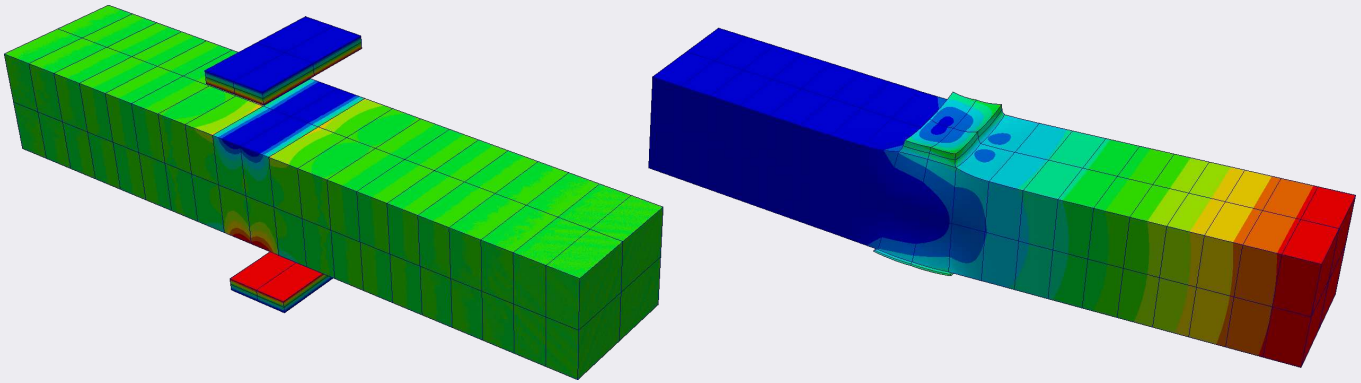
POLITECNICO
DI TORINO
Department of MEchanical
and AeroSpace Engineering

# Free Vibration Analysis of Beams with Piezo-Patches Using a One-Dimensional Model with Node-Dependent Kinematic



Enrico Zappino[1] , E. Carrera, G. Li

[1] MUL$^2$ Team, Department of Mechanical and Aerospace Engineering, Politecnico di Torino, Turin, Italy

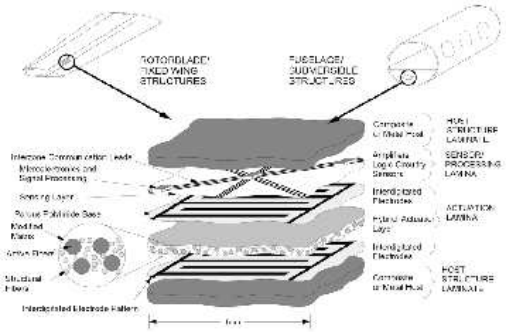
DEMEASS VIII

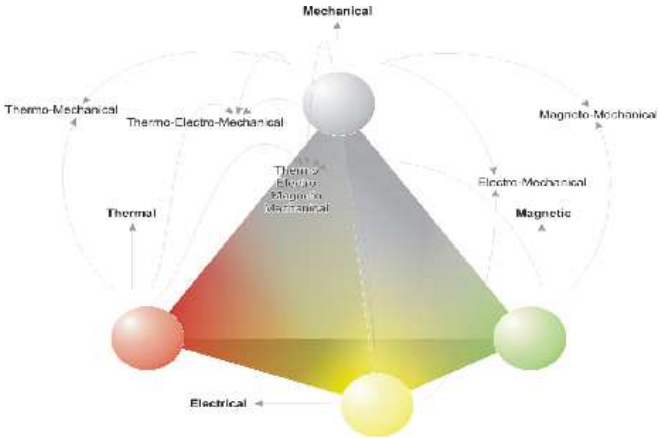
Design, Modelling and Experiments of Advanced Structures and Systems

