_{\infty} V_{\infty}$, taken as reference value. Considering the $j^{\text {th }}$ aerodynamic panel, then:

$$
\Delta \boldsymbol{p}=\frac{\Delta \boldsymbol{p}^{\prime}}{\frac{1}{2} \rho_{\infty} V_{\infty}^{2}}
$$

For a generic panel the Aerodynamic Force, applied at the Load Point of that panel, is obtained by multiplying the dynamic pressure by some geometrical quantities of the panel and by the dimensionless pressure load. This happens because the pressure load on that generic panel has been normalized with respect to the dynamic pressure, in this way generating the dimensionless pressure load. At this point, let us consider the modulus of the Aerodynamic Load of panel $j$ of Wing Segment $S$, given by Eq. 9.49:

$$
\begin{equation*}
\left|L_{j}^{S}\right|=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{j} 2 e_{j} \Delta p_{j}^{S} \tag{9.49}
\end{equation*}
$$

where the product $\Delta x_{j} 2 e_{j}$ represents the $j^{t h}$ panel's surface. The quantity $\Delta x_{j}$ is the average chord of the panel and $e_{j}$ refers to its half length along the wing span, i.e. along the $y^{S}$ local direction.

If the global components of the unit normal vector of the aerodynamic panel $j$ are $n_{x j}$, $n_{y j}$ and $n_{z j}$ (notice that $n_{x j}=0$ because the reference aerodynamic configuration has no angle of attack), the $y$ and $z$ global components of the Aerodynamic Load are:

$$
\begin{align*}
& {\left[L_{j}^{S}\right]_{x}=0} \\
& {\left[L_{j}^{S}\right]_{y}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{j} 2 e_{j} n_{y j} \Delta p_{j}^{S}}  \tag{9.50}\\
& {\left[L_{j}^{S}\right]_{z}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{j} 2 e_{j} n_{z j} \Delta p_{j}^{S}}
\end{align*}
$$

It's obvious that Aerodynamic Load vector is parallel to the normal of panel $j$; in this way the load results to be perpendicular to the wind direction and plays the role of a lift load for that panel.

Since what it has been described applies to all the Wing Segments, then the vector which contains the moduli of the Aerodynamic Forces of all the panels is written as a
product between the dynamic pressure and a matrix $\boldsymbol{I}^{D}$. The matrix depends on the geometrical features of the aerodynamic mesh. For the sake of clarity, let us show how to build this matrix, by considering four aerodynamic panels for instance. In that case, the moduli of Aerodynamic Loads are the following:

$$
\begin{aligned}
\left|L_{1}^{S}\right| & =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{1} 2 e_{1} \Delta p_{1}^{S} \\
\left|L_{2}^{S}\right| & =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{2} 2 e_{2} \Delta p_{2}^{S} \\
\left|L_{3}^{S}\right| & =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{3} 2 e_{3} \Delta p_{3}^{S} \\
\left|L_{4}^{S}\right| & =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \Delta x_{4} 2 e_{4} \Delta p_{4}^{S}
\end{aligned}
$$

It is useful to join the terms in a matrix form:

$$
\left\{\begin{array}{l}
\left|L_{1}^{S}\right| \\
\left|L_{2}^{S}\right| \\
\left|L_{3}^{S}\right| \\
\left|L_{4}^{S}\right|
\end{array}\right\}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\left[\begin{array}{cccc}
\Delta x_{1} 2 e_{1} & 0 & 0 & 0 \\
0 & \Delta x_{2} 2 e_{2} & 0 & 0 \\
0 & 0 & \Delta x_{3} 2 e_{3} & 0 \\
0 & 0 & 0 & \Delta x_{4} 2 e_{4}
\end{array}\right]\left\{\begin{array}{c}
\Delta p_{1}^{S} \\
\Delta p_{2}^{S} \\
\Delta p_{3}^{S} \\
\Delta p_{4}^{S}
\end{array}\right\}
$$

The extension of the same procedure to the general case conducts to the final expression relating the Lift Forces (written as their moduli) to the dimensionless pressures, as given by Eq. 9.51:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \boldsymbol{I}^{D} \cdot \Delta \boldsymbol{p} \tag{9.51}
\end{equation*}
$$

The first step has been completed. Now the Vortex Lattice key equation is resumed. As seen before, Eq. 9.52 describes the dimensionless normalwash as function of the pressures on each aerodynamic panel:

$$
\begin{equation*}
\boldsymbol{w}=\boldsymbol{A}^{D} \cdot \Delta \boldsymbol{p} \tag{9.52}
\end{equation*}
$$

where $\boldsymbol{A}^{D}$ is the Aerodynamic Influence Coefficient Matrix for the aerodynamic panels. This matrix is calculated once by using the geometry of the aerodynamic reference configuration. In fact, it depends only on the aerodynamic discretization.

Considering a generic panel $i$, the normalwash $w_{i}^{\prime}$ is the component of the velocity normal to the reference undeformed surface of the Wing Segment where the panel lies, i.e. normal to the considered aerodynamic panel. Being this surface parallel to the free stream direction $\boldsymbol{V}_{\infty}$, then the normalwash results to be perpendicular to the free stream, too. The normalization of this quantity is with respect to $V_{\infty}$, since it is the reference velocity:

$$
w=\frac{w^{\prime}}{V_{\infty}}
$$

At this point, it would be possible to write an expression linking the lift forces and the dimensionless normalwash of the panels. Moreover, the use of IPS method has previously conducted to Eq. 9.47, where the slopes at Control Point are dependent on the nodal degrees of freedom. Thus, the last step would consist in finding a correlation between such slopes and the dimensionless normalwash.

In the steady case, considering that the structure changes configuration when it deforms, the boundary condition used for the Vortex Lattice formulation is:

$$
\begin{equation*}
\boldsymbol{w}=\frac{\partial \mathcal{Z}_{\text {loc }}}{\mathrm{d} x} \tag{9.53}
\end{equation*}
$$

where all the quantities are real numbers (steady case). In fact, the normalization of the normalwash with respect to $V_{\infty}$ means that the dimensionless normalwash is equal to the tangent of the angle from the velocity of the stream in proximity of the panel, in the deformed configuration, to the free stream direction $\boldsymbol{V}_{\infty}$. As a consequence, considering small angles of deflection because of the model's linearity, the dimensionless normalwash has to equal the slope at the aerodynamic Control Point.

The boundary condition is not only a constraint expressing the coupling between the aerodynamics and the deflection of the structure, but in this case is the convenient interface able to correlate the Lift Forces to the nodal degrees of freedom. As a result, from Eqs. 9.53, 9.52 and 9.47 it is possible to relate the vector containing dimensionless pressures to the nodal degrees of freedom vector (this last is unknown):

$$
\begin{equation*}
\Delta \boldsymbol{p}=\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \boldsymbol{w}=\left[\boldsymbol{A}^{D}\right]^{-1} \boldsymbol{A}_{3} \cdot \boldsymbol{q} \tag{9.54}
\end{equation*}
$$

At last, the vector with the aerodynamic forces is written as:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \boldsymbol{I}^{D} \cdot\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \boldsymbol{A}_{3} \cdot \boldsymbol{q} \tag{9.55}
\end{equation*}
$$

With the help of the definition of a new matrix in which all the terms different from the dynamic pressure and the unknowns have been isolated:

$$
\begin{equation*}
\overline{\boldsymbol{c}}=\boldsymbol{I}^{D} \cdot\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \boldsymbol{A}_{3} \tag{9.56}
\end{equation*}
$$

finally the formula in matrix notation is obtained:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \overline{\boldsymbol{c}} \cdot \boldsymbol{q} \tag{9.57}
\end{equation*}
$$

To be remebered is that Aerodynamic Forces of Eq. 9.57 are applied at Load Points of the aerodynamic panels.

### 9.4.4 The Aeroelastic Stiffness Matrix

At this point, the Aerodynamic Loads are known as concentrated Forces localized on the aerodynamic panels. According to the FE approximation, they have to be transferred to the Structural Nodes. The result will be a vector of equivalent nodal loads, by means of which the construction of the Aeroelstic Stiffness Matrix will be carry out. The procedure to apply utilizes the following algorithm.

From Eq. 9.57 it is possible to extract the Forces which are applied only on the aerodynamic panels of the generic Wing Segment $S$. This operation is particularly easy since the aerodynamic panels are numbered as explained above. In fact, the IDs of the panels are the sole input needed to know the matrix of Eq. 9.58:

$$
\begin{equation*}
\boldsymbol{L}^{S}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \overline{\boldsymbol{c}}^{S} \cdot \boldsymbol{q} \tag{9.58}
\end{equation*}
$$

where $\overline{\boldsymbol{c}}^{S}$ is directly obtained from $\overline{\boldsymbol{c}}$. Considering the properties of the present aerodynamic theory, all these forces are parallel and perpendicular to the surface that represents the

Wing Segment $S$. It means that local $x^{S}, y^{S}$ components of the Aerodynamic Loads are zero; then $\boldsymbol{L}^{S}$ contains not only the moduli of the loads on aerodynamic panels of Wing Segment $S$, but also their local $z^{S}$ components.

The transfer from Loads at the Aerodynamic Points to the Loads at Structural Nodes is performed in the classical way, so that they will be energetically equivalent. For this purpose, the Principle of Virtual Displacements is involved. According to it, the virtual work carried out by the Lift Forces on the virtual variation of displacemenets of Load Points balances the virtual work carried out by the equivalent nodal forces on the virtual variation of nodal degrees of freedom. Now the virtual work is going to be calculated and the expression for the out-of-plane displacements of the Load Points of all the panels contained in Wing Segment $S$ is resumed:

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{\mathrm{loc}}^{S}=\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S} \cdot \boldsymbol{q} \tag{9.59}
\end{equation*}
$$

From its definition it follows that the virtual work $\delta W$ of all the Aerodynamic Forces applied in Wing Segment $S$ is:

$$
\begin{equation*}
\delta W=\left[\delta \tilde{\mathcal{Z}}_{\mathrm{loc}}^{S}\right]^{T} \cdot \boldsymbol{L}^{S}=\left[\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S} \cdot \delta \boldsymbol{q}\right]^{T} \cdot \boldsymbol{L}^{S}=\delta \boldsymbol{q}^{T} \cdot\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \boldsymbol{L}^{S} \tag{9.60}
\end{equation*}
$$

where the virtual variation of Eq. 9.59 is considered. On the contrary, the virtual work of the equivalent nodal forces applied to all Structural Nodes of all the beams (of the entire wing system) is given by:

$$
\begin{equation*}
\delta W=\delta \boldsymbol{q}^{T} \cdot \boldsymbol{L}_{\mathrm{str}}^{S} \tag{9.61}
\end{equation*}
$$

where the virtual variation of nodal degrees of freedom $\boldsymbol{q}$ is considered. The vector $\boldsymbol{L}_{\mathrm{str}}^{S}$ contains the nodal forces on all Structural Nodes. The superscript ${ }^{S}$ is used to indicate that only the Aerodynamic Loads applied to the Load Points of the aerodynamic panels of Wing Segment $S$ have been taken into account.

By comparison of Eqs. 9.60 and 9.61, for the nodal forces equivalent to the Aerodynamic Forces applied on Wing Segment $S$ it is possible to deduce the following relation:

$$
\begin{equation*}
\boldsymbol{L}_{\mathrm{str}}^{S}=\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \boldsymbol{L}^{S}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \overline{\boldsymbol{c}}^{S} \cdot \boldsymbol{q} \tag{9.62}
\end{equation*}
$$

At this point, not only the Lift Forces have been related to the degrees of freedom, but also the equivalent nodal loads are functions of the unknowns vector $\boldsymbol{q}$. If all the contributions of all Wing Segments are added following Eq. 9.62, the loads on the Structural Nodes $\boldsymbol{L}_{\text {str }}$ can be obtained. This operation means that an assembly of the matrices $\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \overline{\boldsymbol{c}}^{S}$ is required. Differently from the previous case, the assembly procedure consists in summing up the various contributions corresponding to different Wing Segments.

The final assembled matrix is named $-\boldsymbol{K}_{\text {aero }}$, where the negative sign is adopted for the sake of convenience. The expression of the Aerodynamic Loads on all Structural Nodes after all Wing Segments have been taken into account is:

$$
\begin{equation*}
\boldsymbol{L}_{\mathrm{str}}=-\boldsymbol{K}_{\text {aero }} \cdot \boldsymbol{q} \tag{9.63}
\end{equation*}
$$

Even if $\boldsymbol{L}_{\text {str }}$ represents a nodal load vector, Eq. 9.63 expresses it as function of the degrees of freedom via the Aerodynamic Stiffness Matrix $\boldsymbol{K}_{\text {aero }}$, coherently with the role of $\boldsymbol{q}$ as
independent variables vector. Therefore, such a term can go to the left hand side of the aeroelastic system equation and summed up to the product due to the Structural Stiffness:

$$
\begin{equation*}
\boldsymbol{K}_{\mathrm{str}} \cdot \boldsymbol{q}=\boldsymbol{L}_{\mathrm{str}}=-\boldsymbol{K}_{\mathrm{aero}} \cdot \boldsymbol{q} \tag{9.64}
\end{equation*}
$$

or

$$
\begin{equation*}
\left[\boldsymbol{K}_{\mathrm{str}}+\boldsymbol{K}_{\mathrm{aero}}\right] \cdot \boldsymbol{q}=\mathbf{0} \tag{9.65}
\end{equation*}
$$

or

$$
\begin{equation*}
\boldsymbol{K}_{\text {aeroelastic }} \cdot \boldsymbol{q}=\mathbf{0} \tag{9.66}
\end{equation*}
$$

The isolation of the Stiffness Matrices in Eq. 9.65 leads to a single term, called Aeroelastic Stiffness Matrix. As a matter of fact, it substitutes the Structural Stiffness Matrix in the FEM system, so that the stiffness of the structure is sensible to and inclusive of the Aerodynamic Loads applied. In this way the deflection due to such Loads is already taken into account directly in the stiffness of the system. This is the meaning of aeroelastic analysis.

### 9.4.5 The Right Hand Side Load Vector

From Eq. 9.66 it appears that there is no motion. And this is the case because the angle of attack so far considered is zero. In fact, in the generic Wing Segment $S$ the aerodynamic mesh lies on its reference surface, which is parallel to the free stream $\boldsymbol{V}_{\infty}$, even if the Wing Segment is not. So, there is no motion unless external non-aerodynamic loads, i.e. some mechanical loads, are applied.

To solve this problem, a given known shape of the structure is assigned. Such a shape can be assigned, for instance, as coordinates of those points having the same $x^{S}$ and $y^{S}$ local coordinates of the Pseudo-Structural Points. Instead, the $z^{S}$ local out-of-plane coordinates are not equal to zero; in fact, these coordinates describe a shape with different from zero angle of attack. In particular, it is the undeformed shape of the structure, so that all the geometrical features are included in the aeroelastic model.

The new points will be denoted as Perturbed Pseudo-Structural Points, precisely because only the local out-of-plane coordinates are perturbed from the null value. Replacing the appropriate quantities in Eq. 9.19, the formula:

$$
\begin{equation*}
\boldsymbol{x}_{\mathrm{loc}}^{\text {pert } S}=\boldsymbol{E}^{S} \cdot\left[\boldsymbol{x}^{\text {pert } S}-\boldsymbol{x}_{2^{S}}\right]=\boldsymbol{E}^{S} \cdot\left[\boldsymbol{J}^{S} \cdot \boldsymbol{x}^{\text {pert }}-\boldsymbol{x}_{2^{S}}\right] \tag{9.67}
\end{equation*}
$$

involves the global coordinates of all the Perturbed Pseudo-Structural Points, which are addressed as $\boldsymbol{x}^{\text {pert }}$. The vector $\boldsymbol{x}_{\mathrm{loc}}^{\text {pert } S}$ contains their local coordinates, whereas the matrices $\boldsymbol{E}^{S}$ and $\boldsymbol{J}^{S}$ and the vector $\boldsymbol{x}_{2^{S}}$ have already been defined in section 9.4.2.

From now on, the algorithm will be very similar to the previous procedure described to find Eq. 9.62. What is aerodynamically relevant is the out-of-plane coordinate which can be easily extracted from Eq. 9.67 (see also Eq. 9.23):

$$
\begin{equation*}
\boldsymbol{Z}_{\mathrm{loc}}^{S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{x}_{\mathrm{loc}}^{\mathrm{pert} S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot\left[\boldsymbol{J}^{S} \cdot \boldsymbol{x}^{\mathrm{pert}}-\boldsymbol{x}_{2^{S}}\right] \tag{9.68}
\end{equation*}
$$

Looking at Eq. 9.68 it is easy to understand that now $\boldsymbol{x}_{\text {loc }}^{\text {pert } S}$ is considered like some sort of a displacements vector. In fact, it represents a configuration different from the reference one. It is also known that the coordinates of the original Pseudo-Structural Points do not
have angle of attack. Hence:

$$
\begin{array}{r}
\mathbf{0}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{x}_{\mathrm{loc}}^{S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot\left[\boldsymbol{J}^{S} \cdot \boldsymbol{x}-\boldsymbol{x}_{2^{S}}\right] \\
\Rightarrow \quad \boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot \boldsymbol{x}_{2^{S}}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot \boldsymbol{J}^{S} \cdot \boldsymbol{x}
\end{array}
$$

Thus,

$$
\begin{equation*}
\boldsymbol{Z}_{\mathrm{loc}}^{S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{x}_{\mathrm{loc}}^{\mathrm{pert} S}=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot \boldsymbol{J}^{S} \cdot\left[\boldsymbol{x}^{\mathrm{pert}}-\boldsymbol{x}\right]=\boldsymbol{I}_{z}^{S} \cdot \boldsymbol{E}^{S} \cdot \boldsymbol{J}^{S} \cdot \Delta \boldsymbol{x} \tag{9.69}
\end{equation*}
$$

where the vector $\Delta \boldsymbol{x}$ is known, being $\boldsymbol{x}$ and $\boldsymbol{x}^{\text {pert }}$ geometrical input data. By using the spline method and Eq. 9.40, it is possible to relate the slopes of the Control Points to the local out-of-plane coordinates of the Pseudo-Structural Points:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}^{S}}{\partial x^{S}}=\mathcal{D}^{S} \cdot \boldsymbol{S}^{S} \cdot \boldsymbol{Z}_{\mathrm{loc}}^{S}=\mathcal{D}^{S} \cdot \boldsymbol{S}^{S} \boldsymbol{I}_{z}^{S} \boldsymbol{E}^{S} \boldsymbol{J}^{S} \cdot \Delta \boldsymbol{x}=\mathcal{D}^{S} \cdot \overline{\boldsymbol{a}}_{3}^{S} \cdot \Delta \boldsymbol{x} \tag{9.70}
\end{equation*}
$$

Equation 9.70 can be written for all Wing Segments and so an assembly procedure is required to have all the local slopes of panels of the entire wing system due to the initial shape. Again, the assembly procedure is different from the classical way followed in structural finite element method. Before, the reason was that adjacent Wing Segments did not share any aerodynamic panels. On the contrary, now adjacent Wing Segments could share some common Pseudo-Structural Points (perturbed or not). In spite of such a remark, potential common points are considered in both the Wing Segments in writing $\boldsymbol{x}$ and $\Delta \boldsymbol{x}$. As a result, assembling $\mathcal{D}^{S} \overline{\boldsymbol{a}}_{3}^{S}$ terms, none of them has to be summed up to construct $\overline{\boldsymbol{A}}_{3}$ of Eq. 9.71:

$$
\begin{equation*}
\frac{\partial \mathcal{Z}_{\mathrm{loc}}}{\partial x}=\overline{\boldsymbol{A}}_{3} \cdot \Delta \boldsymbol{x} \tag{9.71}
\end{equation*}
$$

For the sake of thoroughness, the local out-of-plane coordinate of the Load Points can also be expressed (see Eq. 9.46 and 9.48):

$$
\begin{equation*}
\tilde{\mathcal{Z}}_{\text {loc }}^{S}=\tilde{\mathcal{D}}^{S \star} \overline{\boldsymbol{a}}_{3}^{S} \cdot \Delta \boldsymbol{x} \quad \Rightarrow \quad \tilde{\mathcal{Z}}_{\text {loc }}=\tilde{\overline{\boldsymbol{A}}}_{3}^{\star} \cdot \Delta \boldsymbol{x} \tag{9.72}
\end{equation*}
$$

As previously discussed, the boundary condition has been introduced to connect the slopes at Control Points and the dimensionless normalwash. It leads to Eq. 9.73:

$$
\begin{equation*}
\boldsymbol{w}=\frac{\mathrm{d} \boldsymbol{\mathcal { Z }}_{\mathrm{loc}}}{\mathrm{~d} x}=\overline{\boldsymbol{A}}_{3} \cdot \Delta \boldsymbol{x} \tag{9.73}
\end{equation*}
$$

Now the Vortex Lattice key equation is resumed. As seen before, Eq. 9.52 describes the dimensionless normalwash as function of the pressures on each aerodynamic panel. Let us resume it as follows:

$$
\boldsymbol{w}=\boldsymbol{A}^{D} \cdot \Delta \boldsymbol{p}
$$

Then, the expression relating the Lift Forces (written as their moduli) to the dimensionless pressures is given by Eq. 9.51 , which is written here:

$$
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \boldsymbol{I}_{D} \cdot \Delta \boldsymbol{p}
$$

The combination of the Eqs. 9.73 and 9.52 allows to find:

$$
\begin{equation*}
\Delta \boldsymbol{p}=\left[\boldsymbol{A}^{D}\right]^{-1} \boldsymbol{w}=\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \overline{\boldsymbol{A}}_{3} \cdot \Delta \boldsymbol{x} \tag{9.74}
\end{equation*}
$$

At last, the vector with the Aerodynamic Forces is written coupling Eqs. 9.74 and 9.51:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \boldsymbol{I}_{D} \cdot\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \overline{\boldsymbol{A}}_{3} \cdot \Delta \boldsymbol{x} \tag{9.75}
\end{equation*}
$$

With the help of the definition of a new matrix in which all the terms different from the dynamic pressure and the unknowns have been isolated:

$$
\begin{equation*}
\overline{\overline{\boldsymbol{c}}}=\boldsymbol{I}_{D} \cdot\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \overline{\boldsymbol{A}}_{3} \tag{9.76}
\end{equation*}
$$

finally Eq. 9.77 written in matrix notation is obtained:

$$
\begin{equation*}
\boldsymbol{L}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \overline{\overline{\boldsymbol{c}}} \cdot \Delta \boldsymbol{x} \tag{9.77}
\end{equation*}
$$

At this point, the Aerodynamic Loads are known as concentrated Forces localized on the aerodynamic panels. According to the FE approximation, they have to be transferred to the Structural Nodes.

From Eq. 9.77 it is possible to extract the Forces which are applied only on the aerodynamic panels of the generic Wing Segment $S$. This operation is particularly easy since the aerodynamic panels are numbered as explained above. In fact, the IDs of the panels are the single input needed to know the matrix of Eq. 9.78:

$$
\begin{equation*}
\boldsymbol{L}^{S}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \overline{\bar{c}}^{S} \cdot \Delta \boldsymbol{x} \tag{9.78}
\end{equation*}
$$

where $\overline{\overline{\boldsymbol{c}}}^{S}$ is directly obtained from $\overline{\overline{\boldsymbol{c}}}$. The transfer from Loads at the Aerodynamic Points to the Loads at Structural Nodes is performed so that they will be energetically equivalent. For this purpose, the Principle of Virtual Displacements is involved. Since the virtual variation of out-of-plane displacements of Load Points is going to be computed, it is advantageous to write them as functions of nodal degrees of freedom, resuming Eq. 9.59:

$$
\tilde{\mathcal{Z}}_{\mathrm{loc}}^{S}=\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S} \cdot \boldsymbol{q}
$$

From its definition it follows that the virtual work $\delta W$ of all the Aerodynamic Forces applied in Wing Segment $S$ is formally coincident with Eq. 9.60, given by

$$
\begin{equation*}
\delta W=\left[\delta \tilde{\boldsymbol{\mathcal { Z }}}_{\mathrm{loc}}^{S}\right]^{T} \cdot \boldsymbol{L}^{S}=\left[\tilde{\mathcal{D}}^{S \star} \boldsymbol{a}_{3}^{S} \cdot \delta \boldsymbol{q}\right]^{T} \cdot \boldsymbol{L}^{S}=\delta \boldsymbol{q}^{T} \cdot\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \boldsymbol{L}^{S} \tag{9.79}
\end{equation*}
$$

Again, let us follow the same procedure used to find the Aeroelastic Stiffness matrix. Thus, the nodal forces on all Structural Nodes equivalent to the Aerodynamic Loads applied on the panels of Wing Segment $S$ are collected into the vector $\boldsymbol{L}_{\text {str }}^{S}$. In conclusion, Eq. 9.80 is obtained:

$$
\begin{equation*}
\boldsymbol{L}_{\mathrm{str}}^{S}=\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \boldsymbol{L}^{S}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \overline{\boldsymbol{c}}^{S} \cdot \Delta \boldsymbol{x} \tag{9.80}
\end{equation*}
$$

If all the contributions of all Wing Segments are added following Eq. 9.62, the loads $\boldsymbol{L}_{\text {RHS }}$ on the Structural Nodes can be obtained, where RHS means Right Hand Side. The assembly of the matrices $\frac{1}{2} \rho_{\infty} V_{\infty}^{2}\left[\boldsymbol{a}_{3}^{S}\right]^{T} \cdot\left[\tilde{\mathcal{D}}^{S \star}\right]^{T} \cdot \overline{\overline{\boldsymbol{c}}}^{S}$ is equal to the procedure carried out in the previous section. In fact, it consists in summing up the various contributions corresponding to different Wing Segments.

At the end, the final aeroelastic equation to be solved is:

$$
\begin{equation*}
\left[\boldsymbol{K}_{\mathrm{str}}+\boldsymbol{K}_{\mathrm{aero}}\right] \cdot \boldsymbol{q}=\boldsymbol{L}_{\mathrm{RHS}} \tag{9.81}
\end{equation*}
$$

or

$$
\begin{equation*}
\boldsymbol{K}_{\text {aeroelastic }} \cdot \boldsymbol{q}=\boldsymbol{L}_{\mathrm{RHS}} \tag{9.82}
\end{equation*}
$$

Equation 9.82 allows to compute the vector of unknowns nodal degrees of freedom $\boldsymbol{q}$. Now that the right hand side is different from zero, a non-trivial solution exists.

### 9.5 Effect of cross-section in-plane deformation on the aeroelastic coupling

### 9.5.1 Cross-section in-plane deformation for classical and refined beam models

Classical beam models such as Euler-Bernoulli's and Timoshenko's assume that the section is not deformable, i.e. there is no in-plane cross-section deformation (distortion). An example of wing cross-section distortion is reported in Fig. 9.1. The cross-section indeformability makes classical beam model unsuitable to analyze cases in which the deformation of the cross-section is crucial, such as profiles of smart wings.


Figure 9.1: Distortion of cross section.
A strategy to overcome this difficulty is either to employ more complex models. For this purpose, the hierarchical refined 1D CUF model presented in this thesis is here used to analyze the aeroelastic response of wings taking into account the cross-section distortion.

### 9.5.2 A numerical approach for wing aerodynamic analysis

## Preliminaries

The evaluation of aerodynamic loads can be typically carried out through a CFD code which solves for example either Navier-Stokes equations or Euler equations numerically. This kind of analysis has a high computational cost but under some assumptions it is possible to employ simplified approaches. In the wing cases considered in the present work, the flow field is assumed to be steady and the fluid viscosity is not decisive since the taking place viscous effects can be confined into a small region (boundary layers and wake regions). The fluid can be thus considered as inviscid and the flow field is irrotational, since the curl of the velocity vector $\mathbf{V}(x, y, z)$ is equal to zero:

$$
\begin{equation*}
\nabla \times \mathbf{V}=\mathbf{0} \tag{9.83}
\end{equation*}
$$

In this case the velocity vector $\mathbf{V}(x, y, z)$ can be considered as the gradient of a potential function $\phi$ :

$$
\begin{equation*}
\nabla \phi=\mathbf{V} \tag{9.84}
\end{equation*}
$$

Hence, the analysis of a wing or an airfoil under these conjectures can be performed by potential methods. The potential function describing the flow field around an object can be described as a combination of singularities such as doublets, vortices, sources or uniform flux over the external body surface. According to the detailed exposition in [134], the equation to be used to compute the solution of the aerodynamic problem is the Laplace's equation:

$$
\begin{equation*}
\nabla^{2} \phi=0 \tag{9.85}
\end{equation*}
$$

Laplace's equation describes a potential flow field only if the compressibility effects can be neglected, as occurs for the results presented afterwards where the free stream air velocity is rather low. Otherwise some corrections, e.g. Prandtl-Glauert transformation, are necessary as explained in [135]. The assumptions here introduced lead to an integral-differential equation which expresses the potential function in an arbitrary point of the fluid domain as a combination of singularities. For the sake of completeness, this equation is not reported here but more details can be found in [134]. Among the potential methods, the panel methods can be formulated following a low-order or a high- order approach. Low-order (first-order) panel method employs triangular or quadrilateral panels having constant values of singularities strength such as Hess and Smith approach. Higher-order panel methods instead use higher than first-order panels (e.g. paraboloidal panels) and a varying singularity strength over each panel.

## XFLR5: an implementation of aerodynamic potential methods

XFLR5 is a software developed by Andre Deperrois. It performs viscous and inviscid aerodynamic analysis on airfoils and wings using three potential methods: the LLT (Lifting Line Theory), the VLM (Vortex Lattice Method) and the 3D Panel Method. The LLT method derives from the Prandtl's wing theory and considers the wing as a linear distribution of vortices. The VLM considers a wing as an infinitely thin lifting surface via a distribution of vortices placed over a wing reference surface. This method requires the non-penetration condition on the reference surface as a boundary condition. Hence, the normal component of the induced velocity $\mathbf{V}_{i}$ on the generic $i^{t h}$ aerodynamic panel with normal vector $\mathbf{n}_{i}$ is equal to zero:

$$
\begin{equation*}
\mathbf{V}_{i} \cdot \mathbf{n}_{i}=0 \tag{9.86}
\end{equation*}
$$

Further details on this method can be found in [133]. The 3D Panel Method schematizes the wing surface as a distribution of doublets and sources. The strength of the doublets and sources is calculated to meet the appropriate boundary conditions, which may be of Dirichlet- or Neumann-type. According to the creator of the program, after a trial and error process, the best results can be obtained just using the Dirichlet BC type [136]. The 3D Panel Method employs a low-order panel method. The LLT approach is not able to evaluate the pressure coefficients on the wing surface, but only the lifting loads along the lifting line. The VLM is able to analyze the pressure coefficients but only on the reference surface which is defined as the mean surface between the upper and the lower wing surfaces. The 3D Panel Method is able to calculate the pressure coefficients on both the upper and the lower wing surfaces. Therefore, this method offers the most realistic description of the aerodynamic field in XFLR5.

### 9.5.3 Aeroelastic static response analysis via iterative scheme

In this work the aeroelastic static response of the wing is computed through an iterative procedure based on a coupled CUF-XFLR5 method. Hence, the aerodynamic analysis is
performed through the potential methods available in XFLR5, as previously mentioned, whereas variable kinematic 1D CUF models provide the structural wing deformation with a variable ex-pansion order $N$.

## Iterative procedure



Figure 9.2: Aeroelastic iterative procedure with controllers on aerodynamic coefficients and wing deformation.

Figure 9.2 shows in detail the aeroelastic iterative process, which starts with the evaluation of the pressure coefficients in the undeformed wing configuration. The further steps to be repeated for each iteration are:

1. post-processing calculation of the aerodynamic forces;
2. structural analysis of the wing subject to the aerodynamic forces previously computed;
3. new calculation of the aerodynamic pressure coefficients for the new deformed configuration;
4. post-processing evaluation of the wing deformation and cross-section distortion.

Structural displacements are evaluated in specific sections distributed regularly along the wing span. The cross-section distortion $s$ is defined as the in-plane displacement, i.e. a quantity which expresses the in-plane difference between the deformed shape and the "undeformed" shape of the airfoil cross-section:

$$
\begin{equation*}
s=\sqrt{\Delta u_{x}^{2}+\Delta u_{z}^{2}} \tag{9.87}
\end{equation*}
$$

where $\Delta u_{x}$ and $\Delta u_{z}$ are the cartesian components of the relative displacement vector $\Delta \mathbf{u}$ along the chord direction $x$ and the transversal direction $z$, respectively, between the deformed cross-section and the base section. Given a structural model, the base section corresponds to the undeformed cross-section shifted and rotated in such a way that its


Figure 9.3: Aerolastic iterative convergence.
leading edge and trailing edge points corresponds to the leading edge and trailing edge points of the deformed cross-section obtained by such a structural model.

The iterative process in Fig. 9.2 is stopped once the convergences of the lifting coefficient $C_{L}$, the moment coefficient $C_{M}$, and the cross-section distortion of the wing sections are achieved simultaneously, as can be shown in Fig. 9.3. The description of a similar iterative process can be found also in [137]. The convergence controls are thus:

$$
\begin{align*}
& \frac{\left|C_{L}^{i}-C_{L}^{i+1}\right|}{C_{L}^{i}}<\text { toll } ; \quad \frac{\left|C_{M}^{i}-C_{M}^{i+1}\right|}{C_{M}^{i}}<\text { toll }  \tag{9.88}\\
& \frac{\left|\bar{s}^{i}-\bar{s}^{i+1}\right|}{\bar{s}^{i}}<\text { toll } \tag{9.89}
\end{align*}
$$

where toll is equal to $10^{-4}, C_{L}^{i}$ and $\bar{s}^{i}$ are the lifting coefficient and the average cross-section distortion for the generic $i^{\text {th }}$ iteration, respectively. The average distortion $\bar{s}$ is defined in Eq. 12.3. A linear approach is adopted as usual in classical aeroelasticity: for each iteration the aerodynamic loads computed for the deformed wing configuration are applied to the undeformed configuration, without changing the structural stiffness matrix K of Eq. 3.89.

## Aerodynamic loads computation

The aerodynamic load computed by XFLR5 is a distributed pressure and in this work it is modeled as distributed forces. The generic force acting on the $j^{t h}$ aerodynamic panel is evaluated as:

$$
\begin{equation*}
F_{j}=\frac{1}{2} C_{p}^{j} \rho_{\infty} V_{\infty}^{2} A_{j} \tag{9.90}
\end{equation*}
$$

where $V_{\infty}$ is the free stream velocity and $\rho_{\infty}$ is the air density. $A_{j}$ is the area of the $j^{t h}$ aerodynamic panel which the pressure coefficient $C_{p}^{j}$ refers to. According to XFLR5 notation, normal vectors are considered positive when $C_{p}^{j}$ is negative and their verse is outer-pointing. Each aerodynamic force is transferred from the aerodynamic model to the structural model following the approach described in section 3.2.4 for the generic point $\operatorname{load} \mathbf{P}$.

For each iteration, the three-dimensional deformed configuration of the wing is built using 11 airfoils along the half-wing span at a distance of 0.5 m from each other. The first
section lies at the wing root. The wing is discretized through a lattice of quadrilateral aerodynamic panels. Let $N_{A P}^{x}$ be the number of panels along the chord line and let $N_{A P}^{y}$ be the number of panels along the half-wing span. When the VLM is employed the total number of aerodynamic panels $N_{A P}$ is equal to $2 N_{A P}^{x} N_{A P}^{y}$. For the 3D Panel Method $N_{A P}$ must be calculated as $4 N_{A P}^{x} N_{A P}^{y}+2 N_{A P}^{x}$, where the term $4 N_{A P}^{x} N_{A P}^{y}$ is the number of panels along the wing span on the upper and lower surfaces of the wing. In addition, the term $2 N_{A P}^{x}$ is the number of panels placed on the tip lateral cross-sections. For the sake of convenience, only half-wing is analyzed since the aerodynamic loads are considered to be symmetric with respect to the wing root.

## Chapter 10

## Results: structural response to aerodynamic VLM loadings

Conventional wing configurations, which are beam-like structures, are analyzed and the results are presented here. They are subjected to aerodynamic, bending, and torsional loadings. Clamped boundary condition is accounted for. The beams are considered to have an unconventional airfoil-shaped section or a conventional thin-walled rectangular cross-section. The wing shapes examined below are summarized in Table 10.1. An isotropic material is used. Young's modulus $E$ is equal to $69[\mathrm{GPa}]$ and Poisson's ratio $\nu$ is equal to 0.33 . For all the cases presented, a selective integration of the shape functions along the beam axis is adopted. Moreover, the aerodynamic loadings refer to an air density equal to $1.225\left[\mathrm{~kg} / \mathrm{m}^{3}\right]$.

Table 10.1: Wing configurations adopted to discuss the results.

| Name | A | B | C | D | E | F |
| :--- | ---: | ---: | ---: | ---: | ---: | ---: |
| Section | Airfoil | Airfoil | Airfoil | Wing box | Airfoil | Airfoil |
| $\Lambda$ | $+13.5^{\circ}$ | $0^{\circ}$ | $0^{\circ}$ | $0^{\circ}$ | $-13.5^{\circ}$ | $0^{\circ}$ |
| $\Gamma$ | $0^{\circ}$ | $0^{\circ}$ | $0^{\circ}$ | $10^{\circ}$ | $0^{\circ}$ | $0^{\circ}$ |
| $\lambda$ | 0.25 | 1 | 1 | 0.25 | 0.25 | 0.25 |
| $c_{\text {root }}[\mathrm{m}]$ | 1.6 | 1 | 0.2 | 1.6 | 1.6 | 1.6 |
| $c_{t i p}[\mathrm{~m}]$ | 0.4 | 1 | 0.2 | 0.4 | 0.4 | 0.4 |
| $\bar{c}[\mathrm{~m}]$ | 1 | 1 | 0.2 | 1 | 1 | 1 |
| $L[\mathrm{~m}]$ | 5 | 5 | 4 | 7 | 5 | 5 |
| $L / \bar{c}$ | 5 | 5 | 20 | 7 | 5 | 5 |
| $b[\mathrm{~m}]$ | 10 | 10 | 8 | 14 | 10 | 10 |
| $S_{w}\left[\mathrm{~m}^{2}\right]$ | 10 | 10 | 1.6 | 14 | 10 | 10 |
| $A R$ | 10 | 10 | 40 | 14 | 10 | 10 |

Airfoil: NACA 2415 airfoil profile with 3 cells.
Wing box: thin-walled rectangular cross-section.

| $\Lambda:$ sweep angle | $\Gamma:$ dihedral angle | $\lambda:$ taper ratio | $\bar{c}:$ mean chord |
| :--- | :--- | :--- | :--- |
| $L:$ beam's length | $b:$ wingspan | $S_{w}:$ wing area | $A R:$ aspect ratio |

### 10.1 Aerodynamic model assessment

The first assessment examines to aerodynamic implications closely related to the Vortex Lattice formulation. For this purpose, a first wing shape is chosen and named configuration A. It consists of a swept tapered wing subjected to an aerodynamic load. The NACA 2415 airfoil profile is adopted as cross-section, which is subdivided into three cells. The cells are obtained by inserting two spars along the span-wise direction at $25 \%$ and $75 \%$ of the chord length, see Fig. 10.1a. Their thicknesses are respectively $10 \%$ and $7 \%$ of the maximum airfoil thickness, whereas the percentage is about $4 \%$ for the skin. The aspect ratio $A R$ is defined as the square of the wingspan $b$ divided by the area of the wing planform $S_{w}$ and is equal to 10 for configuration $A$. Thus its half-wing corresponds to a clamped beam which has a span-to-mean chord ratio $L / \bar{c}$ equal to 5 .

(a) NACA 2415 with 3 cells

(b) Thin-walled rectangle

Figure 10.1: Cross-sections used for the wing configurations.
An aerodynamic mesh composed of $4 \times 40$ panels is set on the surface of the structure. The angle of attack $\alpha$ of the wing and the free stream velocity $V_{\infty}$ are considered as free parameters of the first analysis. The former ranges from 30 to $70[\mathrm{~m} / \mathrm{s}]$, whereas the latter is from $1^{\circ}$ to $5^{\circ}$. They represent acceptable values for the hypothesis of VLM. The effect of such parameters on the Total Lift Force $L$ is investigated and shown in Table 10.2. By comparing the values for $\alpha=1^{\circ}$ and $\alpha=5^{\circ}$, the trend of $L$ confirms its correlation with $\tan \alpha$ as exposed in Eq. 10.1, since $\tan (\pi-\alpha)$ is equal to $-\tan \alpha$. For the sake of brevity, the results corresponding to the in-between angles of the range are not reported. As far as the stream velocity is concerned, $L$ is dependent on the square of $V_{\infty}$, as confirmed in Eq. 10.1.

$$
\begin{align*}
\boldsymbol{L} & =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \boldsymbol{I}^{D} \cdot\left[\boldsymbol{A}^{D}\right]^{-1} \cdot \boldsymbol{w}  \tag{10.1}\\
& =\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \tan (\pi-\alpha) \boldsymbol{I}^{D}\left[\boldsymbol{A}^{D}\right]^{-1} \boldsymbol{d}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \tan (\pi-\alpha) \overline{\boldsymbol{b}}
\end{align*}
$$

Configuration $A$ is now discretized by 20 refined $B 4$ elements and the previous study is carried out on the maximum displacement $u_{z \max }$ where the expansion order is $N=3$. The trend of $u_{z \text { max }}$ in Table 10.2 appears to be exactly the same as that obtained by $L$ and expressed as follows:

$$
\begin{gathered}
\frac{L\left(\alpha=5^{\circ}, V=30 \div 70\right)}{L\left(\alpha=1^{\circ}, V=30 \div 70\right)}=\frac{u_{z \max }\left(\alpha=5^{\circ}, V=30 \div 70\right)}{u_{z \max }\left(\alpha=1^{\circ}, V=30 \div 70\right)}=\frac{\tan \left(\alpha=5^{\circ}\right)}{\tan \left(\alpha=1^{\circ}\right)} \cong 5 \\
\frac{L\left(\alpha=1^{\circ} \div 5^{\circ}, V=60\right)}{L\left(\alpha=1^{\circ} \div 5^{\circ}, V=30\right)}=\frac{u_{z \max }\left(\alpha=1^{\circ} \div 5^{\circ}, V=60\right)}{u_{z \max }\left(\alpha=1^{\circ} \div 5^{\circ}, V=30\right)}=\frac{(V=60)^{2}}{(V=30)^{2}}=4
\end{gathered}
$$

Given a reference combination of $\alpha$ and $V_{\infty}$, changing one of either parameters implies applying new aerodynamic loads on the panels which are all multiplied by the same factor with respect to the reference loading. The outcome is that the pressure distribution and the

Table 10.2: Effect of $V_{\infty}$ and $\alpha$ on Total Lift Force and $u_{z \text { max }}$. Configuration $A$.

| Theory : $N=3$ |  | $20 B 4$ elements | $4 \times 40$ panels |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $V_{\infty}[m / s]$ | 30 | 40 | 50 | 60 | 70 |
|  | $237.22^{a}$ | 421.72 | 658.94 | 948.87 | 1291.5 |
| $\alpha=1^{\circ}$ | $0.3816^{b}$ | 0.6784 | 1.0600 | 1.5264 | 2.0776 |
|  | 1189.0 | 2113.8 | 3302.7 | 4756.0 | 6473.4 |
| $\alpha=5^{\circ}$ | 1.9127 | 3.4003 | 5.3129 | 7.6506 | 10.413 |

${ }^{a}$ Total Lift Force $L[\mathrm{~N}]$
${ }^{b}$ Maximum displacement $u_{z \max }[\mathrm{~mm}]$
deflection on the wing do not change in shape, but only in value. From a numerical point of view, the reason is that the aerodynamic mesh is considered to be fixed. In conclusion, the Total Lift Force $L$ and the maximum displacement $u_{z \max }$ are influenced in the same way by the angle of attack $\alpha$ and the free stream velocity $V_{\infty}$.

The effect of the aerodynamic parameters on $L$ is also shown in Fig. 10.2. Because of the model's linearity, only small angles of attack can be considered and the Lift changes almost linearly with $\alpha$. It is noteworthy that the trend is affected neither by the chosen aerodynamic mesh nor by the structural theory involved. On the contrary, it depends only on the used aerodynamic model and is valid for any wing configuration. Although only the clamped right half-wing of the structure is considered, the aerodynamic computation takes into account the wing as a whole. This capability exploits the symmetry condition in the formulation of Vortex Lattice Method and is adopted for the following analyses, unless differently specified.


Figure 10.2: Effect of $V_{\infty}$ and $\alpha$ on Total Lift Force. Aerodynamic mesh: $4 \times 40$ panels. Configuration A.

The second analysis investigates the pressure distribution on the wing. For this purpose, the straight wing named configuration $B$ is used and exposed to a free stream velocity equal to $50[\mathrm{~m} / \mathrm{s}]$ with $\alpha=3^{\circ}$. As before, the structure is modelized via the clamped
right half-wing but the letf one is considered in the computation of aerodynamic loading. Figure 10.3 b shows the resulting distribution of dimensionless pressure with respect to the dynamic pressure $\frac{1}{2} \rho V_{\infty}^{2}$. In particular, the three-dimensional trend is drawn upon the undeformed configuration. Besides, the bidimensional projection of the pressures onto the aerodynamic mesh is shown in colour at the top of the figure. The distribution is based on discrete pressure values corresponding to the chosen discretization of $20 \times 40$ panels. For any wing section, the maximum pressure value acts on its leading edge. By enabling the aerodynamic symmetry, the distribution along the spanwise direction is symmetrical to the longitudinal aircraft plane. Thus it has a maximum value placed on the root cross-section and decreases as $y$ increases according to typical pressure profiles for subsonic aircraft aerodynamics.


Figure 10.3: Dimensionless pressure distribution for enabled and disabled aerodynamic symmetry condition. $V_{\infty}=70 \mathrm{~m} / \mathrm{s}$ and $\alpha=5^{\circ}$. Aerodynamic mesh: $20 \times 40$ panels. Configuration $B$.

It is interesting to note how the dimensionless pressure distribution strongly changes when the aerodynamic symmetry is disabled. In Fig. 10.3a the trend is no longer symmetrical to the longitudinal plane, but it is now symmetrical to the middle plane of the half-wing. In fact now, from an aerodynamic point of view, the free edges exposed to the free stream are the tip and root sides.

Table 10.3 shows how dimensionless pressure values change when the symmetry condition is taken into account and when it is not taken into account ("enabled" or "disabled"). Although for any section the maximum load always acts on its leading edge, now the maximum pressure is no longer at $y=0$, but at $y=\frac{L}{2}$ and its value is lower than before. Moreover, not only is the maximum value different, but the pressure at the mid-cross-section of the half-wing is also lower. This is a clear example of the way the left half-wing influences the aerodynamic field also around the right one. In particular, its presence ha sthe effect of increasing the Total Lift Force.

The third analysis focuses on the effect on the results of the aerodynamic mesh in shape and refinement. At first configuration $A$ is discretized via a uniform lattice of panels with a variable panel ratio $(\mathrm{PR})$. It is a parameter defined as the ratio between the chordwise and spanwise lengths of the used aerodynamic elements. In other words, it represents

Table 10.3: Dimensionless pressure values for enabled and disabled aerodynamic symmetry.

## Aerodynamic symmetry condition

| $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$ | $\alpha=3 \mathrm{deg}$ |  | $20 \times 40$ panels |
| :---: | :--- | :---: | :---: |
| Symmetry | $p_{y=0}$ | $p_{y=L / 2}$ | $p_{y=L}$ |
| Enabled | $2.4417^{\star}$ | 2.3324 | 1.0341 |
| Disabled | 0.9974 | $2.1368^{\star}$ | 0.9974 |

* Maximum dimensionless pressure value
the mesh shape. The number of panels along $x$ axis is the last parameter of the analysis, influencing the total number of elements as a consequence. The aerodynamic field ( $\alpha$ and $V_{\infty}$ ) is considered to be constant in the analysis in order to evaluate only the mesh effect. The structural mesh stays fixed on $20 B 4$ elements and the theory has $N=3$.

Table 10.4: Convergence study: Effect of the aerodynamic mesh on Total Lift Force and $u_{z \max }$. Configuration $A$.


The Total Lift Force and the maximum transverse displacement $u_{z \text { max }}$ are reported in Table 10.4. For each panel ratio, the convergence is investigated as the number of panels changes. As expected, the trends confirm that the convergence of the VLM does not depend on the mesh shape. In particular, both the displacement and the Lift decrease as $N_{A P}$ increases approaching the final value. It is interesting to note that the results are different according to the panels' shape. For instance, considering panel ratio as equal to 0.5 and panel ratio as equal to 2.0 with the same $N_{A P}=40$, the aerodynamic loading and the resulting deflection differ. The same situation recurs for $N_{A P}=160$. The $0.5-\mathrm{mesh}$ is
twice as refined as the 2.0 -mesh along the chord direction, whereas the opposite occurs along the spanwise direction. Since the results are different, it means that the refinement of the aerodynamic mesh leads to solutions which depend on the direction the refinement acts along.

In order to make more detailed considerations, the percentage errors defined in Eq. 10.2 are introduced. For each panel ratio, the reference values $L_{\text {REF }}$ and $u_{z \text { max REF }}$ refer to the Lift and the displacement with 10 panels along the $x$ axis. Their trend versus $N_{A P}$ are summarized in Fig. 10.4.

$$
\begin{equation*}
\% \text { Error } L=\frac{L-L_{\mathrm{REF}}}{L_{\mathrm{REF}}} \quad \% \text { Error } u_{z \max }=\frac{u_{z \max }-u_{z \max \mathrm{REF}}}{u_{z \max \mathrm{REF}}} \tag{10.2}
\end{equation*}
$$

The numerical convergence is more evident on the graph, but it is different for the Lift and the deflection, as can be easily noted. This is true for any panel ratio (PR) involved. The reason is that refining the aerodynamic discretization leads to a more accurate evaluation of the Total Lift Force. However it is the result of a better description of the pressure distribution and its corresponding shape. This means that by increasing $N_{A P}$ the resulting loads not only differ in magnitude, but are also distributed differently. It is interesting to note that the deflection depends not only on the magnitude of the load, but also on its distribution. Thus, the dissimilar trends shown in Fig. 10.4 are plausible since they confirm the obtainment of different pressure profile as the panels increase.


Figure 10.4: Error of Total Lift Force and $u_{z \max }$ as a function of the aerodynamic mesh. Structural mesh: $20 B 4$ elements . Theory: $N=3$. Configuration $A$.

The analysis on the aerodynamic mesh is also extended to configuration $B$ and Table 10.5 reports the results. Globally, the conclusions are very similar to the previous case. In fact, the convergent trend is verified and confirms that it is independent of the panel ratio. The most evident difference is the rate of convergence, which is higher for the swept wing case for every panel ratio. Moreover, the comparison between the values obtained with the same number of panels and different panel ratio suggests a less notable dependence on the mesh shape for configuration $A$. In any case, this point should be analyzed more in detail. Finally it is possible to conclude that the convergence of VLM is ensured as expected, at least for simple wing configurations, but its speed depends on the shape of the panels and the geometry of analyzed structure.

Table 10.5: Convergence study: Effect of the aerodynamic mesh on Total Lift Force and $u_{z \max }$. Configuration $B$.

| $V_{\infty}=5$ |  | 3 deg | Theory | $N=3$ | $20 \mathrm{B4} \mathrm{el}$. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Number of panels along x axis |  |  |  |  |
| Panel <br> ratio | 2 | 4 | 6 | 8 | 10 |
| 0.5 | $(10)^{a}$ | (40) | (90) | (160) | (250) |
|  | $2039.1^{\text {b }}$ | 1995.9 | 1979.2 | 1970.4 | 1965.0 |
|  | $9.4821^{c}$ | 9.0617 | 8.8903 | 8.7985 | 8.7413 |
| 1.0 | (20) | (80) | (180) | (320) | (500) |
|  | 1994.7 | 1970.1 | 1961.2 | 1956.6 | 1953.8 |
|  | 9.0538 | 8.7957 | 8.7010 | 8.6519 | 8.6219 |
| 2.0 | (40) | (160) | (360) | (640) | (1000) |
|  | 1968.4 | 1956.1 | 1951.7 | 1949.4 | 1948.0 |
|  | 8.7845 | 8.6485 | 8.6002 | 8.5753 | 8.5601 |
| ${ }^{a}$ (Total number of panels $N_{A P}$ ) <br> ${ }^{b}$ Total Lift Force $L[\mathrm{~N}]$ <br> ${ }^{c}$ Maximum displacement $u_{z \text { max }}$ |  |  |  |  |  |

### 10.2 Structural model assessment

At this point the considerations closely related to the aerodynamic method are concluded. From now on the analyses deal mainly with the structural behavior of the configurations summarized in Table 10.1. In particular, the fourth assessment of the present chapter discusses the effect of the structural parameters for the first four wings undergoing an aerodynamic load with a pressure distribution similar to Fig. 10.3b.

At first, configuration $B$ rotated with an angle of attack $\alpha=3^{\circ}$ and exposed to a free stream with $V_{\infty}=50[\mathrm{~m} / \mathrm{s}]$ is considered. Referring to the last analysis of Table 10.5, the aerodynamic mesh used is fixed to $4 \times 40$ panels. In this situation, the loading is approximately similar to that obtained with much more refined meshes, but with the advantage of a less expensive numerical cost. In any case, such a choice is arbitrary in that at this point the aim is to evaluate the structural behavior and the conclusions are independent of $N_{A P}$. A structural convergence study is carried out to evaluate the combined effect of the number of Finite Elements $N_{E L}$ and the expansion order $N$ on the solution. The mechanics of the beam is described in terms of the maximum vertical displacement $u_{z \text { max }}$, which is located at the leading edge of the tip cross-section for configuration $B$. This location derives from the coupling of bending and torsional loads applied on the wing exposed to the free stream. Typically, the torsion of a straight wing about $y$ axis tends to lift the leading edge of any cross-section. The results are summarized in Table 10.6.

In accordance with the typical behavior of FEM solutions, the maximum displacement increases and becomes more accurate as $N_{E L}$ increases. Therefore a higher number of elements enhances the flexibility of the structure. An excellent agreement is obtained between NASTRAN and the higher-order model's results. Table 10.6 shows the percentage error in computing the maximum displacement for all the theories and a mesh of 40

Table 10.6: Convergence study: Effect of the number of $B 4$ elements on $\bar{u}_{z \max }[\mathrm{~mm}]$ for different beam models. Configuration $B$.

|  | $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$ |  | $\alpha=3 \mathrm{deg}$ | $4 \times 40$ panels |  |  |
| :---: | :---: | :---: | :---: | ---: | :---: | :---: |
|  |  |  |  |  |  |  |
| $N_{E L}$ | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| 2 | 8.8021 | 8.8119 | 8.8189 | 8.2425 | 8.3508 | 8.3686 |
| 5 | 8.8021 | 8.8119 | 8.8189 | 8.4300 | 8.5586 | 8.5805 |
| 10 | 8.8021 | 8.8119 | 8.8189 | 8.4880 | 8.6202 | 8.6431 |
| 20 | 8.8021 | 8.8119 | 8.8189 | 8.5159 | 8.6485 | 8.6723 |
| 40 | 8.8021 | 8.8119 | 8.8189 | 8.5285 | 8.6607 | 8.6854 |
|  | $+1.451 \%$ | $+1.564 \%$ | $+1.645 \%$ | $-1.702 \%$ | $-0.179 \%$ | $+0.106 \%$ |

NASTRAN (solid): 8.6762
elements. The trend is convergent for each adopted theory as reported in Fig. 10.5a. In particular, when the theory is linear (EBBM, TBM and $N=1$ ), the results are not affected by $N_{E L}$ whereas for $N>1$ the solution approaches a reference value. This shows that the refinment of $B 4$ elements-mesh for a straight wing charged with this kind of distributed load is not the best way to improve the results if a first-order model is involved.


Figure 10.5: Maximum transverse displacement $u_{z \max }$ as a function of the structural mesh and the models involved for straight wings.

As far as the approximation order is concerned, the linear theories give very similar results, although they are slightly different (in the third significant digit). The absolute value of the gap among $N \geq 1$ theories is more evident underlining the importance of increasing $N$ to reach convergent results. However, this gap decreases in absolute value as the expansion order increases. Higher orders than linear approximation yield a more flexible structure. It is interesting to note how $u_{z \text { max }}$ decreases when the theory changes from a linear to a parabolic form. The main reason for this turnaround stands in Poisson's locking correction adopted only for $N=1$. In fact, according to Carrera and Brischetto $[120,121]$ this correction is detrimental for orders higher than the linear one.

At this point a slender wing is introduced and named configuration C. Essentially, its geometry is analogous to configuration $B$, with the sole difference of a higher value for the parameter $L / \bar{c}$. Analysis of the combined convergence is carried out again and the
conclusions are similar to the previous case, see Table 10.7. This time the free stream velocity is lower, equal to $20[\mathrm{~m} / \mathrm{s}]$, in order to avoid large deflections and so to respect the linear approximation. In the matter of numerical mesh, the convergent trend of $u_{z \max }$ as $N_{E L}$ increases is shown in Fig. 10.5b and is the same as for configuration $B$. Therefore, it appears to be the distinctive numerical trend for a straight beam in a load case which is mainly bending, regardless of its slenderness.

Table 10.7: Convergence study: Effect of the number of $B 4$ elements on $\bar{u}_{z \max }[\mathrm{~cm}]$ for different beam models. Configuration $C$.

|  | $V_{\infty}=20 \mathrm{~m} / \mathrm{s}$ |  | $\alpha=3 \mathrm{deg}$ | $4 \times 40$ panels |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| $N_{E L}$ | EBBM | TBM | $N=1$ |  |  |  |
| 2 | 9.4210 | 9.4216 | 9.4220 | 8.9471 | 8.9603 | 8.9615 |
| 5 | 9.4210 | 9.4216 | 9.4220 | 9.1999 | 9.2263 | 9.2291 |
| 10 | 9.4210 | 9.4216 | 9.4220 | 9.2718 | 9.3067 | 9.3108 |
| 20 | 9.4210 | 9.4216 | 9.4220 | 9.3036 | 9.3418 | 9.3466 |
| 40 | 9.4210 | 9.4216 | 9.4220 | 9.3186 | 9.3578 | 9.3628 |
|  | $+1.046 \%$ | $+1.052 \%$ | $+1.056 \%$ | $-0.053 \%$ | $+0.368 \%$ | $+0.422 \%$ |

NASTRAN (solid): 9.3235

As far as expansion order is concerned, the results highlight a similar diversification of the theories. Nevertheless, by decreasing $L / \bar{c}$ the accuracy of the model becomes more and more important. For configuration $C$ less remarkable differences have been found in the case $N>1$ with respect to configuration $B$. As a matter of fact, cubic and fourth-order approximations differ in the fourth significant digit. Besides, the linear theories give the same results, as first-order approximation matches the classical models. Now the linear terms of $u_{x}$ and $u_{z}$ and the shear effects can be neglected, whereas, in the case of short beams, the shear effects are important since the EBBM solution differs from the TBM and first-order results. It should be noted that EBBM presents more stiffness than TBM: $+0.111 \%$ for shorter configuration $B$ versus $+0.006 \%$ for slender configuration $C$. On the contrary, EBBM provides less stiffness than $N=4:-1.344 \%$ for shorter configuration $B$ versus $-0.622 \%$ for slender configuration $C$. An excellent agreement with NASTRAN is again obtained by higher-order models.

After having analyzed straight wings, the fourth assessment continues to focus on two tapered structures. The first is configuration $A$, which has already been introduced. As usual, the geometry exposed to $V_{\infty}=50[\mathrm{~m} / \mathrm{s}]$ with $\alpha=3^{\circ}$ is discretized by $4 \times 40$ aerodynamic panels. Table 10.8 summarizes the corresponding results and Fig. 10.6a shows the trends as the number of elements changes for each theory involved. For any mesh with more than 2 elements, $u_{z \text { max }}$ increases with $N$ when $N$ is greater than 1, to such an extent that no remarkable differences are detected for high-order expansion, so confirming the convergence on $N$. For a poor mesh of 2 elements, the model presents a convergence problem which is evidently due to the swept configuration. As will be clear in the following, this swept wing is subjected to a significant twist of the cross-section. Bearing in mind that linear theories (EBBM, TBM, and $N=1$ ) are not able to handle this mechanical behavior, the Poisson's locking correction is not sufficient to make them effective in computing the maximum displacement. On the contrary, the refined models approach NASTRAN results.


Figure 10.6: Maximum transverse displacement $u_{z \max }$ as a function of the structural mesh and the models involved for tapered wings.

Table 10.8: Convergence study: Effect of the number of $B 4$ elements on $\bar{u}_{z \max }[\mathrm{~mm}]$ for different beam models. Configuration $A$.

|  | $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$ |  | $\alpha=3 \mathrm{deg}$ | $4 \times 40$ panels |  |  |
| :---: | :---: | :---: | :---: | ---: | :---: | :---: |
|  |  |  |  |  |  |  |
| $N_{E L}$ | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| 2 | 4.2085 | 4.2164 | 4.2244 | 3.7847 | 5.2668 | 3.4187 |
| 5 | 3.1274 | 3.1347 | 3.1471 | 3.6050 | 3.8213 | 3.9757 |
| 10 | 2.9928 | 3.0000 | 3.0131 | 3.4188 | 3.5266 | 3.5811 |
| 20 | 2.9604 | 2.9675 | 2.9809 | 3.3586 | 3.4337 | 3.4863 |
| 40 | 2.9524 | 2.9595 | 2.9729 | 3.3411 | 3.3926 | 3.4285 |
|  | $-15.03 \%$ | $-14.83 \%$ | $-14.44 \%$ | $-3.848 \%$ | $-2.366 \%$ | $-1.332 \%$ |

NASTRAN (solid): 3.4748

In relation to the numerical convergence, the trend is different with respect to configurations $B$ and $C$, since the deflection decreases as $N_{E L}$ increases. By discretizing the tapered wing with a poor mesh, the elements close to the tip have cross-sections with dimensions shorter than reality. This leads the analysis to underestimate the moment of inertia and therefore structural stiffness too. This is why the trends of $u_{z \max }$ versus $N_{E L}$ for EBBM, TBM, and $N=1$ are no longer independent of $N_{E L}$. In the case of a swept not tapered wing, such curves would have been as straight as for configurations $B$ and $C$. Hence, the taper ratio causes a remarkable difference on the maximum displacement between $N_{E L}=2$ and $N_{E L}=40:+42.5 \%$ for EBBM versus $+12.6 \%$ for $N=3$.

The fourth assessment completes the analysis of the structural method with a further configuration, named $D$ in Table 10.1. This time, the wing is dihedral with a thin-walled rectangular cross-section so as to simulate a typical wing box, see Fig. 10.1b. The thickness of the skin is $5 \%$ of the rectangle's height. The considered height-to-chord length ratio is 0.1 . The taper ratio is 0.25 as in the swept configuration $A$, whereas the parameter $L / \bar{c}$ is higher. By using the same aerodynamic parameters and boundary conditions as before, Table 10.9 summarizes the combined convergence of $u_{z \max }$ on $N$ and $N_{E L}$. The structural convergence as $N$ increases is guaranteed and the conclusions about Poisson's locking correction formerly introduced are still valid. On the contrary, the gap between
theories is less evident for configuration $D$ than for configuration $A$. This applies mainly because the torsion of the tip cross-section is less significant for the unswept wing than the swept one. This makes the classical theories effective at least in the computation of the maximum displacement compared to $N=4$, which offers the closest result to the NASTRAN solution.

Table 10.9: Convergence study: Effect of the number of $B 4$ elements on $\bar{u}_{z \max }[\mathrm{~mm}]$ for different beam models. Configuration $D$.

|  | $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$ |  | $\alpha=3 \mathrm{deg}$ | $4 \times 40$ panels |  |  |
| :---: | :---: | :---: | :---: | ---: | ---: | ---: |
|  |  |  |  |  |  |  |
| $N_{E L}$ | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| 2 | 3.4704 | 3.4732 | 3.4744 | 3.2120 | 3.2852 | 3.3446 |
| 5 | 2.5787 | 2.5812 | 2.5822 | 2.4893 | 2.5586 | 2.5777 |
| 10 | 2.4675 | 2.4699 | 2.4709 | 2.4065 | 2.4733 | 2.4849 |
| 20 | 2.4407 | 2.4431 | 2.4441 | 2.3885 | 2.4508 | 2.4629 |
| 40 | 2.4340 | 2.4365 | 2.4374 | 2.3848 | 2.4404 | 2.4539 |
|  | $-1.688 \%$ | $-1.587 \%$ | $-1.551 \%$ | $-3.676 \%$ | $-1.430 \%$ | $-0.885 \%$ |

NASTRAN (solid): 2.4758

In the matter of numerical mesh, Fig. 10.6b shows trends which in some respects are similar to Fig. 10.6a. They share the deflection decrease as $N_{E L}$ increases, so confirming that the taper ratio is the dominant parameter on the numerical convergence. As a consequence, a notable gap between $N_{E L}=2$ and $N_{E L}=40$ appears again: $+42.6 \%$ for Euler-Bernoulli's theory and $+36.3 \%$ for fourth-order model. However, the convergence on $N_{E L}$ is achieved, so confirming the method's numerical consistency for dihedral wings as well as straight, swept, and tapered ones.

### 10.3 Load cases analysis

At this stage, the convergence of the structural method has been proved. In the same manner, the results have highlighted the remarkable difference among the theories involved and the capability of higher-order models, too. The analyses now deal mainly with the structural response of wings subjected to different kinds of loading (aerodynamic, bending and torsional) combined together. Such combinations are known as load cases and listed in Table 10.10.

The aim is to simulate and discuss the behavior of different wing configurations under a number of flight conditions. To begin with, load case 1 is a pure bending $P_{z}$ applied to the straight wing $B$, which is exposed to a free stream $V_{\infty}=70[\mathrm{~m} / \mathrm{s}]$ with $\alpha=5^{\circ}$. These values of aerodynamic parameters will remain unchanged for the current fifth assessment, according to Table 10.10. The aerodynamic mesh has $4 \times 40$ VLM panels whereas 20 $B 4$ elements discretize the structure. While the aerodynamic load is distributed, $P_{z}$ is a concentrated load. In particular, it is placed at $50 \%$ of the span and equally split upon the two spars (at the first and the third quarter of the chord). The value of $P_{z}$ shown in the table is negative just because its versus is opposite to $z$ axis. For instance, during the flight it could simulate the effect of an inertial load along $z$ axis due to wing engines, missiles, nacelles, or drop tanks.

Table 10.10: Load cases applied to the wing configurations.

| Load Case ID | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Configuration | B | A | A | E | E | $\mathrm{A}, \mathrm{E}, \mathrm{F}$ |
| $V_{\infty}[\mathrm{m} / \mathrm{s}]$ | 70 | 70 | 70 | 70 | 70 | 70 |
| $\alpha[\mathrm{deg}]$ | 5 | 5 | 5 | 5 | 5 | 5 |
| Bending $P_{z}[\mathrm{kN}]$ | -6.8 | -7.2 | -20 | -7.2 | -20 | - |
| Position $P_{z}^{a}$ | $50 \%$ | $50 \%$ | $30 \%$ | $50 \%$ | $30 \%$ | - |
| Twist $T_{y}[\mathrm{kNm}]$ | - | - | $-5,0,5$ | - | $0,3,6$ | - |
| Position $T_{y}^{b}$ | - | - | $30 \%$ | - | $30 \%$ | - |

${ }^{a}$ Position $y_{P}$ along the spanwise direction: $y_{P} / L$
${ }^{b}$ Position $y_{T}$ along the spanwise direction: $y_{T} / L$

As explained previously and as is well known in the aeroelastic field, the aerodynamic load generates a combination of bending and torsional stress for the wing. In the case of a straight wing, the effect is a counter-clockwise rotation of the structure about $y$ axis, with its peak at the tip cross-section. Hence its leading edge undergoes the maximum transverse displacement of the whole structure $u_{z \text { max }}$. On the contrary, the bending load has a negligible torsional effect due to the fact that its application point does not coincide with the cross-section center of twist. However, $P_{z}$ provides an overall reduction of the transverse displacement, although it entails the appearence of a rising local stress at its application point.

The problem is analyzed at first with the classical Euler-Bernoulli's beam theory and then with the third-order model. The comparison of results is shown in Fig. 10.7, where the tridimensional deflections are drawn by means of a large scale factor to clearly portray their differences. It is evident that the model $N=3$ has the capability to show the torsional effect due, in this case, exclusively to aerodynamics. On the whole, EBBM properly describes the bending behavior, but turns out to be ineffective with torsion thus giving unrealistic solutions.

The load case 2 is again composed of pure bending and the aerodynamic load, this time involving the swept wing $A$. The considerations introduced for the previous case about the load direction and the application point remain true for the current and the following cases. Because of the positive sweep angle, the whole structure would be expected to undergo a clockwise rotation about $y$ axis due to the aerodynamics. When $P_{z}$ is also applied, it generates a counter-clockwise twist able to contrast the aerodynamic effect. There are two reasons: at first the bending load acts behind the profile's center of twist, then the trailing edge is the "weak side" for configuration $A$. In other words, when $\Lambda$ is positive, the trailing edge is more sensitive to a bending load, even if the latter were placed at the profile's center of twist. In the case of a negative $\Lambda$, the same applies to the leading edge. In this case, both give clockwise torsional effects, which are expecially evident at $50 \%$ of the span. The third-order model shows in Fig. 10.8 that this local twist impacts on the overall structure, to such an extent that the tip cross-section is rotated clockwise too. The limits of EBBM are again evident, even for a bending case.

In load case 3, a pure torsional load $T_{y}$ is added to the second case. Since the aim of the current assessment is to simulate flight conditions, it is not possible to neglect loadings such as a pitching moment by pilot intervention, thrust generated by wing engines, or inertial loads along $x$ axis due again to wing engines, missiles, and drop tanks. Such loads imply a


Figure 10.7: Comparison of classical and higher-order models for configuration $B$. Load case 1.


Figure 10.8: Tridimensional deformation of configuration $A$. Load case 2.
twist about $y$ axis, considering that the center of gravity of these objects lies below the wing surface. Besides, in the case of thrust, its bending contribution about $z$ axis is negligible because of the high level of stiffness of the structure about such an axis. Thus, the analysis of the swept wing as the torsion $T_{y}$ increases is made again with a third-order model and its results are in Fig. 10.9. The local deflection near the application points $\left(y_{P}=y_{T}\right)$ becomes more evident as $T_{y}$ increases, whereas $u_{z}$ decreases at the tip. In particular, the twist warps the trailing more than the leading edge of each cross-section, in accordance with the previous remarks about the torsional center and the "weak side". On the contrary, Euler-Bernoulli's theory always provides the same results, unable to detect any twist. The figure underlines once more how $N=3$ model represents a high-performance theory, unlike EBBM, in the evaluation of different flight conditions.

To conclude the fifth assessment, a further wing configuration is introduced and named


Figure 10.9: Effect of the variable torsional load on configuration $A$. Load case 3.
with letter $E$. It has the same properties of configuration $A$, with a difference in the negative sweep angle. It is not a usual wing, but it has been widely studied recently expecially with the introduction of composite materials. With respect to previous geometries, it is structurally more sensitive to aerodynamics since its tip cross-section undergoes a remarkable counter-clockwise rotation about $y$ axis. As far as load case 4 is concerned, the bending load $P_{z}$ reduces the transverse displacement but introduces two opposite twist effects. The application point behind the cross-section's center of twist generates a counter-clockwise rotation, whereas an opposite torsion is due to the fact that now the "weak side" is the leading edge. In conclusion, Fig. 10.10 displays the counter-clockwise twist of the overall wing obtained by means of a higher-order model. As usual, EBBM does not detect the proper deformation even for this basic bending case and underestimates $u_{z \text { max }}$.

Load case 5 involves $T_{y}$ variable in value and placed at $30 \%$ of the span like $P_{z}$. In Fig. 10.11 configuration $E$ shows the increasing local effect on the most stressed crosssection. When the twist is stronger, the transverse displacement at the tip increases for the forward swept wing. Such a behavior is opposite to the swept configuration $A$. In the same manner, a positive $T_{y}$ does not countervail the typical torsion due to the aerodynamics when forward swept geometry is involved. For the generic cross-section, now the realistic center of rotation seems to be placed behind the center of twist. While for wing $A$ the change of twist in value mainly rotates the trailing edge, only slightly involving the leading edge, wing $E$ undergoes a notable variation in $u_{z}$ on the leading edge, too. Such a point is presumably due to the negative sweep angle, which turns the leading edge to the "weak side". As usual, all these comments are possible thanks to the capability of higher-order models to describe the structural behavior for all the load cases proposed. The discussion of results would not have been possibile by relying on only classical beam models.

By considering the most constraining twist ( $T_{y}=6 \mathrm{kNm}$ ) of load case 5 , the transverse


Figure 10.10: Tridimensional deformation of configuration $E$. Load case 4.

$$
\begin{aligned}
N=3: & P_{z}=-20 \mathrm{kN}, \mathrm{~T}_{\mathrm{y}}=0 \mathrm{kNm} \\
\mathrm{~N}=3: & \mathrm{P}_{\mathrm{z}}=-20 \mathrm{kN}, \mathrm{~T}_{\mathrm{y}}=3 \mathrm{kNm} \\
\mathrm{~N}=3: & P_{\mathrm{z}}=-20 \mathrm{kN}, \mathrm{~T}_{\mathrm{y}}=6 \mathrm{kNm} \\
\text { EBBM }: & P_{\mathrm{z}}=-20 \mathrm{kN}, \mathrm{~T}_{\mathrm{y}}=0,3,6 \mathrm{kNm}
\end{aligned}
$$



Figure 10.11: Effect of the variable torsional load on configuration $E$. Load case 5 .
shear stress $\tau_{x z}$ along the charged cross-section is investigated with a third-order model. The distribution is presented in Fig. 10.12. Obviously, Euler-Bernoulli's beam theory fails to detect any transverse shear effect because of the assumptions on the displacement field. It is to be noted how the linear and higher-order terms of $u_{x}$ and $u_{z}$ cannot be neglected. In fact, the shear effects are remarkable in such a constraining load case. In particular, the highest values of $\tau_{x z}$ are placed at the joint points between the airfoil and the spar at $25 \%$ of the chord. Such a spar is highly stressed, reaching both the maximum and minimum values. On the contrary, the rear spar seems to not undergo high values of transverse shear
stress.


Figure 10.12: Distribution of the transvese shear stress $\tau_{x z}$ on the deformed airfoil cross-section (30\% span). Load case 5.

It is worth pointing out the high rate of stress placed at the trailing edge and slightly below the leading edge. It is to be noted that such conclusions have to refer to the particular case in question, which is composed of bending, torsional and aerodynamic loads. Moreover, Fig. 10.12 illustrates how the proposed 1D model is able to portray the cross-section's deformation. The aim of this example is to highlight the significance of a high-performance analysis of stress, achievable by increasing the expansion order without giving up the one-dimensional discretization.


Figure 10.13: Effect of sweep angle on the torsional response of configurations $A, E$, and $F$ subjected to aerodynamic pressure distribution. Load case 6.

The concluding assessment concerns load case 6 , which involves neither bending nor torsional loadings. The purpose is to analyze the effect of sweep angle on wings exposed to a free stream velocity $V_{\infty}=70[\mathrm{~m} / \mathrm{s}]$ with $\alpha=5^{\circ}$. A further unswept configuration $F$ is introduced and compared to $A$ and $E$, with $\Lambda$ equal to $13.5^{\circ}$ and $-13.5^{\circ}$ respectively. This choice is purpose-made, since the wings have all the same geometrical parameters with the exception of sweep angle. The analysis investigates the structural torsion along the
spanwise direction due to the only aerodynamic pressure by means of the quantity $\Delta u_{z}$. It is defined as the difference of $u_{z}$ between leading and trailing edges. The simulation is performed for each case via EBBM and third-order model and Fig. 10.13 shows the results.

At first glance, the twist is more significant when the sweep angle is high. Nevertheless, the unswept wing also undergoes a twist, since the aerodynamic load is a combination of bending and torsion. In particular, the rotation about $y$ axis is positive and its maximum is not placed at the tip cross-section. As expected and explained above, the rotation is positive for the forward swept wing $E$ and negative for the swept $A$. It should be noted that the corresponding lines in Fig. 10.13 are not symmetrical with respect to the horizontal axis. Their maximum absolute values are not at the same $y$ coordinate and this means that the shape of $\Delta u_{z}$ depends on the sweep angle's sign, too.

As expected, Euler-Bernoulli's beam theory neglects any torsion and makes no difference among wings. Instead, Fig. 10.13 could be very helpful to the design and the evaluation of aeroelastic behavior of wings. Again, the capability of higher-order models applied to one-dimensional finite elements via the CUF is proved.

## Chapter 11

## Results: VLM-CUF 1D aeroelastic coupling

As shown in Fig. 11.1, a local cartesian coordinate system composed of $x$ and $z$ axes parallel to the cross-section plane is defined, whereas $y$ represents the out-of-plane coordinate. However, the $y$ axis is not necessarily a centroidal one.


Figure 11.1: One-dimensional structural mesh and two-dimensional aerodynamic mesh of the wing structure

Several wing configurations with different geometries, layout and loadings are considered. Two different types of response solutions are investigated and compared in this chapter. The first one coincides with the static structural analysis, hereinafter referred as SSA, and involves only the structural stiffness matrix by disabling the aerodynamic matrix $\boldsymbol{K}_{\text {aero }}$. Hence, the following system is solved:

$$
\begin{equation*}
\boldsymbol{K}_{\mathrm{str}} \cdot \boldsymbol{q}=\boldsymbol{L}_{\mathrm{RHS}} \tag{11.1}
\end{equation*}
$$

The second solution is the static aeroelastic analysis (SAA) which solves the aeroelastic system (Eq. 11.2, which is equal to Eq. 9.82) by adding the aerodynamic stiffness matrix to the elastic one:

$$
\begin{equation*}
\boldsymbol{K}_{\text {aeroelastic }} \cdot \boldsymbol{q}=\boldsymbol{L}_{\mathrm{RHS}} \tag{11.2}
\end{equation*}
$$

Straight and swept wing configurations as well as rectangular and airfoil shaped crosssections are considered. Unless otherwise specified, they are subjected to a pure aerodynamic loading (vector $\boldsymbol{L}_{\text {RHS }}$ ) and modeled with a one-dimensional mesh of 20 B 4 elements along the $y$ axis. This choice ensues from the conclusions made in previous CUF works on isotropic and orthotropic thin and thick walled structures [122, 138]. Cantilever boundary condition on half-wings is imposed and the symmetry condition is exploited in the aerodynamic computation. Both isotropic and anisotropic composite materials are taken into account and a further analysis evaluates the combined effect of the material lamination and the sweep angle on the divergence speed of a composite wing.

In the following analyses, the wings are exposed to a free stream velocity $\boldsymbol{V}_{\infty}$ parallel to the $x$ axis with an angle of attack $\alpha$. The assessment of the Vortex Lattice formulation in computing $\boldsymbol{L}_{\text {RHS }}$ was carried out in [138]. A result is here retrieved in order to evaluate the aerodynamic pressure distribution along different wing configurations.

An aft-swept wing with $\Lambda=45^{\circ}$ and a forward-swept wing with $\Lambda=-45^{\circ}$ with a constant chord $c=1 \mathrm{~m}$ are considered as untapered reference configurations [133]. The half wing span-to-chord ratio $L / c$ is equal to 2 and the terms $C_{l}$ and $C_{L}$ are introduced as follows:

$$
\begin{gather*}
C_{l}(y)=\frac{L^{\star}(y)}{\frac{1}{2} \rho_{\infty} V_{\infty}^{2} 2 e(y) c(y)} \quad C_{L}=\frac{L_{\mathrm{RHS}}^{\mathrm{tot}}}{\frac{1}{2} \rho_{\infty} V_{\infty}^{2} L \bar{c}}  \tag{11.3}\\
L_{\mathrm{RHS}}^{\mathrm{tot}}= \\
\sum_{i=1}^{N_{A P}}(\boldsymbol{L})_{i}=\sum_{i=1}^{N_{A P}} \frac{1}{2} \rho_{\infty} V_{\infty}^{2} \tan (\pi-\alpha)\left(\boldsymbol{I}^{D}\left[\boldsymbol{A}^{D}\right]^{-1} \boldsymbol{d}\right)_{i} \tag{11.4}
\end{gather*}
$$

where $c(y)$ and $L^{\star}(y)$ are the chord and the Lift Force generated by the pressure acting on the panels with span-length $2 e(y)$ placed at the $y$ coordinate, respectively. $L_{\text {RHS }}^{\text {tot }}$ is the Total Lift Force acting on the right half-wing and Eq. 11.4 refers to Eq. 11.5:

$$
\begin{equation*}
\boldsymbol{L}_{\mathrm{RHS}}=\frac{1}{2} \rho_{\infty} V_{\infty}^{2} \tan (\pi-\alpha) \widetilde{\boldsymbol{A}}_{3}^{\star} \cdot \boldsymbol{I}^{D}\left[\boldsymbol{A}^{D}\right]^{-1} \boldsymbol{d} \tag{11.5}
\end{equation*}
$$

The air density is henceforth assumed to be $\rho_{\infty}=1.225 \mathrm{~kg} / \mathrm{m}^{3}$. The trend of the $\frac{C_{l}}{C_{L}}$ ratio along the $y$ axis is shown for both the swept wings in Fig. 11.2. In the computation two different aerodynamic meshes are involved, differing in shape and in the total number of panels. It is interesting to note the influence of the sweep angle $\Lambda$ on the pressure distribution and the position of the maximum pressure on the spanwise direction. A slight dependence on the aerodynamic mesh used is furthermore detected and notable mainly for the aft-swept case. An excellent agreement with the results obtained by Katz and Plotkin [133] is achieved.

For a straight configuration, it has been verified that the maximum pressure acts on the leading edge of each section, with an overall maximum placed on the root cross-section. The pressure distribution computed by the VLM decreases along the $y$ axis according to the low-speed aerodynamics of aircraft straight wings [133]. It is noteworthy that the quantities $\alpha$ and $V_{\infty}$ affect such a pressure distribution only in value, but not in shape.


Figure 11.2: Effect of the sweep angle on the spanwise loading for two reference untapered wings

### 11.1 Isotropic metallic wings

### 11.1.1 Airfoil-shaped wing

In this section, an isotropic aluminium with Young's modulus $E=69 \mathrm{GPa}$ and Poisson's ratio $\nu=0.33$ is introduced. A straight untapered wing is exposed to a free stream velocity $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$ with $\alpha=3^{\circ}$. The half-wing span $L$ is equal to 5 m and the chord is $c=1$ m . The cross-section adopted is a thin-walled NACA 2415 airfoil, which is subdivided into three cells by two spars along the spanwise direction inserted at $25 \%$ and $75 \%$ of the chord, see Fig. 11.3(a). The front and rear spars' thicknesses are, respectively, $10 \%$ and $7 \%$ of the airfoil height, whereas the percentage is $4 \%$ for the skin. The considered airfoil height-to-chord ratio is 0.15 . An aerodynamic mesh composed of $4 \times 40$ panels is set on the reference surface of the structure.


Figure 11.3: Cross-sections used for the wing configurations
A convergence study is carried out to evaluate the combined effect of the number of B4 finite elements $N_{E L}$ and the expansion order $N$ on the static aeroelastic response (SAA) of the wing. The mechanics of the structure is described in terms of the maximum transverse displacement $u_{z \max }$, which is located at the trailing edge of the tip cross-section. This location derives from the aeroelastic coupling between the twist and the vertical bending of the wing exposed to the free stream. The results for SAA are shown in Table 11.1.

The numerical convergence on $N_{E L}$ is achieved for each adopted theory as reported in Fig. 11.4. A higher number of elements enhances the flexibility of the structure with the

Table 11.1: Convergence study: effect of the number of $B 4$ elements on $u_{z \max }$ [ mm ] for different beam models. Straight metallic airfoil-shaped wing. $V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \alpha=3^{\circ}, 4 \times 40$ panels. SAA.

| $N_{E L}$ | EBBM | TBM | $N=1$ | $N=2$ | $N=3$ | $N=4$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | 8.9407 | 8.9506 | 8.9603 | 8.3904 | 8.4972 | 8.5134 |
| 5 | 8.9407 | 8.9506 | 8.9603 | 8.5810 | 8.7112 | 8.7315 |
| 10 | 8.9407 | 8.9506 | 8.9603 | 8.6393 | 8.7734 | 8.7953 |
| 20 | 8.9407 | 8.9506 | 8.9603 | 8.6673 | 8.8015 | 8.8245 |
| 40 | 8.9407 | 8.9506 | 8.9603 | 8.6800 | 8.8136 | 8.8377 |

exception of first-order theories EBBM, TBM, and $N=1$, in which cases the results are slightly different and not affected by the mesh.


Figure 11.4: Maximum transverse displacement $u_{z \max }$ as a function of the structural mesh and the models involved. Straight metallic airfoil-shaped wing. SAA

The maximum displacement increases with $N$ for any mesh, to such an extent that no remarkable differences are detected for high-order expansion. It is interesting to note how $u_{z \text { max }}$ decreases when the theory changes from a first-order to a second-order form. The main reason for this turnaround stands in Poisson's locking correction adopted only for first-order theories [120]. In fact, by disabling such a correction for first-order theories, $u_{z \text { max }}$ would be even lower than the second-order case. Thus, higher orders than linear approximation yield a more flexible structure.

The comparison of the aeroelastic response of the wing with MD NASTRAN is now faced. While the proposed CUF 1D model is able to easily handle arbitrary cross-section geometries for both SSA and SAA analyses, the choice of an airfoil-shaped wing requires an equivalent procedure in MD NASTRAN for the aeroelastic analysis (sol 144). The procedure here used involves an equivalent flat plate with a rectangular cross-section which is supposed to approximately emulate the bending and torsional behavior of the actual wing. This technique retrieves the results of structural response (SSA) for the same airfoil-shaped wing analyzed in [138] and summarized in Table 11.2. In that case, an equivalent procedure
was not necessary because the MD NASTRAN structural analysis (sol 101) was carried out by charging the front skin of the airfoil-shaped wing (solid elements - $10^{6}$ DOFs) with the pressures due to the pure aerodynamic loading computed by the VLM (vector $\boldsymbol{L}_{\mathrm{RHS}}$ ).

Table 11.2: Comparison of the structural response (SSA) of the airfoil-shaped wing and the equivalent flat plate between MD NASTRAN and $N=4$ (40 B4 elements).

| SSA | MD NASTRAN <br> Airfoil Wing | MD NASTRAN <br> Flat Plate | $N=4$ <br> Flat Plate | $N=4$ <br> Airfoil Wing |
| :---: | :---: | :---: | :---: | :---: |
|  | 8.6762 | 8.6762 | 8.6787 | 8.6854 |
| $\%$ Diff | - | $+0.000 \%$ | $+0.029 \%$ | $+0.106 \%$ |
| DOFs | $10^{6}$ | 1845 | 5445 | 5445 |

The equivalent height $h_{e q}=80.382 \mathrm{~mm}$ is now chosen so that the associated equivalent wing and the airfoil-shaped wing show the same value for $u_{z \text { max }}$ computed by MD NASTRAN (SSA), see second and third columns of Table 11.2. Hence the aeroelastic analysis is carried out only on the equivalent flat plate via MD NASTRAN (sol 144) and the results are compared with the fourth-order model (SAA) in Table 11.3.

Table 11.3: Comparison of the aeroelastic response (SAA) of the airfoil-shaped wing and the equivalent flat plate between MD NASTRAN and $N=4$ (40 B4 elements).

| SAA | MD NASTRAN <br> Flat Plate | $N=4$ <br> Flate Plate | $N=4$ <br> Airfoil Wing |
| :---: | :---: | :---: | :---: |
| $u_{z \max }$ | 8.7088 | 8.7124 | 8.8377 |
| $\%$ Diff | - | $+0.041 \%$ | $+1.480 \%$ |
| DOFs | 1845 | 5445 | 5445 |

The results about the equivalent flat plate in Tables 11.2 and 11.3 show an excellent agreement of the 1D higher-order model with MD NASTRAN (error of $0.029 \%$ for SSA and $0.041 \%$ for SAA). On the contrary, the comparison between the equivalent flat plate via MD NASTRAN and the actual airfoil shaped wing via $N=4$ leads the error to be no more negligible, mainly for the aeroelastic response analysis ( $0.106 \%$ for SSA vs. $1.480 \%$ for SAA). Thus, the procedure used introduces a further relevant approximation of the actual aeroelastic behavior of the structure. In fact, the in-plane deformation of the cross-section plays an important role as proved in [138], where the same NACA 2415 profile is considered. The present 1D model is able to handle even complex cross-section geometries also for the aeroelastic analysis without increasing the number of DOFs. This model does not require the introduction of an equivalent flat plate to study the structural and aeroelastic response of arbitrary wings via a low number of DOFs ( $5445 \mathrm{vs} .10^{6}$ for MD NASTRAN).

### 11.1.2 Rectangular cross-section

A thin rectangle is now taken into account as cross-section for the straight wing introduced above. Referring to Fig. 11.3(b), the considered height-to-chord ratio is 0.02 ( $h=20 \mathrm{~mm}$ ). An aerodynamic mesh composed of $10 \times 50$ panels is set on the reference surface of the structure, which is exposed to an increasing free stream velocity with $\alpha=1^{\circ}$.

Table 11.4 reports a parametric study on the maximum transverse displacement, placed again at the leading edge of the tip cross-section, as the free stream velocity changes. The

Table 11.4: Effect of the free stream velocity $V_{\infty}[\mathrm{m} / \mathrm{s}]$ on $u_{z \max } \mathrm{~mm}$. Straight wing, $c=1 \mathrm{~m}, L=$ $5 \mathrm{~m}, h=20 \mathrm{~mm} .20 \mathrm{~B} 4$ elements. $\alpha=1^{\circ}, 10 \times 50$ panels. SSA vs. SAA.

|  | $V_{\infty}=10$ |  | $V_{\infty}=30$ |  | $V_{\infty}=50$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | SSA | SAA | SSA | SAA | SSA | SAA |
| EBBM | $\begin{array}{r} 7.6277 \\ -0.00 \end{array}$ | $\begin{aligned} & 7.6272 \\ & 30 \%^{*} \end{aligned}$ | $\begin{array}{r} 68.650 \\ -0.05 \end{array}$ | $\begin{aligned} & 68.611 \\ & 57 \% \end{aligned}$ | $\begin{array}{r} 190.70 \\ -0.15 \end{array}$ | $\begin{aligned} & 190.40 \\ & 46 \% \end{aligned}$ |
| TBM | $\begin{array}{r} 7.6276 \\ -0.00 \end{array}$ | $\begin{aligned} & 7.6272 \\ & 62 \% \end{aligned}$ | $\begin{array}{r} 68.649 \\ -0.05 \end{array}$ | $\begin{aligned} & 68.611 \\ & 56 \% \end{aligned}$ | $\begin{array}{r} 190.69 \\ -0.15 \end{array}$ | $\begin{aligned} & 190.40 \\ & 44 \% \end{aligned}$ |
| $N=1$ | $\begin{array}{r} 7.6278 \\ -0.00 \end{array}$ | $\begin{aligned} & 7.6275 \\ & 45 \% \end{aligned}$ | $\begin{gathered} 68.650 \\ -0.04 \end{gathered}$ | $\begin{aligned} & 68.622 \\ & 07 \% \end{aligned}$ | $\begin{array}{r} 190.70 \\ -0.11 \end{array}$ | $\begin{aligned} & 190.48 \\ & 29 \% \end{aligned}$ |
| $N=2$ | $\begin{array}{r} 6.9600 \\ +0.9 \end{array}$ | $\begin{aligned} & 7.0244 \\ & 258 \% \end{aligned}$ | $\begin{aligned} & 62.640 \\ & +8.93 \end{aligned}$ | $\begin{aligned} & 68.236 \\ & 45 \% \end{aligned}$ | $\begin{array}{r} 174.00 \\ +28.9 \end{array}$ | $\begin{aligned} & 224.45 \\ & 97 \% \end{aligned}$ |
| $N=3$ | $\begin{array}{r} 7.4226 \\ +0.9 \end{array}$ | $7.4966$ $60 \%$ | $\begin{array}{r} 66.804 \\ +9.63 \end{array}$ | $\begin{aligned} & 73.241 \\ & 60 \% \end{aligned}$ | $\begin{array}{r} 185.57 \\ +31 . \end{array}$ | $\begin{aligned} & 243.94 \\ & 57 \% \end{aligned}$ |
| $N=4$ | $\begin{array}{r} 7.4385 \\ +0.9 \end{array}$ | $\begin{aligned} & 7.5126 \\ & 59 \% \end{aligned}$ | $\begin{array}{r} 66.947 \\ +9.63 \end{array}$ | $\begin{aligned} & 73.397 \\ & 48 \% \end{aligned}$ | $\begin{array}{r} 185.96 \\ +31 . \end{array}$ | $\begin{gathered} 244.46 \\ 58 \% \end{gathered}$ |
| NASTRAN | - | 7.5446 | - | 73.731 | - | 245.49 |

* Difference between SAA and SSA
structural and the aeroelastic responses are evaluated. While $u_{z \text { max }}$ increases linearly with the square of $V_{\infty}$ for SSA, the same does not occur for SAA. The contribution of $\boldsymbol{K}_{\text {aero }}$ to the system stiffness becomes more evident as $V_{\infty}$ increases and the difference from SSA gets very significant for $V_{\infty}=50 \mathrm{~m} / \mathrm{s}$. This difference increases with the expansion order $N$ by approaching MD NASTRAN results as reported in Table 11.4. It should be noticed that EBBM and TBM here used are unable to handle any torsional behavior since no warping functions, e.g. in Vlasov beam model, have been implemented. However, $N=1$ model takes into account torsional effects in the displacement field but it results to be ineffective especially when the aeroelastic effect on the wing is relevant.

The twist of the tip cross-section $(y=L)$ for the structural and aeroelastic responses is investigated by means of the quantity $\Delta u_{z}$, which is defined as the difference of $u_{z}$ between the leading and trailing edges. The simulation is performed via a fourth-order approximation and the results are shown in Fig. 11.5. It is interesting to note the raising importance of the aeroelastic effect of $V_{\infty}$ on the tip section twist and the excellent agreement between MD NASTRAN (sol 144) and the higher-order model, which is able to properly describe the torsional behavior of the structure unlike first-order beam theories.

The dimensionless quantity $u_{z \text { TIP }}^{\star}$ is introduced to study the bending behavior of the wing. It represents the transverse displacement at the midpoint of the tip cross-section normalized with respect to $\bar{u}_{z \text { TIP }}$, which is the tip-maximum displacement of a cantilever beam subjected to a uniformly distributed load $q=L_{\text {RHS }}^{\text {tot }} / L$ :

$$
\begin{equation*}
\bar{u}_{z \mathrm{TIP}}=\frac{q L^{4}}{8 E I} \quad \Rightarrow \quad u_{z \mathrm{TIP}}^{\star}=\frac{u_{z \mathrm{TIP}}}{\bar{u}_{z \mathrm{TIP}}}=u_{z \mathrm{TIP}} \frac{8 E I}{L_{\mathrm{RHS}}^{\text {tot }} L^{3}} \tag{11.6}
\end{equation*}
$$

The free stream velocity has no effect on $\bar{u}_{z \text { TIP }}$ for SSA. In fact, in this case the raising $V_{\infty}$ modifies the aerodynamic pressure distribution only in value, but not in shape. $L_{\text {RHS }}^{\text {tot }}$ increases linearly with the square of $V_{\infty}$ as well as $u_{z \text { TIP. }}$. The trend of $u_{z \text { TIP }}^{\star}$ vs. $V_{\infty}$ is


Figure 11.5: Effect of $V_{\infty}$ on the twist $\Delta u_{z \text { TIP. }}$. Isotropic material. SSA vs. SAA
therefore a constant straight line as shown in Fig. 11.6. As confirmed in Table 11.4, a $N=4$ theory detects with a good accuracy in comparison with MD NASTRAN (sol 144) that the increasing free stream velocity enhances the fluid-structure coupling effect on the structure for SAA.


Figure 11.6: Effect of $V_{\infty}$ on the dimensionless $u_{z \text { TIP }}^{\star}$. Isotropic material. SSA vs. SAA
A parametric study on the influence of the aspect ratio on the response of the straight wing is carried out and summarized in Table 11.5. The chord length is maintained constant and the half-wing span $L$ ranges from 5 to 25 m whereas the thickness is $h=100 \mathrm{~mm}$. Poisson's locking correction is not sufficient to make first-order approximation models effective in computing the maximum displacement for SSA, since the tip section twist
becomes relevant especially for slender wings. Moreover, the increasing aeroelastic effect with the aspect ratio is detectable only by means of a higher-order model, which accurately approaches the MD NASTRAN solution as $L$ increases (SAA).

Table 11.5: Effect of the length $L[\mathrm{~m}]$ on $u_{z \max }[\mathrm{~mm}] .20 \mathrm{~B} 4$ elements. Straight wing, chord $=1 \mathrm{~m}$, $\mathrm{h}=100 \mathrm{~mm} .20 \mathrm{~B} 4$ elements. $V_{\infty}=70 \mathrm{~m} / \mathrm{s}, \alpha=1^{\circ}, 10 \times 50$ panels. SSA vs. SAA.

|  | $L=5$ |  | $L=10$ |  | $L=20$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | SSA | SAA | SSA | SAA | SSA | SAA |
| EBBM | $\begin{array}{r} 2.9900 \\ -0.00 \end{array}$ | $\begin{aligned} & 2.9899 \\ & 24 \%^{\star} \end{aligned}$ | $\begin{array}{r} 56.353 \\ -0.01 \end{array}$ | $\begin{aligned} & 56.347 \\ & 14 \% \end{aligned}$ | $\begin{array}{r} 1000.24 \\ -0.06 \end{array}$ | $\begin{aligned} & 999.56 \\ & 79 \% \end{aligned}$ |
| TBM | $\begin{array}{r} 2.9911 \\ -0.0 \end{array}$ | $\begin{aligned} & 2.9910 \\ & 24 \% \end{aligned}$ | $-0.0115 \%$ |  | $\begin{array}{r} 1000.25 \\ -0.06 \end{array}$ | $\begin{aligned} & 999.57 \\ & 77 \% \end{aligned}$ |
| $N=1$ | $\begin{array}{r} 2.9928 \\ +0.0 \end{array}$ | $\begin{aligned} & 2.9933 \\ & 37 \% \end{aligned}$ | $\begin{array}{r} 56.366 \\ +0.0 \end{array}$ | $\begin{aligned} & 56.402 \\ & 347 \% \end{aligned}$ | $\begin{array}{r} 1000.28 \\ +0.2 \end{array}$ | $\begin{aligned} & 1002.99 \\ & 05 \% \end{aligned}$ |
| $N=2$ | $\begin{array}{r} 2.8620 \\ +0.3 \end{array}$ | $\begin{aligned} & 2.8731 \\ & 363 \% \end{aligned}$ | $+1.9864 \%$ |  | $+9.6135 \%$ |  |
| $N=3$ | $\begin{array}{r} 2.9192 \\ +0.3 \end{array}$ | $\begin{aligned} & 2.9307 \\ & 45 \% \end{aligned}$ | +2.0004 \% | $\begin{aligned} & 56.465 \\ & 04 \% \end{aligned}$ | +9.6387\% | $\begin{aligned} & 1084.07 \\ & 87 \% \end{aligned}$ |
| $N=4$ | 2.9325 | 2.9443 | 55.478 | 56.611 | 989.72 | 1087.30 |
|  | +0.4020 \% |  | +2.0419 \% |  | +9.8598\% |  |
| NASTRAN | - | 2.9505 | - | 56.723 | - | 1092.77 |

* Difference between SAA and SSA

The span parameter modifies the distribution of aerodynamic pressures acting on the reference surface exposed to the free stream (vector $\boldsymbol{L}_{\text {RHS }}$ ) not only in value, but also in shape. Figure 11.7 emphasizes the role played by the aspect ratio on this aerodynamic pressure, whose trend approaches a uniformly distributed load as $L$ increases. The progress of the dimensionless quantity $u_{z \text { TIP }}^{\star}$ is thus no more a constant straight line for SSA analysis, but rather a curve approaching the unit value, see Fig. 11.8. As far as the aeroelastic response is concerned, the influence of high aspect ratio on $u_{z \text { TIP }}^{\star}$ depicted by the proposed refined method is once more consistent with the MD NASTRAN results.

### 11.2 Anisotropic composite wings

A composite material is introduced to analyze the aeroelastic tailoring on a straight wing with length $L=10 \mathrm{~m}$ and chord $c=1 \mathrm{~m}$. The rectangular cross-section has height $h=100$ mm . Young's modulus along the longitudinal axis $E_{L}$ is equal to 20.5 GPa , whereas those along the transverse directions are equal to 10 GPa . Poisson's ratio $\nu=0.25$ and the shear modulus $G=5 \mathrm{GPa}$ are the same in all directions. Figures 11.9 and 11.10 show the influence of the lamination $\theta$ on the twist $\Delta u_{z}$ and on the bending of the tip cross-section via $N=4$ model. The bending is described via the transverse displacement of the midpoint placed at the tip section $u_{z \text { TIP }}$.

The comparison of SSA and SAA underlines the importance of the contribution of $\boldsymbol{K}_{\text {aero }}$ to evaluate the aeroelastic behavior of composite wings. While the curve of twist related to SSA is essentially anti-symmetric with respect to the $\theta=0^{\circ}$ lamination, the aeroelastic analysis shows a trend which is far from symmetric, see Fig. 11.9. In general,


Figure 11.7: Effect of $L$ on the spanwise aerodynamic loading. Straight wing. Isotropic material. SSA vs. SAA


Figure 11.8: Effect of $L$ on the dimensionless $u_{z \text { TIP }}^{\star}$. Isotropic material. SSA vs. SAA
the aeroelastic analysis leads the twist of the unswept wing to be higher compared to the structural solution as the lamination changes, expecially for negative values of $\theta$. The same result occurs for bending behavior as shown in Fig. 11.10, where only the SSA case obtains an almost symmetric curve. However, the laminations which lead to the minimum and maximum twist conditions are the same for the structural and aeroelastic analyses, whereas they are different for the bending deflection.

The curves describing the tip twist as $\theta$ changes present some particular intersection points in which $\Delta u_{z}$ reaches the same value for SSA and SAA. In general, these orientations


Figure 11.9: Effect of the lamination on the torsion of the tip cross-section. Straight wing. Composite material. SSA vs. SAA


Figure 11.10: Effect of the lamination on the bending of the tip cross-section (at the midpoint). Straight wing. Composite material. SSA vs. SAA
differ from those which lead the tip bending to be the same for the structural and aeroelastic analyses. These lamination values depend on the wing geometry, the material properties and the aerodynamic conditions.

The numerical results summarized in Tables 11.6 and 11.7 reveal the role played by $\theta$ for different structural theories. The use of a composite material leads to move the maximum transverse displacement from the leading to the trailing edge for particular positive laminations (negative twist) unlike the isotropic case. The excellent agreement
between the fourth-order beam model and MD NASTRAN-shell (sol 144) in describing the aeroelastic response of anisotropic wings with generic orientation is again noteworthy, whereas the diminishing expansion order $N$ corresponds to a lower number of DOFs as well as loss of accuracy.

Table 11.6: Effect of the lamination on the twist $\Delta u_{z \text { TIP }}$ [ mm$]$ of the tip cross-section. Straight wing, $c=1 \mathrm{~m}, L=10 \mathrm{~m}, h=100 \mathrm{~mm} .10 \mathrm{~B} 4$ elements. $V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \alpha=1^{\circ}, 10 \times 50$ panels. Composite material. SSA vs. SAA.

| $\theta$ | SSA | SAA | SAA | SAA | SAA |
| ---: | ---: | :---: | ---: | ---: | ---: |
|  | $N=4$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN |
| $-60^{\circ}$ | 2.9918 | $3.6114(+2.00 \%)$ | 3.5076 | $3.5272(-0.38 \%)$ | 3.5407 |
| $-30^{\circ}$ | 4.3163 | $5.5766(+0.65 \%)$ | 5.5127 | $5.5395(-0.02 \%)$ | 5.5408 |
| $0^{\circ}$ | 1.0206 | $1.0445(-2.76 \%)$ | 1.0458 | $1.0704(-0.34 \%)$ | 1.0741 |
| $30^{\circ}$ | -2.4563 | $-2.2294(+3.03 \%)$ | -2.1959 | $-2.1741(-0.48 \%)$ | -2.1638 |
| $60^{\circ}$ | -1.1191 | $-1.1574(+10.9 \%)$ | -1.0845 | $-1.0569(+1.25 \%)$ | -1.0439 |
| $90^{\circ}$ | 1.0390 | $1.0643(-3.81 \%)$ | 1.0656 | $1.0909(-1.41 \%)$ | 1.1065 |
| DOFs | 1395 | 558 | 930 | 1395 | 2135 |

Table 11.7: Effect of the lamination on the bending of the wing: $u_{z \text { TIP }}$ [ mm ] at the midpoint of the tip cross-section. Straight wing, $c=1 \mathrm{~m}, L=10 \mathrm{~m}, h=100 \mathrm{~mm} .10$ B4 elements. $V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \alpha=1^{\circ}, 10 \times 50$ panels. Composite material. SSA vs. SAA.

| $\theta$ | SSA | SAA | SAA | SAA | SAA |
| ---: | ---: | :---: | :---: | :---: | :---: |
|  | $N=4$ | $N=2$ | $N=3$ | $N=4$ | NASTRAN |
| $-60^{\circ}$ | 181.956 | $214.518(-0.31 \%)$ | 214.494 | $214.826(-0.16 \%)$ | 215.175 |
| $-30^{\circ}$ | 131.597 | $168.142(-0.49 \%)$ | 168.557 | $168.898(-0.04 \%)$ | 168.965 |
| $0^{\circ}$ | 96.152 | $100.656(-0.62 \%)$ | 101.071 | $101.246(-0.04 \%)$ | 101.288 |
| $30^{\circ}$ | 127.888 | $112.284(-1.12 \%)$ | 113.212 | $113.488(-0.06 \%)$ | 113.555 |
| $60^{\circ}$ | 179.707 | $167.586(-1.42 \%)$ | 169.346 | $169.753(-0.15 \%)$ | 170.007 |
| $90^{\circ}$ | 197.216 | $206.892(-0.67 \%)$ | 207.548 | $207.869(-0.21 \%)$ | 208.297 |
| DOFs | 1395 | 558 | 930 | 1395 | 2135 |

The aeroelastic response of the straight wing previously analyzed is carried out when a concentrated load applied at the tip cross-section is combined to the aerodynamic pressures. A mesh of $10 B 4$ elements and a higher-order model are involved $(N=4)$. Due to the material anisotropy, the tip transverse force $P_{z}$ has an additional torsional effect along the span-wise direction which could be clockwise or counter-clockwise depending on the lamination angle's sign. As a consequence, the three-dimensional deflection of the wing is strongly affected by $\theta$ as depicted in Fig. 11.11 with a large scale factor.

The effect of tailoring on the divergence speed of a swept-forward wing structure is displayed in Fig. 11.12. Given a lamination, the lower is the sweep angle the lower is the divergence speed. In general, it is confirmed that for a forward-swept wing the divergence instability constitutes a more critical case with respect to an unswept one, due to the wash-in effect. However, for positive lamination angles $V_{D}$ is so high that this critical phenomenon becomes negligigle.

The lamination angle $\theta_{V_{D \text { min }}}$ associated to the minimum divergence speed depends on the sweep angle of the wing as well as the wing geometry. For the straight wing $\theta_{V_{D \text { min }}}$ is the value associated to the maximum tip twist of the aeroelastic response as depicted in Fig. 11.9. It remarks that the correct evaluation of the torsional behavior of a structure


Figure 11.11: 3D deformation of the straight composite wing. Aeroelastic response with tip transverse force $P_{z}=-515.5 \mathrm{~N} . V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \alpha=1^{\circ}$


Figure 11.12: Combined effects of the lamination $\theta$ and the sweep angle $\Lambda$ on the divergence speed $V_{D}$. Composite material

Table 11.8: Effect of the lamination on the divergence speed $V_{D}[\mathrm{~m} / \mathrm{s}]$. Straight wing, $c=1 \mathrm{~m}, L$ $=10 \mathrm{~m}, h=100 \mathrm{~mm} .10 \mathrm{~B} 4$ elements. $\alpha=1^{\circ}, 10 \times 50$ panels. Composite material.

| $\theta$ | $N=4$ | NASTRAN | \% Diff |
| :---: | :---: | :---: | :---: |
| $-90^{\circ}$ | 232.8142 | 231.7779 | +0.4471 |
| $-75^{\circ}$ | 164.5029 | 163.9645 | +0.3284 |
| $-60^{\circ}$ | 128.6913 | 128.4517 | +0.1865 |
| $-45^{\circ}$ | 109.8690 | 109.7624 | +0.0971 |
| $-30^{\circ}$ | 106.6362 | 106.5942 | +0.0394 |
| $-15^{\circ}$ | 124.5134 | 124.5123 | +0.0009 |
| $0^{\circ}$ | 235.1681 | 235.4656 | -0.1263 |
| DOFs | 1395 | 2135 | -34.66 |

is a mandatory issue for the model involved in the aeroelastic analysis. Thus classical beam theories are ineffective in detecting the wing divergence instability whereas the capabilities of higher-order models are proved in comparison with MD NASTRAN results, see Table 11.8. It should be remarked that the refined CUF model requires a limited number of DOFs to detect the wing divergence speed with an error no higher than $0.5 \%$ with respect to the reference MD NASTRAN-shell solution. Although a simple section geometry has been used here for a direct comparison with MD NASTRAN, the number of DOFs necessary to the present model would be remarkably lower especially for complex section geometries.

## Chapter 12

## Results: 3D Panel-1D CUF aeroelastic coupling

### 12.1 Aerodynamic assessment

Firstly an aerodynamic assessment of VLM and 3D Panel Method, which are able to evaluate the pressure coefficients on the wing surface, is performed analyzing the effects of two typical geometrical parameters: the airfoil thickness and the camber line. A straight wing is considered: the wing span is 10 m and the airfoil chord is 1 m long as drawn in Fig. 12.1(a), where the right half-wing is depicted. This wing configuration is also used in the following structural and aeroelastic analyses. The effect of the camber line on the aerodynamic field is evaluated using NACA 2415, 3415 and 4415 airfoils. The analysis of the influence of the airfoil thickness is then carried out using the symmetric NACA 0005, 0010 and 0015 airfoils.

The number of aerodynamic panels is chosen as a compromise between the limit number of panels that can be used in XFLR5 $(=5000)$ [136] and the number of panels required in order to achieve convergence in the aerodynamic results. In the following analyses, the choice of $N_{A P}^{x}=24$ and $N_{A P}^{y}=50$ remains the same.

For the present assessment analysis the free stream velocity is assumed to be $V_{\infty}=50$ $\mathrm{m} / \mathrm{s}$ such that the compressibility effects can be neglected. The air density is assumed to be $\rho_{\infty}=1.225 \mathrm{~kg} / \mathrm{m}^{3}$. The angle of attack $\alpha$ of the wing is equal to 3 deg. In all the following analyses the air density $\rho_{\infty}$ and the angle of attack $\alpha$ will be invariable parameters. The results focus on the variation of the spanwise local lifting coefficient $C_{l}$ along the wing span defined as:

$$
\begin{equation*}
C_{l}(y)=\frac{L(y)}{\frac{1}{2} \rho_{\infty} V_{\infty}^{2} 2 e(y) c(y)} \tag{12.1}
\end{equation*}
$$

where $c(y)$ and $L(y)$ are the chord and the Lift Force generated by the pressure acting on the panels with span-length $2 e(y)$ placed at the $y$ coordinate. More details can be found in [138]. As a first result, the trend of $C_{l}$ along the $y$ axis (right half-wing) is reported in Fig. 12.2(a). This analysis is carried out considering the variation of the airfoil thickness. As expected, the VLM is not able to take into account the variation of airfoil thickness, since it computes aerodynamic pressures on the wing reference surface, and underestimates $C_{l}$ with respect to the 3D Panel Method. On the contrary, the 3D Panel Method is able to evaluate the change of the lifting coefficient as the airfoil thickness increases, as can be seen in Fig. 12.2(a).

Figure 12.2(b) reports the trend of the spanwise local lifting coefficient $C_{l}$ as the camber line changes. It is evident that both aerodynamic methods are able to analyze the influence

(b) NACA 2415 airfoil cross-section with variable thickness and 2 cells

Figure 12.1: Geometrical configuration of the straight wing.
of the camber line. Comparing Figs.12.2(a) and 12.2(b) it should to be noted that the spanwise local lifting coefficient, and thus the aerodynamic pressures, is affected more by the camber line change than the airfoil thickness change. It can be concluded that the 3D Panel Method is able to provide a more realistic evaluation of the pressure distribution on the wing than the VLM. Moreover, the 3D Panel Method affords pressure loads on the actual wing surface, which are fundamental for an accurate study of the actual wing deformation and airfoil distortion, in lieu of loads applied on a fictitious wing reference surface as for the VLM case. These reasons make the 3D Panel Method the recommended classical aerodynamic tool for the following aeroelastic wing analyses.