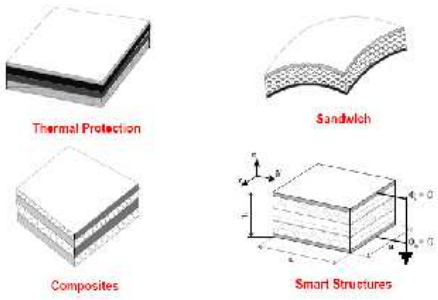
Moscow, May 22$^{nd}$, 2017

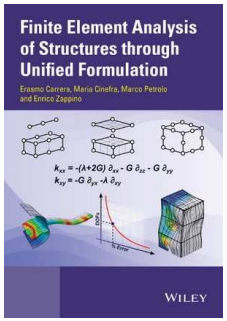
# MUL2 Team

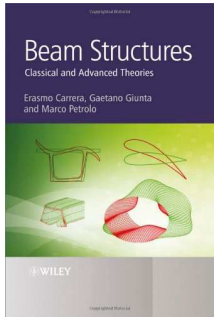
www.MUL2.com

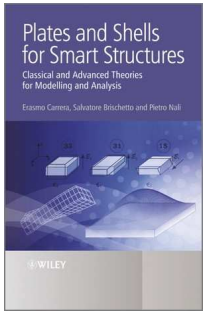


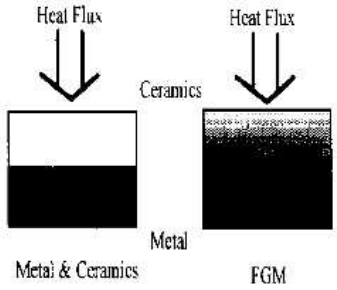


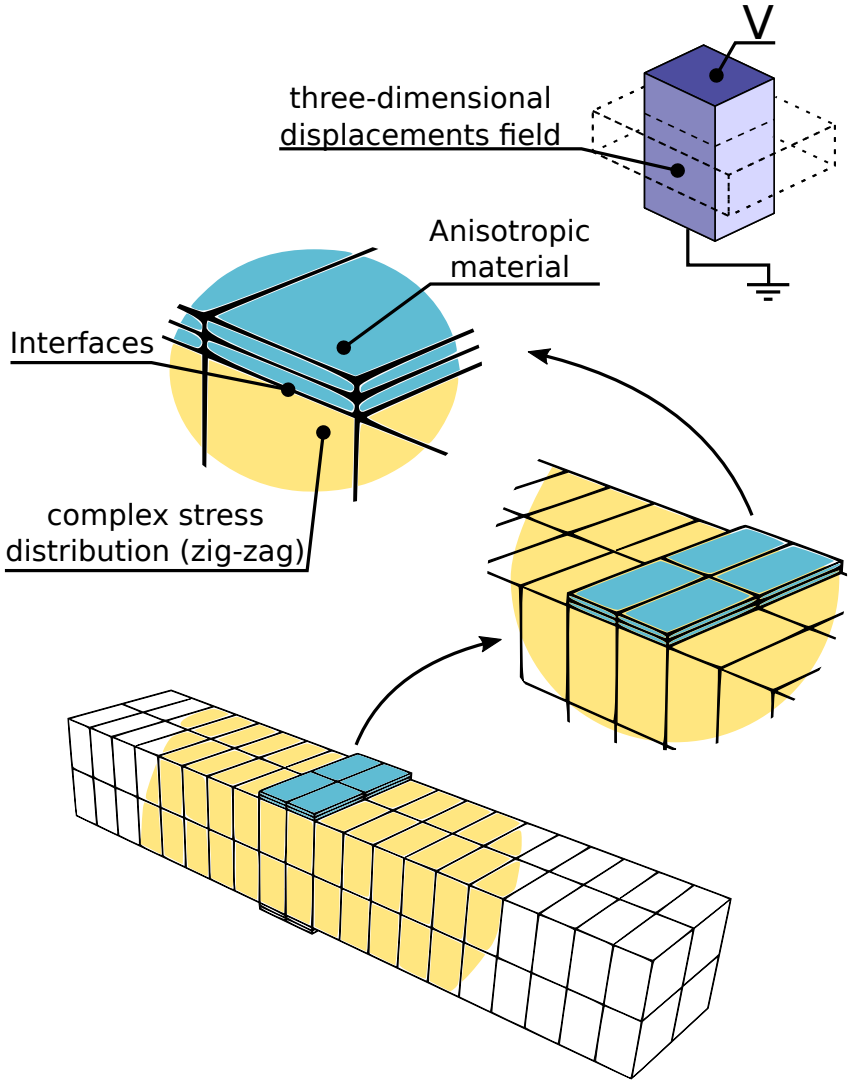

MULtilayered structures+MULtifield interactions

# Introduction

The analysis of piezoelectric devices has to account for many aspects that make it challenging:

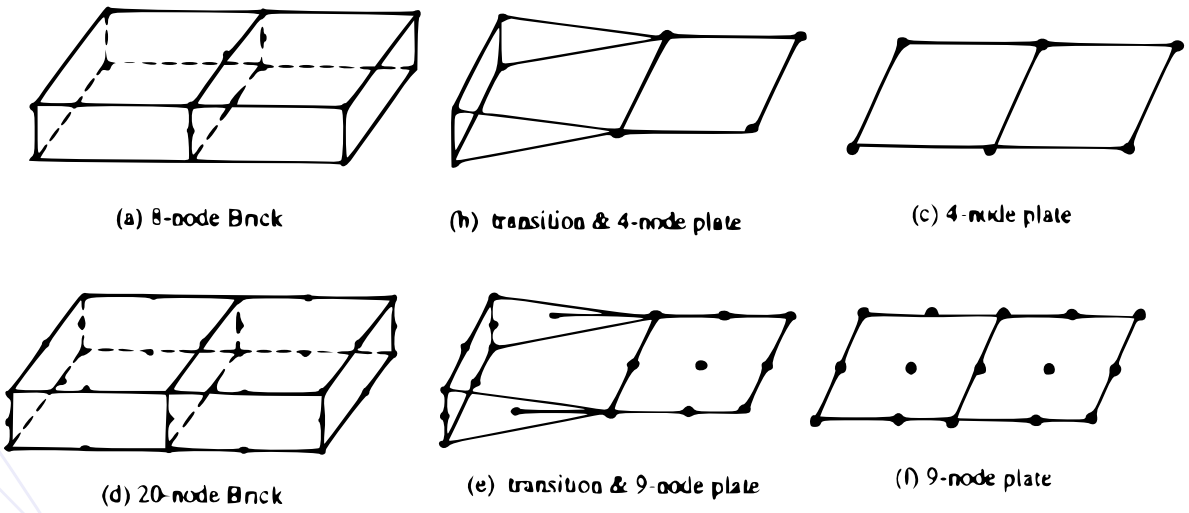
- material anisotropy;
- electrical and mechanical coupled fields;
- complex boundary conditions;
- layered structures (interfaces);
- complex three-dimensional stress distribution.

These features make the use of refined structural models mandatory since classical beam and plate models are not able to deal with complex displacements and strain fields.



# Introduction

The use of a refined model over the whole structural domain requires more computational costs than those necessary. The best solution would be to use refined models only in the region in which they are required and classical models elsewhere. The problem of mixing or joining different structural models is a well-known topic in literature as shown in the following Figure proposed by Kin *et al.*[1].




J. Kim, V.V. Varadan and V.K.. Varadan
Finite element modelling of structures Including piezoelectric active devices
*International journal for numerical methods in engineering* , 40: 817-832, 1997).

The present work has the aim to improve the efficiency of the well-know refined one-dimensional models based on Carrera Unified Formulation, CUF, introducing a node-dependent kinematic formulation able to adopt advanced kinematics only where required. This approach would allow refined models to be used only where they are required, in this case, only where piezo-patches are placed.