### 12.2 Structural assessment

In order to validate the results given by the proposed higher-order 1D CUF approach for wings with a high-deformable cross-section, unlike the cases analyzed in chapters 10 and 11, a comparison of the static structural wing response is here performed with MSC Nastran. Only the right half-wing of the straight configuration introduced in the previous aerodynamic assessment (see Fig. 12.1(a)) is considered here due to loads and structural symmetry. A clamped boundary condition is taken into account for the root cross-section (at $y=0$ ), whereas the tip cross-section is free. The cross-section employed is a 2415 NACA airfoil with constant thickness equal to 2 mm . A spar with a thickness equal to 2 mm is inserted along the spanwise direction at $25 \%$ of the. The isotropic material adopted


Figure 12.2: Effect of the airfoil thickness (a) and camber line (b) on the spanwise local lifting coefficient $C_{l}$ of the straight wing along the $y$ axis. Comparison of VLM and 3D Panel Method. $V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \rho_{\infty}=1.225 \mathrm{~kg} / \mathrm{m}^{3}, \alpha=3 \mathrm{deg}$.
is aluminum: Young's modulus $E=69 \mathrm{GPa}$, Poisson's ratio $\nu=0.33$.
Due to the small thickness and the well-known aspect ratio restrictions typical of solid elements, this wing is modeled in MSC Nastran by means of 214500 solid Hex8 elements and 426852 nodes, corresponding to 127800 degrees of freedom (DOFs). The same structure is analyzed by means of CUF models with a variable expansion order up to $\mathrm{N}=14$ and discretized through a 1D mesh of 10 B 4 finite elements ( 31 nodes). The number of DOFs depends on $N$ as expressed in Eq. 3.24. For instance, with 10 B4 elements DOFs $=11160$ for $N=14$. However, an analysis of the present structure is carried out also through a Nastran shell FE model, but it is not reported herein for the sake of brevity. Nonetheless,
the error obtained between 1D CUF and shell results is comparable with the error obtained between 1D CUF and solid results.

Table 12.1: Pressure distribution on the wing along the spanwise direction for the structural assessment. $V_{\infty}=50 \mathrm{~m} / \mathrm{s}, \rho_{\infty}=1.225 \mathrm{~kg} / \mathrm{m}^{3}, p_{\text {ref }}=1 / 2 \rho_{\infty} V_{\infty}^{2}=551.25 \mathrm{~Pa}$.

| Interval | $p / p_{\text {ref }}$ |
| :---: | :---: |
| $0.00 \leq \mathrm{y} \leq 1.25$ | 1.00 |
| $1.25<\mathrm{y} \leq 2.50$ | 0.75 |
| $2.50<\mathrm{y} \leq 3.75$ | 0.50 |
| $3.75<\mathrm{y} \leq 5.00$ | 0.25 |

A variable pressure distribution step-like along the spanwise direction is applied to the upper and lower wing surfaces in order to simulate a real pressure distribution, see Table 12.1. The static structural response of the wing is evaluated in terms of the distortion $s$ at the tip cross-section. For the upper and lower surfaces, Figs. 12.3(a) and 12.3(b) show the percent error e obtained computing the distortion through 1D CUF models and Nastran solid model, which is taken as reference:

$$
\begin{equation*}
e=100 \cdot \frac{s_{\text {Nastran }}-s_{1 \mathrm{DCUF}}}{s_{\mathrm{Nastran}}} \tag{12.2}
\end{equation*}
$$

As depicted in Figs. 12.3(a) and 12.3(b), the proposed 1D FEs provide a convergent solution by gradually approaching the Nastran solid results as the expansion order increases from 8 to 14 , according to the conclusions made in previous CUF works [118, 130]. For $N=14$ the maximum percent error is about $3 \%$ for the upper surface and about $2.7 \%$ for the lower surface. For the wing configuration considered, the choice of $N=14$ seems hence to be accurate enough to detect the cross-section distortion with an acceptable error with respect to Nastran 3D results and with a remarkable reduction in terms of DOFs (about a $91 \%$ reduction, 11160 vs. 127800 ).

### 12.3 Aeroelastic coupling

This section focuses on the results regarding the equilibrium aeroelastic response of a wing exposed to a free stream velocity $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$ via the iterative CUF-XFLR5 procedure. This analysis aims at evaluating the influence of CUF expansion order $N$ on the aeroelastic behavior of the structure, as the accurate description of the cross-section distortion depends on $N$. The same material and straight wing configuration as those considered in the previous assessment are employed here, see Fig. 12.1(a). In this case the cross-section is the NACA 2415 airfoil depicted in Fig. 12.1(b). The spar thickness $t_{3}$ is constant and equal to 2 mm whereas the skin thickness of upper and lower surfaces varies gradually from 2 $\mathrm{mm}\left(t_{1}\right.$ in Fig. 12.1(b)) to $1 \mathrm{~mm}\left(t_{2}\right.$ in Fig. 12.1(b)) in the zone between the $40 \%$ and the $45 \%$ of the chord. This particular choice is coherent with the purpose of studying a high-deformable nonclassical cross-section.

The 1D structural mesh consists of 10 B 4 elements. For the sake of brevity, a convergent study on the number of mesh elements is not reported here. In fact, the choice of 10 B 4 elements yields a good evaluation of displacements for all the points of the structure, as detailed in $[138,139]$, where a similar structural case in terms of wing configuration and applied aerodynamic loads was studied via the present structural model and successfully assessed with a commercial FE solid model.


Figure 12.3: Percent error obtained by different 1D CUF models in the computation of the distortion along the airfoil upper (a) and lower (b) surfaces at the wing tip cross-section ( $y=5 \mathrm{~m}$ ). Static structural wing response to a variable pressure distribution. Reference solution: Nastran solid.

The aeroelastic analysis is now carried out following the iterative coupled procedure CUF-XFLR5 described in Fig. 9.2 and varying $N$. The convergence process on the lifting and moment coefficients is drawn in Fig. 12.4(a) by means of a dimensionless parameter $C_{L} / C_{L}^{\text {conv }}$ and in Fig. 12.4(b), respectively. $C_{L}^{\text {conv }}$ is the final convergent value of the lifting coefficient which is different for each expansion order employed as well as the final convergent moment coefficient $C_{M}^{\text {conv }}$, as reported in Table 12.2.

Hence, a different choice of $N$ influences the structural response of the wing to the aerodynamic loads and consequently affects also the aerodynamic analysis, due to the aeroelastic coupling. The higher the expansion order employed the more difference appears


Figure 12.4: Convergence of lifting and moment coefficients in the iterative aeroelastic analysis for structural models with different accuracy. Aerodynamic method: 3D Panel. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.
between $C_{L}^{\text {conv }}\left(C_{M}^{\text {conv }}\right)$ and the initial value $C_{L}^{\mathrm{in}}\left(C_{M}^{\text {conv }}\right)$ evaluated for the undeformed wing. For the cases presented in this work, the number of iterations required to achieve the convergence of the lifting coefficient is the same as that one required to achieve the convergence of the moment coefficient. It can be seen that the increase of $N$ corresponds to a different and increasing number of iterations $N_{\mathrm{iter}}^{\text {conv }} \mathrm{C}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}$ required to achieve the convergence of aerodynamic coefficients, as reported in Table 12.3. This tendency will be clearly explained afterwards as a consequence of the introduction of higher-order terms in the model formulation which enriches the displacement field.

An average cross-section distortion $\bar{s}$ is now introduced in order to evaluate the aeroelastic deformation of the cross-section shape along the wing span. Given an airfoil cross-section,

Table 12.2: Convergent values of lifting coefficient $C_{L}^{\text {conv }}$ and moment coefficient $C_{M}^{\text {conv }}$ for different structural models. Static aeroelastic equilibrium response. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}, C_{L}^{\mathrm{in}}=0.4637, C_{M}^{\mathrm{in}}=$ -0.1629 .

| Model | $C_{L}^{\text {conv }}$ | $C_{M}^{\text {conv }}$ | $N_{\text {iter }}^{\text {conv } \mathrm{C}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}}$ | DOFs |
| :--- | :---: | :---: | :---: | :---: |
| $N=1$ | 0.4643 | -0.1633 | 2 | 279 |
| $N=4$ | 0.4641 | -0.1634 | 2 | 1395 |
| $N=8$ | 0.4667 | -0.1659 | 3 | 4185 |
| $N=9$ | 0.4877 | -0.1823 | 6 | 5115 |
| $N=10$ | 0.4953 | -0.1886 | 8 | 6138 |
| $N=12$ | 0.5034 | -0.1950 | 9 | 8463 |
| $N=14$ | 0.5090 | -0.1994 | 10 | 11160 |

Table 12.3: Convergence of the lifting coeffient $C_{L}$ in the iterative aeroelastic analysis for different structural models. Airfoil cross-section at $y=4 \mathrm{~m} . V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.

| Model | Iteration |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $N=1$ | 0.4643 | - | - | - | - | - | - | - | - | - |
| $N=4$ | 0.4641 | - | - | - | - | - | - | - | - | - |
| $N=8$ | 0.4666 | 0.4666 | 0.4666 | 0.4666 | - | - | - | - | - | - |
| $N=9$ | 0.4816 | 0.4861 | 0.4873 | 0.4876 | 0.4877 | 0.4877 | 0.4877 | 0.4877 | - | - |
| $N=10$ | 0.4828 | 0.4904 | 0.4934 | 0.4946 | 0.4950 | 0.4952 | 0.4953 | 0.4953 | 0.4953 | 0.4953 |
| $N=12$ | 0.4850 | 0.4949 | 0.4995 | 0.5016 | 0.5026 | 0.5030 | 0.5032 | 0.5033 | 0.5033 | 0.5034 |
| $N=14$ | 0.4863 | 0.4977 | 0.5034 | 0.5063 | 0.5077 | 0.5084 | 0.5087 | 0.5089 | 0.5090 | 0.5090 |

- : convergence achieved with a tolerance toll $=10^{-4}$
the average distortion $\bar{s}$ is defined as:

$$
\begin{equation*}
\bar{s}=\frac{\oint s \cdot d l}{\oint d l} \tag{12.3}
\end{equation*}
$$

where $l$ is the curvilinear coordinate along the external airfoil surface and $s$ is the distortion of the single point of the external airfoil surface defined in Eq. 9.87. It is noteworthy that $s$ is a positive quantity and a null value for the average distortion $\bar{s}$ means no distortion. Figure 12.5 plots the trend of the average distortion along the wing span showing which are the most in-plane deformed airfoil cross-sections in the static aeroelastic equilibrium response. A remarkable variation in the trend of the average distortion appears depending on the accuracy of the structural model chosen. Models with an expansion order higher than 9 reveal that the section at $y=4 \mathrm{~m}$ appears to be the most distorted section.

For this cross-section, Table 12.4 presents the numerical values of average distortion $\bar{s}$ in the iterative aeroelastic analysis for different structural theories. As occurred for the convergence of aerodynamic coefficients in Table 12.3, the number of iterations $N_{\text {iter }}^{\text {convs }}$ required to achieve the convergence of $\bar{s}$ increases as $N$, and consequently DOFs, increases. In fact, increasing the expansion order $N$, the structural model becomes in general more deformable approaching the real structural behavior. It means that a complete threedimensional displacement field as well as local effects are evaluated with an increasing accuracy, especially for structures with high-deformable cross-sections, see Figs. 12.3(a) and $12.3(\mathrm{~b})$. Since the model accuracy increases, the structural deformation is therefore more sensitive to the variations of aerodynamic loads, which are different for each iteration following the convergent trend in Figs. 12.4(a) and 12.4(b), leading to an increasing $N_{\text {iter }}^{\text {conv s }}$.

Numerical results in Table 12.4 highlight that, given an expansion order, a higher number of iterations is necessary to achieve convergence on structural distortion than convergence on aerodynamic coefficients ( $N_{\text {iter }}^{\text {convs }}>N_{\text {iter }}^{\text {conv } \mathrm{C}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}}$ ), although the tolerance employed is the same.


Figure 12.5: Spanwise distribution of the average distortion $\bar{s}$ of the airfoil cross-sections for different structural models. Static aeroelastic equilibrium response. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.

Table 12.4: Convergence of the average distortion $\bar{s}[\mathrm{~mm}]$ in the iterative aeroelastic analysis for different structural models. Airfoil cross-section at $y=4 \mathrm{~m} . V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.

| Model | Iteration |  |  |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
| $N=1$ | 0.0402 | 0.0403 | 0.0403 | - | - | - | - | - | - | - | - | - | - |
| $N=4$ | 0.0135 | 0.0136 | 0.0136 | - | - | - | - | - | - | - | - | - | - |
| $N=8$ | 0.1729 | 0.1816 | 0.1820 | 0.1821 | 0.1821 | - | - | - | - | - | - | - | - |
| $N=9$ | 1.1441 | 1.4721 | 1.5624 | 1.5868 | 1.5934 | 1.5951 | 1.5956 | 1.5958 | - | - | - | - | - |
| $N=10$ | 1.4177 | 1.9198 | 2.1159 | 2.1930 | 2.2234 | 2.2353 | 2.2400 | 2.2419 | 2.2426 | 2.2429 | - | - | - |
| $N=12$ | 1.6738 | 2.2852 | 2.5542 | 2.6774 | 2.7340 | 2.7600 | 2.7719 | 2.7774 | 2.7799 | 2.7811 | 2.7816 | 2.7818 | - |
| $N=14$ | 1.7925 | 2.4670 | 2.7867 | 2.9456 | 3.0250 | 3.0646 | 3.0844 | 3.0941 | 3.0990 | 3.1014 | 3.1027 | 3.1033 | 3.1035 |

- : convergence achieved with a tolerance toll $=10^{-4}$

For $N>8$ the displacement field becomes accurate enough to relevantly take into account a cross-section distortion for the airfoil case considered, as can be seen also in Fig. 12.6. As previously explained, given a structural model the distortion is computed by comparing the deformed cross-section to the corresponding base section. For the sake of simplicity, only the base section for $N=1$ is plotted in Fig. 12.6.

As expected, low-order models provide a correct evaluation of the bending and torsional structural behavior, but a not exhaustive description of the in-plane deformation. This conclusion is confirmed by Fig. 12.7, where the airfoil distortion s computed by variable kinematic models is depicted along the upper surface at $y=4 \mathrm{~m}$. The maximum distortion value is reached in the part of the cross-section next to the trailing edge since the stiffening effect due to the spar at $25 \%$ of the chord limits the cross-section distortion. Nonetheless, the chordwise position of the maximum distortion points on the airfoil upper and lower surfaces changes depending on the accuracy of the structural model, see Table 12.5. As a consequence, it is worth pointing out that the increase of $N$ reveals to be relevant not


Figure 12.6: Deformation of the airfoil cross-section at $y=4 \mathrm{~m}$ computed for structural models with different accuracy. Static aeroelastic equilibrium response. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.
only for an accurate detection of distortion values but also of the accurate shape-type deformation.


Figure 12.7: Distortion of the airfoil upper surface of the cross-section at $y=4 \mathrm{~m}$ computed for different structural models. Static aeroelastic equilibrium response. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.

In general, improvements of the structural theory yield more realistic deformations of the wing until a good convergence is achieved for $N=14$, according to the conclusions made for Figs. 12.3(a) and 12.3(b) in the structural assessment. In other words, the difference between the results obtained through the generic $(N-1)^{t h}$ and $N^{t h}$ expansion orders decreases and becomes minimal for $N=14$. For this reason it is possible to consider the fourteenth-order model sufficiently accurate to describe the aeroelastic behavior of the structure here considered.

Table 12.5: Convergent average distortion $\bar{s}^{\text {conv }}$ [ mm ] of the cross-section at $y=4 \mathrm{~m}$ for different structural models. Values and chordwise position of the maximum distortions $s_{\max }^{U S}[\mathrm{~mm}]$ and $s_{\max }^{L S}$ [ mm ] on airfoil upper and lower surfaces. Static aeroelastic equilibrium response. $V_{\infty}=30 \mathrm{~m} / \mathrm{s}$.

| Model | $\bar{s}^{\text {conv }}$ | $N_{\text {iter }}^{\text {conv }}$ | $s_{\max }^{U S}$ | $x_{s_{\max }^{U S}} / c$ | $s_{\max }^{L S}$ | $x_{s_{\text {max }}^{L S}} / c$ | DOFs |
| :--- | ---: | ---: | ---: | :---: | :---: | :---: | ---: |
| $N=1$ | 0.0403 | 3 | 0.0718 | 0.33 | 0.0439 | 0.24 | 279 |
| $N=4$ | 0.0136 | 3 | 0.0103 | 0.33 | 0.0251 | 0.23 | 1395 |
| $N=8$ | 0.1821 | 5 | 0.5267 | 0.74 | 0.4797 | 0.75 | 4185 |
| $N=9$ | 1.5958 | 8 | 4.6073 | 0.74 | 4.1253 | 0.75 | 5115 |
| $N=10$ | 2.2429 | 10 | 6.9936 | 0.73 | 5.1626 | 0.79 | 6138 |
| $N=12$ | 2.7818 | 12 | 9.5341 | 0.73 | 5.7456 | 0.82 | 8463 |
| $N=14$ | 3.1035 | 13 | 10.7482 | 0.73 | 6.0178 | 0.82 | 11160 |

### 12.4 Freestream velocity influence

This analysis aims at establishing the influence of the free stream velocity on the wing distortion. The wing configuration employed for this analysis is the same as the that one used in the previous study. According to the conclusion made above, the structural model considered is $N=14$. The free stream velocities considered are 25,30 , and $35 \mathrm{~m} / \mathrm{s}$. As in the previous analysis, the aerodynamic convergence process is presented through the dimensionless parameter $C_{L} / C_{L}^{\text {conv }}$, as illustrated in Fig. 12.8(a). The convergence of the moment coefficient is also shown in Fig. 12.8(a) through the parameter $C_{M} / C_{M}^{\text {conv }}$.


Figure 12.8: Convergence of lifting and moment coefficients in the iterative aeroelastic analysis for different free stream velocities. Structural model: $N=14$. Aerodynamic method: 3D Panel.

In this case, $C_{L}^{\text {conv }}$ and $C_{M}^{\text {conv }}$ represent the final convergent values of the lifting and moment coefficients for a given $V_{\infty}$. As occurred for the previous aeroelastic analysis, the trends do not show any numerical problems such as oscillations. From Figs. 12.8(a) and $12.8(\mathrm{~b})$ it is important to note that the number of iterations $N_{\text {iter }}^{\text {conv }} \mathrm{CLC}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}$ required to achieve the aerodynamic convergence increases as $V_{\infty}$ increases, and the final convergent values are much different from the initial values, as summarized in Table 12.5. The reason of this behavior is easily explained by the fact that an increasing free stream velocity means increasing aerodynamic loads, and consequently higher structural deformations, and lastly a more relevant coupling effect on the aeroelastic response of the wing. In fact, an increasing airfoil distortion for the most deformed cross-section at $y=4 \mathrm{~m}$ is obtained with $V_{\infty}$ according to numerical results in Table 12.6 and airfoil deformed profiles in Fig. 12.9.

Also for velocity values different from $30 \mathrm{~m} / \mathrm{s}$, a higher number of iterations is necessary to achieve convergence on structural distortion than convergence on aerodynamic coefficients $\left(N_{\text {iter }}^{\text {convs }}>N_{\text {iter }}^{\text {conv }} \mathrm{C}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}\right)$, see Table 12.6.

Table 12.6: Convergent values of lifting coeffient $C_{L}^{\text {conv }}$, moment coefficient $C_{M}^{\text {conv }}$, and average distortion $\bar{s}^{\text {conv }}[\mathrm{mm}]$ of cross-section at $y=4 \mathrm{~m}$ for different free stream velocities $V_{\infty}[\mathrm{m} / \mathrm{s}]$. Static aeroelastic equilibrium response. Structural model: $N=14 . C_{L}^{\mathrm{in}}=0.4637, C_{M}^{\mathrm{in}}=-0.1629$.

| $V_{\infty}$ | $C_{L}^{\text {conv }}$ | $C_{M}^{\text {conv }}$ | $N_{\text {iter }}^{\text {conv } \mathrm{C}_{\mathrm{L}} \mathrm{C}_{\mathrm{M}}}$ | $\bar{s}^{\text {conv }}$ | $N_{\text {iter }}^{\text {conv }}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 0.4879 | -0.1827 | 7 | 1.7269 | 9 |
| 30 | 0.5090 | -0.1994 | 10 | 3.1035 | 13 |
| 35 | 0.5608 | -0.2394 | 18 | 6.3296 | 22 |



Figure 12.9: Deformation of the airfoil cross-section at $y=4 \mathrm{~m}$ computed for different free stream velocities. Static aeroelastic equilibrium response. Structural model: $N=14$.

The limitation of distortion close to the airfoil leading edge due to the spar is enhanced for $V_{\infty}=35 \mathrm{~m} / \mathrm{s}$. The trends of distortion on the airfoil upper and lower surfaces, which are indicated as US and LS respectively, are depicted in Fig. 12.10 at $y=4$ for different velocities. It is important to note that deformations of upper and lower surfaces remarkably differ also because of a different aerodynamic pressure distribution. Table 12.7 shows that not only the maximum distortion values on the airfoil upper $\left(s_{\max 1}^{U S}, s_{\max 2}^{U S}\right)$ and lower $\left(s_{\max 1}^{L S}, s_{\max 2}^{L S}\right)$ surfaces changes as $V_{\infty}$ varies, but also their corresponding chordwise positions. This aspect highlights the importance of higher-order models especially for an accurate evaluation of in-plane cross-section distortion of high-deformable structures.


Figure 12.10: Distortion of the airfoil upper and lower surfaces of the cross-section at $y=4 \mathrm{~m}$ computed for different free stream velocities. Static aeroelastic equilibrium response. Structural model: $N=14$.

Table 12.7: Values and chordwise positions of the maximum distortions $s_{\max 1}^{U S}, s_{\max 2}^{U S}, s_{\max 1}^{L S}, s_{\max 2}^{L S}$ [ mm ] on airfoil upper and lower surfaces of the cross-section at $y=4 \mathrm{~m}$ for different free stream velocities $V_{\infty}[\mathrm{m} / \mathrm{s}]$. Static aeroelastic equilibrium response. Structural model: $N=14$.

| $V_{\infty}$ | $s_{\max 1}^{U S}$ | $x_{s_{\max 1}^{U S}} / c$ | $s_{\max 2}^{U S}$ | $x_{s_{\max 2}^{U S}} / c$ | $s_{\max 1}^{L S}$ | $x_{s_{\max 1}^{L S}} / c$ | $s_{\max 2}^{L S}$ | $x_{s_{\max 2}^{L S}} / c$ |
| :--- | ---: | ---: | ---: | :---: | :---: | :---: | :---: | :---: |
| 25 | 6.0892 | 0.72 | 0.8670 | 0.29 | 3.0324 | 0.83 | 1.6285 | 0.46 |
| 30 | 10.7482 | 0.73 | 1.5540 | 0.30 | 6.0178 | 0.82 | 2.6232 | 0.45 |
| 35 | 21.2323 | 0.74 | 3.1618 | 0.31 | 13.9437 | 0.81 | 4.5026 | 0.43 |

## Part III

## Fluid Dynamic Formulation

## Chapter 13

## 1D CUF model for Computational Fluid Dynamics

In this chapter the basic differential equations of fluid mechanics are presented. In particular, the Navier-Stokes and Stokes equations are derived for compressible and incompressible flows in strong and weak forms. Furthermore, the use of the 1D CUF model to describe the fluid behavior on a fixed computational domain is addressed. The advantages of a reduced order model based on the 1D CUF approximation for fluid dynamics will be highlighted and discussed.

### 13.1 Notation

Some notations used which will be adopted in the following sections are here introduced. $\Omega$ is a (three-dimensional) volume domain embodying the region of interest: $\Omega \subset \mathbb{R}^{3}$. It is usually called computational domain or control volume. It is bounded and its bounding surface is denoted by $\partial \Omega$, whereas $\Gamma$ is the symbol used to refer to a generic surface (twodimensional) in the domain. The outwardly oriented unit vector normal to the boundary $\partial \Omega$ is indicated with $\mathbf{n}$. In general, the computational domain can remain fixed in space or can move with the fluid. In this chapter, the flow motion is studied in a fixed domain $\Omega$, since the fluid-structure interaction problem is not considered here. The time interval over which the flow is studied is indicated with $I=\left(t_{0}, t_{1}\right)$, where $t_{0}$ and $t_{1}$ are the initial and the final time instants of the analysis.

A cartesian coordinate system $\left(x_{1}, x_{2}, x_{3}\right)$ is introduced and the position of a point in the three-dimensional cartesian coordinate system is denoted by the vector $\mathbf{x}$, whose components are $\left(x_{1}, x_{2}, x_{3}\right)$. For the sake of clarity, it is here disclosed that the cartesian coordinate system, whose general form is $\left(x_{1}, x_{2}, x_{3}\right)$, will be sometimes indicated also with the more convenient notation $(x, y, z)$ depending on the case. This choice has not to confuse the reader.

If a quantity $f$ (like pressure) takes a scalar value on the domain $\Omega$, the quantity defines a scalar field on $\Omega$ at time instant $t$, which is indicated with $f: \Omega \times I \rightarrow \mathbb{R}$. If instead a quantity $\mathbf{f}$ associates to each point in $\Omega$ a vector (as in the case of velocity) at time instant $t$, it defines a vector field on $\Omega$ and $I$, which is indicated with $\mathbf{f}: \Omega \times I \rightarrow \mathbb{R}^{3}$. Finally, if a quantity $\underline{\mathbf{T}}$ associates to each point in $\Omega$ a $\mathbb{R}^{3 \times 3}$ matrix at time instant $t$, it defines a (second order) tensor field on $\Omega$ and $I$ if it obeys the ordinary transformation rules for tensor [140]. Hence $\underline{\mathbf{T}}: \Omega \times I \rightarrow \mathbb{R}^{3 \times 3}$. Its components will be indicated by either $(\underline{\mathbf{T}})_{i j}$, or simply $T_{i j}$, with $i, j=1,2,3$.

### 13.2 Navier-Stokes equations

The Navier-Stokes equations, named after Claude-Louis Navier and George Gabriel Stokes, are the basic differential equations of fluid mechanics and describe the motion of fluid substances. The Navier-Stokes equations are based on the assumption that the fluid, at the scale of interest, is a continuum. In other words, the fluid is not made up of discrete particles but rather a continuous substance. Another necessary assumption is that all the fields of interest like pressure, velocity, density, temperature and so on are differentiable.

The Navier-Stokes equations for compressible fluids in a fixed domain $\Omega \subset \mathbb{R}^{3}$, for any $t \in I$, are:

$$
\left\{\begin{array}{l}
\frac{\partial \rho}{\partial t}+\sum_{j=1}^{3} \frac{\partial\left(\rho u_{j}\right)}{\partial x_{j}}=0  \tag{13.1}\\
\frac{\partial\left(\rho u_{i}\right)}{\partial t}+\sum_{j=1}^{3}\left[\frac{\partial\left(\rho u_{i} u_{j}\right)}{\partial x_{j}}-\frac{\partial \tau_{i j}}{\partial x_{j}}\right]+\frac{\partial P}{\partial x_{i}}=\rho f_{i} \quad i=1,2,3 \\
\frac{\partial(\rho e)}{\partial t}+\sum_{j=1}^{3}\left[\frac{\partial\left(\rho h u_{j}\right)}{\partial x_{j}}-\frac{\partial\left(\sum_{j=1}^{3} u_{i} \tau_{i j}-q_{j}\right)}{\partial x_{j}}\right]=0
\end{array}\right.
$$

where $\left\{u_{i}, i=1,2,3\right\}$ the components of the velocity vector $\mathbf{u}$ with respect to the cartesian coordinate system ( $x_{1}, x_{2}, x_{3}$ ).

In Eq. 13.1, the first equation describes the conservation of mass, which is also called continuity equation or mass conservation. The second row represents the conservation of linear momentum. In $\mathbb{R}^{3}$, the conservation of linear momentum has to be written for $i=1,2,3$ leading to three equations. The third equation is the conservation of energy.

The variables in Eq. 13.1 have the following meanings: $\mathbf{u}$ is the velocity vector $\mathbf{u}$ : $\Omega \times I \rightarrow \mathbb{R}^{3}$ so that $[\mathbf{u}]=m / s ; \rho$ is the density of the fluid $\rho: \Omega \times I \rightarrow \mathbb{R}$ so that $[\rho]=\mathrm{kg} / \mathrm{m}^{3} ; P$ is the pressure $P: \Omega \times I \rightarrow \mathbb{R}$ so that $[P]=N / \mathrm{m}^{2}=\mathrm{kg} /\left(\mathrm{ms}^{2}\right)$. The term $f_{i}$ is the $i^{\text {th }}$ component, with respect to the cartesian coordinate system ( $x_{1}, x_{2}, x_{3}$ ), of the vector of body forces (per mass unit) applied to the fluid $\mathbf{f}: \Omega \times I \rightarrow \mathbb{R}^{3}$ so that $[\mathbf{f}]=N / k g=m / \mathrm{s}^{2}$. The term $e$ is the total energy per mass unit, equal to the sum of of the internal energy per mass unit $e_{i}$ and the kinetic energy per mass unit of the fluid ( $\left.[e]=m^{2} / s^{2}\right):$

$$
\begin{equation*}
e=e_{i}+\frac{1}{2}|\mathbf{u}|^{2} \tag{13.2}
\end{equation*}
$$

The term $h$ is the total entalpy per mass unit $\left([h]=m^{2} / s^{2}\right)$, which is defined as:

$$
\begin{equation*}
h=e+\frac{P}{\rho}=e_{i}+\frac{P}{\rho}+\frac{1}{2}|\mathbf{u}|^{2} \tag{13.3}
\end{equation*}
$$

To complete the system in Eq. 13.1 it is necessary to link $e($ and $h$ ) to the variables $\rho, P$, u by defining a law $e=e(\rho, P, \mathbf{u})$ which is typically derived from the state equations of the fluid under exam. In particular, the state equations of an ideal gas are:

$$
\begin{equation*}
P=\rho R^{\star} T \quad e_{i}=c_{v} T \tag{13.4}
\end{equation*}
$$

where $R^{\star}=c_{p}-c_{v}$ is the gas constant $\left(\left[R^{\star}\right]=\left[c_{p}\right]=\left[c_{v}\right]=m^{2} /\left(s^{2} K\right)\right)$ and $T$ is the temperature $([T]=K)$. Equation 13.4 provides the required law $e=e(\rho, P, \mathbf{u})$ :

$$
\begin{equation*}
e_{i}=c_{v} T=\frac{c_{v}}{R^{\star}} \frac{P}{\rho}=\frac{P}{\rho(\gamma-1)} \quad \Rightarrow \quad e=\frac{P}{\rho(\gamma-1)}+\frac{1}{2}|\mathbf{u}|^{2} \tag{13.5}
\end{equation*}
$$

where $\gamma=c_{p} / c_{v}$ is the ratio between the specific heats (per mass unit) at constant pressure and volume, respectively. Finally, the thermal flux vector $\mathbf{q}$ has to be written in terms of the variables $\rho, P, \mathbf{u}$. For this purpose, $\mathbf{q}\left([q]=k g / s^{3}\right)$ is usually related to the temperature gradient via the Fourier law:

$$
\begin{equation*}
\mathbf{q}=-\lambda \nabla T=-\frac{\lambda}{c_{v}} \nabla e_{i}=-\lambda \nabla\left(\frac{P}{\rho R^{\star}}\right)=-\frac{\lambda}{c_{v}} \nabla\left(\frac{P}{\rho(\gamma-1)}\right) \tag{13.6}
\end{equation*}
$$

where $\lambda$ is the thermal conductivity $\left([\lambda]=k g m /\left(s^{3} K\right)\right.$ ).
The term $\tau_{i j}$ is the $\{i j\}^{\text {th }}$ component of the deviatoric stress tensor $\underline{\boldsymbol{\tau}}: \Omega \times I \rightarrow \mathbb{R}^{3 \times 3}$. The Navier-Stokes equations in Eq. 13.1 are still incomplete since kinematic hypotheses on the form of the $3 \times 3$ deviatoric stress tensor $\underline{\tau}$ have to be formulated. The deviatoric stress tensor is related to the kinematic quantities (velocity $\mathbf{u}$ ) via the constitutive law, which provides a characterization of the mechanical behavior of the particular fluid under consideration. Here, a Newtonian behavior is assumed for the fluid. In a Newtonian compressible fluid, the deviatoric stress tensor may be written as a linear function of the velocity derivatives [141]:

$$
\begin{equation*}
\tau_{i j}=\mu\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right)+\mu^{v} \delta_{i j} \operatorname{div} \mathbf{u}=\mu\left[\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right)-\frac{2}{3} \delta_{i j} \operatorname{div} \mathbf{u}\right] \tag{13.7}
\end{equation*}
$$

where $\mu$ is the dynamic viscosity of the fluid and is a positive quantity so that $[\mu]=k g /(m s)$. The term $\mu^{v}$ is the second coefficient of viscosity (related to bulk viscosity) and is set to $\mu^{v}=-\frac{2}{3} \mu$ for a Newtonian compressible fluid. The second coefficient of viscosity is often referred as $\lambda$, but here is referred as $\mu^{v}$ in order not to confuse it with the thermal conductivity. For the sake of completeness, $[\tau]=[P]=k g /\left(m s^{2}\right)$. The index $\delta_{i j}$ is the Kronecker delta:

$$
\delta_{i j}= \begin{cases}1 & i=j  \tag{13.8}\\ 0 & i \neq j\end{cases}
$$

For a fluid at rest, it is verified that $\operatorname{div} \boldsymbol{\tau}=\mathbf{0}$.
The Navier-Stokes equation expressed componentwise in Eq. 13.1 are now retrieved and rewritten in other forms. In particular, the continuity equation and the equation of linear momentum conservation are considered. The conservation of mass is written in a compact vectorial notation:

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\operatorname{div}(\rho \mathbf{u})=0 \tag{13.9}
\end{equation*}
$$

where:

$$
\begin{equation*}
\operatorname{div}(\rho \mathbf{u})=\rho \operatorname{div} \mathbf{u}+\mathbf{u} \cdot \nabla \rho \tag{13.10}
\end{equation*}
$$

Substituting Eq. 13.10 into Eq. 13.9, the continuity equation becomes:

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\rho \operatorname{div} \mathbf{u}+\mathbf{u} \cdot \nabla \rho=0 \tag{13.11}
\end{equation*}
$$

In order to write the conservation of linear momentum in a compact vectorial notation, a continuously differentiable symmetric tensor field, called Cauchy stress tensor $\underline{\mathbf{T}}: \Omega \times I \rightarrow$ $\mathbb{R}^{3 \times 3}$ is introduced:

$$
\begin{equation*}
\underline{\mathbf{T}}=-P \underline{\mathbf{I}}+\underline{\boldsymbol{\tau}} \tag{13.12}
\end{equation*}
$$

where $\mathbf{I}$ is the $3 \times 3$ identity matrix and $P$ is the pressure. The generic component of the deviatoric stress tensor $\boldsymbol{\tau}$ has already been expressed in Eq. 13.7. Considering this equation
and Eq. 13.12 , the generic component of the Cauchy stress tensor is:

$$
\begin{equation*}
T_{i j}=-P \delta_{i j}+\tau_{i j}=-P \delta_{i j}+\mu\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right)+\mu^{v} \delta_{i j} \operatorname{div} \mathbf{u} \tag{13.13}
\end{equation*}
$$

Setting $\mu^{v}=-\frac{2}{3} \mu$ for a Newtonian compressible fluid, Eq. 13.13 becomes:

$$
\begin{equation*}
T_{i j}=-P \delta_{i j}+\mu\left[\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right)-\frac{2}{3} \delta_{i j} \operatorname{div} \mathbf{u}\right] \tag{13.14}
\end{equation*}
$$

As mentioned for the deviatoric stress tensor, Eq. 13.14 highlights that in a Newtonian incompressible fluid also the Cauchy stress tensor may be written as a linear function of the velocity derivatives. According to the definition of the Cauchy stress tensor (Eq. 13.12) and its components in Eq. 13.14, the conservation of linear momentum can be rewritten in a compact vectorial notation in terms of $\mathbf{T}$ :

$$
\begin{equation*}
\rho \frac{\partial \mathbf{u}}{\partial t}+\rho(\mathbf{u} \cdot \nabla) \mathbf{u}-\operatorname{div} \underline{\mathbf{T}}=\rho \mathbf{f} \tag{13.15}
\end{equation*}
$$

The non-linear term $\rho(\mathbf{u} \cdot \nabla) \mathbf{u}$ in Eq. 13.15 is called convective term since it describes the convective transport. The components of vector $(\mathbf{u} \cdot \nabla) \mathbf{u}$ are:

$$
\begin{equation*}
((\mathbf{u} \cdot \nabla) \mathbf{u})_{i}=\sum_{j=1}^{3} u_{j} \frac{\partial u_{i}}{\partial x_{j}} \quad i=1,2,3 \tag{13.16}
\end{equation*}
$$

The Navier-Stokes equations in Eq. 13.15 can be rewritten componentwise:

$$
\begin{equation*}
\rho \frac{\partial u_{i}}{\partial t}+\rho \sum_{j=1}^{3} u_{j} \frac{\partial u_{i}}{\partial x_{j}}-\sum_{j=1}^{3} \frac{\partial T_{i j}}{\partial x_{j}}=\rho f_{i} \tag{13.17}
\end{equation*}
$$

It can be proved that Eq. 13.17 exactly corresponds to the second expression in Eq. 13.1.

### 13.2.1 Navier-Stokes equations for incompressible fluids

The assumption that the density $\rho$ of the fluid is constant (like for blood flow, for instance) is now introduced. The equation of mass conservation written for compressible fluids in Eq. 13.11 can be simplified for constant density fluids taking into account that:

$$
\begin{equation*}
\text { constant density fluid }(\rho=\text { const }) \Rightarrow \nabla \rho=\mathbf{0} \tag{13.18}
\end{equation*}
$$

Hence, the continuity equation becomes:

$$
\begin{equation*}
\frac{\partial \rho}{\partial t}+\rho \operatorname{divu}+\mathbf{u} \cdot \nabla \rho=0 \quad \Rightarrow \quad \operatorname{div} \mathbf{u}=0 \tag{13.19}
\end{equation*}
$$

which is the incompressibility constraint. A flow which satisfies the incompressibility constraint is called incompressible, as well as the fluid. From the continuity equation (Eq. 13.19), the following implication is derived:

$$
\begin{equation*}
\text { constant density fluid }(\rho=\text { const }) \quad \Rightarrow \quad \text { incompressible flow }(\operatorname{div} \mathbf{u}=0) \tag{13.20}
\end{equation*}
$$

whereas the opposite statement is not true in general. In fact, there exist incompressible flows ( $\operatorname{div} \mathbf{u}=0$ ) featuring variable density fluids ( $\rho \neq$ const), e.g. stratified fluids.

The Cauchy stress tensor $\mathbf{T}$ has been defined as the sum of the deviatoric stress tensor $\boldsymbol{\tau}$ and the pressure contribution in Eq. 13.12. Its components have been expressed for compressible fluids in Eq. 13.14 according to Eq. 13.7, which describes the generic component $\tau_{i j}$ for a Newtonian compressible fluid. For a Newtonian incompressible fluid, the deviatoric stress tensor may be written as a linear function of the velocity derivatives [141] as done in Eq. 13.7, but the components $\tau_{i j}$ can be simplified taking into account the corresponding continuity equation (Eq. 13.19):

$$
\begin{equation*}
\tau_{i j}=\mu\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right)+\mu^{v} \delta_{i j} \operatorname{div} \bar{u}=\mu\left(\frac{\partial u_{j}}{\partial x_{i}}+\frac{\partial u_{i}}{\partial x_{j}}\right) \tag{13.21}
\end{equation*}
$$

As a consequence, the Cauchy stress tensor for incompressible flow becomes:

$$
\begin{equation*}
\underline{\mathbf{T}}=-P \underline{\mathbf{I}}+\underline{\boldsymbol{\tau}}=-P \underline{\mathbf{I}}+\mu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)=-P \underline{\mathbf{I}}+2 \mu \underline{\mathbf{D}}(\mathbf{u}) \tag{13.22}
\end{equation*}
$$

where the tensor $\underline{\mathbf{D}}$ is the strain rate tensor. Its expression is:

$$
\begin{equation*}
\underline{\mathbf{D}}(\mathbf{u})=\frac{\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)}{2} \tag{13.23}
\end{equation*}
$$

For the sake of completeness, the term $2 \mu \underline{\mathbf{D}}(\mathbf{u})$ in the definition of the Cauchy stress tensor (Eq. 13.22) is often referred as the viscous stress component of the stress tensor. The quantities $\nabla \mathbf{u}$ and $\nabla \mathbf{u}^{T}$ are the following tensors:

$$
\nabla \mathbf{u}=\left[\begin{array}{ccc}
\frac{\partial u_{x}}{\partial x} & \frac{\partial u_{x}}{\partial y} & \frac{\partial u_{x}}{\partial z}  \tag{13.24}\\
\frac{\partial u_{y}}{\partial x} & \frac{\partial u_{y}}{\partial y} & \frac{\partial u_{y}}{\partial z} \\
\frac{\partial u_{z}}{\partial x} & \frac{\partial u_{z}}{\partial y} & \frac{\partial u_{z}}{\partial z}
\end{array}\right] \quad \nabla \mathbf{u}^{T}=\left[\begin{array}{ccc}
\frac{\partial u_{x}}{\partial x} & \frac{\partial u_{y}}{\partial x} & \frac{\partial u_{z}}{\partial x} \\
\frac{\partial u_{x}}{\partial y} & \frac{\partial u_{y}}{\partial y} & \frac{\partial u_{z}}{\partial y} \\
\frac{\partial u_{x}}{\partial z} & \frac{\partial u_{y}}{\partial z} & \frac{\partial u_{z}}{\partial z}
\end{array}\right]
$$

where, for the sake of convenience, it has been preferred to write the components of the velocity vector $\mathbf{u}$ as $u_{x}, u_{y}, u_{z}$ with respect to the cartesian coordinate system ( $x, y, z$ ) instead of $u_{1}, u_{2}, u_{3}$ with respect to the cartesian coordinate system $\left(x_{1}, x_{2}, x_{3}\right)$. The two forms are equivalent, in fact:

$$
\begin{equation*}
u_{1}=u_{x}, \quad u_{2}=u_{y}, \quad u_{3}=u_{z}, \quad x_{1}=x, \quad x_{2}=y, \quad x_{3}=z \tag{13.25}
\end{equation*}
$$

From Eq. 13.24 it is easily proved that the generic component of the strain rate tensor is:

$$
\begin{equation*}
D_{i j}=\frac{1}{2}\left(\frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial u_{j}}{\partial x_{i}}\right) \quad i, j=1,2,3 \tag{13.26}
\end{equation*}
$$

It is reminded here that the divergence is a linear operator. Thus:

$$
\begin{equation*}
\operatorname{div}(a \mathbf{b}+c \mathbf{d}+e \underline{\mathbf{F}}+g \underline{\mathbf{H}})=\operatorname{div}(a \mathbf{b})+\operatorname{div}(c \mathbf{d})+\operatorname{div}(e \underline{\mathbf{F}})+\operatorname{div}(g \underline{\mathbf{H}}) \tag{13.27}
\end{equation*}
$$

Using the property in Eq. 13.27, it is possible to express the divergence of $\underline{\mathbf{T}}$ as follows:

$$
\begin{equation*}
\operatorname{div} \underline{\mathbf{T}}=-\operatorname{div}(P \underline{\mathbf{I}})+2 \operatorname{div}[\mu \underline{\mathbf{D}}(\mathbf{u})] \tag{13.28}
\end{equation*}
$$

The product rule of the divergence operator for tensors is now exploited:

$$
\begin{equation*}
\operatorname{div}(P \underline{\mathbf{I}})=P \underline{\operatorname{di}} \underline{\mathbf{1}} \underline{\mathbf{+}}+\underline{\mathbf{I}} \nabla P=\nabla P \tag{13.29}
\end{equation*}
$$

where $\operatorname{div} \underline{\mathbf{I}}=0$ since $\underline{\mathbf{I}}$ is the $3 \times 3$ identity matrix and therefore a matrix of costant coefficients. The conservation of momentum written in the general form in Eq. 13.15 in terms of the Cauchy stress tensor can thus be formulated in terms of the strain rate tensor $\underline{\mathrm{D}}$ and the pressure $P$ as follows:

$$
\begin{equation*}
\rho \frac{\partial \mathbf{u}}{\partial t}-2 \operatorname{div}[\mu \underline{\mathbf{D}}(\mathbf{u})]+\rho(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla P=\rho \mathbf{f} \tag{13.30}
\end{equation*}
$$

It is convenient to introduce the kinematic viscosity $\nu\left([\nu]=m^{2} / s\right)$ and the scaled pressure $p\left([p]=m^{2} / s^{2}\right):$

$$
\begin{equation*}
\nu=\frac{\mu}{\rho} \quad p=\frac{P}{\rho} \tag{13.31}
\end{equation*}
$$

The second term of Eq. 13.30 can be now expressed in terms of the density $\rho$ and the kinematic viscosity $\nu$ instead of the dynamic viscosity $\mu$ :

$$
\begin{equation*}
\operatorname{div}[\mu \underline{\mathbf{D}}(\mathbf{u})]=\operatorname{div}[\rho \nu \underline{\mathbf{D}}(\mathbf{u})]=\rho \operatorname{div}[\nu \underline{\mathbf{D}}(\mathbf{u})] \tag{13.32}
\end{equation*}
$$

where the assumption $\rho=$ const is used. This assumption is exploited to relate the gradient of pressure $P$ to the gradient of scaled pressure $p$ :

$$
\begin{equation*}
\nabla P=\nabla(\rho p)=\rho \nabla p \tag{13.33}
\end{equation*}
$$

Substituting Eqs. 13.32 and 13.33 in Eq. 13.30 and dividing by the constant density $\rho$, the equation of the conservation of momentum becomes:

$$
\begin{equation*}
\frac{\partial \mathbf{u}}{\partial t}-2 \operatorname{div}[\nu \underline{\mathbf{D}}(\mathbf{u})]+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p=\mathbf{f} \tag{13.34}
\end{equation*}
$$

Recovering the expression of $\underline{\mathbf{D}}$ in Eq. 13.23 the system of conservation equations of momentum and mass for incompressible fluids can be rewritten in terms of the velocity $\mathbf{u}$ and pressure $p$ variables as:

$$
\begin{cases}\frac{\partial \mathbf{u}}{\partial t}-\operatorname{div}\left[\nu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)\right]+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p=\mathbf{f} & \text { in } \Omega, t \in I  \tag{13.35}\\ \operatorname{div} \mathbf{u}=0 & \text { in } \Omega, t \in I\end{cases}
$$

The term $-\operatorname{div}\left[\nu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)\right]$ describes the process of molecular diffusion, whereas it is reminded that $(\mathbf{u} \cdot \nabla) \mathbf{u}$ describes the convective transport and its components are written in Eq. 13.16.

Equation 13.35 describes the Navier-Stokes equations for incompressible fluids. The principal unknowns are the velocity $\mathbf{u}$ and the scaled pressure $p=P / \rho$. It is important to note that the energy equation has disappeared. Even though the conservation of energy (third equation in Eq. 13.1 for compressible fluids) can still be written for incompressible flows, its solution can be carried out independently once the velocity and pressure fields are obtained from the solution of the continuity equation and the equation of linear momentum conservation. Hence, the conservation of energy is not considered here.

## Case of constant viscosity

This section shows how the Navier-Stokes equations for incompressible fluids written in Eq. 13.35 simplifies for the case of constant kinematic viscosity $\nu$.

Before introducing the assumption of constant kinematic viscosity, the molecular diffusion term is split into two terms exploiting the property of the linear divergence operator in Eq. 13.27:

$$
\begin{equation*}
2 \operatorname{div}[\nu \underline{\mathbf{D}}(\mathbf{u})]=\operatorname{div}\left[\nu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)\right]=\operatorname{div}(\nu \nabla \mathbf{u})+\operatorname{div}\left(\nu \nabla \mathbf{u}^{T}\right) \tag{13.36}
\end{equation*}
$$

The first term of Eq. 13.36 is firstly addressed. The divergence of the product of a scalar $(\nu)$ and a tensor $(\nabla \mathbf{u})$ derives from the product of a scalar $(\nu)$ and a vector $(\nabla u)$ :

$$
\begin{equation*}
\operatorname{div}(\nu \nabla u)=\nu \operatorname{div}(\nabla u)+\nabla u \cdot \nabla \nu=\nu \Delta u+\nabla u \cdot \nabla \nu \tag{13.37}
\end{equation*}
$$

where the Laplacian operator $\Delta$ is introduced:

$$
\begin{equation*}
\operatorname{div}(\nabla u)=\frac{\partial^{2} u}{\partial x^{2}}+\frac{\partial^{2} u}{\partial y^{2}}+\frac{\partial^{2} u}{\partial z^{2}}=\Delta u \tag{13.38}
\end{equation*}
$$

Equation 13.37 can be extended to the tensor case:

$$
\begin{equation*}
\operatorname{div}(\nu \nabla \mathbf{u})=\nu \operatorname{div}(\nabla \mathbf{u})+\nabla \mathbf{u} \cdot \nabla \nu=\nu \Delta \mathbf{u}+\nabla \mathbf{u} \cdot \nabla \nu \tag{13.39}
\end{equation*}
$$

where the Laplacian of vector $\mathbf{u}$ is:

$$
\operatorname{div}(\nabla \mathbf{u})=\left[\begin{array}{c}
\frac{\partial^{2} u_{x}}{\partial x^{2}}+\frac{\partial^{2} u_{x}}{\partial y^{2}}+\frac{\partial^{2} u_{x}}{\partial z^{2}}  \tag{13.40}\\
\frac{\partial^{2} u_{y}}{\partial x^{2}}+\frac{\partial^{2} u_{y}}{\partial y^{2}}+\frac{\partial^{2} u_{y}}{\partial z^{2}} \\
\frac{\partial^{2} u_{z}}{\partial x^{2}}+\frac{\partial^{2} u_{z}}{\partial y^{2}}+\frac{\partial^{2} u_{z}}{\partial z^{2}}
\end{array}\right]=\Delta \mathbf{u}
$$

The second term of Eq. 13.36 is now addressed following the same procedure described above:

$$
\begin{equation*}
\operatorname{div}\left(\nu \nabla \mathbf{u}^{T}\right)=\nu \operatorname{div}\left(\nabla \mathbf{u}^{T}\right)+\nabla \mathbf{u}^{T} \cdot \nabla \nu \tag{13.41}
\end{equation*}
$$

The term $\nu \operatorname{div}\left(\nabla \mathbf{u}^{T}\right)$ is explicitly computed:

$$
\nu \operatorname{div}\left(\nabla \mathbf{u}^{T}\right)=\nu\left[\begin{array}{c}
\frac{\partial^{2} u_{x}}{\partial x^{2}}+\frac{\partial^{2} u_{y}}{\partial x \partial y}+\frac{\partial^{2} u_{z}}{\partial x \partial z}  \tag{13.42}\\
\frac{\partial^{2} u_{x}}{\partial x \partial y}+\frac{\partial^{2} u_{y}}{\partial y^{2}}+\frac{\partial^{2} u_{z}}{\partial y \partial z} \\
\frac{\partial^{2} u_{x}}{\partial x \partial z}+\frac{\partial^{2} u_{y}}{\partial y \partial z}+\frac{\partial^{2} u_{z}}{\partial z^{2}}
\end{array}\right]=\nu\left[\begin{array}{c}
\frac{\partial}{\partial x}\left(\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}\right) \\
\frac{\partial}{\partial y}\left(\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}\right) \\
\frac{\partial}{\partial z}\left(\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}\right)
\end{array}\right]
$$

As can be seen in Eq. 13.42:

$$
\begin{equation*}
\nu \operatorname{div}\left(\nabla \mathbf{u}^{T}\right)=\nu \nabla(\operatorname{div} \mathbf{u}) \tag{13.43}
\end{equation*}
$$

As a result, substituting Eq. 13.43 into Eq. 13.41, the second term of Eq. 13.36 finally becomes:

$$
\begin{equation*}
\operatorname{div}\left(\nu \nabla \mathbf{u}^{T}\right)=\nu \operatorname{div}\left(\nabla \mathbf{u}^{T}\right)+\nabla \mathbf{u}^{T} \cdot \nabla \nu=\nu \nabla(\operatorname{div} \mathbf{u})+\nabla \mathbf{u}^{T} \cdot \nabla \nu \tag{13.44}
\end{equation*}
$$

Substituting Eqs. 13.39 and 13.44 into Eq. 13.36, the molecular diffusion term is:

$$
\begin{equation*}
\operatorname{div}\left[\nu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)\right]=\nu[\Delta \mathbf{u}+\nabla(\operatorname{div} \mathbf{u})]+\left[\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right] \cdot \nabla \nu \tag{13.45}
\end{equation*}
$$

The assumption of constant kinematic viscosity $\nu$ is now taken into account:

$$
\begin{equation*}
\nu=\mathrm{const} \quad \Rightarrow \quad \nabla \nu=\mathbf{0} \tag{13.46}
\end{equation*}
$$

When $\nu$ is constant, Eq. 13.45 simplifies:

$$
\begin{equation*}
\operatorname{div}\left[\nu\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right)\right]=\nu[\Delta \mathbf{u}+\nabla(\operatorname{di} \nabla \mathbf{u})]=\nu \Delta \mathbf{u} \tag{13.47}
\end{equation*}
$$

where the continuity equation of incompressible fluids div $\mathbf{u}=0$ is exploited. Finally, the Navier-Stokes equations for incompressible fluids with constant kinematic viscosity are:

$$
\begin{cases}\frac{\partial \mathbf{u}}{\partial t}-\nu \Delta \mathbf{u}+(\mathbf{u} \cdot \nabla) \mathbf{u}+\nabla p=\mathbf{f} & \text { in } \Omega, t \in I  \tag{13.48}\\ \operatorname{div} \mathbf{u}=0 & \text { in } \Omega, t \in I\end{cases}
$$

The system in Eq. 13.48 can be rewritten componentwise:

$$
\left\{\begin{array}{l}
\frac{\partial u_{i}}{\partial t}-\nu \Delta u_{i}+\sum_{j=1}^{3} u_{j} \frac{\partial u_{i}}{\partial x_{j}}+\frac{\partial p}{\partial x_{i}}=f_{i} \quad i=1,2,3  \tag{13.49}\\
\sum_{j=1}^{3} \frac{\partial u_{j}}{\partial x_{j}}=0
\end{array}\right.
$$

The case of fluids with constant density (incompressible flows) and constant kinematic viscosity is the simplified case which will be considered in the following sections of the chapter.

## Initial and boundary conditions

As mentioned, Navier-Stokes equations for incompressible flows (Eq. 13.35) have the velocity $\mathbf{u}$ and the scaled pressure $p=P / \rho$ as principle unknowns of the problem. In order for problem in Eq. 13.35 or 13.48 to be well posed it is necessary to assign the initial condition:

$$
\begin{equation*}
\mathbf{u}\left(\mathbf{x}, t=t_{0}\right)=\mathbf{u}_{0}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega \tag{13.50}
\end{equation*}
$$

which prescribes the initial status of the fluid velocity. The term $\mathbf{u}_{0}$ is a given divergence-free velocity field, in order to satisfy the continuity equation div $\mathbf{u}$ for incompressible flows.