# Refined 1D models

A generic three-dimensional field can be written as:

$$\boldsymbol{u} = \boldsymbol{u}(x, y, z)$$

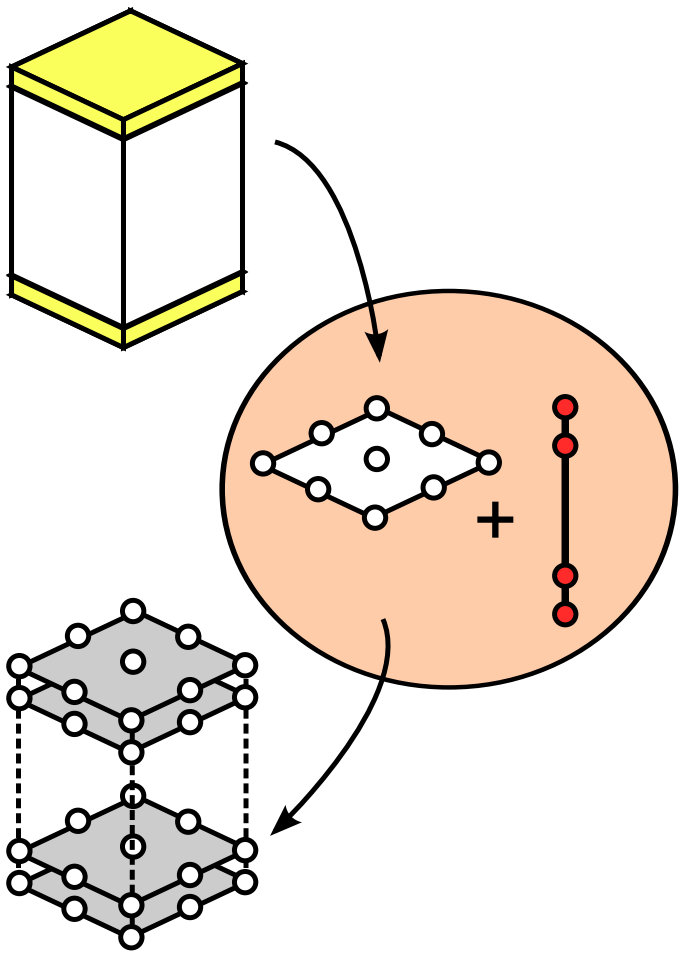
If we introduce a one-dimensional approximation the displacement field becomes:

$$\boldsymbol{u} = u(y) F_\tau(x, z) \boldsymbol{q}_\tau$$

A functions expansion, $F_\tau q_\tau$, is used to represent the cross-section behaviour. The problem on the axis is solved using the classical Finite Element Method, FEM:

$$\boldsymbol{u} = N_i(y) F_\tau(x, z) \boldsymbol{q}_{i\tau}$$

Where $N_i$ are the classical one-dimensional Lagrange shape functions.



# Refined 1D models

The vector $\boldsymbol{u}$ in the mechanical problem contains only the displacement field but it may contain many others information as the potential, $\phi$, in the case of the electro-mechanical case. This means that the displacement and the potential are described using the same kinematic:

$$\boldsymbol{u}^T = (u_x, u_y, u_z, \phi)$$

In the case of one-dimensional problem we can consider many cross-sectional approximation, in this work two expansion have been considered:

**TE- Taylor Expansion**

The Taylor Expansion can be written in the following form:

$$u(x,z) = \sum x^{i_\tau} z^{j_\tau} u_\tau(y)$$

The expansion in centred on the axis of the beam. The unknowns $u_\tau$ has not a clear physical meaning (higher-order moment)

**LE- Lagrange Expansion**

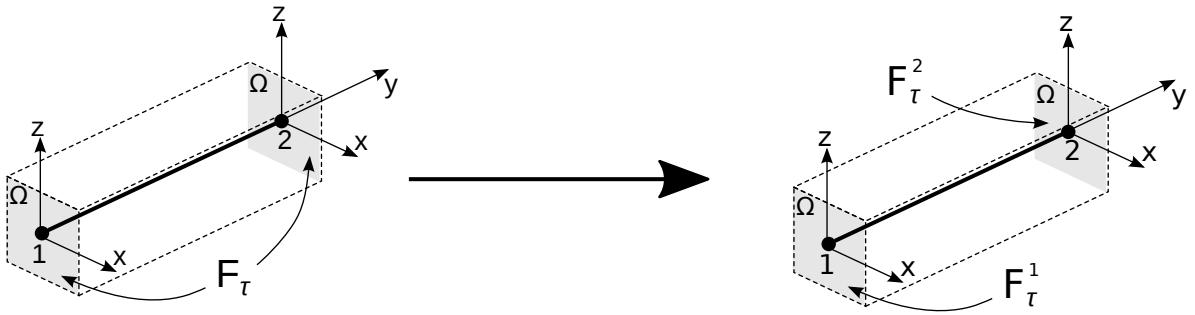
The Lagrange Expansion can be written in the following form:

$$u(x,z) = \sum L_\tau(x,z) u_\tau(y)$$

The expansion use the Lagrange 2D polynomials so each function is centred in one node on the cross section. The unknowns $u_\tau$ are all displacements.