In order to solve the Navier-Stokes equations, it is necessary to provide also appropriate boundary conditions. In fact, Navier-Stokes equations must be supplemented by proper boundary conditions that allow the determination of the velocity field up to the boundary of the computational domain $\Omega$. The more classical boundary conditions which are mathematically compatible with the Navier-Stokes equations are:

1. Prescribed velocity (or Dirichlet boundary condition). A given velocity field $\mathbf{g}^{D}$ : $\Gamma_{D} \times I \rightarrow \mathbb{R}^{3}$ is imposed on a measurable subset of the domain boundary $\Gamma_{D} \subset \partial \Omega$. The Dirichlet boundary condition to be provided to the incompressible Navier-Stokes equations in Eq. 13.35 is:

$$
\begin{equation*}
\mathbf{u}=\mathbf{g}^{D} \quad \text { on } \Gamma_{D} \subset \partial \Omega \tag{13.51}
\end{equation*}
$$

It is observed that if $\Gamma_{D}=\partial \Omega$, the prescribed Dirichlet data $\mathbf{g}^{D}$ must be compatible with the incompressibility constraint, i.e. the continuity equation for compressible flows (Eq. 13.19). Hence:

$$
\begin{equation*}
\int_{\Gamma_{D}=\partial \Omega} \mathbf{g}^{D} \cdot \mathbf{n} \mathrm{~d} \Gamma=\int_{\Omega} \operatorname{div} \mathbf{u} \mathrm{d} \Omega=0 \tag{13.52}
\end{equation*}
$$

which is the divergence theorem, also knows as Gauss's theorem.
2. Applied stresses (or Neumann boundary condition). They represent the forces which are imposed on the control volume through its surface. It is assumed that they may be represented via a vector field $\mathbf{T}^{N}: \Gamma_{N} \times I \rightarrow \mathbb{R}^{3}$ defined on a measurable subset (possibly empty) of the domain boundary $\Gamma_{N} \subset \partial \Omega$ and with dimensions $\left[\mathbf{T}^{N}\right]=N / m^{2}$. As a consequence, under the hypotheses of the Cauchy theorem [140, 141], the boundary condition is:

$$
\begin{equation*}
\underline{\mathbf{T}} \cdot \mathbf{n}=[-P \underline{\mathbf{I}}+2 \mu \underline{\mathbf{D}}(\mathbf{u})] \cdot \mathbf{n}=-P \mathbf{n}+2 \mu \underline{\mathbf{D}}(\mathbf{u}) \cdot \mathbf{n}=\mathbf{T}^{N} \quad \text { on } \Gamma_{N} \subset \partial \Omega \tag{13.53}
\end{equation*}
$$

where the definition of Cauchy stress tensor $\underline{\mathbf{T}}$ for incompressible fluids in Eq. 13.22 is retrieved. According to Eq. 13.35, where the scaled pressure $p$ is used instead of the pressure $P$, the scaled vector field:

$$
\begin{equation*}
\mathbf{t}^{N}=\frac{\mathbf{T}^{N}}{\rho} \tag{13.54}
\end{equation*}
$$

is introduced ( $\left[\mathbf{t}^{N}\right]=[p]=m^{2} / s^{2}$ ). For the sake of convenience, the Neumann boundary condition in Eq. 13.53 can be now rewritten in terms of $\mathbf{u}, p, \mathbf{t}^{N}$ via the kinematic viscosity $\nu$ and the outward unit normal vector $\mathbf{n}$ in the following alternative form:

$$
\begin{equation*}
\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right)=\mathbf{t}^{N} \quad \text { on } \Gamma_{N} \subset \partial \Omega \tag{13.55}
\end{equation*}
$$

Equation 13.55 represents the Neumann boundary condition to be provided to the incompressible Navier-Stokes equations in Eq. 13.35.

It is important to note that $\Gamma_{D}$ and $\Gamma_{N}$ provide a partition of the domain boundary $\partial \Omega$, that is $\Gamma_{D} \cup \Gamma_{N}=\partial \Omega, \dot{\Gamma}_{D} \cap \check{\Gamma}_{N}=\emptyset$.

### 13.3 Stokes equations

The space discretization of the Navier-Stokes equations leads to a non-linear set of ordinary differential equations because of the presence of the convective term $(\mathbf{u} \cdot \nabla) \mathbf{u}$. This makes both the analysis and the numerical solution more difficult. In some cases, when the fluid is highly viscous, the contribution of the non-linear convective term may be neglected.

Let the Reynolds number $R e$ to be defined:

$$
\begin{equation*}
R e=\frac{|\mathbf{U}| L}{\nu} \tag{13.56}
\end{equation*}
$$

where $L$ is a representative length of the domain $\Omega$ (e.g. the length of a tube wherein the fluid flow is studied) and $\mathbf{U}$ is a representative fluid velocity (e.g. the free stream velocity). The Reynolds number measures the extent at which convection dominates over diffusion. When $R e \ll 1$ (for instance, flow in smaller arteries or capillaries) the convective term
$(\mathbf{u} \cdot \nabla) \mathbf{u}$ in Eqs. 13.15, 13.35, or 13.48 can be omitted, and Navier-Stokes equations reduce to the so-called Stokes equations.

The generalized Stokes problem with mixed Dirichlet-Neumann homogeneous boundary conditions is:

$$
\begin{cases}\alpha \mathbf{u}-\nu \Delta \mathbf{u}+\nabla p=\mathbf{f} & \text { in } \Omega  \tag{13.57}\\ \operatorname{div} \mathbf{u}=0 & \text { in } \Omega \\ \mathbf{u}=\mathbf{0} & \text { on } \Gamma_{D} \\ \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}=\mathbf{0} & \text { on } \Gamma_{N}\end{cases}
$$

for a given coefficient $\alpha \geq 0$. The homogeneous Dirichlet boundary condition is imposed on $\Gamma_{D}$. The homogeneous Neumann boundary condition is imposed on $\Gamma_{N}$. The two subsets $\Gamma_{D}$ and $\Gamma_{N}$ of the domain boundary $\partial \Omega$ provide a partition of $\partial \Omega$. In fact, $\Gamma_{D} \cup \Gamma_{N}=\partial \Omega$, $\stackrel{\circ}{\Gamma}_{D} \cap \stackrel{\circ}{\Gamma}_{N}=\emptyset$.

The Stokes problem describes the motion of an incompressible viscous flow in which the (quadratic non-linear) convective term has been neglected. As mentioned previously, this simplification is acceptable when $R e \ll 1$. The derivative with respect to time has been neglected from Navier-Stokes equations. In fact, a Stokes flow is steady and has no dependence on time other than through time-dependent boundary conditions. This means that, given the boundary conditions of a Stokes flow, the flow can be computed without knowledge of the flow at any other time.

For the sake of brevity, from here on the given coefficient $\alpha$ is set equal to 0 : $\alpha=0$. The Stokes equations with homogeneous Dirichlet and homogeneous Neumann boundary conditions thus become:

$$
\begin{cases}-\nu \Delta \mathbf{u}+\nabla p=\mathbf{f} & \text { in } \Omega  \tag{13.58}\\ \operatorname{div} \mathbf{u}=0 & \text { in } \Omega \\ \mathbf{u}=\mathbf{0} & \text { on } \Gamma_{D} \\ \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}=\mathbf{0} & \text { on } \Gamma_{N}\end{cases}
$$

For the sake of simplicity, the Stokes equations in Eq. 13.58 are the differential equations which will be considered in the following sections of the present chapter.

### 13.3.1 Weak formulation of Stokes equations

The Navier-Stokes and Stokes equations which have been introduced so far represent the strong formulation of the fluid flow differential problem. A formulation of the problem alternative to the strong one is now investigated and called weak formulation. The weak formulation of the differential problem allows the order of the derivation required for the unknown solution ( $\mathbf{u}, p$ ) to be reduced.

The weak form of Stokes (or Navier-Stokes) equations is formally obtained by taking the scalar product of the momentum equations with a vector function $\mathbf{v}$ (called test function) belonging to a suitable functional space $V$ (called test function space), integrating over the computational domain $\Omega$ and applying the Green integration formula. Similarly, the continuity equation is multiplied by a scalar test function $q$ belonging to a suitable test functional space $Q$ and integrated over the computational domain $\Omega$. More details about the functional spaces $V$ and $Q$ will be specified later on.

Let the weak formulation of the Stokes problem in Eq. 13.58 to be now presented. In particular, the case of mixed Dirichlet-Neumann homogeneous boundary conditions will be treated in detail. However, for the sake of completeness, a hint of the general weak form of Stokes-equations with mixed Dirichlet-Neumann nonhomogeneous boundary conditions will be also concisely given.

The momentum conservation equation for the Stokes problem (first expression in Eq. 13.58) multiplied by a test function $\mathbf{v}$ and integrated over $\Omega$ is:

$$
\begin{equation*}
\int_{\Omega}[-\nu \Delta \mathbf{u} \cdot \mathbf{v}+\nabla p \cdot \mathbf{v}] \mathrm{d} \Omega=\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega \tag{13.59}
\end{equation*}
$$

The first term of Eq. 13.59 can be rewritten using the Green formula for the Laplacian operator, which in terms of the generic scalar quantities $u$ and $v$ is:

$$
\begin{equation*}
\int_{\Omega}(\Delta u) v \mathrm{~d} \Omega=\int_{\partial \Omega} \frac{\partial u}{\partial n} v \mathrm{~d} \Gamma-\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} \Omega \tag{13.60}
\end{equation*}
$$

Since:

$$
\begin{equation*}
\Delta \mathbf{u} \cdot \mathbf{v}=\sum_{i=1}^{3}\left(\Delta u_{i}\right) v_{i} \tag{13.61}
\end{equation*}
$$

the expression in Eq. 13.60 corresponding to vectors $\mathbf{u}$ and $\mathbf{v}$ is:

$$
\begin{equation*}
\int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} \mathrm{d} \Omega=\sum_{i=1}^{3} \int_{\Omega}\left(\Delta u_{i}\right) v_{i} \mathrm{~d} \Omega=\sum_{i=1}^{3}\left[\int_{\partial \Omega} \frac{\partial u_{i}}{\partial n} v_{i} \mathrm{~d} \Gamma-\int_{\Omega} \nabla u_{i} \cdot \nabla v_{i} \mathrm{~d} \Omega\right] \tag{13.62}
\end{equation*}
$$

In a compact form, Green formula for the Laplacian operator in terms of vectors $\mathbf{u}$ and $\mathbf{v}$ is thus:

$$
\begin{equation*}
\int_{\Omega} \Delta \mathbf{u} \cdot \mathbf{v} \mathrm{d} \Omega=\int_{\partial \Omega} \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{v} \mathrm{d} \Gamma-\int_{\Omega} \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega \tag{13.63}
\end{equation*}
$$

The term $\nabla \mathrm{u}: \nabla \mathrm{v}$ is:

$$
\begin{equation*}
\nabla \mathbf{u}: \nabla \mathbf{v}=\operatorname{tr}\left(\nabla \mathbf{u}^{T} \nabla \mathbf{v}\right) \tag{13.64}
\end{equation*}
$$

where the symbol $t r$ stands for the trace of a square matrix, i.e. the sum of the elements on its main diagonal. The second term of Eq. 13.59 can be rewritten using the Green formula for the divergence operator:

$$
\begin{equation*}
\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega=\int_{\partial \Omega} p \mathbf{v} \cdot \mathbf{n} \mathrm{~d} \Gamma-\int_{\Omega} \nabla p \cdot \mathbf{v} \mathrm{~d} \Omega \tag{13.65}
\end{equation*}
$$

Hence:

$$
\begin{equation*}
\int_{\Omega} \nabla p \cdot \mathbf{v} \mathrm{~d} \Omega=\int_{\partial \Omega} p \mathbf{v} \cdot \mathbf{n} \mathrm{~d} \Gamma-\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega \tag{13.66}
\end{equation*}
$$

Substituting Eqs. 13.63 and 13.66 into Eq. 13.59 and using the fact that $\nu$ is constant for the fluid considered, the weak form of the momentum equation becomes:

$$
\begin{equation*}
\int_{\Omega} \nu \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega-\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega=\int_{\partial \Omega} \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \mathbf{v} \mathrm{d} \Gamma-\int_{\partial \Omega} p \mathbf{v} \cdot \mathbf{n} \mathrm{~d} \Gamma+\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega \tag{13.67}
\end{equation*}
$$

$\forall \mathbf{v} \in V$. More details about the choice of $V$ is faced later on. Collecting the intergral terms over $\partial \Omega$ and using:

$$
\begin{equation*}
p \mathbf{v} \cdot \mathbf{n}=p \mathbf{n} \cdot \mathbf{v} \tag{13.68}
\end{equation*}
$$

Eq. 13.67 becomes:

$$
\begin{equation*}
\int_{\Omega} \nu \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega-\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega=\int_{\partial \Omega}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma+\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega \tag{13.69}
\end{equation*}
$$

$\forall \mathbf{v} \in V$.
The mass conservation Stokes equation (second expression in Eq. 13.58) multiplied by a test function $q$, belonging to a suitable functional space $Q$, and integrated over $\Omega$ is:

$$
\begin{equation*}
-\int_{\Omega} q \operatorname{div} \mathbf{u} \mathrm{~d} \Omega=0 \tag{13.70}
\end{equation*}
$$

$\forall q \in Q$. It is noteworthy that in Eq. 13.70 the negative sign "-" has been included only for the sake of convenience, without changing the result.

## Choice of test function spaces $V$ and $Q$

Let the choice of the test function spaces $V$ and $Q$ to be now investigated. For this purpose, a short digression about some particular functional spaces is addressed. However, more details about functional spaces can be found in [142]. Let the space of square-integrable functions on $\Omega \subset \mathbb{R}^{3}$ to be considered:

$$
\begin{equation*}
\mathrm{L}^{2}(\Omega)=\left\{f: \Omega \rightarrow \mathbb{R} \text { s.t. } \int_{\Omega}(f(\mathbf{x}))^{2} \mathrm{~d} \Omega<+\infty\right\} \tag{13.71}
\end{equation*}
$$

The functions of $\mathrm{L}^{2}(\Omega)$ are a particular distribution. However, it is not guaranteed that their derivatives (in the sense of distributions) are still functions of $\mathrm{L}^{2}(\Omega)$. Let $k$ be a positive integer. The Sobolev space $\mathrm{H}^{k}(\Omega)$ of order $k$ on $\Omega$ is the space formed by the totality of functions of $\mathrm{L}^{2}(\Omega)$ such that all their (distributional) derivatives up to order $k$ belong to $\mathrm{L}^{2}(\Omega)$ :

$$
\begin{equation*}
\mathrm{H}^{k}(\Omega)=\left\{f \in \mathrm{~L}^{2}(\Omega): D^{\boldsymbol{\alpha}} f \in \mathrm{~L}^{2}(\Omega) \quad \forall \boldsymbol{\alpha}:|\boldsymbol{\alpha}| \leq k\right\} \tag{13.72}
\end{equation*}
$$

where the symbol $D^{\boldsymbol{\alpha}}$ stands for the derivation in the sense of distributions, see [142]. Let the space $\mathrm{H}^{1}(\Omega)$ to be defined setting $k=1$ in Eq. 13.72:

$$
\begin{equation*}
\mathrm{H}^{1}(\Omega)=\left\{v: \Omega \rightarrow \mathbb{R} \text { s.t. } v \in \mathrm{~L}^{2},(\Omega): \frac{\partial v}{\partial x_{i}} \in \mathrm{~L}^{2}(\Omega), i=1,2,3\right\} \tag{13.73}
\end{equation*}
$$

As customarily, in the case of scalar test functions $v: \Omega \rightarrow \mathbb{R}$, the test function space $V$ to be used in the weak formulation of a differential problem would be chosen as a subspace of $\mathrm{H}^{1}(\Omega)$ :

$$
\begin{equation*}
V=\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)=\left\{v \in \mathrm{H}^{1}(\Omega):\left.v\right|_{\Gamma_{D}}=0\right\} \tag{13.74}
\end{equation*}
$$

In fact, the test function space $V$ would be chosen in such a way that the test functions $v$ vanish on that boundary portion where a Dirichlet boundary condition (homogeneous or nonhomogeneous) is prescribed on the unknown $u$. Similarly, in the case of vector test functions $v: \Omega \rightarrow \mathbb{R}^{3}$ the test function space $V$ would be chosen as a subspace of $\left[H_{\Gamma_{D}}^{1}(\Omega)\right]^{3}$ :

$$
\begin{equation*}
V=\left[\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)\right]^{3}=\left\{\mathbf{v} \in\left[\mathrm{H}^{1}(\Omega)\right]^{3}:\left.\mathbf{v}\right|_{\Gamma_{D}}=\mathbf{0}\right\} \tag{13.75}
\end{equation*}
$$

In other words, the test function space $V$ is chosen in such a way that the test functions $\mathbf{v}$ vanish on that boundary portion where a Dirichlet boundary condition (homogeneous or
nonhomogeneous) is prescribed on the velocity vector $\mathbf{u}$. In general, the velocity vector $\mathbf{u}$ is such that $\mathbf{u} \in\left[\mathrm{H}^{1}(\Omega)\right]^{3}$. Moreover, $\mathbf{u}=\mathbf{g}^{D}$ on $\Gamma_{D}$ for the nonhomogeneous Dirichlet boundary conditions case. On the contrary, $\mathbf{u} \in\left[\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)\right]^{3}=V$ (see Eq. 13.75) for the homogeneous Dirichlet boundary conditions case (since $\mathbf{u}=\mathbf{0}$ on $\Gamma_{D}$ ). Moreover, if $\mathbf{u}=\mathbf{0}$ on $\Gamma_{D}=\partial \Omega$ (i.e. $\Gamma_{N}=\emptyset$ ), then $\mathbf{u} \in\left[\mathrm{H}_{0}^{1}(\Omega)\right]^{3}=V$ :

$$
\begin{equation*}
V=\left[\mathrm{H}_{0}^{1}(\Omega)\right]^{3}=\left\{\mathbf{v} \in\left[\mathrm{H}^{1}(\Omega)\right]^{3}:\left.\mathbf{v}\right|_{\Gamma_{D}=\partial \Omega}=\mathbf{0}\right\} \tag{13.76}
\end{equation*}
$$

As far as the scalar test function space $Q$ is concerned, if $\Gamma_{N} \neq \emptyset$ then it can be chosen $Q=\mathrm{L}^{2}(\Omega)$ (see Eq. 13.71). Instead, if $\Gamma_{N}=\emptyset$ (i.e. $\Gamma_{D}=\partial \Omega$ ) the following scalar test function space $Q$ is considered for the test functions $q$ as well as the pressure $p \in Q$ :

$$
\begin{equation*}
Q=\mathrm{L}_{0}^{2}(\Omega)=\left\{p \in \mathrm{~L}^{2}(\Omega): \int_{\Omega} p \mathrm{~d} \Omega=0\right\} \tag{13.77}
\end{equation*}
$$

where the pressure $p$ is assumed to have null average over the computational domain $\Omega$. As a summary:

$$
\begin{equation*}
Q=\mathrm{L}^{2}(\Omega) \quad \text { if } \quad \Gamma_{N} \neq \emptyset, \quad Q=\mathrm{L}_{0}^{2}(\Omega) \quad \text { if } \quad \Gamma_{N}=\emptyset \tag{13.78}
\end{equation*}
$$

More details about the choice of test function spaces $V$ and $Q$ can be found in [142].
As far as the choice of functional spaces depending on boundary conditions of the problem is concerned, the Dirichlet conditions are said to be essential as they are imposed explicitly in the functional space in which the problem is set. The Neumann conditions are instead said to be natural, as they are satisfied implicitly by the solution of the problem.

## Weak form of Stokes equations with mixed Dirichlet-Neumann nonhomogeneous boundary conditions

The operations so far carried out to obtain the weak formulation of the Stokes equations have not employed the boundary conditions. Although only the Stokes problem with mixed Dirichlet-Neumann homogeneous boundary conditions have been formulated in Eq. 13.58, these operations are valid still in the more general case of mixed Dirichlet-Neumann nonhomogeneous boundary conditions (Eqs. 13.51 and 13.53):

$$
\begin{cases}\mathbf{u}=\mathbf{g}^{D} & \text { on } \Gamma_{D}  \tag{13.79}\\ \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}=\mathbf{t}^{N} & \text { on } \Gamma_{N}\end{cases}
$$

to be provided to the first two equations of system 13.58. Retrieving the momentum conservation and the mass conservation in Eqs. 13.69 and 13.70 and according to the considerations made about the choice of test function spaces, the weak form of the Stokes equations is therefore:

Find $\mathbf{u} \in\left[\mathrm{H}^{1}(\Omega)\right]^{3}, p \in Q$ such that

$$
\left\{\begin{array}{l}
\int_{\Omega} \nu \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega-\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega=  \tag{13.80}\\
\int_{\partial \Omega}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma+ \\
\quad \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega \quad \forall \mathbf{v} \in V \\
-\int_{\Omega} q \operatorname{div} \mathbf{u} \mathrm{~d} \Omega=0 \quad \forall q \in Q
\end{array}\right.
$$

where $V=\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)$ and $Q$ is given in Eq. 13.78. The weak form presented in Eq. 13.80 is valid for the general case of mixed Dirichlet- Neumann nonhomogeneous boundary conditions. The integral term over $\partial \Omega$ has to be evaluated according to the boundary conditions chosen on $\partial \Omega$, as will be shown in Eq. 13.81. The sample case of mixed Dirichlet-Neumann homogeneous boundary conditions (i.e. $\mathbf{g}^{D}=\mathbf{0}$ and $\mathbf{t}^{N}=\mathbf{0}$ ) will be faced in Eq. 13.82.

## Weak form of Stokes equations with mixed Dirichlet-Neumann homogeneous boundary conditions

When mixed Dirichlet-Neumann homogeneous boundary conditions are considered in the Stokes problem, the corresponding weak form can be seen as a particular case of Eq. 13.80. As previously mentioned, the integral term on the boundary $\partial \Omega$ has to be evaluated and in this section it is computed considering the homogeneous boundary conditions written in Eq. 13.58. First af all, this term (comparing in Eq. 13.69 and 13.80 is split into two integral terms over $\Gamma_{D}$ and $\Gamma_{N}$ :

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma=\int_{\Gamma_{D}}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma+\int_{\Gamma_{N}}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma \tag{13.81}
\end{equation*}
$$

The Dirichlet-Neumann homogeneous boundary conditions are now considered by eliding the integral terms:

$$
\int_{\partial \Omega}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma=\frac{\int_{\Gamma_{D}}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \underbrace{\underbrace{\mathbf{v}}_{\Gamma_{N}} \mathrm{~d} \Gamma_{D}}_{=\mathbf{0} \text { on } \Gamma_{N}}+\underbrace{\int_{\Gamma_{N}}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right)}+\overrightarrow{v d \Gamma}=0}{}=0
$$

As a consequence, for the sample boundary conditions considered:

$$
\begin{equation*}
\int_{\partial \Omega}\left(\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}\right) \cdot \mathbf{v} \mathrm{d} \Gamma=0 \tag{13.83}
\end{equation*}
$$

Finally, the weak form of the Stokes equations with mixed Dirichlet-Neumann homogeneous boundary conditions ( $\Gamma_{D} \neq \emptyset, \Gamma_{N} \neq \emptyset$ ) is:

Find $\mathbf{u} \in V=\left[\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)\right]^{3}, p \in Q=\mathrm{L}^{2}(\Omega)$ such that

$$
\begin{cases}\int_{\Omega} \nu \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega-\int_{\Omega} p \operatorname{div} \mathbf{v} \mathrm{~d} \Omega=\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega & \forall \mathbf{v} \in V  \tag{13.84}\\ -\int_{\Omega} q \operatorname{div} \mathbf{u} \mathrm{~d} \Omega=0 & \forall q \in Q\end{cases}
$$

Let the bilinear forms $a: V \times V \rightarrow \mathbb{R}$ and $b: V \times Q \rightarrow \mathbb{R}$ to be now defined as follows:

$$
\begin{align*}
& a(\mathbf{u}, \mathbf{v})=\int_{\Omega} \nu \nabla \mathbf{u}: \nabla \mathbf{v} \mathrm{d} \Omega  \tag{13.85}\\
& b(\mathbf{u}, q)=-\int_{\Omega} q \operatorname{div} \mathbf{u} \mathrm{~d} \Omega \tag{13.86}
\end{align*}
$$

With these notations, the weak form in Eq. 13.84 can be written in a more compact notation:

$$
\begin{align*}
& \text { Find } \mathbf{u} \in V=\left[\mathrm{H}_{\Gamma_{D}}^{1}(\Omega)\right]^{3}, p \in Q=\mathrm{L}^{2}(\Omega) \text { such that } \\
& \begin{cases}a(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p)=(\mathbf{f}, \mathbf{v}) & \forall \mathbf{v} \in V \\
b(\mathbf{u}, q)=0 & \forall q \in Q\end{cases} \tag{13.87}
\end{align*}
$$

where $(\mathbf{f}, \mathbf{v})=\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} \Omega=\sum_{i=1}^{3} \int_{\Omega} f_{i} v_{i}$.
The bilinear form $a(\cdot, \cdot)$ in Eq. 13.87 is symmetric since the space where the solution u is sought coincides with the space of the test functions ( $\mathbf{u} \in V$ and $\mathbf{v} \in V$ ). For the sake of completeness, it is noticed that a general weak form of the Stokes equations analogous to Eq. 13.87 can be obtained for the mixed Dirichlet-Neumann nonhomogeneous boundary conditions case (Eq. 13.79), even though it is not reported here. In this particular case, the bilinear form $a(\cdot, \cdot)$ would be asymmetrical, since the space where the solution $\mathbf{u}$ is sought would not coincide with the space of the test functions $V$. Thus, a lifting of the Dirichlet boundary datum $\mathbf{g}^{D}$ would be required in order to make $a(\cdot, \cdot)$ symmetric. However, more details about this procedure can be found in [142].

It is important to note that the couple ( $\mathbf{u}, p$ ) solves the Stokes problem 13.87 if and only if it is a saddle-point of the Lagrangian functional:

$$
\begin{equation*}
\mathcal{L}(\mathbf{v}, q)=\frac{1}{2} a(\mathbf{v}, \mathbf{v})+b(\mathbf{v}, q)-(f, \mathbf{v}) \tag{13.88}
\end{equation*}
$$

This means that:

$$
\begin{equation*}
\mathcal{L}(\mathbf{u}, p)=\min _{\mathbf{v} \in V} \max _{q \in Q} \mathcal{L}(\mathbf{v}, q) \tag{13.89}
\end{equation*}
$$

In this regard, the pressure $q$ plays the role of Lagrange multiplier associated to the incompressibility (i.e. divergence-free) constraint.

### 13.3.2 Galerkin approximation

The Galerkin approximation of the Stokes problem in Eq. 13.84 has the following form:
Find $\mathbf{u}_{h} \in V_{h}, p_{h} \in Q_{h}$ such that

$$
\begin{cases}\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h} \mathrm{~d} \Omega-\int_{\Omega} p_{h} \operatorname{div} \mathbf{v}_{h} \mathrm{~d} \Omega=\int_{\Omega} \mathbf{f} \cdot \mathbf{v}_{h} \mathrm{~d} \Omega & \forall \mathbf{v}_{h} \in V_{h}  \tag{13.90}\\ -\int_{\Omega} q_{h} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega=0 & \forall q_{h} \in Q_{h}\end{cases}
$$

According to the compact notation in Eq. 13.87 which involves the bilinear forms, the Galerkin approximation (Eq. 13.87) can be written as:

$$
\text { Find } \mathbf{u}_{h} \in V_{h}, p_{h} \in Q_{h} \text { such that }
$$

$$
\begin{cases}a\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)+b\left(\mathbf{v}_{h}, p_{h}\right)=\left(\mathbf{f}, \mathbf{v}_{h}\right) & \forall \mathbf{v}_{h} \in V_{h}  \tag{13.91}\\ b\left(\mathbf{u}_{h}, q_{h}\right)=0 & \forall q_{h} \in Q_{h}\end{cases}
$$

where $\left\{V_{h} \subset V\right\}$ and $\left\{Q_{h} \subset Q\right\}$ represent two families of finite dimensional subspaces depending on a real positive discretization parameter $h$, such that:

$$
\begin{align*}
& \operatorname{dim} V_{h}=M_{h}^{V}<\infty \quad \forall h>0 \\
& \operatorname{dim} Q_{h}=M_{h}^{Q}<\infty \quad \forall h>0 \tag{13.92}
\end{align*}
$$

The terms $\mathbf{u}_{h}$ and $p_{h}$ in the Galerkin approximation in Eqs. 13.90 and 13.91 are the discrete solutions of the Stokes problem in weak form (Eq. 13.84). An important theorem due to Franco Brezzi guarantees uniqueness and solution of the problem in Eq. 13.91. For the sake of completeness, the original version of this theorem can be found in [143]. The proof of this theorem is also reported in [142] for a saddle-point problem that is more general than the Stokes problem and within a more abstract framework.

In general, let $\left\{\varphi_{j} \in V_{h}, j=1, \ldots, M_{h}^{V}\right\}$ and $\left\{\phi_{j} \in Q_{h}, j=1, \ldots, M_{h}^{Q}\right\}$ to be denoted as the basis functions of the spaces $V_{h}$ and $Q_{h}$, respectively. The discrete solutions $\mathbf{u}_{h}$ and $p_{h}$ with respect to these bases are:

$$
\begin{equation*}
\mathbf{u}_{h}(\mathrm{x})=\sum_{j=1}^{M_{h}^{V}} u_{j} \boldsymbol{\varphi}_{j}(\mathrm{x}) \quad p_{h}(\mathrm{x})=\sum_{j=1}^{M_{h}^{Q}} p_{j} \phi_{j}(\mathrm{x}) \tag{13.93}
\end{equation*}
$$

Equation 13.93 means that the discrete solutions $\mathbf{u}_{h}$ and $p_{h}$ are linear combinations of the bases functions by means of the coefficients $u_{j}$ and $p_{j}$, respectively.

### 13.4 1D CUF model for the approximation of Stokes equations

The strong and weak forms of the Stokes problem have been introduced in section 13.3. The corresponding Galerkin approximation in Eq. 13.90 (or Eq. 13.91) provides a discretization of the problem. The use of the 1D CUF model as the numerical method to be considered in the Galerkin approximation is addressed in this section.

The solution of the elasticity problem by means of the Finite Element Method (FEM) and the Carrera Unified Formulation (CUF) has been presented in chapter 3 and discussed in chapters $4,5,6,7$. Similarly to the formulation of the 1D CUF FE model introduced for the structural problem, this section addresses the numerical modeling of the Stokes problem through the 1D CUF FE model. This discretization method is particularly suitable for the Stokes problem when the computational domain is similar to the computational domains defined in chapter 3. In other words, a perfect example for the use of the 1D CUF FE model is fluid flow in a pipe. As depicted in Fig. 13.1, the computational domain $\Omega$, i.e. the volume in the pipe, can be discretized by means of an arbitrary cross-section $\Gamma_{S}$ and a one-dimensional mesh of $N_{\mathrm{EL}}$ finite elements, according to the procedure followed to discretize structurally slender bodies via the CUF model in chapter 3.


Figure 13.1: An example of computational domain $\Omega$ and cross-section $\Gamma_{S}$.

### 13.4.1 Velocity discretization

According to the framework of Carrera Unified Formulation (CUF) introduced in section 2.4, the discrete velocity field $\mathbf{u}_{h}$ is assumed to be an expansion of a certain class of functions
$F_{\tau}^{U}$, which depend on the cross-section coordinates $x$ and $z$ :

$$
\begin{equation*}
\mathbf{u}_{h}(x, y, z)=F_{\tau}^{U}(x, z) \mathbf{u}_{\tau}(y) \quad \tau=1, \ldots, N_{u}^{U}=N_{u}^{U}\left(N^{U}\right) \tag{13.94}
\end{equation*}
$$

The compact expression is based on Einstein's notation: repeated subscript $\tau$ indicates summation. The vector $\mathbf{u}_{\tau}$ is the $\tau^{\text {th }}$ generalized velocity unknowns vector which contains the components $\left\{\begin{array}{lll}u_{x \tau} & u_{y \tau} & u_{z \tau}\end{array}\right\}^{T}$. The vector $\mathbf{u}_{\tau}$ depends on the single spatial coordinate $y$ and so Eq. 13.94 represents the key formulation of the one-dimensional (1D) CUF model for the velocity field.

The number of expansion terms $N_{u}^{U}$ depends on the expansion order $N^{U}$, which is a free parameter of the formulation. The expansion order depends directly on the choice of the cross-section functions $F_{\tau}^{U}$, which is arbitrary. As mentioned at the end of section 2.4.2, the formulation developed in the present dissertation bases on the choice of Maclaurin polynomials instead of Lagrange polynomials. The generic $\tau^{\text {th }}$ cross-section function is hence a multivariate Maclaurin polynomial given by the multiplication of a monomial in $x$ by a monomial in $z$ as described in Eq. 13.95:

$$
\begin{equation*}
F_{\tau}^{U}(x, z)=x^{h} z^{k} \quad h, k=0, \ldots, N^{U} \quad \tau=1, \ldots, N_{u}^{U}=\frac{\left(N^{U}+1\right)\left(N^{U}+2\right)}{2} \tag{13.95}
\end{equation*}
$$

which corresponds to Eq. 2.35. The relation between indices $\tau, h$, and $k$ in Eq. 13.95 can be found in Table 2.1, where $N_{u}$ and $N$ have to be replaced with $N_{u}^{U}$ and $N^{U}$. The multivariate Maclaurin polynomials considered for the approximation of the velocity on the cross-section as the expansion order $N^{U}$ increases are depicted in Fig. 13.2, which reminds Pascal's triangle, also called Tartaglia's triangle. For the sake of simplicity, the polynomials in Fig. 13.2 refer to a rectangular domain with dimensions equal to $a$ and $b$ along $x$ and $z$ axes, respectively. However, more details can be found in section 2.4.

According to the in-depth analysis carried out in chapter 3, the basic procedure in the isoparametric finite element displacement-based formulation is to express both the element coordinates and the element displacements in the form of interpolations (shape functions) using the natural coordinate system of the element. The same procedure is applied here for the present fluid dynamic formulation where the unknowns are the velocity and pressure fields. The shape functions $N_{i}^{U}$ are therefore used to approximate the velocity unknowns. According to CUF formulation in Eq. 2.34 and finite element coordinates approximation in Eq. 3.2, generic velocities $\mathbf{u}_{\tau}$ lying on the longitudinal axis are expressed as:

$$
\begin{equation*}
\mathbf{u}_{\tau}(y)=N_{i}^{U}(y) \mathbf{q}_{\tau i} \quad i=1, \ldots, N_{N}^{U} \tag{13.96}
\end{equation*}
$$

where again repeated subscript $i$ indicates summation based on Einstein's notation. The generic nodal velocity vector $\mathbf{q}_{\tau i}$ contains the velocity degrees of freedom of the generic $\tau^{\text {th }}$ expansion term corresponding to the $i^{t h}$ element node. The dimensions of this vector are $3 \times 1$ and its components are:

$$
\mathbf{q}_{\tau i}=\left\{\begin{array}{c}
q_{\tau i x}  \tag{13.97}\\
q_{\tau i y} \\
q_{\tau i z}
\end{array}\right\}
$$

Combining the finite element approximation in Eq. 13.96 and CUF formulation in Eq. 13.94, the velocity field described by the present one-dimensional model becomes:

$$
\mathbf{u}_{h}(x, y, z)=F_{\tau}^{U}(x, z) N_{i}^{U}(y) \mathbf{q}_{\tau i} \quad \begin{align*}
& \tau=1, \ldots, N_{u}^{U}=N_{u}^{U}\left(N^{U}\right)  \tag{13.98}\\
& i=1 \ldots N_{N}^{U}
\end{align*}
$$



Figure 13.2: Higher-order Maclaurin polynomials for different values of the velocity expansion order $N^{U}$ or the pressure expansion order $N^{P}$.
where repeated subscripts $\tau$ and $i$ indicate summation based on Einstein's notation. The 1D CUF FE approximation for $\mathbf{u}_{h}$ in Eq. 13.98 is now written explicitly with the summation operators on indices $\tau$ and $i$ :

$$
\begin{aligned}
\mathbf{u}_{h} & =\sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} F_{\tau}^{U} N_{i}^{U} \mathbf{q}_{\tau i}=\sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} F_{\tau}^{U} N_{i}^{U}\left\{\begin{array}{l}
q_{\tau i x} \\
q_{\tau i y} \\
q_{\tau i z}
\end{array}\right\} \\
& =\sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} F_{\tau}^{U} N_{i}^{U}\left[q_{\tau i x}\left\{\begin{array}{l}
1 \\
0 \\
0
\end{array}\right\}+q_{\tau i y}\left\{\begin{array}{l}
0 \\
1 \\
0
\end{array}\right\}+q_{\tau i z}\left\{\begin{array}{l}
0 \\
0 \\
1
\end{array}\right\}\right]
\end{aligned}
$$

$$
\begin{align*}
& =\sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}}\left[q_{\tau i x}\left\{\begin{array}{c}
F_{\tau}^{U} N_{i}^{U} \\
0 \\
0
\end{array}\right\}+q_{\tau i y}\left\{\begin{array}{c}
0 \\
F_{\tau}^{U} N_{i}^{U} \\
0
\end{array}\right\}+q_{\tau i z}\left\{\begin{array}{c}
0 \\
0 \\
F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}\right]  \tag{13.99}\\
& =\sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}}\left[q_{\tau i x} \varphi_{\tau i 1}+q_{\tau i y} \varphi_{\tau i 2}+q_{\tau i z} \varphi_{\tau i 3}\right]=\sum_{e=1}^{3} \sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} q_{\tau i e} \boldsymbol{\varphi}_{\tau i e}
\end{align*}
$$

According to Eq. 13.93, which expresses the discrete solution $\mathbf{u}_{h}$ as a linear combination of the bases functions $\phi_{j}(\mathbf{x})$ by means of the scalar coefficients $u_{j}$, the functions $\varphi_{\tau i e}$ :

$$
\boldsymbol{\varphi}_{\tau i e}(x, y, z)=\left\{\begin{array}{l}
\delta_{1 e} F_{\tau}^{U}(x, z) N_{i}^{U}(y)  \tag{13.100}\\
\delta_{2 e} F_{\tau}^{U}(x, z) N_{i}^{U}(y) \\
\delta_{3 e} F_{\tau}^{U}(x, z) N_{i}^{U}(y)
\end{array}\right\}
$$

are therefore the bases of the space $V_{h}$ due to the 1D CUF approximation whereas $q_{\tau i x}$, $q_{\tau i y}$, and $q_{\tau i z}$ are the unknown scalar velocity coefficients. As a consequence, the term $\mathbf{q}_{\tau i}$ is the vector of the unknown velocity coefficients which identify the discrete solution $\mathbf{u}_{h}$ in the space $V_{h}$. Indices $\delta_{1 e}, \delta_{2 e}$, and $\delta_{3 e}$ in Eq. 13.100 derive from the Kronecker delta:

$$
\delta_{1 e}=\left\{\begin{array}{ll}
1 & e=1  \tag{13.101}\\
0 & e \neq 1
\end{array} \quad \delta_{2 e}=\left\{\begin{array}{ll}
1 & e=2 \\
0 & e \neq 2
\end{array} \quad \delta_{3 e}= \begin{cases}1 & e=3 \\
0 & e \neq 3\end{cases}\right.\right.
$$

### 13.4.2 Pressure discretization

According to the framework of Carrera Unified Formulation (CUF), the discrete pressure field $p_{h}$ is assumed to be an expansion of a certain class of functions $F_{m}^{P}$, which depend on the cross-section coordinates $x$ and $z$ :

$$
\begin{equation*}
p_{h}(x, y, z)=F_{m}^{P}(x, z) p_{m}(y) \quad m=1, \ldots, N_{u}^{P}=N_{u}^{P}\left(N^{P}\right) \tag{13.102}
\end{equation*}
$$

The compact expression is based on Einstein's notation: repeated subscript $m$ indicates summation. The term $p_{m}$ is the $m^{t h}$ generalized pressure unknown. This term depends on the single spatial coordinate $y$ and so Eq. 13.102 represents the key formulation of the one-dimensional (1D) CUF model for the pressure field.

The number of expansion terms $N_{u}^{P}$ depends on the expansion order $N^{P}$, which is a free parameter of the formulation. As done for velocity, Maclaurin polynomials are used to discretize the pressure field. The generic $m^{\text {th }}$ cross-section function is hence a multivariate Maclaurin polynomial given by the multiplication of a monomial in $x$ by a monomial in $z$ as described in Eq. 13.103:

$$
\begin{equation*}
F_{m}^{P}(x, z)=x^{h} z^{k} \quad h, k=0, \ldots, N^{P} \quad \tau=1, \ldots, N_{u}^{P}=\frac{\left(N^{P}+1\right)\left(N^{P}+2\right)}{2} \tag{13.103}
\end{equation*}
$$

which corresponds to Eq. 13.95. The relation between indices $m, h$, and $k$ in Eq. 13.95 can be found in Table 2.1, where $N_{u}$ and $N$ have to be replaced with $N_{u}^{P}$ and $N^{p}$. The multivariate Maclaurin polynomials considered for the approximation of the pressure on the cross-section as the expansion order $N^{P}$ increases are depicted in Fig. 13.2.

The finite element method is used to approximate the pressure field along the longitudinal axis $y$. The shape functions $N_{t}^{P}$ are therefore used to approximate the pressure unknowns. The generic pressure $p_{m}$ lying on the longitudinal axis is expressed as:

$$
\begin{equation*}
p_{m}(y)=N_{t}^{P}(y) p_{m t} \quad i=1, \ldots, N_{N}^{P} \tag{13.104}
\end{equation*}
$$

where again repeated subscript $t$ indicates summation based on Einstein's notation. Combining the finite element approximation in Eq. 13.104 and CUF formulation in Eq. 13.103, the pressure field described by the present one-dimensional model becomes:

$$
p_{h}(x, y, z)=F_{m}^{P}(x, z) N_{t}^{P}(y) p_{m t} \quad \begin{align*}
& m=1, \ldots, N_{u}^{P}=N_{u}^{P}\left(N^{P}\right)  \tag{13.105}\\
& t=1, \ldots, N_{N}^{P}
\end{align*}
$$

where repeated subscripts $m$ and $t$ indicate summation based on Einstein's notation. Equation 13.93 expresses the discrete solution $p_{h}$ as a linear combination of the bases functions $\phi_{j}(\mathrm{x})$ by means of the scalar coefficients $p_{m t}$. Writing the 1D CUF FE approximation for $p_{h}$ in Eq. 13.105 explicitly with the summation operators on $m$ and $t$ and comparing with Eq.13.93:

$$
\begin{equation*}
p_{h}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} F_{m}^{P} N_{t}^{P}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} \phi_{m t} \tag{13.106}
\end{equation*}
$$

it results that the following functions $\phi_{m t}$ :

$$
\begin{equation*}
\phi_{m t}(x, y, z)=F_{m}^{P}(x, z) N_{t}^{P}(y) \tag{13.107}
\end{equation*}
$$

are therefore the bases of the space $Q_{h}$ due to the 1D CUF approximation, whereas the terms $p_{m t}$ are the unknown pressure coefficients which identify the discrete solution $p_{h}$ in the space $Q_{h}$.

### 13.4.3 Refined models with variable velocity-pressure accuracy

Thanks to the hierarchical CUF approach described in chapters 2 and 3, different higherorder theories with a variable order of expansion for the velocity and pressure unknowns and hence with a variable accuracy can be easily developed. According to the CUF FE discretization introduced in the previous sections, the parameters of the analysis are the $N_{N}^{U}, N_{N}^{P}, N^{U}$, and $N^{P}$. No assumptions have been introduced for these parameters. This means that different expansions can be assumed to describe the velocity and pressure fields. For instance, the sample case of $N^{U}=4$ and $N^{P}=2$ is shown in Fig. 13.3.

(a)

(b)

Figure 13.3: Maclaurin polynomials chosen as cross-section functions of the velocity and pressure fields for the sample case of velocity expansion order $N^{U}=4$ and pressure expansion order $N^{P}=2$.

The choice of the analysis parameters $N_{N}^{U}, N_{N}^{P}, N^{U}$, and $N^{P}$ is fundamental to obtain the correct numerical solution of the fluid problem. A key point concerns the stability of the problem. More details will be presented in section 13.4.7.

### 13.4.4 Galerkin approximation through 1D CUF model

According to the 1D CUF approximation for the velocity field (Eqs. 13.98 and 13.99) the generic discrete test function $\mathbf{v}_{h} \in V_{h}$ is:

$$
\begin{equation*}
\mathbf{v}_{h}=\sum_{e=1}^{3} \sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} q_{\tau i e}^{\star} \boldsymbol{\varphi}_{\tau i e} \tag{13.108}
\end{equation*}
$$

where $\mathbf{q}_{\tau i}^{\star}$ is the vector of coefficients which identify the discrete test function $\mathbf{v}_{h}$ in the space $V_{h}$. According to the 1D CUF approximation for the pressure field (Eqs. 13.105 and 13.106 ) the generic discrete test function $q_{h} \in Q_{h}$ is:

$$
\begin{equation*}
q_{h}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t}^{\star} \phi_{m t} \tag{13.109}
\end{equation*}
$$

where $p_{m t}^{\star}$ are the coefficients which identify the discrete test function $q_{h}$ in the space $Q_{h}$. It is sufficient therefore that the Galerkin approximation in Eq. 13.90 is verified for each function of the basis of $V_{h}$ and $Q_{h}$, since all the functions in the space $V_{h}$ and $Q_{h}$ are a linear combination of the basis funcions (see Eq. 13.93). Since the functions $\varphi_{\text {iie }}$ are the bases chosen for the space $V_{h}$, and $\phi_{m t}$ are the bases chosen for the space $Q_{h}$, the following system of equations is thus required to be satisfied:

Find $\mathbf{u}_{h} \in V_{h}, p_{h} \in Q_{h}$ such that

$$
\begin{cases}\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega-\int_{\Omega} p_{h} \operatorname{div} \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega=\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega & \forall \tau, \forall i, \forall e  \tag{13.110}\\ -\int_{\Omega} \phi_{m t} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega=0 & \forall m, \forall t\end{cases}
$$

with $\tau=1, \ldots, N_{u}^{U}, i=1, \ldots, N_{N}^{U}, e=1, \ldots, 3, m=1, \ldots, N_{u}^{P}, t=1, \ldots, N_{N}^{P}$. Equation 13.110 represents a system of $\left(3 N_{u}^{U} N_{N}^{U}+N_{u}^{P} N_{N}^{P}\right)$ equations for the single finite element of the mesh. According to the compact notation in Eq. 13.91 which involves the bilinear forms, the requirement of Eq. 13.110 to be satisfied can be written as follows:

Find $\mathbf{u}_{h} \in V_{h}, p_{h} \in Q_{h}$ such that

$$
\begin{cases}a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i e}\right)+b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)=\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i e}\right) & \forall \tau, \forall i, \forall e  \tag{13.111}\\ b\left(\mathbf{u}_{h}, \phi_{m t}\right)=0 & \forall m, \forall t\end{cases}
$$

Each of the contributions appearing in Eq. 13.110 (i.e. in Eq. 13.111) are now separately considered leading to the construction of the finite element matrices $\underline{\mathbf{A}}, \underline{\mathbf{B}}, \underline{\mathbf{B}}^{T}$ and the vector F related to the Stokes equations with mixed Dirichlet-Neumann homogeneous boundary conditions (Eq. 13.58). For the sake of exposition simplicity, the formulation is written by considering a single finite element of the mesh used to discretize the computational domain $\Omega$, as done for the structural analysis in chapter 3, since Eq. 13.98 refers to the single finite element with $N_{N}^{U}$ nodes and Eq. 13.105 refers to the single finite element with $N_{N}^{P}$ nodes. Additional considerations will be necessary in the finite element assembly procedure, which involves the element matrices and vectors computed, when more than one finite element are considered in the mesh discretization (see section 13.4.7). These considerations are the same as those exposed in detail for the solid mechanics in chapter 3.
$\operatorname{Term} a\left(\mathbf{u}_{h}, \varphi_{\tau i e}\right)$
The first contribution in Eqs. 13.110 and 13.111, which are equivalent expressions to the Galerkin approximation (Eqs. 13.90 and 13.91) of the Stokes problem, is the bilinear form:

$$
\begin{equation*}
a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i e}\right)=\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \varphi_{\tau i e} \mathrm{~d} \Omega \tag{13.112}
\end{equation*}
$$

In order to compute this term which involves the discrete velocity solution and the generic basis function of the space $V_{h}$, the corresponding term which involves the discrete velocity solution and the generic discrete test function of the space $V_{h}$ written in the Galerkin approximation (Eqs. 13.90 and 13.91) is retrieved:

$$
\begin{equation*}
a\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)=\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h} \mathrm{~d} \Omega \tag{13.113}
\end{equation*}
$$

Using Eq. 13.64 , the term $\nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h}$ is:

$$
\begin{equation*}
\nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h}=\operatorname{tr}\left(\nabla \mathbf{u}_{h}^{T} \nabla \mathbf{v}_{h}\right) \tag{13.114}
\end{equation*}
$$

where the symbol tr stands for the trace of a square matrix, i.e. the sum of the elements on its main diagonal. As a consequence:

$$
\begin{equation*}
\nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h}=\operatorname{tr}\left(\nabla \mathbf{u}_{h}^{T} \nabla \mathbf{v}_{h}\right)=\nabla \mathbf{v}_{h}: \nabla \mathbf{u}_{h}=\operatorname{tr}\left(\nabla \mathbf{v}_{h}^{T} \nabla \mathbf{u}_{h}\right) \tag{13.115}
\end{equation*}
$$

The expression of $\operatorname{tr}\left(\nabla \mathbf{v}_{h}^{T} \nabla \mathbf{u}_{h}\right)$ can be written explicitly in the cartesian coordinate system $(x, y, z)$ :

$$
\begin{align*}
\operatorname{tr}\left(\nabla \mathbf{v}_{h}^{T} \nabla \mathbf{u}_{h}\right)= & \frac{\partial v_{h x}}{\partial x} \frac{\partial u_{h x}}{\partial x}+\frac{\partial v_{h y}}{\partial x} \frac{\partial u_{h y}}{\partial x}+\frac{\partial v_{h z}}{\partial x} \frac{\partial u_{h z}}{\partial x}+ \\
& \frac{\partial v_{h x}}{\partial y} \frac{\partial u_{h x}}{\partial y}+\frac{\partial v_{h y}}{\partial y} \frac{\partial u_{h y}}{\partial y}+\frac{\partial v_{h z}}{\partial y} \frac{\partial u_{h z}}{\partial y}+  \tag{13.116}\\
& \frac{\partial v_{h x}}{\partial z} \frac{\partial u_{h x}}{\partial z}+\frac{\partial v_{h y}}{\partial z} \frac{\partial u_{h y}}{\partial z}+\frac{\partial v_{h z}}{\partial z} \frac{\partial u_{h z}}{\partial z}
\end{align*}
$$

Using Eq. 13.116 , Eq. 13.115 can be rewritten in a compact vectorial notation:

$$
\begin{align*}
\nabla \mathbf{u}_{h}: \nabla \mathbf{v}_{h} & =\frac{\partial \mathbf{v}_{h}}{\partial x} \cdot \frac{\partial \mathbf{u}_{h}}{\partial x}+\frac{\partial \mathbf{v}_{h}}{\partial y} \cdot \frac{\partial \mathbf{u}_{h}}{\partial y}+\frac{\partial \mathbf{v}_{h}}{\partial z} \cdot \frac{\partial \mathbf{u}_{h}}{\partial z} \\
& =\left[\frac{\partial \mathbf{v}_{h}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x}+\left[\frac{\partial \mathbf{v}_{h}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y}+\left[\frac{\partial \mathbf{v}_{h}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \tag{13.117}
\end{align*}
$$

Comparing Eqs. 13.112 and 13.113 , it is possible to write the quantity $\nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i e}$ by replacing $\boldsymbol{\varphi}_{\tau i e}$ with $\mathbf{v}_{h}$ in Eq. 13.117:

$$
\begin{align*}
\nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i e} & =\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial x} \cdot \frac{\partial \mathbf{u}_{h}}{\partial x}+\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial y} \cdot \frac{\partial \mathbf{u}_{h}}{\partial y}+\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial z} \cdot \frac{\partial \mathbf{u}_{h}}{\partial z} \\
& =\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x}+\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y}+\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \tag{13.118}
\end{align*}
$$

Hence:

$$
\begin{aligned}
\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega= & \underbrace{\int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x} \mathrm{~d} \Omega}_{\text {TERM } 1}+\underbrace{\int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y} \mathrm{~d} \Omega}_{\text {TERM } 2}+ \\
& \underbrace{\int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i e}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \mathrm{~d} \Omega}_{\text {TERM } 3}
\end{aligned}
$$

Each of the contributions appearing in Eq. 13.119 are now separately considered.
TERM 1. For the sake of convenience, the discrete solution $\mathbf{u}_{h}$ is now written with indices $s$ and $j$ different from indices $\tau$ and $i$ used in the 1D CUF FE approximation in Eq. 13.98:

$$
\begin{equation*}
\mathbf{u}_{h}=\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k} \boldsymbol{\varphi}_{s j k} \tag{13.120}
\end{equation*}
$$

This choice is formally convenient and is made purely for the sake of exposition simplicity. Using Eq. 13.99, the derivative of $\mathbf{u}_{h}$ with respect to $x$ appearing in TERM 1 of Eq. 13.119 becomes:

$$
\begin{align*}
\frac{\partial \mathbf{u}_{h}}{\partial x} & =\frac{\partial}{\partial x}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[q_{s j x} \boldsymbol{\varphi}_{s j 1}+q_{s j y} \boldsymbol{\varphi}_{s j 2}+q_{s j z} \boldsymbol{\varphi}_{s j 3}\right]\right] \\
& =\frac{\partial}{\partial x}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c}
\left.\left.q_{s j x}\left\{\begin{array}{c}
F_{s}^{U} N_{j}^{U} \\
0 \\
0
\end{array}\right\}+q_{s j y}\left\{\begin{array}{c}
0 \\
F_{s}^{U} N_{j}^{U} \\
0
\end{array}\right\}+q_{s j z}\left\{\begin{array}{c}
0 \\
0 \\
F_{s}^{U} N_{j}^{U}
\end{array}\right\}\right]\right] \\
\\
\end{array}\right\} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \frac{\partial}{\partial x}\left\{\begin{array}{l}
q_{s j x} F_{s}^{U} N_{j}^{U} \\
q_{s j y} F_{s}^{U} N_{j}^{U} \\
q_{s j z} F_{s}^{U} \\
N_{j}^{U}
\end{array}\right\}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U}\left\{\begin{array}{c}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\}\right. \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}
\end{align*}
$$

It is important to notice that the derivative of $\mathbf{u}_{h}$ with respect to $x$ has effect only on the cross-section functions $F_{s}^{U}$, since they depend on the cross-section coordinates $x$ and $z$, whereas the shape functions $N_{j}^{U}$ of the present model depend only on coordinate $y$ (see Eq. 13.98). According to Eq. 13.110 (i.e. in Eq. 13.111), TERM 1 has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered and for this purpose it is reminded from Eq. 13.99 that the basis functions chosen for the space $V_{h}$ are:

$$
\boldsymbol{\varphi}_{\tau i 1}=\left\{\begin{array}{c}
F_{\tau}^{U} N_{i}^{U}  \tag{13.122}\\
0 \\
0
\end{array}\right\} \quad \boldsymbol{\varphi}_{\tau i 2}=\left\{\begin{array}{c}
0 \\
F_{\tau}^{U} N_{i}^{U} \\
0
\end{array}\right\} \quad \boldsymbol{\varphi}_{\tau i 3}=\left\{\begin{array}{c}
0 \\
0 \\
F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}
$$

Substituting Eqs. 13.121 and 13.122 into TERM 1 of Eq. 13.119 and expanding on the index $e$, it is obtained:

$$
\begin{align*}
& \int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 1}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x} \mathrm{~d} \Omega=\int_{\Omega} \nu\left[\frac{\partial}{\partial x}\left\{\begin{array}{lll}
F_{\tau}^{U} N_{i}^{U} & 0 & 0
\end{array}\right]\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
F_{\tau, x}^{U} N_{i}^{U} & 0 & 0
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Omega\right.}_{=\mathbf{A}_{1.1}^{\tau s i j}} \begin{array}{lll} 
& 0 & 0
\end{array}\}) \mathbf{q}_{s j} \tag{13.123}
\end{align*}
$$

$$
\begin{align*}
& \int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 2}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x} \mathrm{~d} \Omega=\int_{\Omega} \nu\left[\frac{\partial}{\partial x}\left\{\begin{array}{lll}
0 & F_{\tau}^{U} N_{i}^{U} & 0
\end{array}\right]\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
0 & F_{\tau, x}^{U} N_{i}^{U} & 0
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & \int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Omega & 0
\end{array}\right\}}_{=\mathbf{A}_{1.2}^{\tau s i j}} \mathbf{q}_{s j}  \tag{13.124}\\
& \int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 3}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x} \mathrm{~d} \Omega=\int_{\Omega} \nu\left[\frac{\partial}{\partial x}\left\{\begin{array}{lll}
0 & 0 & F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
0 & 0 & F_{\tau, x}^{U} N_{i}^{U}
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, x}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & 0 & \int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Omega
\end{array}\right\}}_{=\mathbf{A}_{1.3}^{\tau s i j}} \mathbf{q}_{s j} \tag{13.125}
\end{align*}
$$

As occurred for $\mathbf{u}_{h}$, the derivative of $\boldsymbol{\varphi}_{\text {rie }}$ with respect to $x$ has effect only on the crosssection functions $F_{\tau}^{U}$. The quantities $\mathbf{A}_{1.1}^{\tau s i j}, \mathbf{A}_{1.2}^{\tau s i j}, \mathbf{A}_{1.3}^{\tau s i j}$ are three row vectors with dimensions $1 \times 3$ which relate TERM 1 written for $e=1,2,3$ and the generic $\tau$ and $i$ to the nodal velocity vector $\mathbf{q}_{s j}$. Finally, TERM 1 can be reformulated by exploiting a compact notation:

$$
\int_{\Omega} \nu\left\{\begin{array}{l}
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 1}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x}}  \tag{13.126}\\
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 2}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x}} \\
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 3}}{\partial x}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial x}}
\end{array}\right\} \mathrm{d} \Omega=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c}
\mathbf{A}_{1.1}^{\tau s i j} \\
\mathbf{A}_{1.2}^{\tau s i j} \\
\mathbf{A}_{1.3}^{\tau s i j}
\end{array}\right] \mathbf{q}_{s j}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{1}^{\tau s i j} \mathbf{q}_{s j}
$$

where $\underline{\mathbf{A}}_{1}^{\tau s i j}$ is the part of the fundamental nucleus of Matrix $\underline{\mathbf{A}}$ related to TERM 1 of the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\text {тie }}\right)$. Introducing the $3 \times 3$ identity matrix $\underline{\mathbf{I}}$, it is obtained:

$$
\begin{equation*}
\underline{\mathbf{A}}_{1}^{\tau s i j}=\int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Omega \underline{\mathbf{I}} \tag{13.127}
\end{equation*}
$$

This matrix has dimensions $3 \times 3$ and, as it will be explained later on, it has to be expanded for every value of indices $\tau$ and $i$, according to Eq. 13.110 (i.e. in Eq. 13.111).