# Node-dependent kinematic models (1)

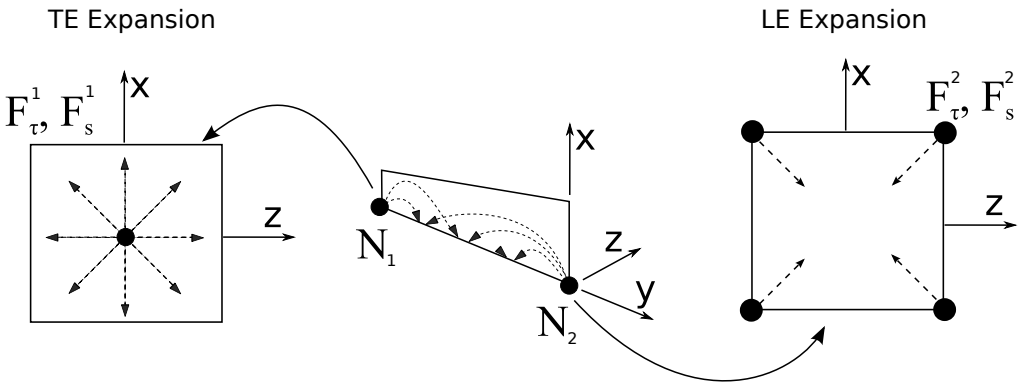
A two-node one-dimensional element is considered. A more refined kinematics can be introduced only in one of the nodes, that is, two different expansions should be used.




$$\mathbf{u}^1 = \mathbf{u}_{1\tau} F_\tau^1, \qquad \tau = 1 \dots M^1$$

$$\mathbf{u}^2 = \mathbf{u}_{2\tau} F_\tau^2, \qquad \tau = 2 \dots M^2$$

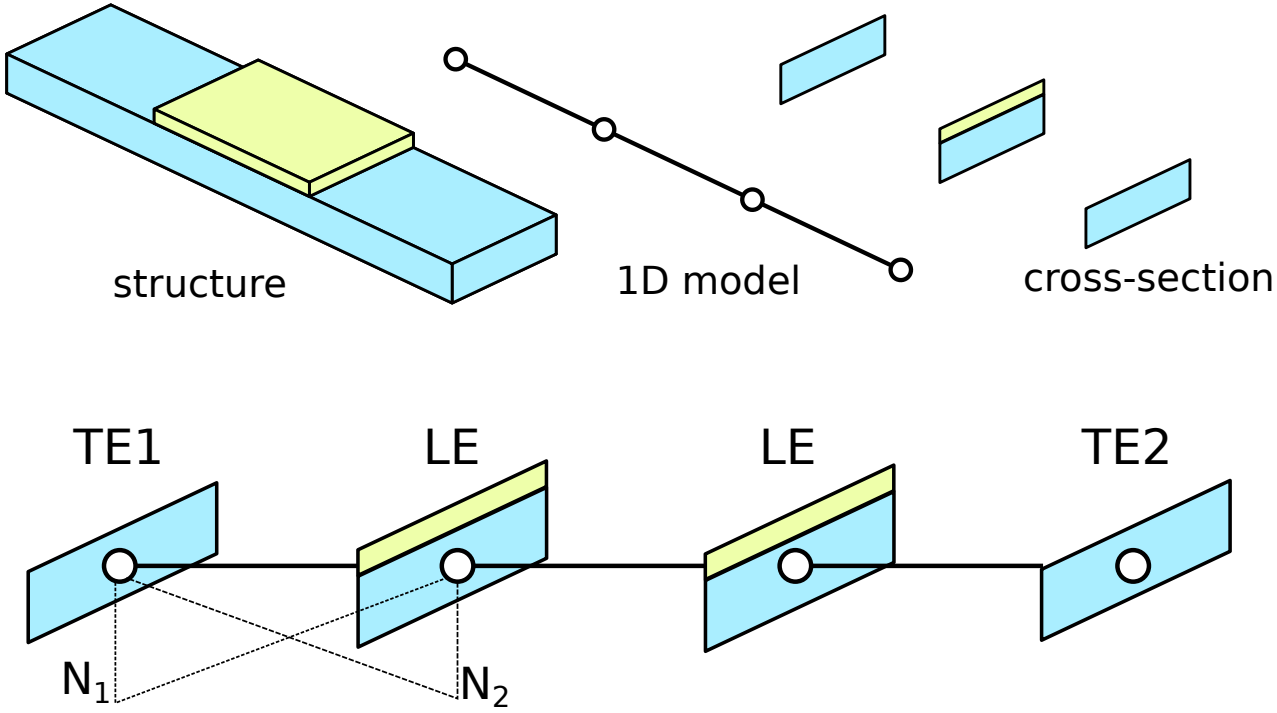
$$\mathbf{u} = \mathbf{u}_{1\tau} N_1 F_\tau^1 + \mathbf{u}_{2\tau} N_2 F_\tau^2$$




$$\mathbf{u} = N_1 \underbrace{\left[\mathbf{u}_1^1 + \mathbf{u}_2^1 x + \mathbf{u}_3^1 z\right]}_{1^{st}\,node} + N_2 \underbrace{\left[\mathbf{u}_1^2 L_1 + \mathbf{u}_2^2 L_2 + \mathbf{u}_3^2 L_3 + \mathbf{u}_4^2 L_4\right]}_{2^{nd}\,node}$$

# Node-dependent kinematic models (2)

An example of node-dependent kinematic model could be the following:




$u = N_1 u^{(TE1)} + N_2 u^{(LE)}$ $\quad$ $u = N_1 u^{(LE)} + N_2 u^{(LE)}$ $\quad$ $u = N_1 u^{(LE)} + N_2 u^{(TE2)}$

$\phi = N_1 \phi^{(LE)} + N_2 \phi^{(LE)}$

The kinematic is considered a property of the nodes, that is, no issues due to the need to couple different kinematics arise using this approach.