TERM 2. The procedure to obtain TERM 2 of Eq. 13.119 is similar to that followed for TERM 1. The discrete solution $\mathbf{u}_{h}$ is again written with indices $s$ and $j$, as done in Eq. 13.120. Using Eq. 13.99, the derivative of $\mathbf{u}_{h}$ with respect to $y$ appearing in TERM 2
of Eq. 13.119 becomes:

$$
\begin{align*}
\frac{\partial \mathbf{u}_{h}}{\partial y} & =\frac{\partial}{\partial y}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[q_{s j x} \boldsymbol{\varphi}_{s j 1}+q_{s j y} \boldsymbol{\varphi}_{s j 2}+q_{s j z} \boldsymbol{\varphi}_{s j 3}\right]\right] \\
& =\frac{\partial}{\partial y}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[q_{s j x}\left\{\begin{array}{c}
F_{s}^{U} N_{j}^{U} \\
0 \\
0
\end{array}\right\}+q_{s j y}\left\{\begin{array}{c}
0 \\
F_{s}^{U} N_{j}^{U} \\
0
\end{array}\right\}+q_{s j z}\left\{\begin{array}{c}
0 \\
0 \\
F_{s}^{U} N_{j}^{U}
\end{array}\right\}\right]\right] \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \frac{\partial}{\partial y}\left\{\begin{array}{c}
q_{s j x} F_{s}^{U} N_{j}^{U} \\
q_{s j y} F_{s}^{U} \\
N_{j}^{U} \\
q_{s j z} F_{s}^{U} \\
N_{j}^{U}
\end{array}\right\}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U}\left\{\begin{array}{c}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\} \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j} \tag{13.128}
\end{align*}
$$

It is important to notice that the derivative of $\mathbf{u}_{h}$ with respect to $y$ has effect only on the shape functions $N_{j}^{U}$, since they depend on coordinate $y$, whereas the cross-section functions $F_{s}^{U}$ of the present model depend only on cross-section coordinates $x$ and $z$ (see Eq. 13.98). According to Eq. 13.110 (i.e. in Eq. 13.111), TERM 2 has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered. Substituting Eqs. 13.128 and 13.122 into TERM 2 of Eq. 13.119 and expanding on the index $e$, it is obtained:

$$
\begin{align*}
\int_{\Omega} \nu\left[\frac{\partial \varphi_{\tau i 1}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y} \mathrm{~d} \Omega & =\int_{\Omega} \nu\left[\frac{\partial}{\partial y}\left\{\begin{array}{lll}
F_{\tau}^{U} N_{i}^{U} & 0 & 0
\end{array}\right]\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
F_{\tau}^{U} N_{i, y}^{U} & 0 & 0
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\int_{\Omega} \nu N_{i, y}^{U} N_{j, y}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Omega\right.}_{=\mathbf{A}_{2.1}^{\tau s i j}} \begin{array}{lll}
0 & 0
\end{array}\} \tag{13.129}
\end{align*} \mathbf{q}_{s j} \quad \text {, }
$$

$$
\int_{\Omega} \nu\left[\frac{\partial \varphi_{\tau i 2}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y} \mathrm{~d} \Omega=\int_{\Omega} \nu\left[\frac{\partial}{\partial y}\left\{\begin{array}{lll}
0 & F_{\tau}^{U} N_{i}^{U} & 0
\end{array}\right\}\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega
$$

$$
=\int_{\Omega} \nu\left\{\begin{array}{lll}
0 & F_{\tau}^{U} N_{i, y}^{U} & 0
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega
$$

$$
=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & \int_{\Omega} \nu N_{i, y}^{U} N_{j, y}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Omega & 0 \tag{13.130}
\end{array}\right\}}_{=\mathbf{A}_{2.2}^{\tau s i j}} \mathbf{q}_{s j}
$$

$$
\begin{align*}
\int_{\Omega} \nu\left[\frac{\partial \varphi_{\tau i 3}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y} \mathrm{~d} \Omega & =\int_{\Omega} \nu\left[\frac{\partial}{\partial y}\left\{\begin{array}{lll}
0 & 0 & F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
0 & 0 & F_{\tau}^{U} N_{i, y}^{U}
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j, y}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & 0 & \int_{\Omega} \nu N_{i, y}^{U} N_{j, y}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Omega
\end{array}\right\}}_{=\mathbf{A}_{2.3}^{\tau s i j}} \mathbf{q}_{s j} \tag{13.131}
\end{align*}
$$

As occurred for $\mathbf{u}_{h}$, the derivative of $\boldsymbol{\varphi}_{\text {rie }}$ with respect to $y$ has effect only on the shape functions $N_{i}^{U}$ and $N_{j}^{U}$. The quantities $\mathbf{A}_{2.1}^{\tau s i j}, \mathbf{A}_{2.2}^{\tau s i j}, \mathbf{A}_{2.3}^{\tau s i j}$ are three row vectors with dimensions $1 \times 3$ which relate TERM 2 written for $e=1,2,3$ and the generic $\tau$ and $i$ to the nodal velocity vector $\mathbf{q}_{s j}$. Finally, TERM 2 can be reformulated by exploiting a compact notation:

$$
\int_{\Omega} \nu\left\{\begin{array}{l}
{\left[\frac{\partial \varphi_{\tau i 1}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y}}  \tag{13.132}\\
{\left[\frac{\partial \varphi_{\tau i 2}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y}} \\
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 3}}{\partial y}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial y}}
\end{array}\right\} \mathrm{d} \Omega=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c}
\mathbf{A}_{2.1}^{\tau s i j} \\
\mathbf{A}_{2.2}^{\tau s i j} \\
\mathbf{A}_{2.3}^{\tau s i j}
\end{array}\right] \mathbf{q}_{s j}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{2}^{\tau s i j} \mathbf{q}_{s j}
$$

where $\underline{\mathbf{A}}_{2}^{\tau s i j}$ is the part of the fundamental nucleus of Matrix $\underline{\mathbf{A}}$ related to TERM 2 of the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\text {тie }}\right)$. Introducing the $3 \times 3$ identity matrix $\underline{\mathbf{I}}$, it is obtained:

$$
\begin{equation*}
\underline{\mathbf{A}}_{2}^{\tau s i j}=\int_{\Omega} \nu N_{i, y}^{U} N_{j, y}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Omega \underline{\mathbf{I}} \tag{13.133}
\end{equation*}
$$

This matrix has dimensions $3 \times 3$ and, as it will be explained later on, it has to be expanded for every value of indices $\tau$ and $i$, according to Eq. 13.110 (i.e. in Eq. 13.111).

TERM 3. The procedure to obtain TERM 3 of Eq. 13.119 is similar to that followed for TERM 1 and TERM 2. The discrete solution $\mathbf{u}_{h}$ is again written with indices $s$ and $j$, as done in Eq. 13.120. Using Eq. 13.99, the derivative of $\mathbf{u}_{h}$ with respect to $z$ appearing in TERM 3 of Eq. 13.119 becomes:

$$
\begin{aligned}
\frac{\partial \mathbf{u}_{h}}{\partial z} & =\frac{\partial}{\partial z}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[q_{s j x} \boldsymbol{\varphi}_{s j 1}+q_{s j y} \boldsymbol{\varphi}_{s j 2}+q_{s j z} \boldsymbol{\varphi}_{s j 3}\right]\right] \\
& \left.\left.=\frac{\partial}{\partial z}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c} 
\\
q_{s j x} \\
0
\end{array}\right\} \begin{array}{c}
F_{s}^{U} N_{j}^{U} \\
0 \\
0
\end{array}\right\}+q_{s j y}\left\{\begin{array}{c}
0 \\
F_{s}^{U} N_{j}^{U} \\
0
\end{array}\right\}+q_{s j z}\left\{\begin{array}{c}
0 \\
0 \\
F_{s}^{U} N_{j}^{U}
\end{array}\right\}\right]\right] \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \frac{\partial}{\partial z}\left\{\begin{array}{c}
q_{s j x} F_{s}^{U} N_{j}^{U} \\
q_{s j y} \\
F_{s j z}^{U} \\
N_{j}^{U} \\
q_{s j}^{U}
\end{array}\right\}=\sum_{s=1}^{N_{j}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U}\left\{\begin{array}{l}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\}
\end{aligned}
$$

$$
\begin{equation*}
=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} \mathbf{q}_{s j} \tag{13.134}
\end{equation*}
$$

It is important to notice that the derivative of $\mathbf{u}_{h}$ with respect to $z$ has effect only on the cross-section functions $F_{s}^{U}$, since they depend on the cross-section coordinates $x$ and $z$, whereas the shape functions $N_{j}^{U}$ of the present model depend only on coordinate $y$ (see Eq. 13.98). According to Eq. 13.110 (i.e. in Eq. 13.111), TERM 3 has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered. Substituting Eqs. 13.134 and 13.122 into TERM 3 of Eq. 13.119 and expanding on the index $e$, it is obtained:

$$
\begin{aligned}
\int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 1}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \mathrm{~d} \Omega & =\int_{\Omega} \nu\left[\frac{\partial}{\partial z}\left\{\begin{array}{lll}
F_{\tau}^{U} N_{i}^{U} & 0 & 0
\end{array}\right\}\right]\left[\begin{array}{lll}
\sum_{s=1}^{U} & \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} \mathbf{q}_{s j}
\end{array}\right] \mathrm{d} \Omega \\
& =\int_{\Omega} \nu\left\{\begin{array}{lll}
F_{\tau, z}^{U} N_{i}^{U} & 0 & 0
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Omega\right.}_{=\mathbf{A}_{3.1}^{\tau s i j}} \begin{array}{lll}
0 & 0
\end{array}\}
\end{aligned} \mathbf{q}_{s j}
$$

$$
\begin{align*}
\int_{\Omega} \nu\left[\frac{\partial \varphi_{\tau i 2}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \mathrm{~d} \Omega & \left.=\int_{\Omega} \nu\left[\begin{array}{lll}
\frac{\partial}{\partial z}\left\{\begin{array}{lll}
0 & F_{\tau}^{U} & N_{i}^{U}
\end{array}\right]
\end{array}\right\}\right]\left[\begin{array}{ll}
\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} & \mathbf{q}_{s j}
\end{array}\right] \mathrm{d} \Omega  \tag{13.135}\\
& =\int_{\Omega} \nu\left\{\begin{array}{llll}
0 & F_{\tau, z}^{U} N_{i}^{U} & 0
\end{array}\right\}\left[\begin{array}{lll}
\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} & \mathbf{q}_{s j}
\end{array}\right] \mathrm{d} \Omega \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & \int_{\Omega} \nu N_{i}^{U} & N_{j}^{U} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Omega \\
0
\end{array}\right\}}_{=\mathbf{A}_{3.2}^{\tau s i j}} \mathbf{q}_{s j} \tag{13.136}
\end{align*}
$$

$$
\int_{\Omega} \nu\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 3}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z} \mathrm{~d} \Omega=\int_{\Omega} \nu\left[\frac{\partial}{\partial z}\left\{\begin{array}{lll}
0 & 0 & F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}\right]\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega
$$

$$
=\int_{\Omega} \nu\left\{\begin{array}{lll}
0 & 0 & F_{\tau, z}^{U} N_{i}^{U}
\end{array}\right\}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s, z}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Omega
$$

$$
=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & 0 & \int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Omega \tag{13.137}
\end{array}\right\}}_{=\mathbf{A}_{3.3}^{\tau s, i j}} \mathbf{q}_{s j}
$$

As occurred for $\mathbf{u}_{h}$, the derivative of $\boldsymbol{\varphi}_{\tau i e}$ with respect to $x$ has effect only on the crosssection functions $F_{\tau}^{U}$. The quantities $\mathbf{A}_{3.1}^{\tau s i j}, \mathbf{A}_{3.2}^{\tau s i j}, \mathbf{A}_{3.3}^{\tau s i j}$ are three row vectors with dimensions $1 \times 3$ which relate TERM 3 written for $e=1,2,3$ and the generic $\tau$ and $i$ to the
nodal velocity vector $\mathbf{q}_{s j}$. Finally, TERM 3 can be reformulated by exploiting a compact notation:

$$
\int_{\Omega} \nu\left\{\begin{array}{l}
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 1}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z}}  \tag{13.138}\\
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 2}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z}} \\
{\left[\frac{\partial \boldsymbol{\varphi}_{\tau i 3}}{\partial z}\right]^{T} \frac{\partial \mathbf{u}_{h}}{\partial z}}
\end{array}\right\} \mathrm{d} \Omega=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c}
\mathbf{A}_{3.1}^{\tau s i j} \\
\mathbf{A}_{3.2}^{\tau s i j} \\
\mathbf{A}_{3.3}^{\tau s i j}
\end{array}\right] \mathbf{q}_{s j}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{3}^{\tau s i j} \mathbf{q}_{s j}
$$

where $\underline{\mathbf{A}}_{3}^{\tau s i j}$ is the part of the fundamental nucleus of Matrix $\underline{\mathbf{A}}$ related to TERM 3 of the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\text {тie }}\right)$. Introducing the $3 \times 3$ identity matrix $\underline{\mathbf{I}}$, it is obtained:

$$
\begin{equation*}
\underline{\mathbf{A}}_{3}^{\tau s i j}=\int_{\Omega} \nu N_{i}^{U} N_{j}^{U} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Omega \underline{\mathbf{I}} \tag{13.139}
\end{equation*}
$$

This diagonal matrix has dimensions $3 \times 3$ and, as it will be explained later on, it has to be expanded for every value of indices $\tau$ and $i$, according to Eq. 13.110 (i.e. in Eq. 13.111).

TERM 1 + TERM $2+$ TERM 3. The expressions of TERMS 1, 2, and 3 appearing in Eq. 13.119 have been explicitly computed in Eqs. 13.123- 13.125, 13.129- 13.131, 13.135-13.137 for $e=1,2,3$. For the sake of convenience, they have been collected in the $3 \times 3$ matrices $\underline{\mathbf{A}}_{1}^{\tau s i j}, \underline{\mathbf{A}}_{2}^{\tau s i j}, \underline{\mathbf{A}}_{3}^{\tau s i j}$, respectively. Collecting and summing these terms, the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\text {Tie }}\right)$ in Eq. 13.119 expanded for $e=1,2,3$ can be finally written in a compact vectorial notation $(\forall \tau, \forall i)$ :

$$
\begin{align*}
& \left\{\begin{array}{l}
a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i 1}\right) \\
a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i 2}\right) \\
a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i 3}\right)
\end{array}\right\}=\left\{\begin{array}{l}
\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i 1} \mathrm{~d} \Omega \\
\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i 2} \mathrm{~d} \Omega \\
\int_{\Omega} \nu \nabla \mathbf{u}_{h}: \nabla \boldsymbol{\varphi}_{\tau i 3} \mathrm{~d} \Omega
\end{array}\right\} \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\underline{\mathbf{A}}_{1}^{\tau s i j}+\underline{\mathbf{A}}_{2}^{\tau s i j}+\underline{\mathbf{A}}_{3}^{\tau s i j}\right] \mathbf{q}_{s j}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}^{\tau s i j} \mathbf{q}_{s j} \tag{13.140}
\end{align*}
$$

where $\underline{\mathbf{A}}^{\tau s i j}$ is the fundamental nucleus of Matrix $\underline{\mathbf{A}}$ related to the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i e}\right)$ $(e=1,2,3)$ of the 1D CUF FE model. The fundamental nucleus multiplies the nodal velocity vector $\mathbf{q}_{s j}$, which is related to the generic $s^{t h}$ velocity expansion term of CUF formulation corresponding to the $j^{\text {th }}$ node of the single finite element, see Eq. 13.98. The fundamental nucleus will be expanded with respect to indices $\tau, s, i$, and $j$ in order to build Matrix $\underline{\mathbf{A}}$ of the single finite element. This nucleus is a diagonal matrix with dimensions $3 \times 3$, since it is the result of the sum of the diagonal matrices previously computed:

$$
\begin{align*}
\underline{\mathbf{A}}^{\tau s i j} & =\underline{\mathbf{A}}_{1}^{\tau s i j}+\underline{\mathbf{A}}_{2}^{\tau s i j}+\underline{\mathbf{A}}_{3}^{\tau s i j} \\
& =\left[\int_{\Omega} \nu\left(N_{i}^{U} N_{j}^{U} F_{\tau, x}^{U} F_{s, x}^{U}+N_{i, y}^{U} N_{j, y}^{U} F_{\tau}^{U} F_{s}^{U}+N_{i}^{U} N_{j}^{U} F_{\tau, z}^{U} F_{s, z}^{U}\right) \mathrm{d} \Omega\right] \underline{\mathbf{I}} \tag{13.141}
\end{align*}
$$

As previously mentioned, it is reminded that, for the moment, the present procedure refers to a single finite element; i.e. here the assembly procedure is not yet considered. As a consequence, the integration in Eq. 13.141 is performed over the volume corresponding to the domain of a single finite element. By definition, the cross-section related to the single one-dimensional finite element is considered to be constant over the element length. Therefore, the integral over the volume in Eq. 13.141 and in the following equations is split into the integral over the cross-section $\Gamma_{S}$ and the integral along the axis (here indicated generically as $l$ ) of the one-dimensional finite element, which has length $L_{\mathrm{EL}}$ :

$$
\begin{equation*}
\int_{\Omega} \ldots \mathrm{d} \Omega=\int_{l} \int_{\Gamma_{S}} \ldots \mathrm{~d} \Gamma \mathrm{~d} y \tag{13.142}
\end{equation*}
$$

By definition in Eq. 13.94, the cross-section functions $F_{\tau}^{U}$ and $F_{s}^{U}$ depend only on the crosssection coordinates. They and their derivatives with respect to $x$ and $z$ can be therefore taken out of the integral along the element length. Similarly, the shape functions of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$, see Eq. 13.102. Hence, in Eq. 13.141 the shape functions are taken out of the integral over the cross-section $\Gamma_{S}$. For these reasons, it is possible to split the integral along the element length and the integral over the element cross-section into two different contributions to be multiplied.

It is important to remind that the Stokes equations whose Galerkin approximation has been here formulated have been derived for the case of costant kinematic viscosity $\nu$, see sections 13.2 and 13.3. Hence, the term $\nu$ can be taken out of the integral over $\Gamma_{S}$ as well as the integral along the element length. The expression of $\underline{\mathbf{A}}^{\tau s i j}$ can be finally written as follows:

$$
\begin{align*}
& \underline{\mathbf{A}}^{\tau s i j}=\left[\nu \int_{l} N_{i}^{U} N_{j}^{U} \mathrm{~d} y \int_{\Gamma_{S}} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Omega+\nu \int_{l} N_{i, y}^{U} N_{j, y}^{U} \mathrm{~d} y \int_{\Gamma_{S}} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Omega+\right. \\
&\left.\nu \int_{l} N_{i}^{U} N_{j}^{U} \mathrm{~d} y \int_{\Gamma_{S}} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Omega\right] \underline{\mathbf{I}} \tag{13.143}
\end{align*}
$$

The integrals of the products of shape functions along the element length in Eq. 13.143 are collected in the following terms:

$$
\begin{equation*}
E_{j}^{i U}=\int_{l} N_{i}^{U} N_{j}^{U} \mathrm{~d} y \quad E_{j, y}^{i, y}=\int_{l} N_{i, y}^{U} N_{j, y}^{U} \mathrm{~d} y \tag{13.144}
\end{equation*}
$$

The three integrals of the products of cross-section functions and their derivatives over $\Gamma_{S}$ are introduced:
$J_{s}^{\tau U}=\int_{\Gamma_{S}} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma \quad J_{s, x}^{\tau, x U}=\int_{\Gamma_{S}} F_{\tau, x}^{U} F_{s, x}^{U} \mathrm{~d} \Gamma \quad J_{s, z}^{\tau, z U}=\int_{\Gamma_{S}} F_{\tau, z}^{U} F_{s, z}^{U} \mathrm{~d} \Gamma$
It is noteworthy to highlight that the three terms in square brackets in Eq. 13.143 are scalar quantities which multiply the identity matrix $\mathbf{I}$. The nine components of the fundamental nucleus $\underline{\mathbf{A}}^{\tau s i j}$ can be computed explicitly:

$$
\begin{gather*}
A_{x x}^{\tau s i j}=A_{y y}^{\tau s i j}=A_{z z}^{\tau s i j}=\nu E_{j}^{i U} J_{s, x}^{\tau, x U}+\nu E_{j, y}^{i, y U} J_{s}^{\tau U}+\nu E_{j}^{i U} J_{s, z}^{\tau, z U}  \tag{13.146}\\
A_{x y}^{\tau s i j}=A_{x z}^{\tau s i j}=A_{y x}^{\tau s i j}=A_{y z}^{\tau s i j}=A_{z x}^{\tau s i j}=A_{z y}^{\tau s i j}=0
\end{gather*}
$$

It is important to note the symmetry of the fundamental nucleus, which derives from the symmetric bilinear form $a(\cdot, \cdot)$ in Eq. 13.87:

$$
\begin{equation*}
\underline{\mathbf{A}}^{\tau s i j}=\underline{\mathbf{A}}^{s \tau j i}=\underline{\mathbf{A}}^{s \tau j i} \tag{13.147}
\end{equation*}
$$

$\operatorname{Term} b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)$
The second contribution in Eqs. 13.110 and 13.111, which are equivalent expressions to the Galerkin approximation (Eqs. 13.90 and 13.91) of the Stokes problem, is the bilinear form:

$$
\begin{equation*}
b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)=-\int_{\Omega} p_{h} \operatorname{div} \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega \tag{13.148}
\end{equation*}
$$

For the sake of completeness, it comes from the bilinear form in Eq. 13.91 (i.e. 13.90) and from the choice of the basis functions $\varphi_{\tau i e}$ of the space $V_{h}$ containing the discrete test functions $\mathbf{v}_{h}$, according to the 1D CUF FE model (Eq. 13.99):

$$
\begin{equation*}
b\left(\mathbf{v}_{h}, p_{h}\right)=-\int_{\Omega} p_{h} \operatorname{div} \mathbf{v}_{h} \mathrm{~d} \Omega \tag{13.149}
\end{equation*}
$$

Retrieving the basis functions chosen for the space $V_{h}$ written in Eq. 13.122, the term $\operatorname{div} \varphi_{\tau i e}$ can be computed for $e=1,2,3$ :

$$
\begin{align*}
& \operatorname{div} \varphi_{\tau i 1}=\operatorname{div}\left\{\begin{array}{c}
F_{\tau}^{U} N_{i}^{U} \\
0 \\
0
\end{array}\right\}=\frac{\partial}{\partial x}\left[F_{\tau}^{U} N_{i}^{U}\right]+\frac{\partial}{\partial y}[0]+\frac{\partial}{\partial z}[0]=F_{\tau, x}^{U} N_{i}^{U}  \tag{13.150}\\
& \operatorname{div} \varphi_{\tau i 2}=\operatorname{div}\left\{\begin{array}{c}
0 \\
F_{\tau}^{U} N_{i}^{U} \\
0
\end{array}\right\}=\frac{\partial}{\partial x}[0]+\frac{\partial}{\partial y}\left[F_{\tau}^{U} N_{i}^{U}\right]+\frac{\partial}{\partial z}[0]=F_{\tau}^{U} N_{i, y}^{U}  \tag{13.151}\\
& \operatorname{div} \varphi_{\tau i 3}=\operatorname{div}\left\{\begin{array}{c}
0 \\
0 \\
F_{\tau}^{U} N_{i}^{U}
\end{array}\right\}=\frac{\partial}{\partial x}[0]+\frac{\partial}{\partial y}[0]+\frac{\partial}{\partial z}\left[F_{\tau}^{U} N_{i}^{U}\right]=F_{\tau, z}^{U} N_{i}^{U} \tag{13.152}
\end{align*}
$$

It is important to notice that in Eqs. 13.150 and 13.152 the derivatives of $\boldsymbol{\varphi}_{\tau i 1}$ and $\boldsymbol{\varphi}_{\tau i 3}$ with respect to $x$ and $z$ have effect only on the cross-section functions $F_{\tau}^{U}$, since they depend on the cross-section coordinates $x$ and $z$. On the contrary, the derivative of $\boldsymbol{\varphi}_{\tau i 2}$ with respect to $y$ has effect only on the shape functions $N_{i}^{U}$, since they depend on coordinate $y$ (see Eq. 13.98).

In order to express Eq. 13.148 in terms of the pressure unknowns, it is reminded from Eq. 13.106 that the discrete solution $p_{h}$ is a linear combination of the basis functions $\phi_{m t}$ :

$$
\begin{equation*}
p_{h}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} \phi_{m t}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} F_{m}^{P} N_{t}^{P} \tag{13.153}
\end{equation*}
$$

According to Eq. 13.110 (i.e. in Eq. 13.111), the term $b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)$ has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered. Subtituting Eqs. 13.150-13.152 and 13.153 into Eq. 13.148 and varying index $e$ :

$$
\begin{align*}
-\int_{\Omega} p_{h} \operatorname{div} \varphi_{\tau i 1} \mathrm{~d} \Omega & =-\int_{\Omega}\left[\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} F_{m}^{P} N_{t}^{P}\right] F_{\tau, x}^{U} N_{i}^{U} \mathrm{~d} \Omega \\
& =\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} \underbrace{\left[-\int_{\Omega} N_{i}^{U} N_{t}^{P} F_{\tau, x}^{U} F_{m}^{P} \mathrm{~d} \Omega\right]}_{=B_{1}^{\tau m i t}} p_{m t} \tag{13.154}
\end{align*}
$$

$$
\begin{align*}
-\int_{\Omega} p_{h} \operatorname{div} \varphi_{\tau i 2} \mathrm{~d} \Omega & =-\int_{\Omega}\left[\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} F_{m}^{P} N_{t}^{P}\right] F_{\tau}^{U} N_{i, y}^{U} \mathrm{~d} \Omega \\
& =\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} \underbrace{\left[-\int_{\Omega} N_{i, y}^{U} N_{t}^{P} F_{\tau}^{U} F_{m}^{P} \mathrm{~d} \Omega\right]}_{=B_{2}^{m_{m i t}}} p_{m t}  \tag{13.155}\\
-\int_{\Omega} p_{h} \operatorname{div} \varphi_{\tau i 3} \mathrm{~d} \Omega & =-\int_{\Omega}\left[\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} p_{m t} F_{m}^{P} N_{t}^{P}\right] F_{\tau, z}^{U} N_{i}^{U} \mathrm{~d} \Omega \\
& =\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} \underbrace{\left[-\int_{\Omega} N_{i}^{U} N_{t}^{P} F_{\tau, z}^{U} F_{m}^{P} \mathrm{~d} \Omega\right]}_{=B_{3}^{\tau m i t}} p_{m t} \tag{13.156}
\end{align*}
$$

The quantities $B_{1}^{\text {qmit }}, B_{2}^{\tau m i t}, B_{3}^{\tau m i t}$ are three scalar terms which relate the bilinear form in Eq. 13.148 written for $e=1,2,3$ and the generic $\tau$ and $i$ to the nodal unknown pressure $p_{m t}$. Collecting Eqs. 13.154-13.156, the bilinear form $b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)$ in Eq. 13.148 expanded for $e=1,2,3$ can be finally written in a compact vectorial notation $(\forall \tau, \forall i)$ :

$$
\begin{align*}
\left\{\begin{array}{l}
b\left(\boldsymbol{\varphi}_{\tau i 1}, p_{h}\right) \\
b\left(\boldsymbol{\varphi}_{\tau i 2}, p_{h}\right) \\
b\left(\boldsymbol{\varphi}_{\tau i 3}, p_{h}\right)
\end{array}\right\} & =\left\{\begin{array}{c}
-\int_{\Omega} p_{h} \operatorname{div} \boldsymbol{\varphi}_{\tau i 1} \mathrm{~d} \Omega \\
-\int_{\Omega} p_{h} \operatorname{div} \boldsymbol{\varphi}_{\tau i 2} \mathrm{~d} \Omega \\
-\int_{\Omega} p_{h} \operatorname{div} \boldsymbol{\varphi}_{\tau i 3} \mathrm{~d} \Omega
\end{array}\right\}  \tag{13.157}\\
& =\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}}\left\{\begin{array}{l}
B_{1}^{\tau m i t} \\
B_{2}^{\tau m i t} \\
B_{3}^{\tau m i t}
\end{array}\right\} p_{m t}=\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} \mathbf{B}^{\tau m i t}{ }^{T} p_{m t}
\end{align*}
$$

where $\mathbf{B}^{\text {tmit }}{ }^{T}$ is the fundamental nucleus of Matrix $\underline{\mathbf{B}}^{T}$ related to the bilinear form $b\left(\boldsymbol{\varphi}_{\tau i e}, p_{h}\right)(e=1,2,3)$ of the 1D CUF FE model. The fundamental nucleus multiplies the nodal unknown pressure $p_{m t}$, which is related to the generic $m^{\text {th }}$ pressure expansion term of CUF formulation corresponding to the $t^{t h}$ node of the single finite element, see Eq. 13.105. The fundamental nucleus will be expanded with respect to indices $\tau, m, i$, and $t$ in order to build Matrix $\underline{\mathbf{B}}^{T}$ of the single finite element. This nucleus is a column vector with dimensions $3 \times 1$ :

$$
\mathbf{B}^{\tau m i t} T=\left\{\begin{array}{l}
B_{1}^{\tau m i t}  \tag{13.158}\\
B_{2}^{\tau m i t} \\
B_{3}^{\tau m i t}
\end{array}\right\}=\left\{\begin{array}{l}
-\int_{\Omega} N_{i}^{U} N_{t}^{P} F_{\tau, x}^{U} F_{m}^{P} \mathrm{~d} \Omega \\
-\int_{\Omega} N_{i, y}^{U} N_{t}^{P} F_{\tau}^{U} F_{m}^{P} \mathrm{~d} \Omega \\
-\int_{\Omega} N_{i}^{U} N_{t}^{P} F_{\tau, z}^{U} F_{m}^{P} \mathrm{~d} \Omega
\end{array}\right\}
$$

As previously mentioned, it is reminded that, for the moment, the present procedure refers to a single finite element; i.e. here the assembly procedure is not yet considered. As a
consequence, the integration in Eq. 13.158 is performed over the volume corresponding to the domain of a single finite element. By definition, the cross-section related to the single one-dimensional finite element is considered to be constant over the element length. Therefore, the integral over the volume in Eq. 13.158 and in the following equations is split into the integral over the cross-section $\Gamma_{S}$ and the integral along the axis of the one-dimensional finite element, which has length $L_{\text {EL }}$, according to Eq. 13.142.

As done for the computation of the fundamental nucleus $\underline{\mathbf{A}}^{\tau s i j}$, the cross-section functions $F_{m}^{P}, F_{\tau}^{U}$ and their derivatives can be taken out of the integral over the crosssection, since tey depend on $x$ and $z$. The shape functions $N_{i}^{U}, N_{t}^{P}$ and their derivatives can be instead taken out of the integral along the axis of the one-dimensional finite element. Hence, it is convenient to split the terms in 13.158 into two different contributions to be multiplied. The integrals of the products of shape functions along the element length in Eq. 13.158 are collected in the following terms:

$$
\begin{equation*}
E_{t}^{i U P}=\int_{l} N_{i}^{U} N_{t}^{P} \mathrm{~d} y \quad E_{t}^{i, y^{U P}}=\int_{l} N_{i, y}^{U} N_{t}^{P} \mathrm{~d} y \tag{13.159}
\end{equation*}
$$

The three integrals of the products of cross-section functions and their derivatives over $\Gamma_{S}$ are introduced:

$$
\begin{equation*}
J_{m}^{\tau U P}=\int_{\Gamma_{S}} F_{\tau}^{U} F_{m}^{P} \mathrm{~d} \Gamma \quad J_{m}^{\tau, x U P}=\int_{\Gamma_{S}} F_{\tau, x}^{U} F_{m}^{P} \mathrm{~d} \Gamma \quad J_{m}^{\tau, z U P}=\int_{\Gamma_{S}} F_{\tau, z}^{U} F_{m}^{P} \mathrm{~d} \Gamma \tag{13.160}
\end{equation*}
$$

The three components of the fundamental nucleus $\mathbf{B}^{\tau m i t}{ }^{T}$ can be computed explicitly:

$$
\begin{align*}
B_{x}^{\tau m i t^{T}} & =B_{1}^{\tau m i t}=-E_{t}^{i U P} J_{m}^{\tau, x U P} \\
B_{y}^{\tau m i t^{T}} & =B_{2}^{\tau m i t}=-E_{t}^{i, y^{U P}} J_{m}^{\tau U P}  \tag{13.161}\\
B_{z}^{\tau m i t^{T}} & =B_{3}^{\tau m i t}=-E_{t}^{i U P} J_{m}^{\tau, z U P}
\end{align*}
$$

$\operatorname{Term} b\left(\mathbf{u}_{h}, \phi_{m t}\right)$
The third contribution in Eqs. 13.110 and 13.111, which are equivalent expressions to the Galerkin approximation (Eqs. 13.90 and 13.91) of the Stokes problem, is the bilinear form:

$$
\begin{equation*}
b\left(\mathbf{u}_{h}, \phi_{m t}\right)=-\int_{\Omega} \phi_{m t} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega \tag{13.162}
\end{equation*}
$$

For the sake of completeness, it comes from the bilinear form in Eq. 13.91 (i.e. 13.90) and from the choice of the basis functions $\phi_{m t}$ of the space $Q_{h}$ containing the discrete test functions $q_{h}$, according to the 1D CUF FE model (Eq. 13.106):

$$
\begin{equation*}
b\left(\mathbf{u}_{h}, q_{h}\right)=-\int_{\Omega} q_{h} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega \tag{13.163}
\end{equation*}
$$

For the sake of convenience, the discrete solution $\mathbf{u}_{h}$ is now written with indices $s$ and $j$ different from indices $\tau$ and $i$ used in the 1D CUF FE approximation in Eq. 13.98:

$$
\begin{equation*}
\mathbf{u}_{h}=\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k} \boldsymbol{\varphi}_{s j k} \tag{13.164}
\end{equation*}
$$

This choice, already made for the bilinear form $a\left(\mathbf{u}_{h}, \boldsymbol{\varphi}_{\tau i e}\right)$, is formally convenient and is made purely for the sake of exposition simplicity. Using Eq. 13.99, the divergence of $\mathbf{u}_{h}$ is:

$$
\begin{align*}
\operatorname{div} \mathbf{u}_{h} & =\operatorname{div}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[q_{s j x} \boldsymbol{\varphi}_{s j 1}+q_{s j y} \boldsymbol{\varphi}_{s j 2}+q_{s j z} \boldsymbol{\varphi}_{s j 3}\right]\right]  \tag{13.165}\\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\left(\operatorname{div} \boldsymbol{\varphi}_{s j 1}\right) q_{s j x}+\left(\operatorname{div} \boldsymbol{\varphi}_{s j 2}\right) q_{s j y}+\left(\operatorname{div} \boldsymbol{\varphi}_{s j 3}\right) q_{s j z}\right]
\end{align*}
$$

where the linear property of the divergence operator has been used (see Eq. 13.27). The divergence of the basis functions $\boldsymbol{\varphi}_{s j 1}, \boldsymbol{\varphi}_{s j 2}$, and $\boldsymbol{\varphi}_{s j 3}$ have been previously computed in Eqs. $13.150,13.151$ and 13.152 , respectively. The only difference in these equations is the formal use of different indices $\tau$ and $i$ instead of indices $s$ and $j$. Thus:

$$
\begin{align*}
\operatorname{div} \mathbf{u}_{h} & =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[F_{s, x}^{U} N_{j}^{U} q_{s j x}+F_{s}^{U} N_{j, y}^{U} q_{s j y}+F_{s, z}^{U} N_{j}^{U} q_{s j z}\right] \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left\{\begin{array}{lll}
F_{s, x}^{U} N_{j}^{U} & F_{s}^{U} N_{j, y}^{U} & F_{s, z}^{U} N_{j}^{U}
\end{array}\right\}\left\{\begin{array}{c}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\} \tag{13.166}
\end{align*}
$$

As far as the basis functions $\phi_{m t}$ of the discrete test funtions space $Q_{h}$ are concerned, Eq. 13.107 is here retrieved:

$$
\begin{equation*}
\phi_{m t}=F_{m}^{P} N_{t}^{P} \tag{13.167}
\end{equation*}
$$

Substituting Eqs. 13.166 and 13.167 into Eq. 13.162 , the bilinear form becomes $(\forall m, \forall t)$ :

$$
\begin{align*}
&-\int_{\Omega} \phi_{m t} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega=-\int_{\Omega} {\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left\{\begin{array}{llll}
F_{s, x}^{U} N_{j}^{U} & F_{s}^{U} N_{j, y}^{U} & \left.F_{s, z}^{U} N_{j}^{U}\right\}
\end{array}\right\}\left\{\begin{array}{l}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\}\right] } \\
& F_{m}^{P} N_{t}^{P} \mathrm{~d} \Omega \tag{13.168}
\end{align*}
$$

Equation 13.168 can be written in a compact vectorial notation and it is reminded that the following equation has to be expanded for all the value of indices $m$ and $t$ according to Eq. 13.110 (i.e. in Eq. 13.111):

$$
\begin{align*}
b\left(\mathbf{u}_{h}, \phi_{m t}\right)= & -\int_{\Omega} \phi_{m t} \operatorname{div} \mathbf{u}_{h} \mathrm{~d} \Omega \\
= & \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left\{-\int_{\Omega} N_{t}^{P} N_{j}^{U} F_{m}^{P} F_{s, x}^{U} \mathrm{~d} \Omega \quad-\int_{\Omega} N_{t}^{P} N_{j, y}^{U} F_{m}^{P} F_{s}^{U} \mathrm{~d} \Omega\right. \\
& \left.\quad-\int_{\Omega} N_{t}^{P} N_{j}^{U} F_{m}^{P} F_{s, z}^{U} \mathrm{~d} \Omega\right\}\left\{\begin{array}{l}
q_{s j x} \\
q_{s j y} \\
q_{s j z}
\end{array}\right\}  \tag{13.169}\\
= & \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \mathbf{B}^{m s t j} \mathbf{q}_{s j}
\end{align*}
$$

where $\mathbf{B}^{m s t j}$ is the fundamental nucleus of Matrix $\underline{\mathbf{B}}$ related to the bilinear form $b\left(\mathbf{u}_{h}, \phi_{m t}\right)$ of the 1D CUF FE model. This fundamental nucleus multiplies the nodal velocity vector $\mathbf{q}_{s j}$, which is related to the generic $s^{t h}$ velocity expansion term of CUF formulation corresponding to the $j^{\text {th }}$ node of the single finite element, see Eq. 13.98. The fundamental nucleus will be expanded with respect to indices $m, s, t$, and $j$ in order to build Matrix $\underline{\mathbf{B}}$ of the single finite element. This nucleus is a row vector with dimensions $1 \times 3$ :

$$
\mathbf{B}^{m s t j}=\left\{\begin{array}{c}
B_{x}^{m s t j}  \tag{13.170}\\
B_{y}^{m s t j} \\
B_{z}^{m s t j}
\end{array}\right\}^{T}=\left\{\begin{array}{l}
-\int_{\Omega} N_{t}^{P} N_{j}^{U} F_{m}^{P} F_{s, x}^{U} \mathrm{~d} \Omega \\
-\int_{\Omega} N_{t}^{P} N_{j, y}^{U} F_{m}^{P} F_{s}^{U} \mathrm{~d} \Omega \\
-\int_{\Omega} N_{t}^{P} N_{j}^{U} F_{m}^{P} F_{s, z}^{U} \mathrm{~d} \Omega
\end{array}\right\}
$$

As previously mentioned, it is reminded that, for the moment, the present procedure refers to a single finite element; i.e. here the assembly procedure is not yet considered. As a consequence, the integration in Eq. 13.170 is performed over the volume corresponding to the domain of a single finite element. By definition, the cross-section related to the single one-dimensional finite element is considered to be constant over the element length. Therefore, the integral over the volume in Eq. 13.170 and in the following equations is split into the integral over the cross-section $\Gamma_{S}$ and the integral along the axis of the one-dimensional finite element, which has length $L_{\text {EL }}$, according to Eq. 13.142.

As done for the computation of the fundamental nuclei $\underline{\mathbf{A}}^{\tau s i j}$ and $\mathbf{B}^{\tau m i t} T$, the crosssection functions $F_{m}^{P}, F_{s}^{U}$ and their derivatives can be taken out of the integral over the cross-section, since tey depend on $x$ and $z$. The shape functions $N_{j}^{U}, N_{m}^{P}$ and their derivatives can be instead taken out of the integral along the axis of the one-dimensional finite element. Hence, it is convenient to split the terms in 13.170 into two different contributions to be multiplied. The integrals of the products of shape functions along the element length in Eq. 13.170 are collected in the following terms:

$$
\begin{equation*}
E_{t}^{j P U}=\int_{l} N_{t}^{P} N_{j}^{U} \mathrm{~d} y \quad E_{t}^{j, y P U}=\int_{l} N_{t}^{P} N_{j, y}^{U} \mathrm{~d} y \tag{13.171}
\end{equation*}
$$

The three integrals of the products of cross-section functions and their derivatives over $\Gamma_{S}$ are introduced:

$$
\begin{equation*}
J_{m}^{s P U}=\int_{\Gamma_{S}} F_{s}^{U} F_{m}^{P} \mathrm{~d} \Gamma \quad J_{m}^{s, x P U}=\int_{\Gamma_{S}} F_{s, x}^{U} F_{m}^{P} \mathrm{~d} \Gamma \quad J_{m}^{s, z P U}=\int_{\Gamma_{S}} F_{s, z}^{U} F_{m}^{P} \mathrm{~d} \Gamma \tag{13.172}
\end{equation*}
$$

The three components of the fundamental nucleus $\mathbf{B}^{m s t j}$ can be computed explicitly:

$$
\begin{align*}
B_{x}^{m s t j} & =-E_{t}^{j P U} J_{m}^{s, x P U} \\
B_{y}^{m s t j} & =-E_{t}^{j, y P U} J_{m}^{s P U}  \tag{13.173}\\
B_{z}^{m s t j} & =-E_{t}^{j P U} J_{m}^{s, z P U}
\end{align*}
$$

## $\operatorname{Term}\left(\mathbf{f}, \varphi_{\tau i e}\right)$

The fourth contribution in Eqs. 13.110 and 13.111, which are equivalent expressions to the Galerkin approximation (Eqs. 13.90 and 13.91) of the Stokes problem, is the term:

$$
\begin{equation*}
\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i e}\right)=\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega \tag{13.174}
\end{equation*}
$$

For the sake of completeness, it comes from the term in Eq. 13.91 (i.e. Eq. 13.90)

$$
\begin{equation*}
\left(\mathbf{f}, \mathbf{v}_{h}\right)=\int_{\Omega} \mathbf{f} \cdot \mathbf{v}_{h} \mathrm{~d} \Omega \tag{13.175}
\end{equation*}
$$

and from the choice of the basis functions $\boldsymbol{\varphi}_{\text {rie }}$ of the space $V_{h}$ containing the discrete test functions $\mathbf{v}_{h}$, according to the 1D CUF FE model (Eq. 13.99). Equation 13.174 is reformulated exploiting the scalar product as follows:

$$
\begin{equation*}
\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i e}\right)=\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Omega=\int_{\Omega} \boldsymbol{\varphi}_{\tau i e} \cdot \mathbf{f} \mathrm{~d} \Omega=\int_{\Omega} \boldsymbol{\varphi}_{\tau i e}^{T} \mathbf{f} \mathrm{~d} \Omega \tag{13.176}
\end{equation*}
$$

Retrieving the basis functions chosen for the space $V_{h}$ written in Eq. 13.122, their transpose vectors for $e=1,2,3$ are:

$$
\begin{align*}
\boldsymbol{\varphi}_{\tau i 1}^{T} & =\left\{\begin{array}{lll}
F_{\tau}^{U} N_{i}^{U} & 0 & 0
\end{array}\right\}^{T}  \tag{13.177}\\
\boldsymbol{\varphi}_{\tau i 2}^{T} & =\left\{\begin{array}{lll}
0 & F_{\tau}^{U} N_{i}^{U} & 0
\end{array}\right\}^{T}  \tag{13.178}\\
\boldsymbol{\varphi}_{\tau i 3}^{T} & =\left\{\begin{array}{llll}
0 & 0 & F_{\tau}^{U} & N_{i}^{U}
\end{array}\right\}^{T} \tag{13.179}
\end{align*}
$$

It is remindend that $\mathbf{f}$ is the $3 \times 1$ vector of body forces (per mass unit) applied to the fluid, which have been defined in Eq. 13.1. Its components are referred as $f_{x}, f_{y}, f_{z}$. According to Eq. 13.110 (i.e. in Eq. 13.111), the term ( $\mathbf{f}, \boldsymbol{\varphi}_{\text {Tie }}$ ) has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered. Subtituting Eqs. 13.177-13.179 into Eq. 13.176 and varying index $e$, a compact vectorial notation is obtained:

$$
\left\{\begin{array}{l}
\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i 1}\right)  \tag{13.180}\\
\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i 2}\right) \\
\left(\mathbf{f}, \boldsymbol{\varphi}_{\tau i 3}\right)
\end{array}\right\}=\left\{\begin{array}{c}
\int_{\Omega} \boldsymbol{\varphi}_{\tau i 1}^{T} \mathbf{f} \mathrm{~d} \Omega \\
\int_{\Omega} \boldsymbol{\varphi}_{\tau i 2}^{T} \mathbf{f} \mathrm{~d} \Omega \\
\int_{\Omega} \boldsymbol{\varphi}_{\tau i 3}^{T} \mathbf{f} \mathrm{~d} \Omega
\end{array}\right\}=\left\{\begin{array}{c}
\int_{\Omega} F_{\tau}^{U} N_{i}^{U} f_{x} \mathrm{~d} \Omega \\
\int_{\Omega} F_{\tau}^{U} N_{i}^{U} f_{y} \mathrm{~d} \Omega \\
\int_{\Omega} F_{\tau}^{U} N_{i}^{U} f_{z} \mathrm{~d} \Omega
\end{array}\right\}=\left\{\begin{array}{c}
F_{x}^{\tau i} \\
F_{y}^{\tau i} \\
F_{z}^{\tau i}
\end{array}\right\}=\mathbf{F}^{\tau i}
$$

where $\mathbf{F}^{\tau i}$ is the fundamental nucleus of the Vector of Equivalent Nodal Forces $\mathbf{F}$ related to the term $\left(\mathbf{f}, \boldsymbol{\varphi}_{\text {Tie }}\right)(e=1,2,3)$ of the 1D CUF FE model. This nucleus is written for the generic values of the indices $\tau$ and $i$ related to the cross-section functions $F_{\tau}^{U}$ and the shape functions $N_{i}^{U}$ involved in the basis functions $\varphi_{\text {iie }}$ used to approximate the discrete test function $v_{h}$, see Eq. 13.100. The fundamental nucleus (with dimensions $3 \times 1$ ) will be expanded with respect to indices $\tau$ and $i$ in order to build the Vector of Equivalent Nodal Forces $\mathbf{F}$ related to the single finite element. The quantities $F_{x}^{\tau i}, F_{y}^{\tau i}, F_{z}^{\tau i}$ defined in Eq. 13.180 are the three components of the column vector $\mathbf{F}^{\tau i}$.

In general, the three components of the vector of body forces (per mass unit) applied to the fluid $\mathbf{f}$ can be expressed as functions dependent on the coordinates $(x, y, z)$ of the three-dimensional computational space. As a consequence, in general it is not possible to take these components out of the integral on $\Omega$ in Eq. 13.180.

For the sake of completeness, the case of constant body forces (per mass unit) applied to the fluid on $\Omega$ (i.e. $\mathbf{f}=$ const) is here addressed. As previously mentioned, it is reminded that, for the moment, the present procedure refers to a single finite element; i.e. here the assembly procedure is not yet considered. As done for the computation of the fundamental nuclei $\underline{\mathbf{A}}^{\tau s i j}, \mathbf{B}^{\tau m i t}{ }^{T}$, and $\mathbf{B}^{m s t j}$, the integral on the computational
domain $\Omega$ in Eq. 13.180 can therefore be split into the integral over the cross-section $\Gamma_{S}$ and the integral along the axis of the one-dimensional finite element, which has length $L_{\mathrm{EL}}$, according to Eq. 13.142. The cross-section function $F_{\tau}^{U}$ is integrated over $\Gamma_{S}$ and the following term is defined:

$$
\begin{equation*}
E_{i}^{U}=\int_{l} N_{i}^{U} \mathrm{~d} y \tag{13.181}
\end{equation*}
$$

On the contraty, the shape function $N_{i}^{U}$ is integrated along $y$ and the following term is defined:

$$
\begin{equation*}
J^{\tau U}=\int_{\Gamma_{S}} F_{\tau}^{U} \mathrm{~d} \Gamma \tag{13.182}
\end{equation*}
$$

Finally, for the present simplified case of constant $\mathbf{f}$ (i.e. $f_{x}=$ const, $f_{y}=$ const, $f_{z}=$ const on $\Omega$ ), the three components of the fundamental nucleus $\mathbf{B}^{\tau m i t}{ }^{T}$ can be finally computed as follows:

$$
\begin{align*}
& F_{x}^{\tau i}=E_{i}{ }^{U} J^{\tau U} f_{x} \\
& F_{y}^{\tau i}=E_{i}{ }^{U} J^{\tau U} f_{y}  \tag{13.183}\\
& F_{z}^{\tau i}=E_{i}{ }^{U} J^{\tau U} f_{z}
\end{align*}
$$

### 13.4.5 System of algebraic governing equations

Substituting all the contributions appearing in Eq. 13.110 (i.e. in Eq. 13.111), the discrete solution of the Galerkin approximation of the Stokes problem can be computed solving the following algebraic equations in terms of the fundamental nuclei introduced previously:

$$
\begin{cases}\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}^{\tau s i j} \mathbf{q}_{s j}+\sum_{m=1}^{N_{u}^{P}} \sum_{t=1}^{N_{N}^{P}} \mathbf{B}^{\tau m i t}{ }^{T} p_{m t}=\mathbf{F}^{\tau i} & \forall \tau, \forall i  \tag{13.184}\\ \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \mathbf{B}^{m s t j} \mathbf{q}_{s j}=\mathbf{0} & \forall m, \forall t\end{cases}
$$

As can be seen in Eq. 13.184, the fundamental nuclei $\underline{\mathbf{A}}^{\tau s i j}, \mathbf{B}^{\tau m i t}{ }^{T}, \mathbf{B}^{\text {mstj}}$, and $\mathbf{F}^{\tau i}$ have to be expanded on the indices $\tau, i, m$, and $t$. This expansion leads to the construction of the finite element matrices associated to the Galerkin approximation of the Stokes problem following the scheme depicted in Fig. 13.4.

The construction of Matrix $\underline{\mathbf{A}}$, Matrix $\underline{\mathbf{B}}^{T}$, Matrix $\underline{\mathbf{B}}$, and the Vector of Equivalent Nodal Forces $\mathbf{F}$ to be carried out here is equivalent to the procedure described for solid mechanics in chapter 3. It is very important to bear in mind that the formulation leading to Eq. 13.184 has been written so far considering a single finite element of the mesh used to discretize the computational domain $\Omega$, since Eq. 13.98 refers to the single finite element with $N_{N}^{U}$ nodes and Eq. 13.105 refers to the single finite element with $N_{N}^{P}$ nodes.

When more than one finite element are considered in the 1D mesh used for discretization, the procedure so far described has to be carried out for each finite element. Hence, at first the matrices $\underline{\mathbf{A}}_{\mathrm{EL}}, \underline{\mathbf{B}}_{\mathrm{EL}}^{T}, \underline{\mathbf{B}}_{\mathrm{EL}}$, and vector $\mathbf{F}_{\mathrm{EL}}$ corresponding to the single finite element are constructed following Eq. 13.184 and Fig. 13.4 for each of the $N_{\text {EL }}$ finite elements (similarly to the procedure used to build $\mathbf{K}_{\mathrm{EL}}, \mathbf{M}_{\mathrm{EL}}$, and $\mathbf{F}_{\mathrm{EL}}$ in chapter 3). Then, a typical assembly procedure (equivalent to the procedure used in chapter 3 for matrices $\mathbf{K}$ and $\mathbf{M}$ and vector $\mathbf{F}$, see section 3.4) is followed in order to build Matrix $\underline{\mathbf{A}}$, Matrix $\underline{\mathbf{B}}^{T}$, Matrix B, and the Vector of Equivalent Nodal Forces $\mathbf{F}$ associated to the whole domain,


Figure 13.4: Procedure to build the final system of equations and the finite element matrices and vectors expanding the fundamental nuclei.
i.e all the $N_{\text {EL }}$ elements of the mesh, starting from the single element matrices and vectors $\underline{\mathbf{A}}_{\mathrm{EL}}, \underline{\mathbf{B}}_{\mathrm{EL}}^{T}, \underline{\mathbf{B}}_{\mathrm{EL}}$, and $\mathbf{F}_{\mathrm{EL}}$. For more details see section 13.4.7.

Expanding Eq. 13.184 for all the values of indices $\tau, i, m$, and $t$ and for all the finite elements and considering the finite element matrices and vectors above introduced, the final system of equations is, according to Fig. 13.4, as follows:

$$
\left\{\begin{array}{l}
\underline{\mathbf{A}} \mathbf{q}+\underline{\mathbf{B}}^{T} \mathbf{p}=\mathbf{F}  \tag{13.185}\\
\underline{\mathbf{B}} \mathbf{q}=\mathbf{0}
\end{array}\right.
$$

It is interesting to note that Matrix $\underline{\mathbf{B}}^{T}$, which comes from the fundamental nucleus $\mathbf{B}^{\tau m i t}{ }^{T}$, is exactly the transpose matrix of Matrix $\underline{\mathbf{B}}$, which comes from the fundamental nucleus $\mathbf{B}^{m s t j}$, thanks to the following relation between the nuclei:

$$
\begin{equation*}
\mathbf{B}^{\tau m i t}=\mathbf{B}^{m s t j^{T}} \tag{13.186}
\end{equation*}
$$

which is formally true aside from the use of different indices. The system of Eq. 13.185 can be written collecting Matrices $\underline{\mathbf{A}}, \underline{\mathbf{B}}^{T}$, and $\underline{\mathbf{B}}$ in a single symmetric matrix $\underline{\mathbf{S}}$, collecting the unknowns $\mathbf{q}$ and $\mathbf{p}$ in a single vector of unknowns $\mathbf{q}^{\star}$, and collecting the column vectors $\mathbf{F}$ and $\mathbf{0}$ in a single column vector $\mathbf{F}^{\star}$ following the scheme in Fig. 13.5.