# Node-dependent kinematic models (3)

The approach introduced in the previous section can be easily included in the CUF formulation and extended to any order beam model. The displacement field of the one-dimensional element with node-dependent kinematic can be written including two main novelties:

**Node-dependent kinematic introduction**

$$F_\tau(x,z) \longrightarrow F_\tau^i(x,z)$$

$$M \longrightarrow M^i$$

The first equation, states that the function expansion is not a property of the element, but of the nodes, that is, the index $i$ is included in the notation. The second equation remarks that the number of terms in the expansion, $M$, can be different at each node, and the notation $M^i$ is used to underline this aspect. The generic displacement field can be written as:

**Node-dependent kinematic element displacement field**

$$\mathbf{u} = \mathbf{u}_{i\tau} N_i(y) F_\tau^i(x,z), \qquad \tau = 1 \ldots M^i; \qquad i = 1 \ldots N_n.$$

# Stiffness and Mass Matrix

The elastic work is generally expressed as:

$$\delta L_{int} = \int_V (\delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}) dV = \int_V \delta \mathbf{u}_{js}^T \Big[ \tilde{\boldsymbol{D}}^T \Big( N_j F_s^j \boldsymbol{I} \Big) \Big] \tilde{\boldsymbol{C}} \Big[ \tilde{\boldsymbol{D}} \Big( N_i F_\tau^i \boldsymbol{I} \Big) \Big] \mathbf{u}_{\tau i} dV$$

**Stiffness Matrix Fundamental Nucleus**

$$\delta L_{int} = \delta \mathbf{u}_{js}^T \underbrace{\int_V \Big[ N_j F_s^j \boldsymbol{I} \underbrace{\overbrace{\begin{bmatrix} 4 \times 9 \end{bmatrix}}^{\tilde{\boldsymbol{D}}^T} \overbrace{\begin{bmatrix} 9 \times 9 \end{bmatrix}}^{\tilde{\boldsymbol{C}}} \overbrace{\begin{bmatrix} 9 \times 4 \end{bmatrix}}^{\tilde{\boldsymbol{D}}}}_{\begin{bmatrix} 4 \times 4 \end{bmatrix}} \boldsymbol{I} F_\tau^i N_i \Big] dV}_{Fundamental \quad Nucleus \quad [\boldsymbol{k}^{ij\tau s}]} \mathbf{u}_{i\tau}$$

$\tilde{\boldsymbol{D}}$ and $\tilde{\boldsymbol{C}}$ include both the mechanical and the electrical problem.
The work made by the Inertial forces can be written in the following form:

$$\delta L_{ine} = \int_V (\rho \delta \boldsymbol{u}^T \ddot{\boldsymbol{u}}) dV = \int_V \rho \delta \boldsymbol{u}_{js}^T N_i (F_s^j \boldsymbol{I}) (F_\tau^i \boldsymbol{I}) N_j \ddot{\boldsymbol{u}}_{i\tau} dV$$

# Electro-mechanical Fundamental nucleus

$$\delta L_{int} = \begin{Bmatrix} \delta u_x \\ \delta u_y \\ \delta u_z \\ \\ \delta\phi \end{Bmatrix}^T \overbrace{\begin{bmatrix} \begin{bmatrix} \ddots & & \\ & MM & \\ & & \ddots \end{bmatrix} & \begin{bmatrix} \vdots \\ ME \\ \vdots \end{bmatrix} \\ \\ \begin{bmatrix} \cdots & EM & \cdots \end{bmatrix} & \begin{bmatrix} EE \end{bmatrix} \end{bmatrix}}^{\boldsymbol{K}^{ij\tau s}} \begin{Bmatrix} u_x \\ u_y \\ u_z \\ \\ \phi \end{Bmatrix} \tag{1}$$

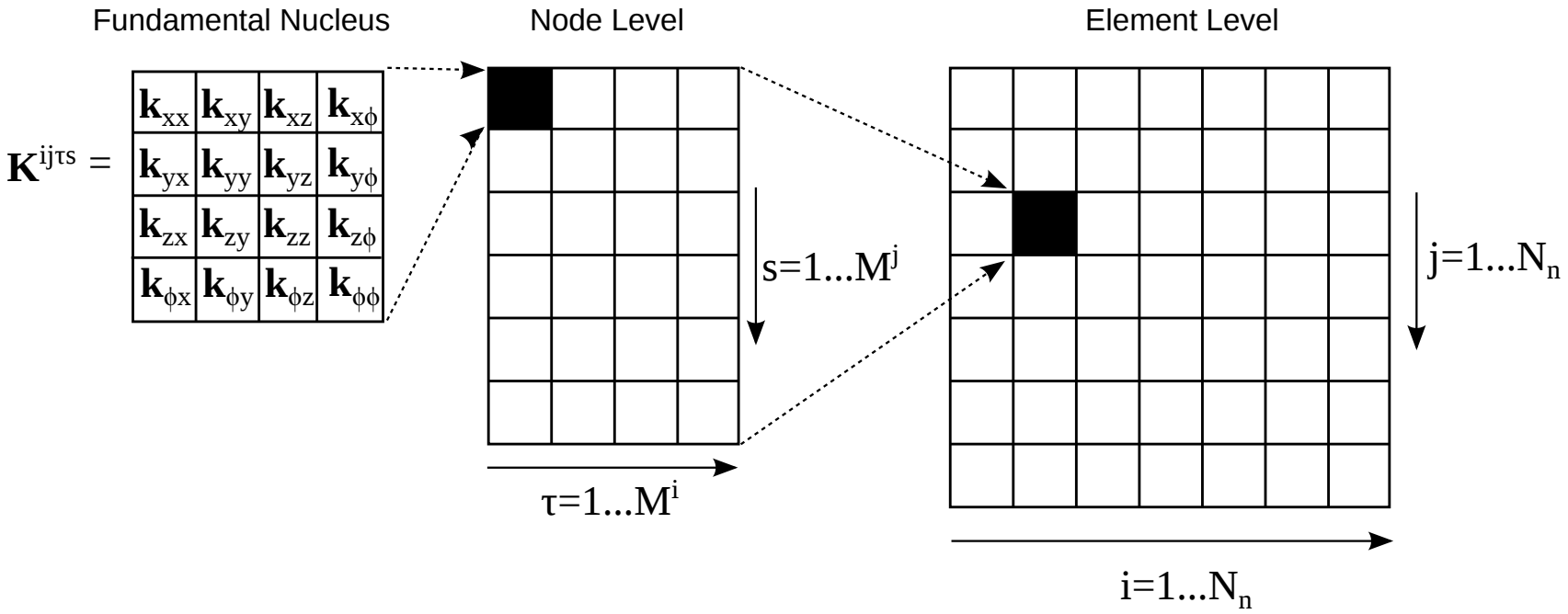
Where:

- MM $(3 \times 3)$ Mechanical problem;
- EE $(1 \times 1)$ Electric problem;
- EM,ME $(3 \times 1)$ Coupled Electro-Mechanical problem;

The complete matrix $(4 \times 4)$ can be used to solve the Electro-Mechanical problem.

# Assembly procedure

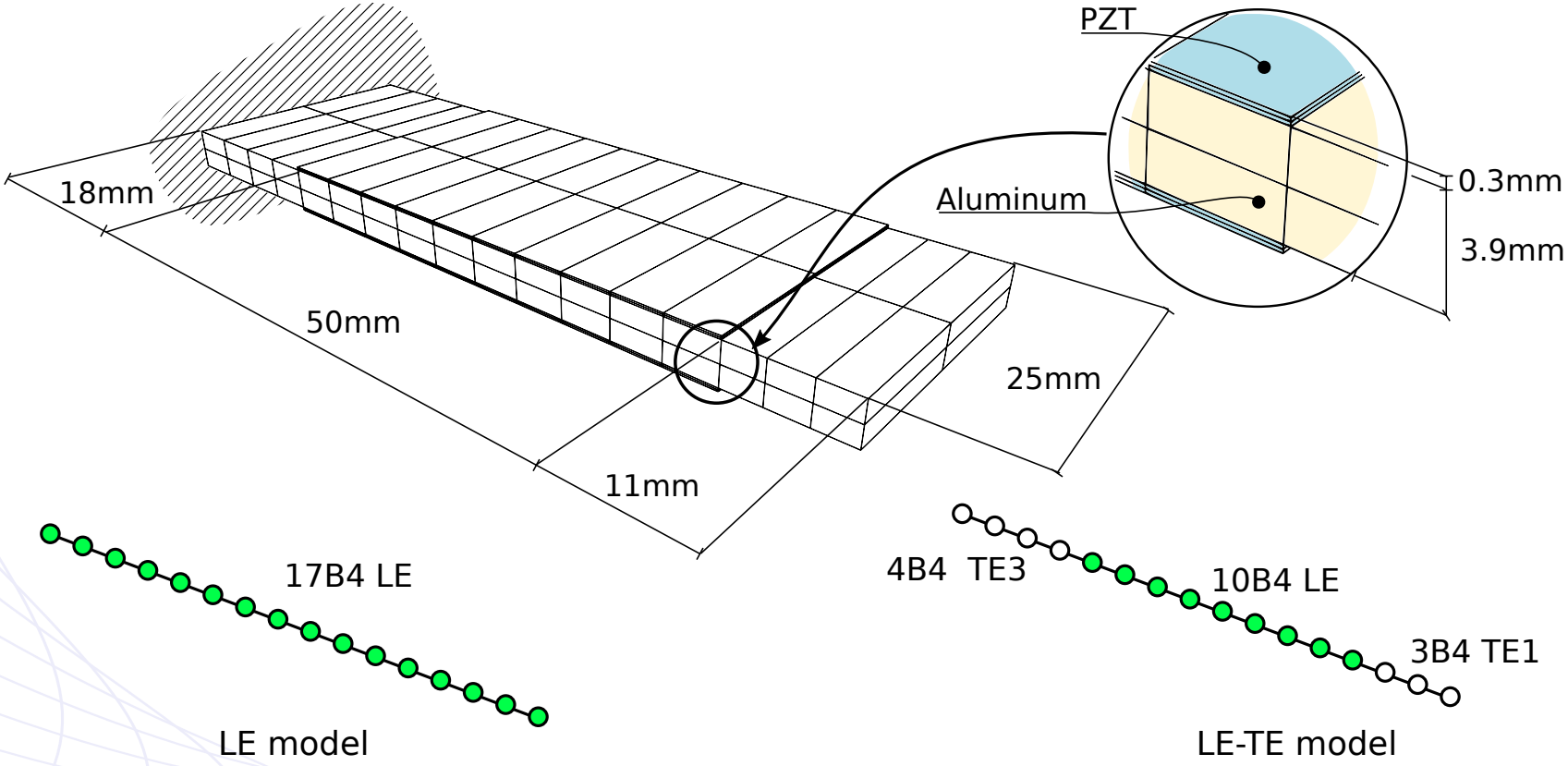
The Carrera Unified Formulation offers a unified approach for the matrix derivation. The fundamental nucleus has a standard formulation for all the structural models, LW or ESL. It works as a 'brick' in the construction of the matrices.




$$
\begin{aligned}
k_{xx}^{ij\tau s} = \quad & C_{22} \int_{\Omega} F_{\tau,x}^{i} F_{s,x}^{j} d\Omega \int_{l} N_i N_j dy + C_{66} \int_{\Omega} F_{\tau,z}^{i} F_{s,z}^{j} d\Omega \int_{l} N_i N_j dy + \\
& C_{44} \int_{\Omega} F_{\tau}^{i} F_{s}^{j} d\Omega \int_{l} N_{i,y} N_{j,y} dy \\
k_{xy}^{ij\tau s} = \quad & C_{23} \int_{\Omega} F_{\tau,x}^{i} F_{s}^{j} d\Omega \int_{l} N_i N_{j,y} dy + C_{44} \int_{\Omega} F_{\tau}^{i} F_{s,x}^{j} d\Omega \int_{l} N_{i,y} N_j dy
\end{aligned}
$$

# Assessment of the piezo-mechanic model

The model has been assessed considering a cantilevered beam with two piezoelectric-patches placed at the top and bottom surfaces. A free vibration analysis has been performed. The results, obtained using a full LE model and a LE/TE model, have been compared with those form literature.