The final condensed system of equations obtained without including boundary conditions is thus:

$$
\begin{equation*}
\underline{\mathbf{S}} \mathbf{q}^{\star}=\mathbf{F}^{\star} \tag{13.187}
\end{equation*}
$$



Figure 13.5: Scheme of finite element matrices and vectors collected in the final condensed system of equations, without including boundary conditions.

### 13.4.6 Boundary conditions

The final system and the final condensed system of equations written in Eqs. 13.185 and 13.187 do not include boundary conditions prescribed on the boundary of the threedimensional computational domain. This section shortly presents how to impose boundary
conditions to the governing equations corresponding to the Galerkin approximation of the Stokes problem through the 1D CUF FE model.

According to the 1D CUF approach previously described, the boundary of the computational domain $(\partial \Omega)$ can be split into boundary cross-sections and lateral surface, see Fig. 13.1. The boundary cross-section is indicated with $\Gamma_{D}^{\text {cs }}$ if a Dirichlet boundary condition is prescribed on it. On the contrary, the boundary cross-section is indicated with $\Gamma_{N}^{\text {cs }}$ if a Neumann boundary condition is prescribed on it. In a similar way, the lateral surface is indicated with $\Gamma_{D}^{1}$ or $\Gamma_{N}^{1}$ if a Dirichlet or a Neumann boundary condition is prescribed on it, respectively. As a consequence, $\Gamma_{D}=\Gamma_{D}^{\text {cs }} \cup \Gamma_{D}^{1}$ and $\Gamma_{N}=\Gamma_{N}^{\text {cs }} \cup \Gamma_{N}^{1}$.

An example is given for the simplified pipe in Fig. 13.6, where the two boundary cross-sections are defined as inlet and outlet sections depending on the fluid flow direction. For this pipe, a Dirichlet boundary condition is applied on the inlet section (hence indicated with $\Gamma_{D}^{\mathrm{in}}$ ), a Neumann boundary condition is applied on the outlet section (hence indicated with $\Gamma_{N}^{\text {out }}$ ) and a Dirichlet boundary condition is applied on the lateral surface (hence indicated with $\Gamma_{D}^{1}$ ).


Figure 13.6: Pipe with a circular cross-section. Inlet, outlet and lateral surfaces.
Homogeneous and nonhomogeneous Dirichlet boundary conditions on boundary crosssections or lateral surface are taken into account in this section, whereas for the sake of brevity, nonhomogeneous Neumannn boundary conditions are not considered here. As previously mentioned, homogeneous Neumann boundary conditions are instead "naturally" (implicitly) satisfied by the solution of the problem in Eq. 13.185.

## Dirichlet boundary conditions on cross-sections (inlet and outlet)

A Dirichlet boundary condition is prescribed on a boundary cross-section $\Gamma_{D}^{\text {cs }}$ at $y=y_{g}$ when the velocity profile, i.e. the trend of the components of the velocity vector $\mathbf{u}$, is given a priori over this section. The generic nonhomogeneous Dirichlet boundary condition is indicated as follows:

$$
\begin{equation*}
\left.\mathbf{u}\right|_{\Gamma_{D}^{c s}}=\mathbf{g}^{D} \tag{13.188}
\end{equation*}
$$

where $\mathbf{g}^{D}$ is an arbitrary two-dimensional known function $\mathbf{g}^{D}(x, z)$ with three components along $x, y$, and $z$ axes:

$$
\mathbf{g}^{D}=\left\{\begin{array}{l}
g_{x}^{D}  \tag{13.189}\\
g_{y}^{D} \\
g_{z}^{D}
\end{array}\right\}
$$

Although the boundary cross-section $\Gamma_{D}^{\text {cs }}$ can be either the inlet $\left(y_{g}=0\right)$ or the outlet $\left(y_{g}=L\right)$ section as depicted in Fig. 13.6, the following formulation of the Dirichlet boundary condition (Eq. 13.188) imposition is written in a more general way such that it can be
followed to impose additional Dirichlet velocity conditions even over a cross-section different from the inlet and outlet sections $\left(y_{g} \neq 0, y_{g} \neq L\right)$.

According to the 1D CUF FE velocity discretization introduced in Eq. 13.98, the discrete velocity field $\mathbf{u}_{h}$ over $\Gamma_{D}^{\text {cs }}$ (i.e. at $y=y_{g}$ ) is written as:

$$
\mathbf{u}_{h}\left(x, y_{g}, z\right)=F_{s}^{U}(x, z) N_{j}^{U}\left(y_{g}\right) \mathbf{q}_{s j} \quad \begin{align*}
& s=1, \ldots, N_{u}^{U}  \tag{13.190}\\
& j=1, \ldots, N_{N}^{U}
\end{align*}
$$

where repeated subscripts $s$ and $j$ indicate summation based on Einstein's notation and replace subscripts $\tau$ and $i$ purely for the sake of exposition convenience. Index $s$ corresponds to the $N_{u}^{U}$ cross-section functions whereas index $j$ corresponds to the $N_{N}^{U}$ shape functions over the single finite element $\mathrm{EL}_{\mathrm{cs}}$ of the mesh used to discretize the computational domain $\Omega$ which passes through the cross-section $\Gamma_{D}^{\text {cs }}$ (i.e. at $y=y_{g}$ ) over which the condition is imposed.

Hence, the boundary condition imposition in Eq. 13.188 becomes an imposition on the nodal velocity vectors $\mathbf{q}_{s j}$ for the single finite element $E L_{\text {cs }}$. These vectors are computed by looking for the orthogonal projection of $\mathbf{g}^{D}$ over the space $V_{h}$ due to the 1D CUF approximation. As will be clear in the following section about the boundary conditions on the lateral surface, the orthogonal projection method is equivalent to the treatment of the Dirichlet boundary in a weak form. As a consequence:

$$
\begin{align*}
\left(\left[\mathbf{g}^{D}-\mathbf{u}_{h}\right] \mid \varphi_{\text {rie }}\right)=0 \quad \text { on } \Gamma_{D}^{\text {cs }} \subset \Gamma_{D} & \forall i=1, \ldots, N_{N}^{U}  \tag{13.191}\\
& \forall e=1, \ldots, 3
\end{align*}
$$

where the scalar product $(\mid)$ of two generic vector functions $\mathbf{f}$ and $\mathbf{g}$ is defined as:

$$
\begin{equation*}
(\mathbf{f} \mid \mathbf{g})=\int_{\Gamma_{D}^{c s}} \mathbf{f} \cdot \mathbf{g} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{c s}} \mathbf{f}^{T} \mathbf{g} \mathrm{~d} \Gamma \tag{13.192}
\end{equation*}
$$

For the sake of completeness, the expression in Eq. 13.192 is equivalent to those of Eqs. 13.174 and 13.175. The imposition in Eq. 13.191 has to be written for each basis function $\varphi_{\text {Tie }}$ of the space $V_{h}$ due to the 1D CUF approximation for the element $\mathrm{EL}_{\mathrm{cs}}$. These basis functions have been previously defined in Eq. 13.100, i.e. Eq. 13.122. A system of equations has therefore to be written for all $e$ indices (correspoding to the three directions $x, y$, and $z$ ), for all $\tau$ indices (corresponding to the $N_{u}^{U}$ cross-section functions), and for all $i$ indices (corresponding to the $N_{N}^{U}$ shape functions over the element $E L_{\mathrm{cs}}$ ).

The 1D CUF FE approximation for $\mathbf{u}_{h}$ in Eq. 13.190 is now written explicitly with the summation operators on indices $s$ and $j$ and by means of the bases $\boldsymbol{\varphi}_{s j k}$ of the space $V_{h}$ due to the 1D CUF approximation:

$$
\begin{equation*}
\mathbf{u}_{h}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j}=\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k} \boldsymbol{\varphi}_{s j k} \tag{13.193}
\end{equation*}
$$

Substituting Eq. 13.193 into Eq. 13.191, it is obtained:

$$
\left(\left[\mathbf{g}^{D}-\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k} \boldsymbol{\varphi}_{s j k}\right] \mid \varphi_{\tau i e}\right)=0 \quad \text { on } \Gamma_{D}^{\mathrm{cs}} \subset \Gamma_{D} \quad \begin{align*}
& \forall \tau=1, \ldots, N_{u}^{U}  \tag{13.194}\\
& \\
& \forall i=1, \ldots, N_{N}^{U} \\
& \\
& \forall e=1, \ldots, 3
\end{align*}
$$

The left hand side term can be split into two terms and Eq. 13.194 becomes:

$$
\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k}\left(\boldsymbol{\varphi}_{s j k} \mid \boldsymbol{\varphi}_{\text {rie }}\right)=\left(\mathrm{g}^{D} \mid \boldsymbol{\varphi}_{\text {rie }}\right) \quad \text { on } \Gamma_{D}^{\mathrm{cs}} \subset \Gamma_{D} \quad \begin{align*}
& \forall \tau=1, \ldots, N_{u}^{U}  \tag{13.195}\\
& \\
& \forall i=1, \ldots, N_{N}^{U} \\
& \\
& \forall e=1, \ldots, 3
\end{align*}
$$

The scalar product $\left(\varphi_{s j k} \mid \varphi_{\tau i e}\right)$ over $\Gamma_{D}^{\text {cs }}$ is written as follows:

$$
\begin{align*}
\left(\boldsymbol{\varphi}_{s j k} \mid \varphi_{\tau i e}\right) & =\int_{\Gamma_{D}^{\mathrm{cs}}} \boldsymbol{\varphi}_{s j k} \cdot \varphi_{\tau i e} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{\mathrm{cs}}} \varphi_{\tau i e} \cdot \boldsymbol{\varphi}_{s j k} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{\mathrm{cs}}} \varphi_{\tau i e}^{T} \boldsymbol{\varphi}_{s j k} \mathrm{~d} \Gamma= \\
& =\left\{\begin{array}{ll}
N_{i}^{U}\left(y_{g}\right) N_{j}^{U}\left(y_{g}\right) \int_{\Gamma_{D}^{\mathrm{cs}}} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma & e=k \\
0 & e \neq k
\end{array} \quad \text { on } \Gamma_{D}^{\mathrm{cs}} \subset \Gamma_{D}\right. \tag{13.196}
\end{align*}
$$

The scalar product $\left(\mathrm{g}^{D} \mid \boldsymbol{\varphi}_{\tau i e}\right)$ is computed for the three directions $x, y$, and $z$ and written in the following compact form:

$$
\begin{align*}
&\left\{\begin{array}{l}
\left(\mathrm{g}^{D} \mid \boldsymbol{\varphi}_{\tau i 1}\right) \\
\left(\mathrm{g}^{D} \mid \boldsymbol{\varphi}_{\tau i 2}\right) \\
\left(\mathrm{g}^{D} \mid \boldsymbol{\varphi}_{\tau i 3}\right)
\end{array}\right\}=\left\{\begin{array}{l}
\int_{\Gamma_{D}^{\mathrm{cs}}} \boldsymbol{\varphi}_{\tau i 1}^{T} \mathbf{g}^{D} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{\mathrm{cs}}} \boldsymbol{\varphi}_{\tau i 2}^{T} \mathbf{g}^{D} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{\mathrm{cs}}} \boldsymbol{\varphi}_{\tau i 3}^{T} \mathbf{g}^{D} \mathrm{~d} \Gamma
\end{array}\right\} \\
&=\left\{\begin{array}{l}
N_{i}^{U}\left(y_{g}\right) \int_{\Gamma_{D}^{\mathrm{cs}}} F_{\tau}^{U} g_{x}^{D} \mathrm{~d} \Gamma \\
N_{i}^{U}\left(y_{g}\right) \int_{\Gamma_{D}^{\mathrm{cs}}} F_{\tau}^{U} g_{y}^{D} \mathrm{~d} \Gamma \\
N_{i}^{U}\left(y_{g}\right) \int_{\Gamma_{D}^{\mathrm{cs}}} F_{\tau}^{U} g_{z}^{D} \mathrm{~d} \Gamma
\end{array}\right\}=\left\{\begin{array}{l}
F_{B C x}^{\mathrm{cs} \tau i} \\
F_{B C y}^{\mathrm{cs} \tau i} \\
F_{B C z}^{\mathrm{cs} \tau i}
\end{array}\right\}=\mathbf{F}_{B C}^{\mathrm{cs} \tau i}  \tag{13.197}\\
& \text { on } \Gamma_{D}^{\mathrm{cs}} \subset \Gamma_{D}
\end{align*}
$$

where $\mathbf{F}_{B C}^{\mathrm{cs} \tau i}$ is the fundamental nucleus of the Vector of Equivalent Nodal Forces $\mathbf{F}_{B C}^{\mathrm{cs}}$ el associated to the nonhomogeneous Dirichlet boundary condition imposed on the crosssection $\Gamma_{D}^{\text {cs }}$. This nucleus has dimension $3 \times 1$ and is written for the generic values of the indices $\tau$ and $i$ related to the cross-section functions $F_{\tau}^{U}$ and the shape functions $N_{i}^{U}$ for the element $\mathrm{EL}_{\text {cs }}$ which passes through $\Gamma_{D}^{\mathrm{cs}}$. The quantities $F_{B C x}^{\mathrm{cs} \tau i}, F_{B C y}^{\mathrm{cs} \tau i}, F_{B C z}^{\mathrm{cs} \tau i}$ defined in Eq. 13.197 are the three components of the column vector $\mathbf{F}_{B C}^{\mathrm{cs} \tau i} .$.

The system of equations in Eq. 13.195 becomes the system of equations in Eq. 13.198, which already includes the expansion over index $e$, by means of Eqs. 13.196 and 13.197:
where $\underline{\mathbf{I}}$ is the $3 \times 3$ identity matrix. Retrieving Eq. $13.145, \underline{\mathbf{A}}_{B C}^{\mathrm{cs} \tau s i j}$, i.e. the fundamental nucleus of Matrix $\underline{\mathbf{A}}_{B C \text { eL }}^{\mathrm{cs}}$ associated to the nonhomogeneous Dirichlet boundary condition
imposed on the cross-section $\Gamma_{D}^{\text {cs }}$, becomes:

$$
\begin{equation*}
\underline{\mathbf{A}}_{B C}^{\mathrm{cs} \tau s i j}=N_{i}^{U}\left(y_{g}\right) N_{j}^{U}\left(y_{g}\right) J_{s}^{\tau U} \underline{\mathbf{I}} \tag{13.199}
\end{equation*}
$$

This nucleus is a diagonal matrix with dimension $3 \times 3$ and in Eq. 13.198 it multiplies the nodal velocity vector $\mathbf{q}_{s j}$, which is related to the generic $s^{t h}$ velocity expansion term of CUF formulation corresponding to the $j^{t h}$ node of the element $\mathrm{EL}_{\mathrm{cs}}$.

The imposition of the nonhomogeneous Dirichlet boundary condition on the cross-section $\Gamma_{D}^{c s}$ is thefore given by the following system of equations:

$$
\begin{equation*}
\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{B C}^{\mathrm{cs} \tau s i j} \mathbf{q}_{s j}=\mathbf{F}_{B C}^{\mathrm{cs} \tau i} \quad \forall \tau=1, \ldots, N_{u}^{U} \tag{13.200}
\end{equation*}
$$

The terms of Eq. 13.200 have to be expanded on the indices $\tau$ and $i$. This expansion leads to the construction of a finite element matrix $\underline{\mathbf{A}}_{B C \text { EL }}^{c s}$ and a finite element Vector of Equivalent Nodal Forces $\mathbf{F}_{B C}^{\mathrm{cs}}$ el associated to the nonhomogeneous Dirichlet boundary condition imposed on the cross-section $\Gamma_{D}^{\mathrm{cs}}$ through which the single finite element $\mathrm{EL}_{\mathrm{cs}}$ passes.

Matrix $\underline{\mathbf{A}}_{B C \text { EL }}^{\mathrm{cs}}$ has dimensions $\mathrm{DOFs}_{\mathrm{EL}}^{U} \times \mathrm{DOFs}_{\mathrm{EL}}^{U}$ as well as Matrix $\underline{\mathbf{A}}_{\mathrm{EL}}$. Matrix $\underline{\mathbf{A}}_{B C \text { EL }}^{\mathrm{cs}}$ is then included into a null matrix $\underline{\mathbf{A}}_{B C}^{\mathrm{cs}}$ (with dimensions DOFs ${ }^{U} \times \mathrm{DOFs}^{U}$ as well as Matrix $\underline{\mathbf{A}}$ ) in the position corresponding to the position of the finite element $\mathrm{EL}_{\mathrm{cs}}$ along the 1 D mesh. In this way, $\underline{\mathbf{A}}_{B C}^{\mathrm{cs}}$ presents different-from-zero terms only in the rows and columns corresponding to the finite element $\mathrm{EL}_{\mathrm{cs}}$. As a consequence, when the Dirichlet boundary condition is prescribed for the inlet cross-section, $\underline{\mathbf{A}}_{B C}^{\mathrm{cs}}$ has different-from-zero terms only in the rows and columns corresponding to the first node $(i, j=1)$ of the first finite element (in Eq. $13.199 N_{i}^{U}\left(y_{g}=0\right)=1$ for $i=1, N_{i}^{U}\left(y_{g}=0\right)=0$ for $i \neq 1$, $N_{j}^{U}\left(y_{g}=0\right)=1$ for $j=1, N_{j}^{U}\left(y_{g}=0\right)=0$ for $\left.j \neq 1\right)$. When the Dirichlet boundary condition is instead prescribed for the outlet cross-section, $\underline{\mathbf{A}}_{B C}^{\mathrm{cs}}$ has different-from-zero terms only in the rows and columns corresponding to the last node $(i, j=2)$ of the last finite element (in Eq. $13.199 N_{i}^{U}\left(y_{g}=L\right)=1$ for $i=2, N_{i}^{U}\left(y_{g}=L\right)=0$ for $i \neq 2$, $N_{j}^{U}\left(y_{g}=L\right)=1$ for $j=2, N_{j}^{U}\left(y_{g}=L\right)=0$ for $\left.j \neq 2\right)$.

Vector $\mathbf{F}_{B C \text { EL }}^{\mathrm{cs}}$ has dimensions DOFs ${ }_{\mathrm{EL}}^{U} \times 1$ as well as Vector $\mathbf{F}_{\mathrm{EL}}$. Vector $\mathbf{F}_{B C \text { EL }}^{\mathrm{cs}}$ is then included into a null vector $\mathbf{F}_{B C}^{\mathrm{cs}}$ (with dimensions DOFs ${ }^{U} \times 1$ as well as Vector $\mathbf{F}$ ) in the position corresponding to the position of the finite element $\mathrm{EL}_{\mathrm{cs}}$ along the 1 D mesh. In this way, $\mathbf{F}_{B C}^{\mathrm{cs}}$ presents different-from-zero terms only in the rows corresponding to the finite element $\mathrm{EL}_{\mathrm{cs}}$. As a consequence, when the Dirichlet boundary condition is prescribed for the inlet cross-section, $\mathbf{F}_{B C}^{\mathrm{cs}}$ has different-from-zero terms only in the rows corresponding to the first node $(i=1)$ of the first finite element (in Eq. $13.198 N_{i}^{U}\left(y_{g}=0\right)=1$ for $i=1$, $N_{i}^{U}\left(y_{g}=0\right)=0$ for $\left.i \neq 1\right)$. When the Dirichlet boundary condition is instead prescribed for the outlet cross-section, $\mathbf{F}_{B C}^{\mathrm{cs}}$ has different-from-zero terms only in the rows corresponding to the last node $(i=2)$ of the last finite element (in Eq. $13.198 N_{i}^{U}\left(y_{g}=L\right)=1$ for $i=2$, $N_{i}^{U}\left(y_{g}=L\right)=0$ for $\left.i \neq 2\right)$.

Finally, the system of equations associated to the imposition of the nonhomogeneous Dirichlet boundary condition on the cross-section $\Gamma_{D}^{\mathrm{cs}}$ is as follows:

$$
\begin{equation*}
\underline{\mathbf{A}}_{B C}^{\mathrm{cs}} \mathbf{q}=\mathbf{F}_{B C}^{\mathrm{cs}} \tag{13.201}
\end{equation*}
$$

It is interesting to note that Matrix $\underline{\mathbf{A}}_{B C}^{\mathrm{cs}}$ is symmetric.

## Dirichlet boundary conditions on the lateral surface

A Dirichlet boundary condition is prescribed on the lateral surface $\Gamma_{D}^{1}$ when the values of the components of the velocity vector $\mathbf{u}$ are given a priori over this surface. The generic nonhomogeneous Dirichlet boundary condition is indicated as follows:

$$
\begin{equation*}
\left.\mathbf{u}\right|_{\Gamma_{D}^{1}}=\mathbf{g}^{L} \tag{13.202}
\end{equation*}
$$

where $\mathbf{g}^{L}$ is an arbitrary three-dimensional known function $\mathbf{g}^{L}(x, y, z)$ with three components along $x, y$, and $z$ axes:

$$
\mathbf{g}^{L}=\left\{\begin{array}{c}
g_{x}^{L}  \tag{13.203}\\
g_{y}^{L} \\
g_{z}^{L}
\end{array}\right\}
$$

The Dirichlet boundary condition (in general nonhomogeneous) on the lateral surface is treated in a weak form, or better, with the corresponding Galerkin approximation consistent with the 1D CUF finite element approach previously described. The Galerkin approximation of the boundary condition in Eq. 13.202 has the following form:

Find $\mathbf{u}_{h} \in V_{h}$ such that

$$
\begin{equation*}
\int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \mathbf{v}_{h} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \mathbf{v}_{h} \mathrm{~d} \Gamma \quad \forall \mathbf{v}_{h} \in V_{h} \tag{13.204}
\end{equation*}
$$

For the sake of convenience, the discrete solution $\mathbf{u}_{h}$ is now written with indices $s$ and $j$ different from indices $\tau$ and $i$ used in the 1D CUF FE approximation in Eq. 13.98, purely for the sake of exposition convenience:

$$
\begin{equation*}
\mathbf{u}_{h}=\sum_{k=1}^{3} \sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} q_{s j k} \boldsymbol{\varphi}_{s j k}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \tag{13.205}
\end{equation*}
$$

According to the 1D CUF approximation for the velocity field (Eqs. 13.98 and 13.99) the generic discrete test function $\mathbf{v}_{h} \in V_{h}$ is reported in Eq. 13.206 (i.e. Eq. 13.108):

$$
\begin{equation*}
\mathbf{v}_{h}=\sum_{e=1}^{3} \sum_{\tau=1}^{N_{u}^{U}} \sum_{i=1}^{N_{N}^{U}} q_{\tau i e}^{\star} \boldsymbol{\varphi}_{\tau i e} \tag{13.206}
\end{equation*}
$$

As done for the formulation of the Galerkin approximation of the Stokes problem, it is sufficient that the Galerkin approximation of the Dirichlet boundary condition in Eq. 13.204 is verified for each function of the basis of $V_{h}$, since all the functions in the space $V_{h}$ are a linear combination of the basis funcions (see Eq. 13.93). Since the functions $\varphi_{\tau i e}$ are the bases chosen for the space $V_{h}$ the following system of equations is thus required to be satisfied:

Find $\mathbf{u}_{h} \in V_{h}$ such that

$$
\int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \varphi_{\tau i e} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Gamma \quad \begin{array}{ll} 
& \forall \tau=1, \ldots, N_{u}^{U}  \tag{13.207}\\
& \forall i=1, \ldots, N_{N}^{U} \\
& \forall e=1, \ldots, 3
\end{array}
$$

Substituting Eq. 13.205 into the left hand side term of Eq. 13.207, it becomes:

$$
\begin{align*}
\int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \boldsymbol{\varphi}_{\tau i e} \mathrm{~d} \Gamma & =\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i e} \cdot \mathbf{u}_{h} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i e}^{T} \mathbf{u}_{h} \mathrm{~d} \Gamma \\
& =\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i e}^{T}\left[\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j}\right] \mathrm{d} \Gamma=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i e}^{T} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma \tag{13.208}
\end{align*}
$$

Let Eq. 13.208 to be written explicitly by varying the index $e$ :

$$
\begin{align*}
& \int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \boldsymbol{\varphi}_{\tau i 1} \mathrm{~d} \Gamma=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 1}^{T} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}}\left\{F_{\tau}^{U} N_{i}^{U} \quad 0 \quad 0\right\} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma  \tag{13.209}\\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\int_{\Gamma_{D}^{1}} N_{i}^{U} N_{j}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma \quad 0 \quad 0\right.}_{=\mathbf{A}_{1 B C}^{1+\tau i j}}\} \\
& \int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \varphi_{\tau i 2} \mathrm{~d} \Gamma=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 2}^{T} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}}\left\{\begin{array}{lll}
0 & F_{\tau}^{U} N_{i}^{U} & 0
\end{array}\right\} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma  \tag{13.210}\\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & \int_{\Gamma_{D}^{1}} N_{i}^{U} N_{j}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma \quad 0
\end{array}\right\}}_{=\mathbf{A}_{2 B C}^{1+s i j}} \mathbf{q}_{s j} \\
& \int_{\Gamma_{D}^{1}} \mathbf{u}_{h} \cdot \boldsymbol{\varphi}_{\tau i 3} \mathrm{~d} \Gamma=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 3}^{T} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma \\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \int_{\Gamma_{D}^{1}}\left\{\begin{array}{lll}
0 & 0 & F_{\tau}^{U} N_{i}^{U}
\end{array}\right\} F_{s}^{U} N_{j}^{U} \mathbf{q}_{s j} \mathrm{~d} \Gamma  \tag{13.211}\\
& =\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underbrace{\left\{\begin{array}{lll}
0 & 0 & \int_{\Gamma_{D}^{1}} N_{i}^{U} N_{j}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma
\end{array}\right\}}_{=\mathbf{A}_{3}^{1} \tau s C} \mathbf{q}_{s j}
\end{align*}
$$

The quantities $\mathbf{A}_{1 B C}^{1 \tau s i j}, \mathbf{A}_{2 B C}^{1 \tau s i j}, \mathbf{A}_{3 B C}^{1 \tau s i j}$ are three row vectors with dimensions $1 \times 3$ which relate the integrals over the lateral surface for $e=1,2,3$ and the generic $\tau$ and $i$ to the nodal velocity vector $\mathbf{q}_{s j}$. Finally, the matrix $\underline{\mathbf{A}}_{B C}^{1 \tau s i j}$, i.e. the fundamental nucleus of Matrix $\underline{\mathbf{A}}_{B C}^{1}$ associated to the nonhomogeneous Dirichlet boundary condition imposed on
the lateral surface $\Gamma_{D}^{1}$, can be defined by exploiting a compact notation $(\forall \tau, \forall i)$ :

$$
\int_{\Gamma_{D}^{1}}\left\{\begin{array}{l}
\mathbf{u}_{h} \cdot \boldsymbol{\varphi}_{\tau i 1}  \tag{13.212}\\
\mathbf{u}_{h} \cdot \varphi_{\tau i 2} \\
\mathbf{u}_{h} \cdot \varphi_{\tau i 3}
\end{array}\right\} \mathrm{d} \Gamma=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}}\left[\begin{array}{c}
\mathbf{A}_{1 B C}^{1 \tau s i j} \\
\mathbf{A}_{2 B C}^{1 \tau s i j} \\
\mathbf{A}_{3 B C}^{1 \tau s i j}
\end{array}\right] \mathbf{q}_{s j}=\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{B C}^{1 \tau s i j} \mathbf{q}_{s j}
$$

This nucleus is a diagonal matrix with dimension $3 \times 3$ and in Eq. 13.212 it multiplies the nodal velocity vector $\mathbf{q}_{s j}$, which is related to the generic $s^{t h}$ velocity expansion term of CUF formulation corresponding to the $j^{\text {th }}$ node of the single finite element, see Eq. 13.98. Introducing the $3 \times 3$ identity matrix $\mathbf{I}$, it is obtained:

$$
\begin{equation*}
\underline{\mathbf{A}}_{B C}^{1 \tau s i j}=\int_{\Gamma_{D}^{1}} N_{i}^{U} N_{j}^{U} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \Gamma \underline{\mathbf{I}} \tag{13.213}
\end{equation*}
$$

The present procedure refers to a single finite element; i.e. here the assembly procedure is not yet considered. The fundamental nucleus $\underline{\mathbf{A}}_{B C}^{1 \tau s i j}$ will be expanded with respect to indices $\tau, s, i$, and $j$ in order to build Matrix $\underline{\mathbf{A}}_{B C}^{1}$ of the single finite element.

The integration in Eq. 13.213 is performed over the lateral surface $\Gamma_{D}^{1}$ corresponding to a single finite element. By definition, the cross-section related to the single one-dimensional finite element is considered to be constant over the element length. Therefore, the integral over the lateral surface in Eq. 13.213 and in the following equations is split into the integral along the contour of the cross-section $\gamma_{S}$ and the integral along the axis (here indicated generically as $l$ ) of the one-dimensional finite element, which has length $L_{\mathrm{EL}}$ :

$$
\begin{equation*}
\int_{\Gamma_{D}^{1}} \ldots \mathrm{~d} \Gamma=\int_{l} \int_{\gamma_{S}} \ldots \mathrm{~d} \gamma \mathrm{~d} y \tag{13.214}
\end{equation*}
$$

By definition in Eq. 13.94, the cross-section functions $F_{\tau}^{U}$ and $F_{s}^{U}$ depend only on the cross-section coordinates. They can be therefore taken out of the integral along the element length. Similarly, the shape functions of the present one-dimensional finite element model are independent of cross-section coordinates $x$ and $z$, see Eq. 13.102. Hence, in Eq. 13.141 the shape functions are taken out of the integral along the cross-section contour $\gamma_{S}$. For these reasons, it is possible to split the integral along the element length and the integral along the element cross-section contour into two different contributions to be multiplied. The expression of $\underline{\mathbf{A}}_{B C}^{1 \tau s i j}$ can be finally written as follows:

$$
\begin{equation*}
\underline{\mathbf{A}}_{B C}^{1 \tau s i j}=\int_{l} N_{i}^{U} N_{j}^{U} \mathrm{~d} y \int_{\gamma_{S}} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \gamma \underline{\mathbf{I}}=E_{j}^{i U} \int_{\gamma_{S}} F_{\tau}^{U} F_{s}^{U} \mathrm{~d} \gamma \underline{\mathbf{I}} \tag{13.215}
\end{equation*}
$$

where Eq. 13.144 has been used. Let the right hand side term of Eq. 13.207 to be evaluated:

$$
\begin{equation*}
\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \varphi_{\tau i e} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{1}} \varphi_{\tau i e} \cdot \mathbf{g}^{L} \mathrm{~d} \Gamma=\int_{\Gamma_{D}^{1}} \varphi_{\tau i e}^{T} \mathbf{g}^{L} \mathrm{~d} \Gamma \tag{13.216}
\end{equation*}
$$

According to Eq. 13.207 (i.e. in Eq. 13.111), the expression of Eq. 13.216 has to be expanded for $\forall \tau, \forall i, \forall e$. Let the expansion on index $e=1,2,3$ to be now considered. Retrieving the basis functions chosen for the space $V_{h}$ written in Eq. 13.122 and varying index $e$, a
compact vectorial notation is obtained:

$$
\left\{\begin{array}{l}
\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \boldsymbol{\varphi}_{\tau i 1} \mathrm{~d} \Gamma  \tag{13.217}\\
\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \boldsymbol{\varphi}_{\tau i 2} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{1}} \mathbf{g}^{L} \cdot \boldsymbol{\varphi}_{\tau i 3} \mathrm{~d} \Gamma
\end{array}\right\}=\left\{\begin{array}{l}
\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 1}^{T} \mathbf{g}^{L} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 2}^{T} \mathbf{g}^{L} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{1}} \boldsymbol{\varphi}_{\tau i 3}^{T} \mathbf{g}^{L} \mathrm{~d} \Gamma
\end{array}\right\}=\left\{\begin{array}{l}
\int_{\Gamma_{D}^{1}} F_{\tau}^{U} N_{i}^{U} g_{x}^{L} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{1}} F_{\tau}^{U} N_{i}^{U} g_{y}^{L} \mathrm{~d} \Gamma \\
\int_{\Gamma_{D}^{1}} F_{\tau}^{U} N_{i}^{U} g_{z}^{L} \mathrm{~d} \Gamma
\end{array}\right\}=\left\{\begin{array}{l}
F_{B C x}^{1 \tau i} \\
F_{B C y}^{1} \tau i \\
F_{B C z}^{1 \tau i}
\end{array}\right\}=\mathbf{F}_{B C}^{1 \tau \tau i}
$$

where $\mathbf{F}_{B C}^{1 \tau i}$ is the fundamental nucleus of the Vector of Equivalent Nodal Forces $\mathbf{F}_{B C}^{1}$ EL associated to the nonhomogeneous Dirichlet boundary condition imposed on the lateral surface $\Gamma_{D}^{1}$. This nucleus has dimension $3 \times 1$ and is written for the generic values of the indices $\tau$ and $i$ related to the cross-section functions $F_{\tau}^{U}$ and the shape functions $N_{i}^{U}$ involved in the basis functions $\varphi_{\text {rie }}$ used to approximate the discrete test function $v_{h}$, see Eq. 13.100. The quantities $F_{B C x}^{1 \tau i}, F_{B C y}^{1 \tau i}, F_{B C z}^{1 \tau i}$ defined in Eq. 13.217 are the three components of the column vector $\mathbf{F}_{B C}^{1}{ }^{1 i}$. In Eq. 13.217, it is important to note that Eq. 13.214 has not been exploited to split the integral over $\Gamma_{D}^{1}$, since $\mathbf{g}^{L}$ is an arbitrary three-dimensional known function $\mathbf{g}^{L}(x, y, z)$.

The imposition of the nonhomogeneous Dirichlet boundary condition on the the lateral surface $\Gamma_{D}^{1}$ is thefore given by the following system of equations:

$$
\sum_{s=1}^{N_{u}^{U}} \sum_{j=1}^{N_{N}^{U}} \underline{\mathbf{A}}_{B C}^{1 \tau s i j} \mathbf{q}_{s j}=\mathbf{F}_{B C}^{1 \tau i} \quad \begin{array}{ll}
1 / 2 & \forall \tau=1, \ldots, N_{u}^{U}  \tag{13.218}\\
& \forall i=1, \ldots, N_{N}^{U}
\end{array}
$$

The terms of Eq. 13.218 have to be expanded on the indices $\tau$ and $i$. It is very important to bear in mind that the formulation leading to Eq. 13.184 has been written so far considering a single finite element of the mesh used to discretize the computational domain $\Omega$, since Eq. 13.98 refers to the single finite element with $N_{N}^{U}$ nodes.

When more than one finite element are considered in the 1D mesh used for discretization, the procedure so far described has to be carried out for each finite element. Hence, at first Matrix $\underline{\mathbf{A}}_{B C \text { EL }}^{1}$ (with dimensions $\mathrm{DOFs}_{\mathrm{EL}}^{U} \times \mathrm{DOFs}_{\mathrm{EL}}^{U}$ as well as Matrix $\underline{\mathbf{A}}_{\mathrm{EL}}$ ) and Vector $\mathbf{F}_{B C \text { EL }}^{\mathrm{l}}$ (with dimensions DOFs ${ }_{\mathrm{EL}}^{U} \times 1$ as well as Vector $\mathbf{F}_{\text {EL }}$ ) corresponding to the single finite element are constructed following Eq. 13.218 for each of the $N_{\text {EL }}$ finite elements (similarly to the procedure used to build $\underline{\mathbf{A}}_{\mathrm{EL}}$ and $\mathbf{F}_{\mathrm{EL}}$ in subsection 13.4.5). Then, a typical assembly procedure (equivalent to the procedure used in chapter 3 for matrices $\mathbf{K}$ and $\mathbf{M}$ and vector $\mathbf{F}$, see section 3.4) is followed in order to build Matrix $\underline{\mathbf{A}}_{B C}^{1}$ (with dimensions DOFs ${ }^{U} \times$ DOFs $^{U}$ as well as Matrix $\underline{\mathbf{A}}$ ) and Vector of Equivalent Nodal Forces $\mathbf{F}_{B C}^{1}$ (with dimensions DOFs ${ }^{U} \times 1$ as well as Vector $\mathbf{F}$ ) associated to the nonhomogeneous Dirichlet boundary condition imposed on the lateral surface $\Gamma_{D}^{1}$, starting from the single element matrices $\underline{\mathbf{A}}_{B C \text { EL }}^{1}$ and vectors $\mathbf{F}_{B C \text { EL }}^{1}$.

Finally, the system of equations associated to the imposition of the nonhomogeneous Dirichlet boundary condition on the lateral surface $\Gamma_{D}^{1}$ is as follows:

$$
\begin{equation*}
\underline{\mathbf{A}}_{B C}^{\mathrm{l}} \mathbf{q}=\mathbf{F}_{B C}^{\mathrm{l}} \tag{13.219}
\end{equation*}
$$

It is interesting to note that $\underline{\mathbf{A}}_{B C}^{1}$ is a symmetric and singular matrix. The reason on this singularity is due to the fact that the equations in Eq. 13.218 are not all linear indepenedent, since the Dirichlet boundary condition on $\Gamma_{D}^{1}$ does not "block" (properly) the solution over all the computational domain, but only prescribes relations for the nodal velocity vectors $\mathbf{q}_{s j}$ in order to be satisfied on $\Gamma_{D}^{1}$.

It is remarkably noteworthy that the same procedure used here to impose Dirichlet boundary conditions on the lateral surface of the computational domain can be followed also for solid mechanics in chapter 3 to prescribe imposed displacements also on lateral surface (not only on cross-sections) of the structure.

### 13.4.7 System of algebraic governing equations including boundary conditions

The systems of equations associated to the imposition of the nonhomogeneous Dirichlet boundary conditions on the cross-section $\Gamma_{D}^{\mathrm{cs}}$ and on the lateral surface $\Gamma_{D}^{1}$ have been presented in Eqs. 13.201 and 13.219. These equations are imposed to the final system of governing equations associated to the Galerkin approximation of the Stokes problem through the 1D CUF FE model of Eq. 13.185 by means of a penalization method, i.e. via two high penalty values $\alpha$ and $\beta$ :

$$
\left\{\begin{array} { l } 
{ \underline { \mathbf { A } } \mathbf { q } + \underline { \mathbf { B } } ^ { T } \mathbf { p } = \mathbf { F } }  \tag{13.220}\\
{ \underline { \mathbf { A } } _ { B C } ^ { \mathrm { l } } \mathbf { q } = \mathbf { F } _ { B C } ^ { \mathrm { l } } } \\
{ \underline { \mathbf { A } } _ { B C } ^ { \mathrm { cs } } \mathbf { q } = \mathbf { F } _ { B C } ^ { \mathrm { cs } } } \\
{ \underline { \mathbf { B } } \mathbf { q } = \mathbf { 0 } }
\end{array} \Rightarrow \left\{\begin{array}{l}
\underline{\mathbf{A} \mathbf{q}+\alpha \underline{\mathbf{A}}_{B C}^{\mathrm{l}} \mathbf{q}+\beta \underline{\mathbf{A}}_{B C}^{\mathrm{cs}} \mathbf{q}+\underline{\mathbf{B}}^{T} \mathbf{p}=\mathbf{F}+\alpha \mathbf{F}_{B C}^{\mathrm{l}}+\beta \mathbf{F}_{B C}^{\mathrm{cs}}} \begin{array}{l}
\underline{\mathbf{B}} \mathbf{q}=\mathbf{0}
\end{array} \\
\end{array}\right.\right.
$$

Finally, Eq. 13.220 can be written in a more compact notation:

$$
\left\{\begin{array}{l}
{\left[\underline{\mathbf{A}}+\alpha \underline{\mathbf{A}}_{B C}^{1}+\beta \underline{\mathbf{A}}_{B C}^{\mathrm{cs}}\right] \mathbf{q}+\underline{\mathbf{B}}^{T} \mathbf{p}=\left[\mathbf{F}+\alpha \mathbf{F}_{B C}^{\mathrm{l}}+\beta \mathbf{F}_{B C}^{\mathrm{cs}}\right]}  \tag{13.221}\\
\underline{\mathbf{B}} \mathbf{q}=\mathbf{0}
\end{array}\right.
$$

It is interesting to note that Matrix $\left[\underline{\mathbf{A}}+\alpha \underline{\mathbf{A}}_{B C}^{1}+\beta \underline{\mathbf{A}}_{B C}^{\mathrm{cs}}\right]$ is a symmetric, non-singular, positive-definite matrix.

Similarly to the scheme of Fig. 13.5 followed to construct Eq. 13.187, the system of Eq. 13.221 can be solved once collecting the matrices of the left hand side term in a single symmetric matrix $\widetilde{\mathbf{S}}$, collecting the unknowns $\mathbf{q}$ and $\mathbf{p}$ in a single vector of unknowns $\mathbf{q}^{\star}$, and collecting the column vectors of the right hand side term in a single column vector $\widetilde{\mathbf{F}}^{\star}$. As a result, the final condensed system of equations (including the boundary conditions) to be solved is:

$$
\begin{equation*}
\underline{\widetilde{S}} \mathrm{q}^{\star}=\widetilde{\mathbf{F}}^{\star} \tag{13.222}
\end{equation*}
$$

Matrix $\underline{\widetilde{\mathbf{S}}}$ is a block simmetric and indefinite matrix, featuring real eigenvalues with variable sign (either positive and negative). More details about its singularity or non-singularity are given in the following paragraph about the solution stability. The symmetric indefinite $\mathbf{L} \mathbf{D} \mathbf{L}^{T}$ factorization described in section 3.3.1 is used in the present doctoral research to solve the final condensed system of equations in Eq. 13.222 and obtain the fluid dynamics results presented in chapter 14.

## Stability

The key point of the solution stability is now briefly discussed. When the Galerkin approximation of the Stokes equations is solved, it is necessary to choose stable, or compatible, finite dimensional spaces $V_{h}$ and $Q_{h}$. In particular, it means that these spaces have to satisfy the inf-sup condition (see [143, 142]) related to the saddle-point problem described in Eqs. 13.88 and 13.89. As Stokes equations are first order with respect to $p$
and second order with respect to $\mathbf{u}$, generally speaking it makes sense to use polynomials of degree $k \geq 1$ for the velocity space $V_{h}$ and of degree $k-1$ for the space $Q_{h}$. Nonetheless, this "natural" choice does not always work. The larger the velocity space $V_{h}$, the higher the probability that the inf-sup condition be satisfied. Otherwise said, the space $V_{h}$ should be "rich" enough with respect to the space $Q_{h}$. More in general, finite elements of the same polynomial degree $k \geq 1$ for both velocity and pressures are unstable.

Unstable solutions present spurious pressure modes, also known as parasitic modes. Their presence inhibits the pressure solution to be unique, yielding numerical instabilities. For this reason, thise finite dimensional subspaces that violate the compatibility condition, i.e. the inf-sup condition, are said to be unstable, or incompatible. From the inf-sup condition, it follows that Matrix $\underline{\widetilde{\mathbf{S}}}$ is non-singular if and only if no eigenvalue is null. For the proof of the stability results and convergence analysis, see [144]. More details about the stability via the finite element method can be found in [142].

## Degrees of freedom

For the sake of completeness, a discussion of the dimensions of the finite element matrices and vector introduced so far and the degrees of freedom involved is faced here. As mentioned above and depicted in Fig. 13.1, through the 1D CUF FE model introduced so far the computational domain $\Omega$ can be discretized by means of an arbitrary cross-section $\Gamma_{S}$ and a one-dimensional mesh of finite elements. The number of finite elements the mesh is made of is indicated as $N_{\mathrm{EL}}$. It is pointed out that the number of elements $N_{\mathrm{EL}}^{U}$ used for the velocity discretization and the number of elements $N_{\mathrm{EL}}^{P}$ used for the pressure discretization are set to be equal: $N_{\mathrm{EL}}=N_{\mathrm{EL}}^{U}=N_{\mathrm{EL}}^{P}$. Nonetheless, the number of nodes $N_{N}^{U}$ of each element used for velocity discretization is in general different from the number of nodes $N_{N}^{P}$ of each element used for pressure discretization. Other parameters of the analysis are the expansion orders $N^{U}$ and $N^{P}$, as mentioned in section 13.4.3.

The number of degrees of freedom related to the single element for velocity discretization is referred as $\mathrm{DOFs}_{\mathrm{EL}}^{U}$ and its numerical value is:

$$
\begin{equation*}
\operatorname{DOFs}_{\mathrm{EL}}^{U}=3 N_{u}^{U} N_{N}^{U} \tag{13.223}
\end{equation*}
$$

since $\mathbf{q}_{\tau i}$ contains three components, see Eq. 13.97. Given a mesh of $N_{\text {EL }}^{U}$ connected elements with $N_{N}^{U}$ nodes per element, the total number of nodes $N_{N \text { TOT }}^{U}$ of the FE mesh for velocity discretization is computed as:

$$
\begin{equation*}
N_{N \mathrm{TOT}}^{U}=\left(N_{N}^{U}-1\right) N_{\mathrm{EL}}^{U}+1 \tag{13.224}
\end{equation*}
$$

This formula takes into account the fact that two connected finite elements share the same node. All the $N_{u}^{U}$ generic nodal velocity vectors $\mathbf{q}_{\tau i}$ for each of the $N_{N \text { TOT }}^{U}$ nodes of the mesh will be collected in the nodal velocity vector $\mathbf{q}$ for all values of $\tau\left(\tau=1, \ldots, N_{u}^{U}\right)$ and $i$ $\left(i=1, \ldots, N_{N}^{U}\right)$ of each element. As a consequence, vector $\mathbf{q}$ contains all the nodal velocity degrees of freedom of the present fluid dynamic model, which are commonly referred as $\mathrm{DOFs}^{U}$. Its dimensions are $\left(3 N_{u}^{U} N_{N \text { TOT }}^{U}\right) \times 1$, therefore:

$$
\begin{equation*}
\mathrm{DOFs}^{U}=\operatorname{DOFs}_{N}^{U} N_{N \mathrm{TOT}}^{U}=3 N_{u}^{U} N_{N \mathrm{TOT}}^{U} \tag{13.225}
\end{equation*}
$$

where $\mathrm{DOFs}_{N}^{U}$ is the number of degrees of freedom related to the single element node for velocity discretization. Retrieving Eq. 13.95:

$$
\begin{equation*}
\mathrm{DOFs}^{U}=3 N_{u}^{U} N_{N \text { TOT }}^{U}=3 \frac{\left(N^{U}+1\right)\left(N^{U}+2\right)}{2}\left[\left(N_{N}^{U}-1\right) N_{\mathrm{EL}}^{U}+1\right] \tag{13.226}
\end{equation*}
$$

It is important to note that $\mathrm{DOFs}^{U}$ depend on the expansion order $N^{U}$, the type $\left(N_{N}^{U}\right)$, and the number of finite elements $N_{\mathrm{EL}}^{U}$ in the mesh used for the velocity discretization.

An analogous approach is carried out for pressure quantities. The number of degrees of freedom related to the single element for pressure discretization is referred as DOFs ${ }_{\mathrm{EL}}^{P}$ and its numerical value is:

$$
\begin{equation*}
\operatorname{DOFs}_{\mathrm{EL}}^{P}=N_{u}^{P} N_{N}^{P} \tag{13.227}
\end{equation*}
$$

since coefficients $p_{m t}$ are scalar, see Eq. 13.104. Given a mesh of $N_{\mathrm{EL}}^{P}$ connected elements with $N_{N}^{P}$ nodes per element, the total number of nodes $N_{N \text { TOT }}^{P}$ of the FE mesh for pressure discretization is computed as:

$$
\begin{equation*}
N_{N \mathrm{TOT}}^{P}=\left(N_{N}^{P}-1\right) N_{\mathrm{EL}}^{P}+1 \tag{13.228}
\end{equation*}
$$

This formula takes into account the fact that two connected finite elements share the same node. All the $N_{u}^{P}$ generic nodal pressure coefficients $p_{m t}$ for each of the $N_{N}^{P}$ тот nodes of the mesh will be collected in the nodal pressure vector $\mathbf{p}$ for all values of $m\left(m=1, \ldots, N_{u}^{P}\right)$ and $t\left(t=1, \ldots, N_{N}^{P}\right)$ of each element. As a consequence, vector $\mathbf{p}$ contains all the nodal pressure degrees of freedom of the present fluid dynamic model, which are commonly referred as DOFs ${ }^{P}$. Its dimensions are $\left(N_{u}^{P} N_{N \text { TOT }}^{P}\right) \times 1$, therefore:

$$
\begin{equation*}
\operatorname{DOFs}^{P}=\operatorname{DOFs}_{N}^{P} N_{N \text { тот }}^{P}=N_{u}^{P} N_{N \text { тот }}^{P} \tag{13.229}
\end{equation*}
$$

where $\mathrm{DOFs}_{N}^{P}$ is the number of degrees of freedom related to the single element node for pressure discretization. Retrieving Eq. 13.103:

$$
\begin{equation*}
\operatorname{DOFs}^{P}=N_{u}^{P} N_{N \mathrm{TOT}}^{P}=\frac{\left(N^{P}+1\right)\left(N^{P}+2\right)}{2}\left[\left(N_{N}^{P}-1\right) N_{\mathrm{EL}}^{P}+1\right] \tag{13.230}
\end{equation*}
$$

It is important to note that $\mathrm{DOFs}^{P}$ depend on the expansion order $N^{P}$, the type $\left(N_{N}^{P}\right)$, and the number of finite elements $N_{\mathrm{EL}}^{P}$ in the mesh used for the pressure discretization. In conclusion, the total number of nodal degrees of freedom DOFs of the present fluid dynamic model is the result of the summation of velocity and pressure nodal degrees of freedom:

$$
\begin{align*}
\mathrm{DOFs}^{\mathrm{DOFs}} \mathrm{DOF}^{U}+\mathrm{DOFs}^{P}= & 3 \frac{\left(N^{U}+1\right)\left(N^{U}+2\right)}{2}\left[\left(N_{N}^{U}-1\right) N_{\mathrm{EL}}^{U}+1\right]+ \\
& \frac{\left(N^{P}+1\right)\left(N^{P}+2\right)}{2}\left[\left(N_{N}^{P}-1\right) N_{\mathrm{EL}}^{P}+1\right] \tag{13.231}
\end{align*}
$$

Table 13.1 summarizes the dimensions, i.e. the number of rows $N_{\text {rows }}$ and the number of columns $N_{\text {cols }}$ of the finite element matrices and vector introduced so far, which are related to the degrees of freedom defined previously.

Table 13.1: Dimensions of finite element matrices and vectors for the 1D CUF FE fluid dynamic model.

|  | $N_{\text {rows }}$ | $N_{\text {cols }}$ |
| :---: | :---: | :---: |
| $\underline{\mathbf{A}}_{\text {EL }}$ | DOFs ${ }_{\text {EL }}^{U}$ | DOFs ${ }_{\text {EL }}^{U}$ |
| $\underline{B}_{\text {EL }}^{T}$ | DOFs ${ }_{\text {EL }}^{U}$ | DOFs ${ }_{\text {EL }}^{P}$ |
| $\underline{B}_{\text {EL }}$ | DOFs ${ }_{\text {EL }}^{P}$ | DOFs ${ }_{\text {EL }}^{U}$ |
| $\underline{A}_{B C \text { cL }}^{\text {cs }}$ | DOFs ${ }_{\text {EL }}^{U}$ | DOFs ${ }_{\text {EL }}^{U}$ |
| $\underline{\mathbf{A}}^{1}{ }_{B C \text { EL }}$ | DOFs ${ }_{\text {EL }}^{U}$ | DOFs ${ }_{\text {EL }}^{U}$ |
| A | DOFs ${ }^{U}$ | DOFs ${ }^{U}$ |
| $\underline{B}^{T}$ | DOFs ${ }^{U}$ | DOFs ${ }^{P}$ |
| B | DOFs ${ }^{P}$ | DOFs ${ }^{\text {U }}$ |
| $\underline{0}$ | DOFs ${ }^{P}$ | DOFs ${ }^{P}$ |
| $\underline{\mathbf{A}}_{B C}^{\text {cs }}$ | DOFs ${ }^{U}$ | DOFs ${ }^{U}$ |
| $\underline{\mathbf{A}}^{1}{ }_{B C}$ | DOFs ${ }^{U}$ | DOFs ${ }^{U}$ |
| $\underline{\text { S }}$ | DOFs | DOFs |
| $\underline{\widetilde{S}}$ | DOFs | DOFs |
| $\mathbf{F}_{\text {EL }}$ | DOFs ${ }_{\text {EL }}^{U}$ | 1 |
| $\mathbf{F}_{B C \text { el }}^{\text {cs }}$ | DOFs ${ }_{\text {EL }}^{U}$ | 1 |
| $\mathbf{F}_{B C \text { eL }}^{\text {l }}$ | DOFs ${ }_{\text {EL }}^{U}$ | 1 |
| F | DOFs ${ }^{U}$ | 1 |
| $\mathbf{F}_{B C}^{\text {cs }}$ | DOFs ${ }^{U}$ | 1 |
| $\mathbf{F}_{B C}^{1}$ | DOFs ${ }^{U}$ | 1 |
| 0 | DOFs ${ }^{P}$ | 1 |
| F* | DOFs | 1 |
| $\widetilde{\mathbf{F}}^{\star}$ | DOFs | 1 |
| q | DOFs ${ }^{U}$ | 1 |
| p | DOFs ${ }^{P}$ | 1 |
| $\mathrm{q}^{\star}$ | DOFs | 1 |

## Chapter 14

## Results

This chapter presents some numerical fluid dynamic results achieved through the onedimensional CUF finite element model described in chapter 13. Despite the present model is a reduced order model, the following results show the accuracy in analyzing Stokes flows in pipes with a low computational cost, in terms of DOFs. Several analyses are carried out to assess and highlight the advantages of higher-order 1D CUF models with respect to classical one-dimensional models. Velocity and pressure trends are investigated in the computational domain for fluid flows with different boundary conditions and the use of a different expansion order and a different FE mesh for velocity and pressure is evaluated in order to achieve stable solutions.

### 14.1 Pipe and fluid data

For all the analyses carried out in this chapter, a pipe with a circular section is considered. In particular, this circular cross-section is constant along the longitudinal direction $y$ and its radius is equal to 1 m . For the sake of simplicity, the origin of the cartesian coordinate system $(x, y, z)$ lies on the center of the circular inlet section. The length $L$ of the pipe is equal to 6 m . The fluid flow passes through the pipe as sketched in Fig. 14.1.


Figure 14.1: Pipe with a circular cross-section. Inlet, outlet and lateral surfaces.
The boundary of the computational domain $(\partial \Omega)$ can be split into an inlet section $\Gamma_{D}^{\text {in }}$, an outlet section $\Gamma_{N}^{\text {out }}$ and a lateral surface $\Gamma_{D}^{1}$. The inlet section is the section through which the fluid flows into the pipe, whereas the outlet section is the section through which the fluid comes out of the pipe. As can be seen in Fig. 14.1 the following expression describes the geometrical boundary:

$$
\begin{equation*}
\partial \Omega=\Gamma_{D}^{\text {in }} \cup \Gamma_{N}^{\text {out }} \cup \Gamma_{D}^{1} \tag{14.1}
\end{equation*}
$$

In all the following analyses, a homogeneous Dirichlet boundary condition is prescribed on the lateral surface which means a no-slip condition at the pipe wall:

$$
\begin{equation*}
\left.\mathbf{u}\right|_{\Gamma_{D}^{1}}=\mathbf{0} \tag{14.2}
\end{equation*}
$$

A homogeneous Neumann boundary condition is instead prescribed on the outlet section:

$$
\begin{equation*}
\nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}}-p \mathbf{n}=\mathbf{0} \quad \text { on } \Gamma_{N}^{\text {out }}=\Gamma_{N} \tag{14.3}
\end{equation*}
$$

The two above boundary conditions remain the same for all the cases studied. On the contrary, a nonhomogeneous Dirichlet boundary condition (different for each case) is assigned to the inlet section. Hence, $\Gamma_{D}=\Gamma_{D}^{\mathrm{in}} \cup \Gamma_{D}^{1}$ and $\partial \Omega=\Gamma_{D} \cup \Gamma_{N}$, according to the remarks about proper boundary conditions made in chapter 13. No body forces are applied to the fluid and thus the following condition is taken into account in the Stokes equations:

$$
\begin{equation*}
\mathbf{f}=\mathbf{0} \tag{14.4}
\end{equation*}
$$

The computational domain $\Omega$ is a cylinder and, according to the procedure to be followed by 1D CUF FE model and described in chapter 13, it is discretized via a circular cross-section $\Gamma_{S}$ and a one-dimensional finite element mesh along the longitudinal direction $y$. All the following cases are studied via a mesh of 10 finite elements, whose type ( $B 2, B 3$, or $B 4$ elements) for velocity and pressure fields is a free parameter of the analysis. For the sake of brevity, convergence analyses on the number of finite elements to be used in the mesh are not reported here, whereas particular attention is payed to evaluate the proper type (i.e. the accuracy) of elements able to properly predict the solution.

The fluid used has a kinematic viscosity $\nu$ equal to $10^{-2} \mathrm{~m}^{2} / \mathrm{s}$. For the sake of clarity, this value may represent the kinematic viscosity of a particularly viscous fluid, such as honey. As far as the Reynold number is concerned, in chapter 13 it is explained that the Stokes problem makes sense when the condition $R e \ll 1$ holds. Retrieving the expression of the Reynolds number in Eq. 13.56, it depends on a representative length $L$ of the domain $\Omega$ and a representative fluid velocity $U$. The radius of the circular cross-section of the pipe is chosen as representative length $(L=1 \mathrm{~m})$, whereas the maximum value of the velocity profile prescribed on the inlet section may be taken as representative velocity $U$. As mentioned, the velocity will be prescribed differently for each case, but its maximum will be equal to $1 \cdot 10^{-4}$ or $2 \cdot 10^{-4} \mathrm{~m} / \mathrm{s}$. As a consequence, the Reynolds number for the following analyses will be equal to 0.01 or 0.02 and will be therefore consistent with the condition of physical validity $R e \ll 1$ of Stokes equations. As a consequence, the hypotheses of Stokes equations are satisfied and the analyses presented are physically plausible.

### 14.2 Poiseuille flow

The first case studied represents the Poiseuille flow. The Poiseuille flow is a laminar flow in pipes with circular section that occurs for low values of Reynolds number. This flow represents a good benchmark and therefore it is introduced here to assess the 1D CUF FE model. In the Poiseuille flow, the fluid flow is axisymmetric and the velocity $\mathbf{u}$ does not depend on the position along the longitudinal axis $y$. In particular, the axial velocity $u_{y}$ is parabolic, or better paraboloidal on the two-dimensional cross-section, and the maximum axial velocity $u_{y \text { max }}$ occurs at point $(x=0, z=0)$ of each section (i.e. along the line $y=0$ ). In order to simulate this kind of flow, the following nonhomogeneous Dirichlet
boundary condition is given on the inlet section:

$$
\left\{\begin{array}{l}
u_{x}=0  \tag{14.5}\\
u_{y}=10^{-4}\left(1-x^{2}-z^{2}\right) \quad \text { on } \Gamma_{D}^{\text {in }} \\
u_{z}=0
\end{array}\right.
$$

According to the other boundary conditions prescribed (Eqs. 14.2 and 14.3), this paraboloidal inlet velocity profile should remain constant along the $y$ axis. According to the Poiseuille flow, over the cross-section an accuracy of the second-order for the velocity and of the "zero-order" (constant trend) for pressure should be sufficient to detect the solution. As far as the trend along $y$ is concerned, a "zero-order" (constant trend) for velocity and a forst-order approximation for pressure should be sufficient. For the sake of simplicity, the parameters of the CUF analyses are set to $N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=4, N^{P}=3$, following different accuracy-orders for $\mathbf{u}$ and $p$ in order to obtain a stable solution (more details are discussed later on). The velocity profiles on some sections of the pipe are depicted in Fig. 14.2. As expected, the paraboloidal trend remains the same as $y$ increases, according to the Poiseuille flow.


Figure 14.2: Velocity profiles $u_{y}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6$ $[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=4, N^{P}=3$. Poiseuille flow.

The pressure profiles on some sections of the pipe are depicted in Fig. 14.3. The trend of pressure is constant over each section, but its value decreases along the longitudinal axis $y$. In fact, the maximum pressure value is observed at the inlet section and the minimum value is observed at the outlet section. In particular, it is important to note that the pressure is equal to zero at the outlet section, exactly as prescribed by Poiseuille.


Figure 14.3: Pressure profiles at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6[\mathrm{~m}] . N_{N}^{U}=3$, $N_{N}^{P}=2, N^{U}=4, N^{P}=3$. Poiseuille flow.

The zero vale of pressure at the outlet section can be clearly shown in Fig. 14.4. This figure illustrates the values of pressure at $z=0$ along the axis $x$ at different significant sections. As already mentioned in comments for Fig. 14.3, the pressure on each section is constant and the decrease rate along $y$ is constant, i.e. the pressure follows a linear decrease. For the sake of completeness, it is noteworthy that the pressure value at the middle section $(y=3 \mathrm{~m})$ is exactly the mean value between the pressure value (maximum) at the inlet section and the pressure value (zero) at the outlet section.


Figure 14.4: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=4, N^{P}=3$. Poiseuille flow.

The linear decrease of the pressure along the longitudinal axis $y$ is well highlighted in Fig. 14.5(a). Figure 14.5 presents the pressure solution obtained with a different choice of the element types for velocity and pressure unknowns. The key point of the solution stability is hence discussed from results. As mentioned in chapter 13, finite elements of the same polynomial degree for both velocity and pressures are in general unstable. This statement is here verified for Figs. 14.5(b) and 14.5(d), where the same polynomial order for the finite element shape functions $\left(N_{N}^{U}=N_{N}^{P}=3\right.$ or $N_{N}^{U}=N_{N}^{P}=4$, i.e. B3 elements or $B 4$ elements) is chosen both for velocity and pressure fields. In fact, Figs. 14.5(b) and $14.5(\mathrm{~d})$ show typical spurious pressure modes. Of course, the shape of these spurious modes depends on the accuracy chosen (i.e. number of nodes per element $N_{N}^{P}$ ) for the pressure discretization along $y$ and differs for $N_{N}^{P}=3$ and $N_{N}^{P}=4$. On the contrary, a different choice of $N_{N}^{U}$ and $N_{N}^{P}$ leads to stable solutions and this is the case depicted in Figs. 14.5 (a) and 14.5 (c). In fact, spurious pressure modes do not appear and this means that the space $V_{h}$ is "rich" enough with respect to the space $Q_{h}$.

As a result, the 1D CUF FE model is able to accurately simulate the Poiseuille flow, which is an exact solution of flow in a pipe. Moreover, the use of higher-order terms in modeling velocity and pressure fields does not introduce numerical problems and a stable solution is obtained. The typical considerations about stability of spaces based on the finite element method are still valid also for the present model.


Figure 14.5: Pressure at point $(x=0, z=0)$ along the $y$ axis for different combinations of $N_{N}^{U}$ and $N_{N}^{P} . N_{U}=4, N^{P}=3$. Examples of stable ((a), (c)) and unstable solutions ((b), (d)). Poiseuille flow.

### 14.3 Fourth-order inlet velocity profile $a$

The previous case of Poiseuille flow represents a first assessment of the 1D CUF FE model. Nonetheless, the Poiseuille flow is the most "natural" and simple flow in a pipe due to its second-order axisymmetric velocity profile constant along $y$. Moreover, the condition of constant pressure over the section is commonly taken into account by classical one-dimensional flui dynamics models. Hence, the capabilities of the present model in describing accurately more complex flows is now faced for other inlet velocity profiles.

The second case studied represents a different inlet velocity profile. In particular, it is given by the square of the parabolidal inlet profile of the Poiseuille flow:

$$
\left\{\begin{array}{l}
u_{x}=0  \tag{14.6}\\
u_{y}=10^{-4}\left(1-x^{2}-z^{2}\right)^{2} \quad \text { on } \Gamma_{D}^{\text {in }} \\
u_{z}=0
\end{array}\right.
$$

The only nonzero component is again the axial velocoty $u_{y}$ and the profile is again axisymmetric with its maximum at the centre of the section $(x=0, z=0)$. The expansion order for velocity is now set to $N^{U}=6$, since the fourth-order inlet velocity profile requires at least a fourth-order expansion on $\Gamma_{S}$ to be properly detected. According to the
considerations made previously about the choice of stable spaces, the expansion order $N^{P}$ for pressure is set to a value lower than $N^{U}$; in particular, $N^{P}=4$. As far as the accuracy along the $y$ axis is concerned, the mesh is not changed with respect to the previous case. Hence, $N_{N}^{U}$ is set equal to 3 and $N_{N}^{P}$ is set equal to 2 .

The velocity profiles $u_{y}$ on some sections lying in the initial part of the pipe are depicted in Fig. 14.6. As expected, the velocity profiles remain axisymmetric along the pipe axis and the maximum axial velocity $u_{y \text { max }}$ occurs at point $(x=0, z=0)$ of each section (i.e. along the line $y=0$ ), as occurred for the Poiseuille flow. Differently, the velocity $\mathbf{u}$ now depends on the position along the longitudinal axis $y$, as can be seen in Fig. 14.6. In particular, the profiles seem to approach a paraboloidal profile, which corresponds to the "natural" Poiseuille flow studied in the previous section. The trend of $u_{y}$ can be seen


Figure 14.6: Velocity profiles $u_{y}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6$ $[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.
more accurately in Fig. 14.7, where the axial velocity at $z=0$ is depicted along the axis $x$ at different significant sections. As expected, the flow approaches its natural condition, given by the boundary conditions at the outlet section, corresponding to the Poiseuille flow. The fourth-order inlet velocity profile changes along $y$ and gradually transforms to the paraboloidal profile of Poiseuille. The transition area of the pipe is the initial part, since at $y=L / 5$ the profile has already become parabolic.

It is very important to note that the velocity flux $Q$, also known as flow rate, given by:

$$
\begin{equation*}
Q=\int_{\Gamma_{S}} u_{y} \mathrm{~d} \Gamma \tag{14.7}
\end{equation*}
$$

remains constant along the longitudinal axis in order to respect the continuity equation. More details can be found in [145]. It can be numerically demonstrated that the value of the integral of the axial velocity $u_{y}$, whose two-dimensional profiles on $\Gamma_{S}$ are given by the revolution of the curves depicted in Fig. 14.7, remain constant along $y$. The capability of the CUF model to easily detect complex flows as the present case is not typical for "standard" 1D reduced order models.

The profiles of pressure $p$ over some section lying in the initial part of the pipe are shown in Fig. 14.8. As expected, the pressure trend is axisymmetric as well as the velocity one and a transition is observed. Differently from the Poiseuille flow (see Fig.14.4), the pressure is no more constant. As occurred for axial velocity, the pressure behavior approaches the "natural" pressure behavior given by Poiseuille flow, i.e. constant profiles over the sections.

The transition from the inlet pressure profile to the constant pressure profile is well shown in Fig. 14.9, where the values of pressure at $z=0$ along the axis $x$ at different significant sections are considered. These curves describe completely the pressure field


Figure 14.7: Velocity $u_{y}$ at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.


Figure 14.8: Pressure profiles at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6[\mathrm{~m}] . N_{N}^{U}=3$, $N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.
because of the axisymmetry detected in Fig. 14.8. In the transition volume, the maximum of the pressure placed at $(x=0, z=0)$ decreases as $y$ increases and the pressure profile, not constant along $x$ at the inlet section, approaches gradually the constant profile typical of Poiseuille flow. This constant pressure field over the pipe section continues with a constant rate up to the zero value at the outlet section.

The behavior of the velocity along directions different from the axial one is now briefly presented. Figs. 14.10 and 14.11 show the profiles of velocities $u_{x}$ and $u_{z}$ over the inlet section, the outlet section and other sections lying in the transition part of the pipe. The first consideration concerns the fact that these profiles are antisymmetric with respect to $z$ axis (for $u_{x}$ ) and $x$ axis (for $u_{z}$ ). This is a prove of the fact that the flow is completely axisymmetric, as expected. Although the inlet and outlet velocity maps appear coloured, paying attention to the colour box it is easily realized that the corresponding values are practically zeros, or better numerical zeros, with respect to the representative values corresponding to the other sections (in the transition pipe area). In fact, it has to be


Figure 14.9: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.
reminded that the boundary conditions are imposed numerically via the typical numerical procedure of penalization [42]. As a consequence, an acceptable (i.e. negligible) numerical approximation is taken into account, since in general it does not affect significantly the results. It is verified thus that the homogeneous Dirichlet boundary condition for $u_{x}$ and $u_{z}$ on the inlet section (Eq. 14.6) is not violated as well as the homogeneous Dirichlet boundary conditions for $u_{x}, u_{y}$, and $u_{z}$ on the lateral surface of the pipe.

### 14.4 Fourth-order inlet velocity profile $b$

The third case studied represents an other fourth-order inlet profile for the axial velocity $u_{y}$, given by the product of two quadratic terms. The Dirichlet boundary conditions imposed on the inlet section is:

$$
\left\{\begin{array}{l}
u_{x}=0  \tag{14.8}\\
u_{y}=10^{-4}\left(1-x^{2}-z^{2}\right)\left(2+x^{2}+z^{2}\right) \quad \text { on } \Gamma_{D}^{\text {in }} \\
u_{z}=0
\end{array}\right.
$$

A letter $b$ is employed in order to differentiate the present case and the previous fourth-order inlet profile case, which instead is denoted with letter $a$. The only nonzero component is again the axial velocoty $u_{y}$ and the profile is again axisymmetric with its maximum at the centre of the section $(x=0, z=0)$. The expansion orders for velocity and pressure fields are set to the same values as the previous fourth-order inlet velocity case. The expansion order for velocity is therefore set to $N^{U}=6$, since the fourth-order inlet velocity profile requires at least a fourth-order expansion on $\Gamma_{S}$ to be properly detected. The expansion order $N^{P}$ for pressure is instead set to a value lower than $N^{U}$; in particular, $N^{P}=4$. As far as the accuracy along the $y$ axis is concerned, the mesh is not changed with respect to the previous cases. Hence, $N_{N}^{U}$ is set equal to 3 and $N_{N}^{P}$ is set equal to 2 .


Figure 14.10: Velocity profiles $u_{x}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45$ (d), $y=0.6$ (e), $y=6(\mathrm{f})[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.

The velocity profiles $u_{y}$ on some sections lying in the initial part of the pipe are depicted in Fig. 14.12. Since the inlet profile is axisymmetric, the geometrical shape of the pipe is axisymmetric (circular cross-section) and the boundary condition on the lateral surface $\Gamma_{D}^{1}$ are axisymmetric (homogeneous Dirichlet BC), the flow is again expected to be axisymmetric. In fact, the velocity profiles remain axisymmetric along the pipe axis and the maximum axial velocity $u_{y \text { max }}$ occurs at point $(x=0, z=0)$ of each section (i.e. along the line $y=0$ ). Differently from the Poiseuille and similarly to what observed for the fourth-order inlet profile $a$, the velocity $\mathbf{u}$ changes along the longitudinal axis $y$, as can be seen in Fig. 14.12. In particular, the profiles seem again to approach a paraboloidal profile, which corresponds to the "natural" Poiseuille flow studied in the previous section.

The transition from the fourth-order profile given in Eq. 14.8 and the paraboloidal


Figure 14.11: Velocity profiles $u_{z}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0(\mathrm{a}), y=0.15$ (b), $y=0.3$ (c), $y=0.45(\mathrm{~d}), y=0.6(\mathrm{e}), y=6(\mathrm{f})[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $a$ for velociy $u_{y}$.
profile for velocity is highlighted in Fig. 14.13, where the axial velocity at $z=0$ is depicted along the axis $x$ at different significant sections. As expected, the flow gradually approaches its natural condition, given by the boundary conditions at the outlet section, corresponding to the Poiseuille flow. At $y=L / 5$ the profile has already become parabolic and this shape remains constant up to the outlet section.

From Fig. 14.13, it can be observed that the flow evolves in the pipe in a way such that the velocity flux $Q$ remains constant, not violating the conservation of mass (continuity equation). The value of the intergal of the axial velocity $u_{y}$, whose two-dimensional profiles on $\Gamma_{S}$ are given by the revolution of the curves depicted in Fig. 14.13, remain constant along $y$ and this is an important assessment for the 1D CUF FE model, which proves its capability to predict transition of complex flows not violating the equations governing the


Figure 14.12: Velocity profiles $u_{y}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6$ $[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.


Figure 14.13: Velocity $u_{y}$ at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.
fluid dynamics, despite its reduced order and the corresponding low computation cost.
The profiles of pressure $p$ over some section lying in the initial part of the pipe are shown in Fig. 14.14. As expected, the pressure trend is axisymmetric as well as the velocity one and a transition is observed. Differently from the Poiseuille flow (see Fig.14.4), the pressure is no more constant. As occurred for axial velocity, the pressure behavior approaches the "natural" pressure behavior given by Poiseuille flow, i.e. constant profiles over the sections.

The transition from the inlet pressure profile to the constant pressure profile is well shown in Fig. 14.15, where the values of pressure at $z=0$ along the axis $x$ at different significant sections are considered. These curves describe completely the pressure field because of the axisymmetry detected in Fig. 14.14. In the transition volume, the minimum of the pressure (it was the maximum for the fourt-order inlet profile $a$ ) placed at ( $x=0, z=0$ ) decreases as $y$ increases and the pressure profile, not constant along $x$ at the inlet section, approaches gradually the constant profile typical of Poiseuille flow. This constant pressure field over the pipe section continues with a constant rate up to the zero value at the outlet


Figure 14.14: Pressure profiles at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6[\mathrm{~m}] . N_{N}^{U}=3$, $N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.
section.


Figure 14.15: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.

Figures 14.16 and 14.17 present the coloured maps of transversal velocities $u_{x}$ and $u_{z}$ for the inlet section, the outlet section and other sections lying in the transition part of the pipe. As observed for the fourth-order inlet profile $a$, the transversal velocity fields result antisymmetric with respect to $z$ axis (for $u_{x}$ ) and $x$ axis (for $u_{z}$ ). The flow is again completely axisymmetric, as already concluded by the axial velocity profiles (see Fig. 14.12). The same numerical consideration made in the previous case concerning the inlet and outlet profiles are still valid. According to an acceptable numerical approximation of boundary conditions, the profiles of $u_{x}$ and $u_{z}$ on the inlet and outlet sections are constant despite the coloured maps in subfigures (a) and (f). Moreover, the homogeneous Dirichlet boundary condition for $u_{x}$ and $u_{z}$ on the inlet section (Eq. 14.8) is verificated as well as the homogeneous Dirichlet boundary conditions for $u_{x}, u_{y}$, and $u_{z}$ on the lateral surface of the pipe.In conclusion, the choice of the parameters $N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$
has led to stable solutions not affected by spurious pressure modes.


Figure 14.16: Velocity profiles $u_{x}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45$ (d), $y=0.6$ (e), $y=6$ (f) $[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.

### 14.5 Fifth-order inlet velocity profile

The fourth case studied considers a more complex velocity inlet profile. In particular, a fifth-order nonhomogeneous Dirichlet boundary condition is prescribed on the axial velocity $u_{y}$ as follows:

$$
\left\{\begin{array}{l}
u_{x}=0  \tag{14.9}\\
u_{y}=10^{-4}\left(1-x^{2}-z^{2}\right)\left(1 / 4+x z+x^{3}\right) \quad \text { on } \Gamma_{D}^{\text {in }} \\
u_{z}=0
\end{array}\right.
$$



Figure 14.17: Velocity profiles $u_{z}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45(\mathrm{~d}), y=0.6(\mathrm{e}), y=6(\mathrm{f})[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=4$. Fourth-order inlet profile $b$ for velociy $u_{y}$.

This boundary condition is again homgeneous for the transversal components of the inlet velocity. Unlike the previous three cases, the inlet velocity profile $u_{y}$ for the present case is no more axisymmetric. As a consequence, the flow obtained through this boundary condition is expected to be not axisymmetric. The expansion order for velocity is set to $N^{U}=6$, since the fifth-order inlet velocity profile requires at least a fifth-order expansion on $\Gamma_{S}$ to be properly detected. According to the considerations made previously about the choice of stable spaces, the expansion order $N^{P}$ for pressure is set to a value lower than $N^{U}$; in particular, $N^{P}=4$. As far as the accuracy along the $y$ axis is concerned, the mesh is not changed with respect to the previous cases. Hence, $N_{N}^{U}$ is set equal to 3 and $N_{N}^{P}$ is set equal to 2 .

The velocity profiles $u_{y}$ on some sections lying in the initial part of the pipe are depicted
in Fig. 14.6. As expected, the velocity profiles are not axisymmetric along the pipe axis and a flow transition occurs downline of the inlet section. Nonetheless, the profiles of velocity $u_{y}$ gradually smooth and seem to approach the more natural condition of axisymmetry, given the outlet Neumann boundary condition (Eq. 14.3) and the lateral homogeneous Dirichelt boundary condition (that is axysimmetric). Similarly to what observed for the fourth-order inlet profiles $a$ and $b$, the velocity $\mathbf{u}$ changes along the longitudinal axis $y$ and approach the paraboloidal behavior typical of the Poiseuille flow, as can be seen in Fig. 14.12.


Figure 14.18: Velocity profiles $u_{y}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6$ $[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.

The transition from the fifth-order profile given in Eq. 14.9 and the paraboloidal profile for axial velocity $u_{y}$ is highlighted in Fig. 14.19, where the axial velocity at $z=0$ is depicted along the axis $x$ at different significant sections. As expected, the flow gradually approaches its natural condition, given by the boundary conditions at the outlet section and corresponding to the Poiseuille flow. The transition area appears more extended with respect to the previous fourth-order cases $a$ and $b$, since at $y=L / 5$ the profile has not become parabolic yet. For the sake of completeness, it is important to note that the maximum value of $u_{x}$ decreases along $y$. For increasing values of $y$ the profile becomes parabolic and then conserves its shape up to the outlet section according to the Poiseuille flow.

As highlighted for the fourth-order inlet velocity cases, it can be numerically demonstrated that the value of the flux $Q$, given by the integral of $u_{y}$ over the section (see Eq. 14.7), is conserved along $y$, not violating the continuity equation. Since the flow is now not axisymmetric, the velocity profiles $u_{y}$ are no more given by the revolution of curves depicted in Fig. 14.19. This is an important assessment for the 1D CUF FE model, which proves its capability to predict the evolution of complex flows (not violating the equations governing the fluid dynamics) also in case of non-axisymmetric flows. The key point is that this feature is not typical for "standard" 1D reduced order models and, furthermore, the accuracy desired in the analysis is a free parameter of the CUF model.

The profiles of pressure $p$ over some section lying in the transition part of the pipe are shown in Fig. 14.20. As expected, the pressure trend is non-axisymmetric as well as the velocity one. Since the expansion order for pressure is set to $N^{P}=4$, at maximum a fourth-order approximation of the pressure over $\Gamma_{S}$ is taken into account. As occurred for the previous cases and for the axial velocity of the present case, the profiles in Fig. 14.20 approach the "natural" pressure behavior given by Poiseuille flow, i.e. constant profiles over the sections.

The transition from the inlet pressure profile to the constant pressure profile is well shown in Fig. 14.21, where the values of pressure at $z=0$ along the axis $x$ at different


Figure 14.19: Velocity $u_{y}$ at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.


Figure 14.20: Pressure profiles at sections $y=0, y=0.15, y=0.3, y=0.45, y=0.6[\mathrm{~m}] . N_{N}^{U}=3$, $N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.
significant sections are considered. Again, these curves do not provide the complete trend of $p$ over the section since it is not axisymmetric, unlike the previous cases. The pressure approaches gradually the constant profile typical of Poiseuille flow, firstly uniforming its trend and then decreasing linearly up to the outlet.

Figures $14.22,14.23$ and 14.24 present the coloured maps of velocities $u_{x}, u_{y}$ and $u_{z}$ for the inlet section, the outlet section and other sections lying in the transition part of the pipe. This flow case is very interesting, since it presents different features con the three components of the velocity field. The only component of the velocity which presents a antisymmetry is $u_{z}$, i.e. the component along the $z$ axis; in fact, it is antisymmetric with respect to the $z$ axis. This symmetrical behavior was the same as that observed for transversal velocities $u_{x}$ and $u_{z}$ in the previous fourth-order cases $a$ and $b$. On the contrary, the component of the velocity along the $x$ axis, i.e. $u_{x}$, shows a symmetric behavior with respect to the $z$ axis. Finally, the component $u_{y}$ does not present symmetry (as $u_{x}$ ), axisymmetry (as $u_{y}$ in the previous cases), or antisymmetry (as $u_{z}$ ). This arbitrary


Figure 14.21: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.
behavior has been already discussed and is coherent with Figs. 14.18 and 14.19. It is interesting to note that $u_{y}$ is completely paraboidal at the outlet section, see subfigure (f) of Fig. 14.23. The same numerical considerations made in the previous cases concerning the inlet and outlet profiles are still valid for $u_{x}$ and $u_{y}$. According to an acceptable numerical approximation of boundary conditions, the profiles of $u_{x}$ and $u_{z}$ on the inlet and outlet sections are constant despite the coloured maps in subfigures (a) and (f).

For the sake of completeness, the pressure field is presented also by Fig. 14.25, which shows the coloured maps of pressure for the inlet section, the outlet section and other sections lying in the transition part of the pipe. Unlike the previous inlet velocity cases, $p$ does not present symmetry, axisymmetry, or antisymmetry (as $u_{z}$ ). This fact is due to the particular Dirichlet boundary condition imposed (for velocity) on the inlet section. Again, the pressure decreases up to the outlet section, where its value is, according to an acceptable numerical approximation, constant and equal to zero.

The last result concerns the choice of stable spaces $V_{h}$ and $Q_{h}$. In genearl, the stability is obtained by means of an appropriate choice of the parameters $N_{N}^{U}, N_{N}^{P}, N^{U}$, and $N^{P}$. The stability dependent on paramenters $N_{N}^{U}$ and $N_{N}^{P}$ has been already faced in the section of Poiseuille flow of this chapter. For the sake of brevity, the problem of stability dependent of parameters $N^{U}$ and $N^{P}$. Figure 14.26 presents the pressure solution at $z=0$ along the $x$ axis (as done in Fig. 14.19) obtained with a different choice of the expansion order for pressure field, by taking constant $N_{N}^{U}=6$. It is briefly reminded that the expansion order ( $N^{U}$ for velocity and $N^{P}$ for pressure) identifies the accuracy chosen to approximate the solution of the problem only over the cross-section $\Gamma_{S}$. This is the key point of the CUF model.

As occurred for the choice of the finite element types, in general the choice of an equal expansion order for velocity and pressure unknowns leads to unstable solutions. The corresponding case is depicted in Fig. 14.26(f), where $N_{N}^{U}=N_{N}^{P}=6$. For the present particular case, where the choice of $N_{N}^{P}=6$ is identical to $N_{N}^{P}=5$, an unstable solution


Figure 14.22: Velocity profiles $u_{x}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45$ (d), $y=0.6$ (e), $y=6$ (f) [m]. $N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.
is not clearly detectable for $N_{N}^{U}=N_{N}^{P}=6$. In fact, with a choice of $N_{N}^{U}=N_{N}^{P}=5$, the solution would become unstable, with the presence of spurious pressure modes, as can be seen in Fig. 14.27.

For the sake of completeness, the non-recommended cases of an expansion order for pressure higher than the expansion order for velocity is illustrated in Figs. 14.26(g) and $14.26(\mathrm{~h})$. This choice leads to unstable solutions, since spurious pressure modes appear.



Figure 14.24: Velocity profiles $u_{z}\left[10^{-4} \mathrm{~m} / \mathrm{s}\right]$ at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45$ (d), $y=0.6$ (e), $y=6$ (f) [m]. $N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.


Figure 14.25: Pressure profiles at sections $y=0$ (a), $y=0.15$ (b), $y=0.3$ (c), $y=0.45$ (d), $y=0.6$ (e), $y=6(\mathrm{f})[\mathrm{m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=6, N^{P}=3$. Fifth-order inlet profile for velociy $u_{y}$.


Figure 14.26: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2$. Examples of stable ((a), (b), (c), (d), (e), (f)) and unstable solutions ((g), (h)). Fifth-order inlet profile for velociy $u_{y}$.


Figure 14.27: Pressure at $z=0$ along the $x$ axis at sections $y=0, y=0.15, y=0.3, y=0.45$, $y=0.6, y=1.2, y=3, y=6[\mathrm{~m}] . N_{N}^{U}=3, N_{N}^{P}=2, N^{U}=5, N^{P}=5$ Example of unstable solution with spurious pressure modes. Fifth-order inlet profile for velociy $u_{y}$.

## Chapter 15

## Conclusions

### 15.1 Remarks on free vibration analysis

The free vibration analysis of nonhomogeneous cylindrical shells through refined onedimensional models has been addressed in Chapter 4. Hierarchical 1D finite elements were formulated on the basis of Carrera Unified Formulation (CUF) and assessed by comparison with exact solutions of the three-dimensional governing elasticity equations and solid finite element solutions. As far as the use of higher-order 1D models is concerned, the following main conclusions can be drawn:

1. a different higher-order expansion is required depending on the kind of vibrational mode investigated. Some structural models are not even able to detect all the kinds of vibrational modes because of the a priori displacement field used. For instance, none of the lobe-type modes was computed by first- and second-order theories;
2. the introduction of higher-order terms in the displacement field is not always necessary. For example, the first-order model was accurate enough to provide correct natural frequency values for radial and axial modes;
3. classical beam theories are not able to detect radial and lobe-type vibrational modes, whereas axial frequencies are correctly computed. Although Euler-Bernoulli's and Timoshenko's are basically bending beam theories, for this thin-walled short structure they are not able to accurately detect even bending natural frequencies;
4. the enrichment of the displacement field enables the structure to deform in a more realistic way and thus leads to capture vibrational modes that require in-plane and warping deformation to be detected. Higher-order models are required especially for the evaluation of lobe-type modes, which are typical of shell structures and not detectable by standard one-dimensional models;
5. in general, an increase of the expansion order corresponds to a decrease of the overall structural stiffness and thus to a reduction of the frequency values by approaching the three-dimensional results.

As far as the present hierarchical one-dimensional approach is concerned, the results point out that:
a. CUF is the ideal tool to easily compare different higher-order theories. The expansion order of the model, i.e. its accuracy, is a free parameter of the analysis by exploiting a systematic procedure that leads to governing FE matrices whose form does not
depend on the order of expansion used for the displacement unknowns over the cross-section;
b. despite its one-dimensional approach, the CUF model proved its accuracy in the free vibration analysis of even short, nonhomogeneous thin- and thick-walled shell structures. Both the modal shapes and natural frequency values are in well agreement with those obtained by three-dimensional models;
c. a convergent trend of natural frequency values approaching the three-dimensional results as the expansion order increases is achieved. This proves that the proposed hierarchical model does not introduce additional numerical problems in the free vibration analysis of layered structures with respect to classical beam theories;
d. the refined 1D CUF model presents a sizeable reduction in computational cost in terms of DOFs with respect to the solid FE model.

The accurate dynamic study of thin- and thick-walled layered structures can be faced through the present refined 1D approach. The excellent agreement with exact and quasiexact solutions of the three-dimensional elasticity equations highlights the shell-type capabilities of the 1D CUF model and the importance of refining the axiomatic displacement field with higher-order terms.

The free vibration analysis of classical and joined wings, based on higher-order beam theories, has been presented in Chapter 5. Carrera Unified Formulation, CUF, has been used for the systemic implementation of refined models. According to CUF, the element stiffness and mass matrices are obtained in a compact form, named fundamental nucleus, that does not depend on the theory approximation order, that is, the order of the model is assumed as a free parameter of the modeling. Elements based on classical theories have been derived as particular cases. A preliminary static analysis has been conducted to validate the present formulation in comparison with 3D solid element models. The analysis conducted has shown an excellent match between the models. The numerical analysis has been the conducted for the investigation of dynamic behavior in terms of natural frequencies and vibration modes. Comparisons with shell and solid wing models of commercial FE codes have been made. The following main conclusions can be drawn.

The present one-dimensional formulation permits to deal in an unified manner with:

1. arbitrary cross-sections geometries;
2. compact and thin-walled structures;
3. straight as well as arbitrary orientated structures;
4. unconventional joined wing configurations;

The use of higher-order theories has permitted classical beam model limitations to be overcome. The comparison with shell and solid models has shown the shell capabilities of the refined beam theories, that is, accurate modal shapes for thin-walled structures can be obtained by means of significantly less cumbersome 1D elements. Furthermore, the effects of the higher-order terms become significant when:

1. thin walled cross-section geometries are adopted;
2. the beam is not slender;
3. the aim of the structural dynamics analysis is the proper prediction of vibration modes such as torsional ones.

The use of the proposed beam models appears suitable for aeroeleatic applications that include airfoil in-plane deformations, and for extensions to wings made of advanced composite materials.

### 15.2 Remarks on static and dynamic response analysis

Chapter 6 has presented the extension of refined one-dimensional models to the dynamic response analysis of isotropic thin-walled structures. The static and dynamic analysis of structures with arbitrary cross-section geometries and nonhomogeneous materials through refined one-dimensional models has been addressed in Chapter 7. Variable kinematic 1D finite elements were formulated on the basis of Carrera Unified Formulation, CUF, by exploiting a systematic procedure that leads to governing FE matrices whose form does not depend on the order of expansion used for the displacement unknowns over the cross-section.

Several analyses were carried out to assess and enhance the advantages of higher-order 1 CUF models in evaluating the dynamic response of slender structures. In particular, a cylinder with a thin-walled circular cross-section subjected to harmonic out of phase loadings was studied. Comparing results with three-dimensional elasticity solutions and shell-type solutions obtained by commercial FE software, the following main conclusions can be drawn:

1. the effectiveness of higher-order terms over the cross-section deformation is enhanced when thin-walled geometries are adopted;
2. since classical beam theories assume an undeformed section, they become ineffective in the case of thin-walled geometries;
3. in-plane deformations due to time-dependent internal loadings are accurately detected by the proposed 1D models.
As far as numerical implementation is concerned:
a. the proposed models did not introduce additional numerical problems in direct time integration with respect to classical beam theories;
b. the convergence of the Newmark method was achieved for both the classical and higher-order FE models.

According to works involving 1D CUF models, the results presented confirm that the increase of the expansion order is not only important for static and free vibration analyses; it is also crucial for the dynamic response of beam-like structures. The implementation of 1D CUF models in the Newmark time integration scheme has revealed the shell-type capabilities of such refined models in accurately describing the dynamic behavior of thinwalled structures with a sizeable reduction in computational cost.

As far as the use of 1D higher-order models to analyze nonhomogeneous structures is concerned, the following main conclusions can be drawn:

1. the introduction of higher-order terms in the displacement field is important even for the analysis of structures with conventional cross sections. Higher-order models are required especially for structures with significant material nonhomogeneity and arbitrary geometry;
2. classical beam theories were completely ineffective in studying the kind of structures considered. Although Euler-Bernoulli's and Timoshenko's are basically bending beam theories, they were not able to accurately detect even the axial strain and stress, which are not neglected by the kinematic hypotheses of undeformed cross-section shape they are based on;
3. a convergent trend of displacement, strain and stress values approaching the threedimensional results as the expansion order increases is achieved. This proves that the proposed 1D hierarchical model does not introduce additional numerical problems in the analysis of arbitrary nonhomogeneous structures with respect to classical beam theories.

As far as the present hierarchical one-dimensional approach is concerned, the results point out that:
a. the enrichment of the displacement field enables the structure to deform in a more realistic way and a very good agreement with the three-dimensional solution was achieved; in-plane cross-section deformations are well-described by the present 1D CUF models.
b. despite its one-dimensional approach, the proposed higher-order formulation proved its accuracy in the analysis of even short structures made of homogeneous or nonhomogeneous materials with classical or arbitrary cross-section geometries. Local effects and complete three-dimensional displacement, strain and stress fields were computed in well agreement with those obtained by three-dimensional models;
c. the refined 1D CUF model shows a remarkable reduction in computational cost in terms of DOFs with respect to the solid FE model.

Comparing results with three-dimensional solutions, the present 1D finite element formulation proved to be a valid alternative to shell and solid methods, which necessarily require a higher computational cost, and a promising numerical tool for the analysis of arbitrary nonhomogeneous structures in biomechanical applications. In this respect, further work should be done in order to take into account material anisotropy and nonlinearity typical of biological soft tissues.

### 15.3 Remarks on aeroelastic analysis

Chapter 11 has proposed the aeroelastic model that couples the Vortex Lattice Method and a refined one-dimensional structural model with in-plane warping and plate/shell capabilities. The model has been assessed in excellent agreement with available results from literature as well as MD NASTRAN software. Classical beam theories such as EulerBernoulli and Timoshenko have been obtained as particular cases. Therefore, this has made the comparison with higher-order models to be easily achieved for the structural and aeroelastic response as well as the divergence instability identification. Moreover, the CUF model has been able to detect in which case the first-order beam theories are accurate enough to describe the structural behavior of a beam-like system and in which case the use of refined theories is mandatory.

In general, the model here formulated can handle arbitrary cross-section geometries for moderate and high aspect ratio beam-like structures, with no limitations on the composite lamination properties. However, for multilayered cross-sections an expansion of Lagrange
polynomials for CUF and a layerwise approach may be preferred to detect local effects of composite materials. Both the static structural and aeroelastic analyses can be performed with a limitated number of DOFs in a good agreement with commercial software. Future works will investigate even more the method's capabilities especially in case of complex cross-section layouts and unconventional wing configurations. The choice of the crosssection geometrical shape will not affect the number of DOFs of the present model, unlike shell and solid finite elements codes.

In Chapter 12 an innovative approach for the static aeroelastic analysis of wings is presented. The coupled analyses here reported are performed by means of a panel method for the evaluation of the aerodynamic loads and by means of refined beam model (CUF 1D) for the evaluation if the structural static response. The influence of the expansion order N and the influence of the freestream velocity has been investigated. Few conclusions can be drawn:

1. in the case the wing is rather flexible the wing distortion has a great impact on the alteration of the aerodynamic loads compared with the undeformed wing configuration;
2. CUF 1D model is able to evaluate properly the distortion of a wing but since a particular expansion order, in this case the order is the fourteenth;
3. higher the freestream velocity is much more marked the distortion effect is.

The linear approach for the analysis of the distortion of a cross-section of a wing can profitably employed in order to evaluate the perfomance and the structural response of an aircraft. Comparing to the usual approch (CFD analysis coupled with a shell FE analysis) the use of panel method allows a reduction of computationl cost which is further reduced thanks to the use of CUF 1D model. This approach offers an effective tool for the analysis of a smart wing. The recent scientific literature starts to consider the so called smart wing configuration. The possible mechanisms that can be used employ actuators realized with piezoelectric materials (PZM), shape memory alloys (SMA) or shape memory polymer (SMP). Carrera Unified Formulation has the possibility to include in its formulation the induced displacement by one of usually employed actuators: for this reason CUF offers a performing tool for the analysis for such wing configurations.

Another possibilty offered by the CUF 1D is the reduction of computational cost in case a optimization process is needed. The optimization process of a complex structure such as a wing in general requires high computational cost: it requires a lot of executions in order to find the desidered optimum condition given the constraint conditions. The reduction of computational cost in terms of reduction of DOFs offered by the CUF allows to implement an accurate and light optimization tool.

## Appendix A

## Material and physical coordinate systems

A material coordinate system $(1,2,3)$ and a physical coordinate system $(x, y, z)$ have been introduced in section 2.3. Relations between the components of the stress and strain vectors referred to the material system and the components referred to the physical system have been presented in Eqs. 2.20 and 2.22, respectively. The aim of this appendix chapter is to show how Eqs. 2.20 and 2.22 can be obtained starting from the coordinate transformation equations between material and physical systems.

According to Fig. 2.4, material axes 2 and 3 are rotated by a positive counterclockwise angle $\theta$ about the $z$ axis, coincident to axis 1 , from physical $x$ and $y$ axes. The coordinate transformation from the physical system to the material system occurs by means of a matrix, identified as $\mathbf{L}$. Its dimensions are $3 \times 3$ since the space used is three-dimensional. The construction of $\mathbf{L}$ is carried out introducing three points $A, B$, and $C$ as can be seen in Fig. A.1.


Figure A.1: Coordinate transformation from the physical reference system to the material reference system.

Their coordinates in the two reference systems are:

$$
\text { Point A : }\left\{\begin{array}{l}
x=1 \\
y=0 \\
z=0
\end{array}\right\} ;\left\{\begin{array}{l}
x_{1}=0 \\
x_{2}=\cos \theta \\
x_{3}=-\sin \theta
\end{array}\right\}
$$

$$
\begin{align*}
& \text { Point B : }\left\{\begin{array}{l}
x=0 \\
y=1 \\
z=0
\end{array}\right\} ;\left\{\begin{array}{l}
x_{1}=0 \\
x_{2}=\sin \theta \\
x_{3}=\cos \theta
\end{array}\right\}  \tag{A.1}\\
& \text { Point C : }\left\{\begin{array}{l}
x=0 \\
y=0 \\
z=1
\end{array}\right\} ;\left\{\begin{array}{l}
x_{1}=1 \\
x_{2}=0 \\
x_{3}=0
\end{array}\right\}
\end{align*}
$$

The coordinates of points A, B, and C in the material system are the first, second, and third columns of matrix $\mathbf{L}$. As a consequence, the coordinates (components) ( $x_{1}, x_{2}, x_{3}$ ) of a generic point (vector) written in the material system ( $\boldsymbol{x}_{m}$ ) can be derived from its coordinates (components) $(x, y, z)$ written in the physical system $\left(\boldsymbol{x}_{p}\right)$ by means of matrix L as follows:

$$
\begin{equation*}
\boldsymbol{x}_{m}=\mathbf{L} \boldsymbol{x}_{p} \tag{A.2}
\end{equation*}
$$

which is:

$$
\left\{\begin{array}{l}
x_{1}  \tag{A.3}\\
x_{2} \\
x_{3}
\end{array}\right\}=\left[\begin{array}{ccc}
0 & 0 & 1 \\
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0
\end{array}\right]\left\{\begin{array}{l}
x \\
y \\
z
\end{array}\right\}
$$

Similarly, the coordinate transformation from the material system to the physical system occurs by means of a $3 \times 3$ matrix, identified as $\mathbf{L}^{-1}$. The construction of $\mathbf{L}^{-1}$ is carried out introducing three points D, E, and E as can be seen in Fig. A.2.


Figure A.2: Coordinate transformation from the material reference system to the physical reference system.

Their coordinates in the two reference systems are:

$$
\begin{align*}
& \text { Point D : }\left\{\begin{array}{l}
x_{1}=1 \\
x_{2}=0 \\
x_{3}=0
\end{array}\right\} ;\left\{\begin{array}{l}
x=0 \\
y=0 \\
z=1
\end{array}\right\} \\
& \text { Point E : }\left\{\begin{array}{l}
x_{1}=0 \\
x_{2}=1 \\
x_{3}=0
\end{array}\right\} ;\left\{\begin{array}{l}
x=\cos \theta \\
y=\sin \theta \\
z=0
\end{array}\right\} \\
& \text { Point F : }\left\{\begin{array}{l}
x_{1}=0 \\
x_{2}=0 \\
x_{3}=1
\end{array}\right\} ;\left\{\begin{array}{l}
x=-\sin \theta \\
y=\cos \theta \\
z=0
\end{array}\right\} \tag{A.4}
\end{align*}
$$

The coordinates of points $D, E$, and $F$ in the physical system are the first, second, and third columns of matrix $\mathbf{L}^{-1}$. As a consequence, the coordinates (components) $(x, y, z)$ of a generic point (vector) written in the physical system $\left(\boldsymbol{x}_{p}\right)$ can be derived from its coordinates (components) $\left(x_{1}, x_{2}, x_{3}\right)$ written in the material system ( $\boldsymbol{x}_{m}$ ) by means of matrix $\mathbf{L}^{-1}$ as follows:

$$
\begin{equation*}
\boldsymbol{x}_{p}=\mathbf{L}^{-1} \boldsymbol{x}_{m}=\mathbf{L}^{T} \boldsymbol{x}_{m} \tag{A.5}
\end{equation*}
$$

which is:

$$
\left\{\begin{array}{l}
x  \tag{A.6}\\
y \\
z
\end{array}\right\}=\left[\begin{array}{ccc}
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta \\
1 & 0 & 0
\end{array}\right]\left\{\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right\}
$$

As expected, it can be demonstrated that $\mathbf{L}^{-1}$ is the inverse of matrix $\mathbf{L}$. Furthermore, $\mathbf{L}$ is an orthogonal matrix and its transpose $\mathbf{L}^{T}$ is equal to its inverse $\mathbf{L}^{-1}$, as shown in Eq. A.5.

Now the relationship between the components of stress in physical and material coordinate systems is considered. Let the $3 \times 3$ arrays $\boldsymbol{\sigma}_{m}^{\star}$ and $\boldsymbol{\sigma}_{p}^{\star}$ of the stress components in the material and physical coordinate systems to be introduced:

$$
\begin{gather*}
\boldsymbol{\sigma}_{m}^{\star}=\left[\begin{array}{lll}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{array}\right]=\mathbf{L} \boldsymbol{\sigma}_{p}^{\star} \mathbf{L}^{T}  \tag{A.7}\\
\boldsymbol{\sigma}_{p}^{\star}=\left[\begin{array}{lll}
\sigma_{x x} & \sigma_{x y} & \sigma_{x z} \\
\sigma_{x y} & \sigma_{y y} & \sigma_{y z} \\
\sigma_{x z} & \sigma_{y z} & \sigma_{z z}
\end{array}\right]=\mathbf{L}^{T} \boldsymbol{\sigma}_{m}^{\star} \mathbf{L} \tag{A.8}
\end{gather*}
$$

Their relationships via matrices $\mathbf{L}$ and $\mathbf{L}^{T}$ derive from the transformations between secondorder tensors. More details can be found in [95]. Equations A. 7 and A. 8 hold for any general coordinate transformation, and hence it holds for the particular transformation in Fig. 2.4, i.e. Figs. A. 1 and A.2. Carrying out the matrix multiplications in Eq. A.8, Eq. A. 9 is obtained:

$$
\boldsymbol{\sigma}_{p}^{\star}=\left[\begin{array}{lll}
\sigma_{22} \cos ^{2} \theta-\sigma_{23} \sin 2 \theta & \sigma_{22} \sin \theta \cos \theta+\sigma_{23} \cos ^{2} \theta & \sigma_{12} \cos \theta  \tag{A.9}\\
+\sigma_{33} \sin ^{2} \theta & -\sigma_{23} \sin ^{2} \theta-\sigma_{33} \sin \theta \cos \theta & -\sigma_{13} \sin \theta \\
\sigma_{22} \sin \theta \cos \theta+\sigma_{23} \cos ^{2} \theta & \sigma_{22} \sin ^{2} \theta+\sigma_{23} \sin 2 \theta & \sigma_{12} \sin \theta \\
-\sigma_{23} \sin ^{2} \theta-\sigma_{33} \sin \theta \cos \theta & +\sigma_{33} \cos ^{2} \theta & +\sigma_{13} \cos \theta \\
\sigma_{12} \cos \theta-\sigma_{13} \sin \theta & \sigma_{12} \sin \theta+\sigma_{13} \cos \theta & \sigma_{11}
\end{array}\right]
$$

Rearranging the components of $\boldsymbol{\sigma}_{p}^{\star}$ in Eq. A. 9 in terms of the single-subscript stress components, the (physical) stress vector $\boldsymbol{\sigma}$ and the material stress vector $\boldsymbol{\sigma}_{m}$ defined in section 2.3 are obtained:

$$
\left\{\begin{array}{l}
\sigma_{y y}  \tag{A.10}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\cos ^{2} \theta & \sin ^{2} \theta & 0 & 0 & 0 & \sin 2 \theta \\
\sin ^{2} \theta & \cos ^{2} \theta & 0 & 0 & 0 & -\sin 2 \theta \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & -\sin \theta & 0 \\
0 & 0 & 0 & \sin \theta & \cos \theta & 0 \\
-\sin \theta \cos \theta & \sin \theta \cos \theta & 0 & 0 & 0 & \cos ^{2} \theta-\sin ^{2} \theta
\end{array}\right]\left\{\begin{array}{l}
\sigma_{33} \\
\sigma_{22} \\
\sigma_{11} \\
\sigma_{21} \\
\sigma_{31} \\
\sigma_{23}
\end{array}\right\}
$$

In a compact vectorial notation, Eq. A. 10 is:

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathbf{T} \boldsymbol{\sigma}_{m} \tag{A.11}
\end{equation*}
$$

where $\mathbf{T}$ is the $6 \times 6$ transformation matrix, already defined in Eq. 2.21. Equation A. 11 corresponds to Eq. 2.20. For the sake of completeness, the transformation of the stress vector components from the physical system to the material system is:

$$
\begin{equation*}
\boldsymbol{\sigma}_{m}=\mathbf{T}^{-1} \boldsymbol{\sigma} \tag{A.12}
\end{equation*}
$$

where $\mathbf{T}^{-1}$ is the inverse of the transformation matrix $\mathbf{T}$.
The relationship between the components of strain in physical and material coordinate systems is obtained through a procedure analogous to that above followed for stress components. Let the $3 \times 3$ arrays $\varepsilon_{m}^{\star}$ and $\varepsilon_{p}^{\star}$ of the strain components in the material and physical coordinate systems to be introduced:

$$
\begin{align*}
& \varepsilon_{m}^{\star}=\left[\begin{array}{lll}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33}
\end{array}\right]=\mathbf{L} \boldsymbol{\varepsilon}_{p}^{\star} \mathbf{L}^{T}  \tag{A.13}\\
& \varepsilon_{p}^{\star}=\left[\begin{array}{lll}
\varepsilon_{x x} & \varepsilon_{x y} & \varepsilon_{x z} \\
\varepsilon_{x y} & \varepsilon_{y y} & \varepsilon_{y z} \\
\varepsilon_{x z} & \varepsilon_{y z} & \varepsilon_{z z}
\end{array}\right]=\mathbf{L}^{T} \varepsilon_{m}^{\star} \mathbf{L} \tag{A.14}
\end{align*}
$$

Their relationships via matrices $\mathbf{L}$ and $\mathbf{L}^{T}$ are the same as the relationships for stress components in Eqs. A. 7 and A.8. Carrying out the matrix multiplications in Eq. A.13, Eq. A. 15 is obtained:

$$
\varepsilon_{m}^{\star}=\left[\begin{array}{lll}
\varepsilon_{z z} & \varepsilon_{x z} \cos \theta+\varepsilon_{y z} \sin \theta & -\varepsilon_{x z} \sin \theta+\varepsilon_{y z} \cos \theta  \tag{A.15}\\
\varepsilon_{x z} \cos \theta & \varepsilon_{x x} \cos ^{2} \theta+\varepsilon_{x y} \sin 2 \theta & -\varepsilon_{x x} \sin \theta \cos \theta+\varepsilon_{x y} \cos ^{2} \theta \\
+\varepsilon_{y z} \sin \theta & +\varepsilon_{y y} \sin ^{2} \theta & -\varepsilon_{x y} \sin ^{2} \theta+\varepsilon_{y y} \sin \theta \cos \theta \\
-\varepsilon_{x z} \sin \theta & -\sigma_{x x} \sin \theta \cos \theta+\sigma_{x y} \cos ^{2} \theta & \varepsilon_{x x} \sin ^{2} \theta-\varepsilon_{x y} \sin 2 \theta \\
+\varepsilon_{y z} \cos \theta & -\sigma_{x y} \sin ^{2} \theta+\sigma_{y y} \sin \theta \cos \theta & +\varepsilon_{y y} \cos ^{2} \theta
\end{array}\right]
$$

Rearranging the components of $\varepsilon_{m}^{\star}$ in Eq. A. 15 in terms of the single-subscript strain components, the vectorial relationships between the physical "modified" strain vector $\bar{\varepsilon}_{p}$ and the material "modified" strain vector $\bar{\varepsilon}_{m}$ are obtained:

$$
\left\{\begin{array}{l}
\varepsilon_{33}  \tag{A.16}\\
\varepsilon_{22} \\
\varepsilon_{11} \\
\varepsilon_{21} \\
\varepsilon_{31} \\
\varepsilon_{23}
\end{array}\right\}=\left[\begin{array}{cccccc}
\cos ^{2} \theta & \sin ^{2} \theta & 0 & 0 & 0 & -\sin 2 \theta \\
\sin ^{2} \theta & \cos ^{2} \theta & 0 & 0 & 0 & \sin 2 \theta \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\
0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\
\sin \theta \cos \theta & -\sin \theta \cos \theta & 0 & 0 & 0 & \cos ^{2} \theta-\sin ^{2} \theta
\end{array}\right]\left\{\begin{array}{l}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
\varepsilon_{x z} \\
\varepsilon_{y z} \\
\varepsilon_{x y}
\end{array}\right\}
$$

which is:

$$
\begin{equation*}
\bar{\varepsilon}_{m}=\mathbf{T}^{-1} \bar{\varepsilon}_{p} \tag{A.17}
\end{equation*}
$$

where $\mathbf{T}^{-1}$ is the inverse of the transformation matrix $\mathbf{T}$. Rearranging the components of $\bar{\varepsilon}_{m}$ and $\bar{\varepsilon}_{p}$ in Eq. A. 16 in order to introduce the material strain vector $\varepsilon_{m}$ and the
(physical) strain vector $\varepsilon$ defined in section 2.3 , the following relationship is obtained:

$$
\left\{\begin{array}{c}
\varepsilon_{33}  \tag{A.18}\\
\varepsilon_{22} \\
\varepsilon_{11} \\
2 \varepsilon_{21} \\
2 \varepsilon_{31} \\
2 \varepsilon_{23}
\end{array}\right\}=\left[\begin{array}{cccccc}
\cos ^{2} \theta & \sin ^{2} \theta & 0 & 0 & 0 & -\sin \theta \cos \theta \\
\sin ^{2} \theta & \cos ^{2} \theta & 0 & 0 & 0 & \sin \theta \cos \theta \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \cos \theta & \sin \theta & 0 \\
0 & 0 & 0 & -\sin \theta & \cos \theta & 0 \\
\sin 2 \theta & -\sin 2 \theta & 0 & 0 & 0 & \cos ^{2} \theta-\sin ^{2} \theta
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

In a compact vectorial notation, Eq. A. 18 is:

$$
\begin{equation*}
\boldsymbol{\varepsilon}_{m}=\mathbf{T}^{T} \boldsymbol{\varepsilon} \tag{A.19}
\end{equation*}
$$

where $\mathbf{T}^{T}$ is the transpose of the transformation matrix $\mathbf{T}$. Equation A. 19 corresponds to Eq. 2.22.

## Appendix B

## Poisson's locking correction

Constant and linear approximations of the displacement along the negligible direction do not yield accurate models even in the case of slender beams if full material constitutive equations are considered. Classical plates and beam theories adopt a modified version of such equations, in which the stiffness coefficients are opportunely reduced. The lack of accuracy is due to the coupling among the normal deformations along the spatial directions $x, y$, and $z$ quantified by Poisson's coefficients:

$$
\begin{equation*}
\nu_{i j}=-\frac{\varepsilon_{j j}}{\varepsilon_{i i}} \quad i, j=x, y, z \quad i \neq j \tag{B.1}
\end{equation*}
$$

This phenomenon is thus known as Poisson's Locking (PL). In the case of plates modelling, it is also named as Thickness Locking, see Carrera and Brischetto [120, 121]. Classical plates theories correct the PL by imposing the out-of-plane normal stress to be zero. This hypothesys yields reduced material stiffness coefficients to be taken into account in the Hooke's law for the in-plane stress and strain components. According to Carrera and Giunta [44], the same Poisson's locking correction is obtained for the 1D CUF model (i.e. beam theory) in a similar way, assuming the transversal stress components $\sigma_{x x}$ and $\sigma_{z z}$ equal to zero in the Hooke's law, for both orthotropic and isotropic materials.

Thanks to this correction, classical models and the refined first-order theory yield accurate results in the case of slender beams (compatibly to their kinematic limitative hypotheses).