# Assessment of the piezo-mechanic model

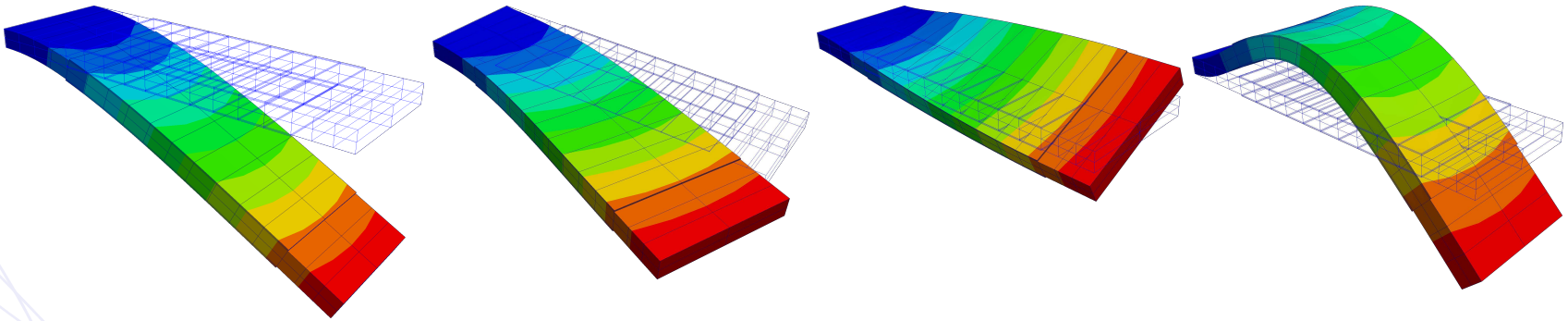
The following tables report the natural frequencies for a full LE model, a LE/TE model and the frequency published by Chevallier *et al.*[1].

Open circuit

| DOFs | LE model 9170 | LE-TE model 7950 | Ref.[1] 29001 |
|---|---|---|---|
| 1 | 493.28 $^{0.04\%}$ | 494.29 $^{0.25\%}$ | 493.07 |
| 2 | 2803.9 $^{0.21\%}$ | 2804.7 $^{0.24\%}$ | 2797.9 |
| 3 | 3112.0 $^{2.23\%}$ | 3157.6 $^{3.73\%}$ | 3044.1 |
| 4 | 3247.2 $^{-0.06\%}$ | 3256.9 $^{0.24\%}$ | 3249.0 |

Short circuit

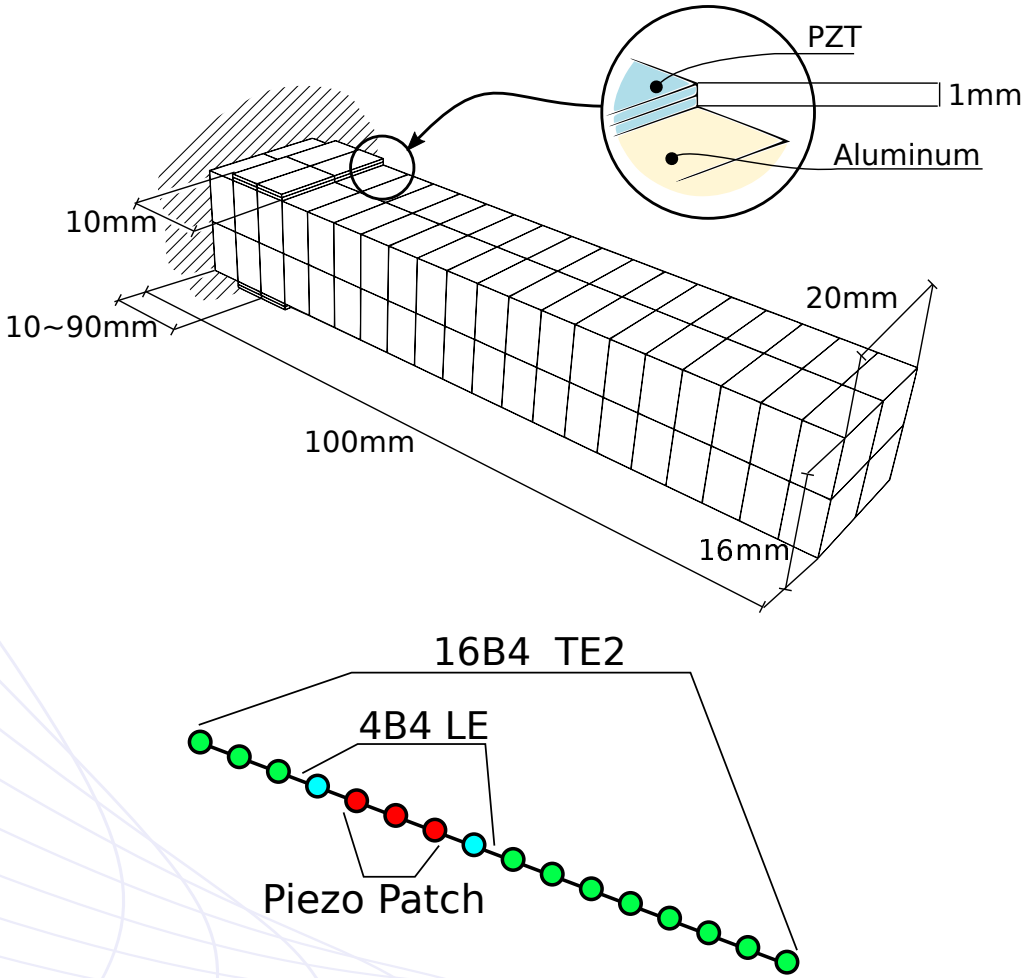
| DOFs | LE model 9170 | LE-TE model 7950 | Ref.[1] 29001 |
|---|---|---|---|
| 1 | 496.69 $^{0.22\%}$ | 497.73 $^{0.43\%}$ | 495.61 |
| 2 | 2812.1 $^{0.51\%}$ | 2813.0 $^{0.54\%}$ | 2797.9 |
| 3 | 3113.6 $^{2.28\%}$ | 3160.7 $^{3.83\%}$ | 3044.1 |
| 4 | 3321.2 $^{2.22\%}$ | 3334.5 $^{2.63\%}$ | 3249.0 |