## B. 1 Orthotropic Material

The vectorial notation of the stress-strain relations referred to the physical coordinate system of Eq. 2.25 for the orthotropic material case is retrieved here:

$$
\left\{\begin{array}{l}
\sigma_{y y}  \tag{B.2}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\widetilde{C}_{33} & \widetilde{C}_{23} & \widetilde{C}_{13} & 0 & 0 & \widetilde{C}_{36} \\
\widetilde{C}_{23} & \widetilde{C}_{22} & \widetilde{C}_{12} & 0 & 0 & \widetilde{C}_{26} \\
\widetilde{C}_{13} & \widetilde{C}_{12} & \widetilde{C}_{11} & 0 & 0 & \widetilde{C}_{16} \\
0 & 0 & 0 & \widetilde{C}_{44} & \widetilde{C}_{45} & 0 \\
0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{55} & 0 \\
\widetilde{C}_{36} & \widetilde{C}_{26} & \widetilde{C}_{16} & 0 & 0 & \widetilde{C}_{66}
\end{array}\right\}\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

The transversal stress components $\sigma_{x x}$ and $\sigma_{z z}$ are imposed to zero. According to Eq. B.2, they are:

$$
\left\{\begin{array}{l}
\sigma_{x x}=\widetilde{C}_{12} \varepsilon_{z z}+\widetilde{C}_{22} \varepsilon_{x x}+\widetilde{C}_{23} \varepsilon_{y y}+\widetilde{C}_{26}\left(2 \varepsilon_{x y}\right)=0  \tag{B.3}\\
\sigma_{z z}=\widetilde{C}_{11} \varepsilon_{z z}+\widetilde{C}_{12} \varepsilon_{x x}+\widetilde{C}_{13} \varepsilon_{y y}+\widetilde{C}_{16}\left(2 \varepsilon_{x y}\right)=0
\end{array}\right.
$$

An algebraic linear system in $\varepsilon_{x x}$ and $\varepsilon_{y y}$ is obtained, whose solution is:

$$
\left\{\begin{array}{l}
\epsilon_{x x}=\frac{\widetilde{C}_{12} \widetilde{C}_{13}-\widetilde{C}_{23} \widetilde{C}_{11}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}} \varepsilon_{y y}+\frac{\widetilde{C}_{12} \widetilde{C}_{16}-\widetilde{C}_{26} \widetilde{C}_{11}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}\left(2 \varepsilon_{x y}\right)  \tag{B.4}\\
\epsilon_{z z}=\frac{\widetilde{C}_{12} \widetilde{C}_{23}-\widetilde{C}_{13} \widetilde{C}_{22}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}} \varepsilon_{y y}+\frac{\widetilde{C}_{12} \widetilde{C}_{26}-\widetilde{C}_{16} \widetilde{C}_{22}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}\left(2 \varepsilon_{x y}\right)
\end{array}\right.
$$

Substituting Eq. B. 4 into the Hooke's equation for $\sigma_{y y}$ from Eq. B.2,

$$
\begin{equation*}
\sigma_{y y}=\widetilde{C}_{33} \varepsilon_{y y}+\widetilde{C}_{23} \varepsilon_{x x}+\widetilde{C}_{13} \varepsilon_{z z}+\widetilde{C}_{36}\left(2 \varepsilon_{x y}\right) \tag{B.5}
\end{equation*}
$$

it is obtained:

$$
\begin{equation*}
\sigma_{y y}=\widetilde{Q}_{33} \varepsilon_{y y}+\widetilde{Q}_{36}\left(2 \varepsilon_{x y}\right) \tag{B.6}
\end{equation*}
$$

where the corrected elastic coefficients $\widetilde{Q}_{33}$ and $\widetilde{Q}_{36}$ are:

$$
\begin{align*}
& \widetilde{Q}_{33}=\widetilde{C}_{33}+\widetilde{C}_{13} \frac{\widetilde{C}_{12} \widetilde{C}_{23}-\widetilde{C}_{13} \widetilde{C}_{22}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}+\widetilde{C}_{23} \frac{\widetilde{C}_{12} \widetilde{C}_{13}-\widetilde{C}_{23} \widetilde{C}_{11}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}  \tag{B.7}\\
& \widetilde{Q}_{36}=\widetilde{C}_{36}+\widetilde{C}_{13} \frac{\widetilde{C}_{12} \widetilde{C}_{26}-\widetilde{C}_{16} \widetilde{C}_{22}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}+\widetilde{C}_{23} \frac{\widetilde{C}_{12} \widetilde{C}_{16}-\widetilde{C}_{26} \widetilde{C}_{11}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}} \tag{B.8}
\end{align*}
$$

From Eqs. B. 3 and B. 4 the stress component $\sigma_{z z}$ can be rewritten as:

$$
\begin{equation*}
\sigma_{z z}=\widetilde{Q}_{13} \varepsilon_{y y}+\widetilde{Q}_{16}\left(2 \varepsilon_{x y}\right) \tag{B.9}
\end{equation*}
$$

where the corrected elastic coefficients $\widetilde{Q}_{13}$ and $\widetilde{Q}_{16}$ are:

$$
\begin{align*}
& \widetilde{Q}_{13}=0  \tag{B.10}\\
& \widetilde{Q}_{16}=0 \tag{B.11}
\end{align*}
$$

Similarly, from Eqs. B. 3 and B. 4 the stress component $\sigma_{x x}$ can be rewritten as:

$$
\begin{equation*}
\sigma_{x x}=\widetilde{Q}_{23} \varepsilon_{y y}+\widetilde{Q}_{26}\left(2 \varepsilon_{x y}\right) \tag{B.12}
\end{equation*}
$$

where the corrected elastic coefficients $\widetilde{Q}_{23}$ and $\widetilde{Q}_{26}$ are:

$$
\begin{align*}
& \widetilde{Q}_{23}=0  \tag{B.13}\\
& \widetilde{Q}_{26}=0 \tag{B.14}
\end{align*}
$$

As a consequence, it can be demonstrated that the stress component $\sigma_{x y}$, whose expression from Eq. B. 2 is:

$$
\begin{equation*}
\sigma_{x y}=\widetilde{C}_{36} \varepsilon_{y y}+\widetilde{C}_{26} \varepsilon_{x x}+\widetilde{C}_{16} \varepsilon_{z z}+\widetilde{C}_{66}\left(2 \varepsilon_{x y}\right) \tag{B.15}
\end{equation*}
$$

can be rewritten in the following corrected form:

$$
\begin{equation*}
\sigma_{x y}=\widetilde{Q}_{36} \varepsilon_{y y}+\widetilde{Q}_{26} \varepsilon_{y y}+\widetilde{Q}_{16} \varepsilon_{z z}+\widetilde{Q}_{66}\left(2 \varepsilon_{x y}\right) \tag{B.16}
\end{equation*}
$$

where the corrected elastic coefficient $\widetilde{Q}_{66}$ is:

$$
\begin{equation*}
\widetilde{Q}_{66}=\widetilde{C}_{66}+\widetilde{C}_{16} \frac{\widetilde{C}_{12} \widetilde{C}_{26}-\widetilde{C}_{16} \widetilde{C}_{22}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}}+\widetilde{C}_{26} \frac{\widetilde{C}_{12} \widetilde{C}_{16}-\widetilde{C}_{26} \widetilde{C}_{11}}{\widetilde{C}_{11} \widetilde{C}_{22}-\widetilde{C}_{12}^{2}} \tag{B.17}
\end{equation*}
$$

Summarizing the "corrected" equations written above for the stress components, when Poisson's locking correction is enabled the stress-strain relations of Eq. B. 2 become:

$$
\left\{\begin{array}{l}
\sigma_{y y}  \tag{B.18}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\widetilde{Q}_{33} & 0 & 0 & 0 & 0 & \widetilde{Q}_{36} \\
\widetilde{Q}_{23}=0 & 0 & 0 & 0 & 0 & \widetilde{Q}_{26}=0 \\
\widetilde{Q}_{13}=0 & 0 & 0 & 0 & 0 & \widetilde{Q}_{16}=0 \\
0 & 0 & 0 & \widetilde{C}_{44} & \widetilde{C}_{45} & 0 \\
0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{55} & 0 \\
\widetilde{Q}_{36} & \widetilde{Q}_{26}=0 & \widetilde{Q}_{16}=0 & 0 & 0 & \widetilde{Q}_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

The following corrected material stiffness matrix $\widetilde{\mathbf{Q}}$ can be thus defined as a summary of Poisson's locking correction:

$$
\begin{equation*}
\sigma=\widetilde{\mathbf{Q}} \varepsilon \tag{B.19}
\end{equation*}
$$

that is:

$$
\left\{\begin{array}{c}
\sigma_{y y}  \tag{B.20}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\widetilde{Q}_{33} & 0 & 0 & 0 & 0 & \widetilde{Q}_{36} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \widetilde{C}_{44} & \widetilde{C}_{45} & 0 \\
0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{55} & 0 \\
\widetilde{Q}_{36} & 0 & 0 & 0 & 0 & \widetilde{Q}_{66}
\end{array}\right] \quad\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

In conclusion, the following instructions for the orthotropic elastic coefficients have to be prescribed in order to correct Poisson's locking:

$$
\left\{\begin{array}{l}
\widetilde{C}_{33} \Rightarrow \widetilde{Q}_{33}  \tag{B.21}\\
\widetilde{C}_{23} \Rightarrow \widetilde{Q}_{23}=0 \\
\widetilde{C}_{13} \Rightarrow \widetilde{Q}_{13}=0 \\
\widetilde{C}_{36} \Rightarrow \widetilde{Q}_{36} \\
\widetilde{\widetilde{C}}_{22} \Rightarrow \widetilde{Q}_{22}=0 \\
\widetilde{C}_{12} \Rightarrow \widetilde{Q}_{12}=0 \\
\widetilde{C}_{26} \Rightarrow \widetilde{Q}_{26}=0 \\
\widetilde{C}_{11} \Rightarrow \widetilde{Q}_{11}=0 \\
\widetilde{C}_{16} \Rightarrow \widetilde{Q}_{16}=0 \\
\widetilde{C}_{66} \Rightarrow \widetilde{Q}_{66}
\end{array}\right.
$$

As can be noticed in Eq. B. 21 the elastic coefficients $\widetilde{C}_{44}, \widetilde{C}_{45}$, and $\widetilde{C}_{55}$ of the transformed material stiffness matrix $\widetilde{\mathbf{C}}$ of Eq. B. 2 are not affected by Poisson's locking correction.

## B. 2 Isotropic Material

When an isotropic material is used, Poisson's locking correction is easily obtained as a particular case of the procedure previously described for orthotropic materials. First of all, the Hooke's law for an isotropic material is here retrieved from Eq. 2.29:

$$
\left\{\begin{array}{l}
\sigma_{y y}  \tag{B.22}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\
C_{23} & C_{22} & C_{12} & 0 & 0 & 0 \\
C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

Following the same procedure explained for the orthotropic material case, the corrected elastic coefficients $Q_{33}$ and $Q_{36}$, which are particular cases of $\widetilde{Q}_{33}$ in Eq. B. 7 and $\widetilde{Q}_{36}$ in Eq. B.8, become:

$$
\begin{align*}
Q_{33} & =E  \tag{B.23}\\
Q_{36} & =0 \tag{B.24}
\end{align*}
$$

On the contrary, the corrected elastic coefficient $Q_{66}$, which is a particular case of $\widetilde{Q}_{66}$ in Eq. B.17, remains unchanged:

$$
\begin{equation*}
Q_{66}=C_{66}=G \tag{B.25}
\end{equation*}
$$

It can be demonstrated that the other corrected elastic coefficients are equal to zero. The corrected material stiffness matrix $\mathbf{Q}$ can be thus defined as a summary of Poisson's locking correction for the isotropic material case:

$$
\begin{equation*}
\boldsymbol{\sigma}=\mathbf{Q} \varepsilon \tag{B.26}
\end{equation*}
$$

that is:

$$
\left\{\begin{array}{c}
\sigma_{y y}  \tag{B.27}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
E & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

In conclusion, the following instructions for the isotropic elastic coefficients have to be prescribed in order to correct Poisson's locking:

$$
\left\{\begin{array}{l}
C_{33} \Rightarrow Q_{33}=E  \tag{B.28}\\
C_{23} \Rightarrow Q_{23}=0 \\
C_{13} \Rightarrow Q_{13}=0 \\
C_{22} \Rightarrow Q_{22}=0 \\
C_{12} \Rightarrow Q_{12}=0 \\
C_{11} \Rightarrow Q_{11}=0
\end{array}\right.
$$

As can be noticed in Eq. B. 28 the isotropic elastic coefficients $C_{44}, C_{55}$, and $C_{66}$ of the material stiffness matrix $\mathbf{C}$ of Eq. B. 22 are not affected by Poisson's locking correction.

## Appendix C

## Shape functions integration

The integrals of the products of shape functions along the finite element length $L_{\mathrm{EL}}$ in Eq. 3.41 have been introduced and collected in Eq. 3.44. The present appendix chapter addresses the different kinds of numerical integrations of these quantities. In particular, full, reduced or selective numerical integrations are faced.

## C. 1 Integration along natural coordinate

The expressions of integrals $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ in Eq. 3.44 are written with respect to the physical coordinate $y$. The integration of the products of shape functions results very easier along the natural coordinate $r$ than the physical coordinate $y$, which is related to the geometrical position of the finite element in the mesh (i.e. $y_{1}$ and $y_{2}$ ). As mentioned in Chapter 3, shape functions $N_{i}(y)$ or $N_{j}(y)$ can be transformed to the equivalent forms $N_{i}(r)$ or $N_{j}(r)$ referred to the natural coordinate $r$ by means of the coordinate transformation from the natural coordinate $r$ to the physical coordinate $y$ :

$$
\begin{equation*}
y=\frac{y_{1}}{2}(1-r)+\frac{y_{2}}{2}(1+r)=y_{1}+\frac{L_{\mathrm{EL}}}{2}(1+r) \tag{C.1}
\end{equation*}
$$

which is computed inverting Eq. 3.1. For the sake of cmpleteness, the 1D Lagrange polynomials with respect to the natural coordinate $r$ have been defined in Eq. 3.4 and also written for $B 2, B 3$, and $B 4$ elements in Eqs. 3.7, 3.10, and 3.13.

Let the first term $E_{j}^{i}$ to be considered:

$$
\begin{equation*}
E_{j}^{i}=\int_{l} N_{i} N_{j} \mathrm{~d} y=\int_{y_{1}}^{y_{2}} N_{i}(y) N_{j}(y) \mathrm{d} y \tag{C.2}
\end{equation*}
$$

where $y_{2}=y_{1}+L_{\mathrm{EL}}$. The integral $E_{j}^{i}$ is now transformed from the physical coordinate $y$ to the natural coordinate $r$ considering Eq. C. 1 and computing the Jacobian of the transofmation:

$$
\begin{equation*}
\frac{\mathrm{d} y}{\mathrm{~d} r}=-\frac{y_{1}}{2}+\frac{y_{2}}{2}=\frac{y_{2}-y_{1}}{2}=\frac{L_{\mathrm{EL}}}{2} \Rightarrow \mathrm{~d} y=\frac{L_{\mathrm{EL}}}{2} \mathrm{~d} r \tag{C.3}
\end{equation*}
$$

Thus, Eq. C. 2 becomes in a form independent of $y_{1}$ and $y_{2}$, but only dependent on the element length $L_{\mathrm{EL}}$ :

$$
\begin{equation*}
E_{j}^{i}=\frac{L_{\mathrm{EL}}}{2} \int_{-1}^{1} N_{i}(r) N_{j}(r) \mathrm{d} r \tag{C.4}
\end{equation*}
$$

where the boundary values:

$$
\begin{align*}
& y=y_{1} \quad \Rightarrow \quad r=-1  \tag{C.5}\\
& y=y_{2} \quad \Rightarrow \quad r=1
\end{align*}
$$

are computed exploiting the coordinate transformation from the physical coordinate $y$ along the finite element to the natural coordinate $r$ in Eq. 3.1.

The second term $E_{j, y}^{i}$ is now considered:

$$
\begin{equation*}
E_{j, y}^{i}=\int_{l} N_{i} N_{j, y} \mathrm{~d} y=\int_{y_{1}}^{y_{2}} N_{i}(y) \frac{\mathrm{d} N_{j}(y)}{\mathrm{d} y} \mathrm{~d} y \tag{C.6}
\end{equation*}
$$

The derivative of $N_{j}(y)$ with respect to $y$ is:

$$
\begin{equation*}
\frac{\mathrm{d} N_{j}(y)}{\mathrm{d} y}=\frac{\mathrm{d} N_{j}(r)}{\mathrm{d} r} \frac{\mathrm{~d} r}{\mathrm{~d} y}=\frac{\mathrm{d} N_{j}(r)}{\mathrm{d} r} \frac{2}{L_{\mathrm{EL}}} \tag{C.7}
\end{equation*}
$$

Thus, Eq. C. 6 with respect to $r$ becomes:

$$
\begin{equation*}
E_{j, y}^{i}=\frac{L_{\mathrm{E} \ell}}{2} \frac{2}{L_{\mathrm{EL}}} \int_{-1}^{1} N_{i}(r) \frac{\mathrm{d} N_{j}(r)}{\mathrm{d} r} \mathrm{~d} r=\int_{-1}^{1} N_{i}(r) N_{j, r}(r) \mathrm{d} r \tag{C.8}
\end{equation*}
$$

Similarly to Eq. C.8, the third term $E_{j}^{i, y}$ with respect to the natural coordinate $r$ is:

$$
\begin{equation*}
E_{j}^{i, y}=\int_{l} N_{i, y} N_{j} \mathrm{~d} y=\int_{-1}^{1} N_{i, r}(r) N_{j}(r) \mathrm{d} r \tag{C.9}
\end{equation*}
$$

The fourth term $E_{j, y}^{i, y}$ is:

$$
\begin{equation*}
E_{j, y}^{i, y}=\int_{l} N_{i, y} N_{j, y} \mathrm{~d} y=\int_{y_{1}}^{y_{2}} \frac{\mathrm{~d} N_{i}(y)}{\mathrm{d} y} \frac{\mathrm{~d} N_{j}(y)}{\mathrm{d} y} \mathrm{~d} y \tag{C.10}
\end{equation*}
$$

Using Eq. C.7, the derivative of $N_{i}(y)$ with respect to $y$ is:

$$
\begin{equation*}
\frac{\mathrm{d} N_{i}(y)}{\mathrm{d} y}=\frac{\mathrm{d} N_{i}(r)}{\mathrm{d} r} \frac{\mathrm{~d} r}{\mathrm{~d} y}=\frac{\mathrm{d} N_{i}(r)}{\mathrm{d} r} \frac{2}{L_{\mathrm{EL}}} \tag{C.11}
\end{equation*}
$$

Considering Eqs. C. 7 and C.11, Eq. C. 10 with respect to $r$ becomes:

$$
\begin{equation*}
E_{j, y}^{i, y}=\frac{L_{\mathrm{E} \ell}}{2} \frac{2}{L_{\mathrm{EL}}} \frac{2}{L_{\mathrm{EL}}} \int_{-1}^{1} \frac{\mathrm{~d} N_{i}(r)}{\mathrm{d} r} \frac{\mathrm{~d} N_{j}(r)}{\mathrm{d} r} \mathrm{~d} r=\frac{2}{L_{\mathrm{EL}}} \int_{-1}^{1} N_{i, r}(r) N_{j, r}(r) \mathrm{d} r \tag{C.12}
\end{equation*}
$$

The following section faces the numerical integration of integrals in Eqs. C.4, C.8, C.9, and C. 12 .

## C. 2 Gauss numerical one-dimensional integration

A numerical integration procedure typically evaluates a generic integral numerically by means of weighting factors. Considering for the sake of simplicity the generic onedimensional integral of the generic function $p(r)$ along the coordinate $r$ in the interval from $a$ to $b$ :

$$
\begin{equation*}
\int_{a}^{b} p(r) \mathrm{d} r=\sum_{k} w_{k} p\left(r_{k}\right) \tag{C.13}
\end{equation*}
$$

the integral is computed numerically as a linear combination (summation over index $k$ ) of the generic quantities $p\left(r_{k}\right)$, i.e. the values of function $p(r)$ evaluated at particular sampling points, through the corresponding weights $w_{k}$. Important issues are the integration accuracy that is needed, i.e. the number of sampling points required in the element formation, the positions of the sampling points, and hence the corresponding error of numerical integration.

Gauss quadrature is a very useful numerical integration procedure in which both the positions of the sampling points and the weights are optimized. According to Eq. C.13, the expression to compute numerically an integral via Gauss procedure is:

$$
\begin{equation*}
\int_{a}^{b} p(r) \mathrm{d} r=\sum_{k=1}^{N_{G P}} w_{k} p\left(r_{k}\right) \tag{C.14}
\end{equation*}
$$

where $N_{G P}$ is the number of Gauss points, i.e. sampling points, to be considered in the integration. In the present section the particular case of a polynomial function $p(r)$ is considered, since the one-dimensional functions to be integrated in Eqs. C.4, C.8, C.9, and C. 12 are products of shape functions (i.e. Lagrange polynomials) and hence are themselves polynomial functions. More details about integration of non-polynomial functions can be found in [42], where also the weights and the position of Gauss points to be used in Gauss quadrature procedure are presented.

Given a number of Gauss points $N_{G P}$, a polynomial $p(r)$ of order at most $\left(2 N_{G P}-1\right)$ is integrated exactly. Polynomials of orders less than $\left(2 N_{G P}-1\right)$ are also integrated exactly. As a consequence, given a polynomial $p(r)$ of order $n$, the minimum number of Gauss points $N_{G P}$ necessary to compute exactly the integral of $p(r)$ is given by the following equation:

$$
\begin{equation*}
N_{G P}=1+\left\lfloor\frac{n}{2}\right\rfloor \tag{C.15}
\end{equation*}
$$

where the floor function $\lfloor x\rfloor$ maps a real number $x$ to the largest integer not greater than $x$.

## C.2.1 Full numerical integration

When the full integration is active, all the integrals of the products of shape functions $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ are computed numerically with a number of Gauss points $N_{G P}$ sufficient to compute integrals in Eqs. C.4, C.8, C.9, and C. 12 exactly. For these integrals, the number of Gauss points $N_{G P}$ is set to $N_{N}$, i.e. 2 for $B 2$ elements, 3 for $B 3$ elements and 4 for $B 4$ elements. In fact, the order of the shape functions of an element is equal to ( $N_{N}-1$ ) and thus the products of two shape functions (or derivatives of shape functions) to be integrated is at most $2\left(N_{N}-1\right)$. Replacing $n$ in Eq. C. 15 with $2\left(N_{N}-1\right)$, it is verified that choice of $N_{G P}=N_{N}$ guarantees the exact integral computation:

$$
\begin{equation*}
N_{G P}=1+\left\lfloor\frac{2\left(N_{N}-1\right)}{2}\right\rfloor=1+\left\lfloor N_{N}-1\right\rfloor=N_{N} \tag{C.16}
\end{equation*}
$$

## C.2.2 Reduced numerical integration

The displacement formulation of finite element analysis yelds a strain energy smaller than the exact strain energy of the mechanical model being considered, and physically, a displacement formulation results in overestimating the system stiffness. Therefore, it is expected that by not evaluating the displacement-based element stiffness matrices accurately in the numerical integration, better overall solution results can be obtained. This should be the case if the error in the numerical integration compensates appropriately for the
overestimation of structural stiffness due to the finite element discretization. In other words, a reduction in the order of the numerical integration from the order that is required to evaluate the element stiffness matrices exactly may be expected to lead to improve results. When such a reduction is used for all the integrals of the products of shape functions, this procedure is called reduced integration. More details can be found in [42].

When the reduced integration is active, all the integrals of the products of shape functions $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ are therefore computed numerically with a number of Gauss points $N_{G P}$ one lower than the number of Gauss points necessary to compute integrals in Eqs. C.4, C.8, C.9, and C. 12 exactly. In particular, for these integrals, the number of Gauss points $N_{G P}$ is set to $\left(N_{N}-1\right)$, i.e. 1 for $B 2$ elements, 2 for $B 3$ elements and 3 for $B 4$ elements, differently from full numerical integration.

For the sake of simplicity, the terms $E_{j}^{i \star}, E_{j, y}^{i \star}, E_{j}^{i, y \star}$, and $E_{j, y}^{i, y \star}$ integrated via reduced integration are introduced instead of the terms $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$, which are fully (exactly) integrated. Hence, the latter terms have to be replaced with the former terms in Eqs. 3.115, 3.116, 3.124, and 3.125. It can be verified that:

$$
\begin{gather*}
E_{j}^{i \star} \leq E_{j}^{i} \\
E_{j, y}^{i, y}=E_{j, y}^{i} \\
E_{j}^{i, y \star}=E_{j}^{i, y}  \tag{C.17}\\
E_{j, y}^{i, y \star}=E_{j, y}^{i, y}
\end{gather*}
$$

since the polynomial functions to be integrated in $E_{j, y}^{i}, E_{j}^{i, y}$ have one order lower than the polynomial functions to be integrated in $E_{j}^{i}$ (i.e. $2\left(N_{N}-1\right)-1$ instead of $2\left(N_{N}-1\right)$ ) and thus $N_{G P}=\left(N_{N}-1\right)$ is sufficient to compute the integrals exactly. Similarly, the polynomial functions to be integrated in $E_{j, y}^{i, y}$ have two orders lower than the polynomial functions to be integrated in $E_{j}^{i}$ (i.e. $2\left(N_{N}-1\right)-2$ instead of $2\left(N_{N}-1\right)$ ) and $N_{G P}=\left(N_{N}-1\right)$ is therefore sufficient to compute the integrals exactly.

## C.2.3 Selective numerical integration

This is the numerical quadrature procedure used to obtain the results presented in this dissertation. A selective integration is widely used in finite element analysis as numerical correction to the well-known problem of shear locking. For more details see [42]. When the selective integration is active, strain terms are integrated with different orders of integration. In particular, the technique here presented uses a reduced integration for all the terms in fundamental nuclei which derive from shear strain or virtual shear strain quantities in the expression of virtual strain energy $\delta L_{\text {int }}$ (Eqs. 3.33 and 3.38) and a full integration for all the other terms.

For the orthotropic material case, the integrals $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ in the terms appearing in the fundamental nucleus components of Eqs. 3.115 and 3.124 which involve the red-coloured elastic coefficients of Eq. C. 18 (cf. Eq. 2.25) have to be integrated via reduced integration.

$$
\left\{\begin{array}{c}
\sigma_{y y}  \tag{C.18}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
\widetilde{C}_{33} & \widetilde{C}_{23} & \widetilde{C}_{13} & 0 & 0 & \widetilde{C}_{36} \\
\widetilde{C}_{23} & \widetilde{C}_{22} & \widetilde{C}_{12} & 0 & 0 & \widetilde{C}_{26} \\
\widetilde{C}_{13} & \widetilde{C}_{12} & \widetilde{C}_{11} & 0 & 0 & \widetilde{C}_{16} \\
0 & 0 & 0 & \widetilde{C}_{44} & \widetilde{C}_{45} & 0 \\
0 & 0 & 0 & \widetilde{C}_{45} & \widetilde{C}_{55} & 0 \\
\widetilde{C}_{36} & \widetilde{C}_{26} & \widetilde{C}_{16} & 0 & 0 & \widetilde{C}_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

The red-coloured elastic coefficients above mentioned are distributed in matrices $\widetilde{\mathbf{C}}_{p n}, \widetilde{\mathbf{C}}_{n p}$, and $\widetilde{\mathbf{C}}_{n n}$ as follows:

$$
\begin{gather*}
\widetilde{\mathbf{C}}_{p n}=\left[\begin{array}{ccc}
0 & \widetilde{C}_{16} & \widetilde{C}_{13} \\
0 & \widetilde{C}_{26} & \widetilde{C}_{23} \\
\widetilde{C}_{45} & 0 & 0
\end{array}\right] ; \quad \widetilde{\mathbf{C}}_{n p}=\left[\begin{array}{ccc}
0 & 0 & \widetilde{C}_{45} \\
\widetilde{C}_{16} & \widetilde{C}_{26} & 0 \\
\widetilde{C}_{13} & \widetilde{C}_{23} & 0
\end{array}\right] ; \\
\widetilde{\mathbf{C}}_{n n}=\left[\begin{array}{ccc}
\widetilde{C}_{55} & 0 & 0 \\
0 & \widetilde{C}_{66} & \widetilde{C}_{36} \\
0 & \widetilde{C}_{36} & \widetilde{C}_{33}
\end{array}\right] \tag{C.19}
\end{gather*}
$$

For the isotropic material case, the integrals $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ in the terms appearing in the fundamental nucleus components of Eqs. 3.116 and 3.125 which involve the redcoloured elastic coefficients of Eq. C. 20 (cf. Eq. 2.29) have to be integrated via reduced integration.

$$
\left\{\begin{array}{c}
\sigma_{y y}  \tag{C.20}\\
\sigma_{x x} \\
\sigma_{z z} \\
\sigma_{x z} \\
\sigma_{y z} \\
\sigma_{x y}
\end{array}\right\}=\left[\begin{array}{cccccc}
C_{33} & C_{23} & C_{13} & 0 & 0 & 0 \\
C_{23} & C_{22} & C_{12} & 0 & 0 & 0 \\
C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{array}\right]\left\{\begin{array}{c}
\varepsilon_{y y} \\
\varepsilon_{x x} \\
\varepsilon_{z z} \\
2 \varepsilon_{x z} \\
2 \varepsilon_{y z} \\
2 \varepsilon_{x y}
\end{array}\right\}
$$

The red-coloured elastic coefficients above mentioned are distributed in matrix $\widetilde{\mathbf{C}}_{n n}$ as follows:

$$
\mathbf{C}_{n n}=\left[\begin{array}{ccc}
C_{55} & 0 & 0  \tag{C.21}\\
0 & C_{66} & 0 \\
0 & 0 & C_{33}
\end{array}\right]
$$

As a consequence, some terms $E_{j}^{i}, E_{j, y}^{i}, E_{j}^{i, y}$, and $E_{j, y}^{i, y}$ have to be replaced with terms $E_{j}^{i \star}$, $E_{j, y}^{i \star}, E_{j}^{i, y \star}$, and $E_{j, y}^{i, y \star}$ (integrated via reduced integration) in Eqs. 3.115, 3.116, 3.124, and 3.125 , paying attention to the important note in Eq. C.17.

For a cross-section made of homogeneous orthotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ in Eq. 3.115 are here written in Eq. C. 22 with indication of the terms integrated via full or reduced integration (written in red):

$$
\begin{align*}
& K_{x x}^{\tau s i j}=\widetilde{C}_{22} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{44} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j, y}^{i \star} J_{s}^{\tau, x}+\widetilde{C}_{26} E_{j}^{i, y \star} J_{s, x}^{\tau}+\widetilde{C}_{66} E_{j, y}^{i, y \star} J_{s}^{\tau} \\
& K_{x y}^{\tau s i j}=\widetilde{C}_{23} E_{j, y}^{i} J_{s}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i \star} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j}^{i \star} J_{s, x}^{\tau, x}+\widetilde{C}_{36} E_{j, y}^{i, y \star} J_{s}^{\tau}+\widetilde{C}_{66} E_{j}^{i, y \star} J_{s, x}^{\tau} \\
& K_{x z}^{\tau s i j}=\widetilde{C}_{12} E_{j}^{i} J_{s, z}^{\tau, x}+\widetilde{C}_{44} E_{j}^{i} J_{s, x}^{\tau, z}+\widetilde{C}_{45} E_{j, y}^{i \star} J_{s}^{\tau, z}+\widetilde{C}_{16} E_{j}^{i, y \star} J_{s, z}^{\tau} \\
& K_{y x}^{\tau s i j}=\widetilde{C}_{23} E_{j}^{i, y} J_{s, x}^{\tau}+\widetilde{C}_{45} E_{j}^{i \star} J_{s, z}^{\tau, z}+\widetilde{C}_{26} E_{j}^{i \star} J_{s, x}^{\tau, x}+\widetilde{C}_{36} E_{j, y}^{i, y \star} J_{s}^{\tau}+\widetilde{C}_{66} E_{j, y}^{i \star} J_{s}^{\tau, x} \\
& K_{y y}^{\tau s i j}=\widetilde{C}_{33} E_{j, y}^{i, y} J_{s}^{\tau}+\widetilde{C}_{55} E_{j}^{i \star} J_{s, z}^{\tau, z}+\widetilde{C}_{36} E_{j, y}^{i \star} J_{s}^{\tau, x}+\widetilde{C}_{36} E_{j}^{i, y \star} J_{s, x}^{\tau}+\widetilde{C}_{66} E_{j}^{i \star} J_{s, x}^{\tau, x} \\
& K_{y z}^{\tau s i j}=\widetilde{C}_{13} E_{j}^{i, y} J_{s, z}^{\tau}+\widetilde{C}_{55} E_{j, y}^{i \star} J_{s}^{\tau, z}+\widetilde{C}_{45} E_{j}^{i \star} J_{s, x}^{\tau, z}+\widetilde{C}_{16} E_{j}^{i \star} J_{s, z}^{\tau, x} \\
& K_{z x}^{\tau s i j}=\widetilde{C}_{12} E_{j}^{i} J_{s, x}^{\tau, z}+\widetilde{C}_{44} E_{j}^{i} J_{s, z}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i, y \star} J_{s, z}^{\tau}+\widetilde{C}_{16} E_{j, y}^{i \star} J_{s}^{\tau, z} \\
& K_{z y}^{\tau s i j}=\widetilde{C}_{13} E_{j, y}^{i} J_{s}^{\tau, z}+\widetilde{C}_{55} E_{j}^{i, y \star} J_{s, z}^{\tau}+\widetilde{C}_{45} E_{j}^{i \star} J_{s, z}^{\tau, x}+\widetilde{C}_{16} E_{j}^{i \star} J_{s, x}^{\tau, z} \\
& K_{z z}^{\tau s i j}=\widetilde{C}_{11} E_{j}^{i} J_{s, z}^{\tau, z}+\widetilde{C}_{44} E_{j}^{i} J_{s, x}^{\tau, x}+\widetilde{C}_{55} E_{j, y}^{i, y \star} J_{s}^{\tau}+\widetilde{C}_{45} E_{j, y}^{i \star} J_{s}^{\tau, x}+\widetilde{C}_{45} E_{j}^{i, y \star} J_{s, x}^{\tau} \tag{C.22}
\end{align*}
$$

For a cross-section made of homogeneous isotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ in Eq. 3.116 are here written in Eq. C. 23 with indication of the terms integrated via full or reduced integration (written in red):

$$
\begin{align*}
K_{x x}^{\tau s i j} & =C_{22} E_{j}^{i} J_{s, x}^{\tau, x}+C_{44} E_{j}^{i} J_{s, z}^{\tau, z}+C_{66} E_{j, y}^{i, y \star} J_{s}^{\tau} \\
K_{x y}^{\tau s i j} & =C_{23} E_{j, y}^{i} J_{s}^{\tau, x}+C_{66} E_{j}^{i, y \star} J_{s, x}^{\tau} \\
K_{x z}^{\tau s i j} & =C_{12} E_{j}^{i} J_{s, z}^{\tau, x}+C_{44} E_{j}^{i} J_{s, x}^{\tau, z} \\
K_{y x}^{\tau s i j} & =C_{23} E_{j}^{i, y} J_{s, x}^{\tau}+C_{66} E_{j, y}^{i \star} J_{s}^{\tau, x} \\
K_{y y}^{\tau s i j} & =C_{33} E_{j, y}^{i, y} J_{s}^{\tau}+C_{55} E_{j}^{i \star} J_{s, z}^{\tau, z}+C_{66} E_{j}^{i \star} J_{s, x}^{\tau, x}  \tag{C.23}\\
K_{y z}^{\tau s i j} & =C_{13} E_{j}^{i, y} J_{s, z}^{\tau}+C_{55} E_{j, y}^{i \star} J_{s}^{\tau, z} \\
K_{z x}^{\tau s i j} & =C_{12} E_{j}^{i} J_{s, x}^{\tau, z}+C_{44} E_{j}^{i} J_{s, z}^{\tau, x} \\
K_{z y}^{\tau s i j} & =C_{13} E_{j, y}^{i} J_{s}^{\tau, z}+C_{55} E_{j}^{i, y \star} J_{s, z}^{\tau} \\
K_{z z}^{\tau s i j} & =C_{11} E_{j}^{i} J_{s, z}^{\tau, z}+C_{44} E_{j}^{i} J_{s, x}^{\tau, x}+C_{55} E_{j, y}^{i, y \star} J_{s}^{\tau}
\end{align*}
$$

For a cross-section made of nonhomogeneous orthotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ in Eq. 3.124 are here written in Eq. C. 24 with indication of the terms integrated via full or reduced integration (written in red):

$$
\begin{aligned}
& K_{x x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{22} F_{s, x} \triangleright_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{44} F_{s, z} \triangleright{ }_{\Omega}+E_{j, y}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s} \triangleright_{\Omega}+ \\
& E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{26} F_{s, x} \triangleright_{\Omega}+E_{j, y}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{66} F_{s} \triangleright_{\Omega} \\
& K_{x y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{23} F_{s} \triangleright_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, z} \triangleright_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s, x} \triangleright_{\Omega}+ \\
& E_{j, y}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s} \triangleright{ }_{\Omega}+E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{66} F_{s, x} \triangleright{ }_{\Omega} \\
& K_{x z}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{12} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{44} F_{s, x} \triangleright{ }_{\Omega}+E_{j, y}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s} \triangleright_{\Omega}+ \\
& E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{16} F_{s, z} \triangleright{ }_{\Omega} \\
& K_{y x}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{23} F_{s, x} \triangleright{ }_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{26} F_{s, x} \triangleright{ }_{\Omega}+ \\
& E_{j, y}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s} \triangleright{ }_{\Omega}+E_{j, y}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{66} F_{s} \triangleright{ }_{\Omega} \\
& K_{y y}^{\tau s i j}=E_{j, y}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{33} F_{s} \triangleright_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{55} F_{s, z} \triangleright_{\Omega}+E_{j, y}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{36} F_{s} \triangleright{ }_{\Omega}+ \\
& E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{36} F_{s, x} \triangleright_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{66} F_{s, x} \triangleright_{\Omega} \\
& K_{y z}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} \widetilde{C}_{13} F_{s, z} \triangleright{ }_{\Omega}+E_{j, y}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{55} F_{s} \triangleright{ }_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{45} F_{s, x} \triangleright \Omega+ \\
& E_{j}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{16} F_{s, z} \triangleright{ }_{\Omega} \\
& K_{z x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{12} F_{s, x} \triangleright_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{44} F_{s, z} \triangleright_{\Omega}+E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{45} F_{s, z} \triangleright_{\Omega}+ \\
& E_{j, y}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{16} F_{s} \triangleright{ }_{\Omega} \\
& K_{z y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{13} F_{s} \triangleright{ }_{\Omega}+E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{55} F_{s, z} \triangleright{ }_{\Omega}+E_{j}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{45} F_{s, z} \triangleright{ }_{\Omega}+ \\
& E_{j}^{i \star} \triangleleft F_{\tau, z} \widetilde{C}_{16} F_{s, x} \triangleright{ }_{\Omega}
\end{aligned}
$$

$$
\begin{align*}
K_{z z}^{\tau s i j}= & E_{j}^{i} \triangleleft F_{\tau, z} \widetilde{C}_{11} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} \widetilde{C}_{44} F_{s, x} \triangleright \Omega+E_{j, y}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{55} F_{s} \triangleright \Omega+ \\
& E_{j, y}^{i \star} \triangleleft F_{\tau, x} \widetilde{C}_{45} F_{s} \triangleright \Omega+E_{j}^{i, y \star} \triangleleft F_{\tau} \widetilde{C}_{45} F_{s, x} \triangleright \Omega \tag{C.24}
\end{align*}
$$

For a cross-section made of nonhomogeneous isotropic material, the components of the fundamental nucleus $\mathbf{K}^{\tau s i j}$ in Eq. 3.125 are here written in Eq. C. 25 with indication of the terms integrated via full or reduced integration (written in red):

$$
\begin{align*}
& K_{x x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} C_{22} F_{s, x} \triangleright{ }_{\Omega}+E_{j}^{i} \triangleleft F_{\tau, z} C_{44} F_{s, z} \triangleright{ }_{\Omega}+E_{j, y}^{i, y \star} \triangleleft F_{\tau} C_{66} F_{s} \triangleright \Omega \\
& K_{x y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, x} C_{23} F_{s} \triangleright \Omega+E_{j}^{i, y \star} \triangleleft F_{\tau} C_{66} F_{s, x} \triangleright \Omega \\
& K_{x z}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, x} C_{12} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, z} C_{44} F_{s, x} \triangleright \Omega \\
& K_{y x}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} C_{23} F_{s, x} \triangleright \Omega+E_{j, y}^{i \star} \triangleleft F_{\tau, x} C_{66} F_{s} \triangleright \Omega \\
& K_{y y}^{\tau s i j}=E_{j, y}^{i, y} \triangleleft F_{\tau} C_{33} F_{s} \triangleright \Omega+E_{j}^{i \star} \triangleleft F_{\tau, z} C_{55} F_{s, z} \triangleright \Omega+E_{j}^{i \star} \triangleleft F_{\tau, x} C_{66} F_{s, x} \triangleright \Omega \\
& K_{y z}^{\tau s i j}=E_{j}^{i, y} \triangleleft F_{\tau} C_{13} F_{s, z} \triangleright \Omega+E_{j, y}^{i \star} \triangleleft F_{\tau, z} C_{55} F_{s} \triangleright \Omega \\
& K_{z x}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, z} C_{12} F_{s, x} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} C_{44} F_{s, z} \triangleright \Omega \\
& K_{z y}^{\tau s i j}=E_{j, y}^{i} \triangleleft F_{\tau, z} C_{13} F_{s} \triangleright \Omega+E_{j}^{i, y \star} \triangleleft F_{\tau} C_{55} F_{s, z} \triangleright \Omega \\
& K_{z z}^{\tau s i j}=E_{j}^{i} \triangleleft F_{\tau, z} C_{11} F_{s, z} \triangleright \Omega+E_{j}^{i} \triangleleft F_{\tau, x} C_{44} F_{s, x} \triangleright \Omega+E_{j, y}^{i, y \star} \triangleleft F_{\tau} C_{55} F_{s} \triangleright \Omega \tag{C.25}
\end{align*}
$$

## Appendix D

## Constraints

Boundary conditions of different kind can be handled by the present one-dimensional CUF model. For the sake of brevity, the imposition of the classical constraints widely used in beam modeling is presented here. These classical constraints represents boundary conditions on a boundary cross-section lying on the beam with length $L$ either at $y_{c}=0$ or at $y_{c}=L$. Nonetheless, general conditions on the displacement field over any arbitrary cross-section of the structure can be imposed.

Moreover, it is important to remark that with the 1D CUF FE model, based on the choice of Maclaurin polynomials as cross-section functions (see chapter 3), it is possible to impose Dirichlet boundary conditions, i.e. prescribed displacements, not only on crosssections, but also on the lateral surface of the structural domain. For this purpose, the procedure described in section 13.4.6 for the Galerkin approximation of fluid dynamic Stokes equations through the 1D CUF FE model has to be followed. This capability is not standard in classical beam modeling and supports the promising potentiality of the proposed reduced order approach.

The displacement field described by 1D CUF model combined to the finite element approximation is here retrieved (see Eq. 3.20):

$$
\mathbf{u}(x, y, z)=F_{\tau}(x, z) N_{i}(y) \mathbf{q}_{\tau i} \quad \begin{align*}
& \tau=1, \ldots, N_{u}  \tag{D.1}\\
& \\
& i=1, \ldots, N_{N}
\end{align*}
$$

where repeated subscripts $\tau$ and $i$ indicate summation based on Einstein's notation and the dependence on time is neglected. For the sake of simplicity, a constraint on the displacement of the cross-section at $y_{c}=0$ is considered here. The constrained cross-section at $y_{c}=0$ lies thus on the first node of the first finite element of the 1D mesh used to discretize the structure. According to Eq. 3.3, the numerical value of the generic $i^{\text {th }}$ shape function $N_{i}$ is:

$$
N_{i}\left(y=y_{c}=0\right)= \begin{cases}1 & i=1  \tag{D.2}\\ 0 & i \neq 1\end{cases}
$$

Using Eq. D.2, the displacement field over the constrained cross-section at $y_{c}=0$ becomes:

$$
\begin{equation*}
\mathbf{u}\left(x, y=y_{c}=0, z\right)=F_{\tau}(x, z) \mathbf{q}_{\tau 1} \quad \tau=1, \ldots, N_{u} \tag{D.3}
\end{equation*}
$$

The most used classical constraints in beam modeling, such as clamped, hinge and roller supports, prescribe different conditions on the displacement field of the cross-section to be constrained in Eq. D.3. Their treatment via the 1D CUF approach acts directly on the nodal displacement unknowns and is now briefly presented.

## D. 1 Clamped cross-section



Figure D.1: Clamped cross-section.
The clamped cross-section constraint depicted in Fig. D. 1 prescribes a null displacement of the cross-section at $y_{c}=0$ along all the three directions $x, y, z$, i.e. $\mathbf{u}\left(x, y=y_{c}=0, z\right)=$ 0. Hence, at $y_{c}=0$ :

$$
\left\{\begin{array}{l}
u_{x}=0  \tag{D.4}\\
u_{y}=0 \\
u_{z}=0
\end{array}\right.
$$

According to the displacement field in Eq. D. 3 of the cross-section at $y_{c}=0$, the clamped cross-section constraint is obtained by the following imposition on the nodal displacement unknowns:

$$
\begin{cases}q_{u_{\tau} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}  \tag{D.5}\\ q_{u_{y} \tau 1}=0 & \forall \tau=1, \ldots, N_{u} \\ q_{u_{z} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}\end{cases}
$$

## D. 2 Hinge supported cross-section



Figure D.2: Hinge supported cross-section.
According to the displacement field in Eq. D. 3 of the cross-section at $y_{c}=0$, a first condition to obtain the hinge supported constraint depicted in Fig. D. 2 is imposed on the nodal displacement unknowns as follows:

$$
\begin{cases}q_{u_{x} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}  \tag{D.6}\\ q_{u_{y} 11}=0 & \\ q_{u_{z} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}\end{cases}
$$

It is noteworthy that with the constraint in Eq. D. 6 the cross-section at $y_{c}=0$ can rotate remaining planar (since $q_{u_{y} 21}$ and $q_{u_{y} 31}$ are not constrained) but can also assume a non-planar configuration due to the higher-order terms when the expansion order $N$ is
higher than 1 . In fact, also the components $q_{u_{y} \tau 1}$ for $\tau=4, \ldots, N_{u}$ are not constrained. In order to simulate a planar deformation of the cross-section at $y_{c}=0$ in the case of an expansion order $N>1$ (higher-order theories), the following condition on the $y$-component of the nodal displacement vectors has to be added to the conditions of Eq. D.6:

$$
\begin{equation*}
q_{u_{y} \tau 1}=0 \quad \forall \tau=4, \ldots, N_{u} \tag{D.7}
\end{equation*}
$$

Moreover, in order to simulate a pure rotation of the cross-section at $y_{c}=0$ about the transversal $x$ axis, the following condition on $q_{u_{y} 21}$ has to be added to the conditions of Eqs. D. 6 and D.7:

$$
\begin{equation*}
q_{u_{y} 21}=0 \tag{D.8}
\end{equation*}
$$

On the contrary, in order to simulate a pure rotation of the cross-section at $y_{c}=0$ about the transversal $z$ axis, the following condition on $q_{u_{y} 31}$ has to be added to the conditions of Eqs. D. 6 and D.7:

$$
\begin{equation*}
q_{u_{y} 31}=0 \tag{D.9}
\end{equation*}
$$

## D. 3 Roller supported cross-section



Figure D.3: Roller supported cross-section.
The treatment of this constrain is very similar to the treatment of the hinge supported constraint in section D.2. According to the displacement field in Eq. D. 3 of the cross-section at $y_{c}=0$, a first condition to obtain the roller supported constraint depicted in Fig. D. 3 is imposed on the nodal displacement unknowns as follows:

$$
\begin{cases}q_{u_{x} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}  \tag{D.10}\\ q_{u_{z} \tau 1}=0 & \forall \tau=1, \ldots, N_{u}\end{cases}
$$

It is noteworthy that with the constraint in Eq. D. 10 the cross-section at $y_{c}=0$ can traslate along the $y$ direction (since $q_{u_{y} 11}$ is not constrained), can rotate remaining planar (since $q_{u_{y} 21}$ and $q_{u_{y} 31}$ are not constrained), but can also assume a non-planar configuration due to the higher-order terms when the expansion order $N$ is higher than 1 . In fact, also the components $q_{u_{y} \tau 1}$ for $\tau=4, \ldots, N_{u}$ are not constrained. In order to simulate a planar deformation of the cross-section at $y_{c}=0$ in the case of an expansion order $N>1$ (higher-order theories), the following condition on the $y$-component of the nodal displacement vectors has to be added to the conditions of Eq. D.10:

$$
\begin{equation*}
q_{u_{y} \tau 1}=0 \quad \forall \tau=4, \ldots, N_{u} \tag{D.11}
\end{equation*}
$$

Moreover, in order to simulate a pure rotation of the cross-section at $y_{c}=0$ about the transversal $x$ axis, the following condition on $q_{u_{y} 21}$ has to be added to the conditions of Eqs. D. 10 and D.11:

$$
\begin{equation*}
q_{u_{y} 21}=0 \tag{D.12}
\end{equation*}
$$

On the contrary, in order to simulate a pure rotation of the cross-section at $y_{c}=0$ about the transversal $z$ axis, the following condition on $q_{u_{y} 31}$ has to be added to the conditions of Eqs. D. 10 and D.11:

$$
\begin{equation*}
q_{u_{y} 31}=0 \tag{D.13}
\end{equation*}
$$

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## Curriculum Vitae

## Alberto VARELLO

## Personal details

## Date of birth September 25th, 1984

Place of birth Torino, Italy

## Academic education

2010 - 2012 Ph.D. student in Aerospace Engineering Politecnico di Torino, Italy<br>Laboratory MUL2, Department of Mechanical and Aerospace Engineering<br>Collaboration CMCS - Chair of Modeling and Scientific Computing, École Polytechnique Fédérale de Lausanne (EPFL), Switzerland<br>Collaborators Prof. Alfio Quarteroni, Dr. Luca Dedè, Prof. Davide Ambrosi Advisor Prof. Erasmo Carrera<br>2009 Research assistant (MICRO.COST project)<br>Politecnico di Torino, Italy<br>Laboratory MUL2, Department of Aerospace Engineering<br>Project Numerical simulation of wind turbine blades<br>Advisor Prof. Erasmo Carrera<br>2007-2009 Master of Science in Aerospace Engineering Politecnico di Torino, Italy<br>Grade 110/110 with honors - Graduated on July 22nd, 2009<br>Laboratory MUL2, Department of Aerospace Engineering<br>Collaboration Computational Aeroelasticity and Structural Dynamics Lab, San Diego State University (SDSU), California, USA<br>Thesis An advanced aeroelastic model for the analysis of beam-like structures<br>Advisors Prof. Erasmo Carrera, Prof. Luciano Demasi<br>2003-2007 Bachelor of Science in Aerospace Engineering Politecnico di Torino, Italy<br>Grade 110/110 with honors - Graduated on March 14th, 2007<br>Laboratory Department of Aerospace Engineering, Department of Mathematics<br>Thesis Innovative methodology in racing bobsleigh design: integration between CAD and CFD<br>Advisors Prof. Paolo Maggiore, Prof. Stefano Berrone

## Scientific publications

## Book

- E. Carrera, M. Petrolo, A. Varello, and E. Zappino. Classical Aeroelasticity of Lifting Body by Finite Element Methods, John Wiley \& Sons, in preparation, 2013.


## International journals

- E. Carrera, A. Varello, and L. Demasi. A refined structural model for static aeroelastic response and divergence of metallic and composite wings. CEAS Aeronautical Journal, 4(2): 175-189, 2013.
- E. Carrera and A. Varello. Dynamic response of thin-walled structures by variable kinematic one-dimensional models. Journal of Sound and Vibration, 331(24): 52685282, 2012.
- E. Carrera, M. Petrolo, and A. Varello. Advanced beam formulations for free vibration analysis of conventional and joined wings. Journal of Aerospace Engineering, 25(2): 282-293, 2012.
- A. Varello, E. Carrera, and L. Demasi. Vortex lattice method coupled with advanced one-dimensional structural models. Journal of Aeroelasticity and Structural Dynamics, 2(2): 53-78, 2011.
- A. Varello and E. Carrera. Free vibration response of thin and thick shells by refined one-dimensional analysis. Submitted, 2013.
- A. Varello and E. Carrera. Nonhomogeneous atherosclerotic plaque analysis via enhanced 1D structural models. Submitted, 2013.
- A. Varello, A. Lamberti, and E. Carrera. Static aeroelastic response of wing-structures accounting for cross-section in-plane deformation. Submitted, 2013


## Conference papers and proceedings

- A. Varello and E. Carrera. Accurate 1D structural models for the analysis of nonhomogeneous biomechanical structures. In Proceedings of the 5th International Symposium on Design, Modelling and Experiments of Advanced Structures and Systems (DeMEASS V), Ulrichsberg, Austria, 28-31 October 2012.
- A. Varello and E. Carrera. Static and dynamic analysis of a thin-walled layered cylinder by refined 1D theories. In Proceedings of the 10th World Congress on Computational Mechanics (WCCM 2012), São Paulo, Brazil, 8-13 July 2012.
- A. Varello and E. Carrera. Refined kinematics for dynamic analysis of thin-walled composite beams. In Proceedings of the 1st International Conference on Mechanics of Nano, Micro and Macro Composite Structures (ICMNMMCS 2012), Torino, Italy, 18-20 June 2012.
- E. Carrera, A. Varello, and A. Lamberti. Effect of warping due to aerodynamic loadings in composite wings. In Proceedings of the 1st International Conference on Mechanics of Nano, Micro and Macro Composite Structures (ICMNMMCS 2012), Torino, Italy, 18-20 June 2012.
- A. Varello, M. Petrolo, and E. Carrera. A refined 1D FE model for application to aeroelasticity of composite wing. In Proceedings of the IV International Conference on Computational Methods for Coupled Problems in Science and Engineering (COUPLED 2011), Kos Island, Greece, 20-22 June 2011.
- A. Varello, L. Demasi, E. Carrera, and G. Giunta. An improved beam formulation for aeroelastic applications. In Proceedings of the 51st AIAA / ASME / ASCE / AHS / ASC Structures, Structural Dynamics \& Material Conference (AIAA SDM 2010), Orlando, FL, USA, 12-15 April 2010.


## Main honors and awards

June 2010 Agusta Westland: Best Thesis 2009 Award (in memory of Eng. Mario Pellerei)
M.Sc. thesis awarded as the most groundbreaking thesis 2009 in Aerospace Engineering at Politecnico di Torino by the Anglo-Italian helicopter company Agusta Westland.

July 2010 Umberto Montalenti Award (in memory of Eng. Umberto Montalenti)
M.Sc. thesis awarded as the thesis with features of major innovation among graduated in Aerospace, Mechanical and Mechatronic Engineering at Politecnico di Torino.

June 2011 Second Place in Liviu Librescu Award
(in memory of Prof. Liviu Librescu)
M.Sc. thesis awarded as the second-best Italian thesis in Aerospace Structures and Aeroelasticity among all the graduated in Italian universities in 2009.

July 2009 Pegasus Award
This award is granted in recognition of my special achievements in European Cooperation through proficiency in English language and successful participation for at least 5 months in a research project conducted in an international environment abroad (visiting scholar at SDSU, California, USA).

## Research projects

2012 SARISTU - Smart Intelligent Aircraft Structures Funded by Seventh Framework Programme
Activity Implementation of multifield stick model elements into a wing aeroelastic model to integrate adaptive droop nose, trailing edge and winglets. In collaboration with Alenia Aermacchi

2011 STEPS - Systems and Technologies for Space Exploration Funded by the European Union and Regione Piemonte, Italy
Activity Development of advanced kinematics models for the multifield analysis of multilayered composite structures.
In collaboration with Thales Alenia Space

## 2009 MICRO.COST

Funded by Regione Piemonte, Italy
Activity Analysis, design, development, and production methods of low-cost eolic generators for microgeneration.

## COMMISSIONE GIUDICATRICE INGEGNERIA AEROSPAZIALE

Il dott. Alberto VARELLO ha discusso in data 12 aprile 2013 presso il Dipartimento di Ingegneria Meccanica e Aerospaziale del Politecnico di Torino la tesi di Dottorato avente il seguente titolo:
Advanced higher-order one-dimensional models for fluid-structure interaction analysis

Le ricerche oggetto della tesi sono di interesse nel settore
Le metodologie appaiono adeguate
I risultati sono interessanti ed analizzati con elevato senso critico.
Nel colloquio il candidato dimostra ottima conoscenza delle problematiche trattate.
La Commissione unanime giudica in modo eccellente il lavoro svolto
e propone che al dott. Alberto VARELLO venga conferito il titolo di Dottore di Ricerca.
Data, 12-04-2013
Prof. Michele GRASSI (Presidente)
Prof. Sergio DE ROSA (Componente)
Prof. Francesco LAROCCA (Segretario)