G. Chevallier, S. Ghorbel and A. Benjeddou
A benchmark for free vibration and effective coupling of thick piezoelectric smart structures
*Smart Materials and Structures* , 17(6): 1-11, 2008.

# Analysis of a beam with piezo-patches

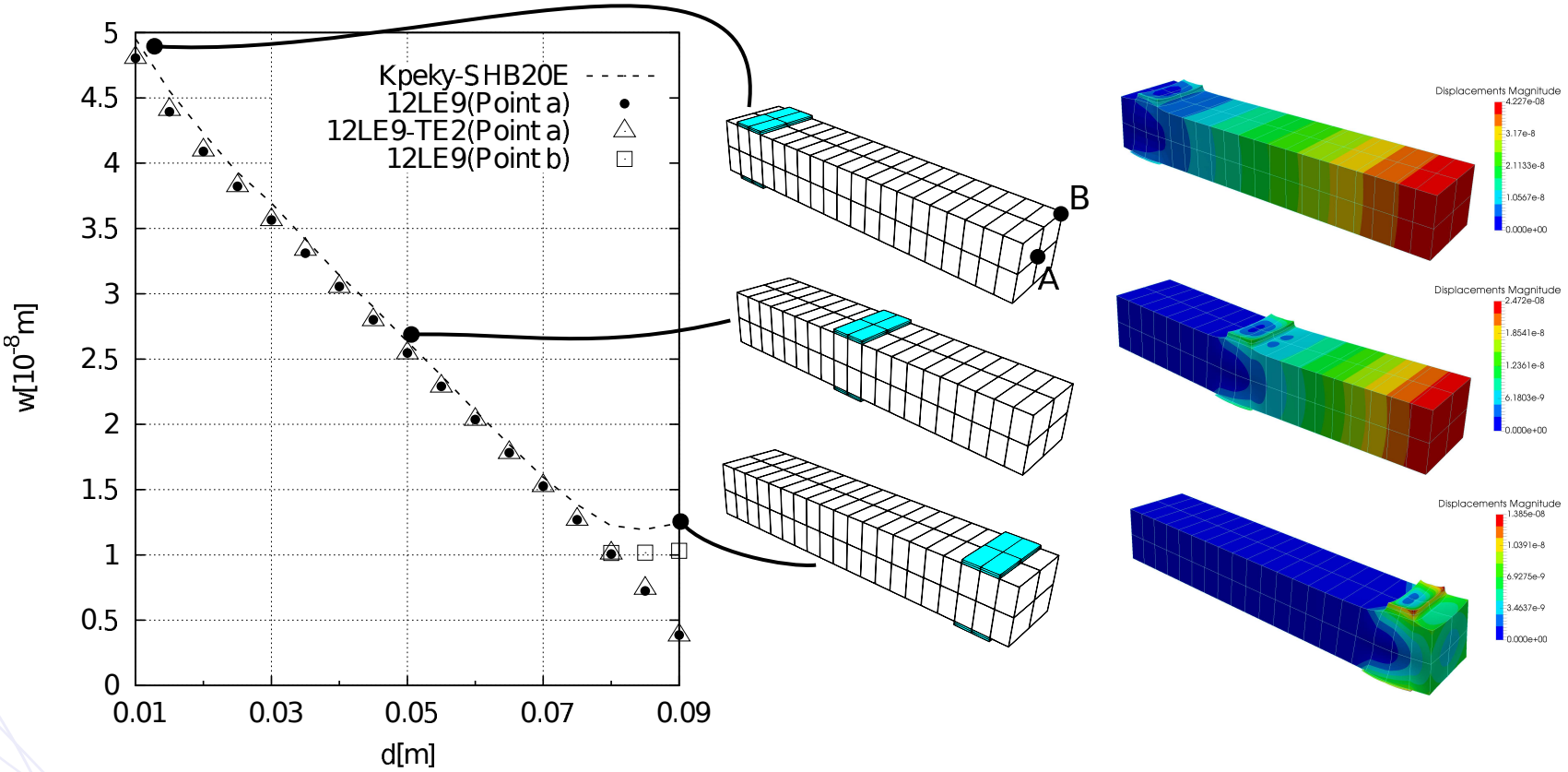
## Model Description




A cantilevered beam with a local piezo-patch has been considered. Two models have been taken into account, the first uses only LE models, while the second uses both TE and LE models. Static, dynamic and frequency response analyses have been performed. The model has been assessed using results from literature.

# Analysis of a beam with piezo-patches

## Static analysis




F. Kpeky, F. Abed-Meraim , H. Boudaoud and E.M. Daya

Linear and quadratic solidshell finite elements SHB8PSE and SHB20E for the modeling of piezoelectric sandwich structures

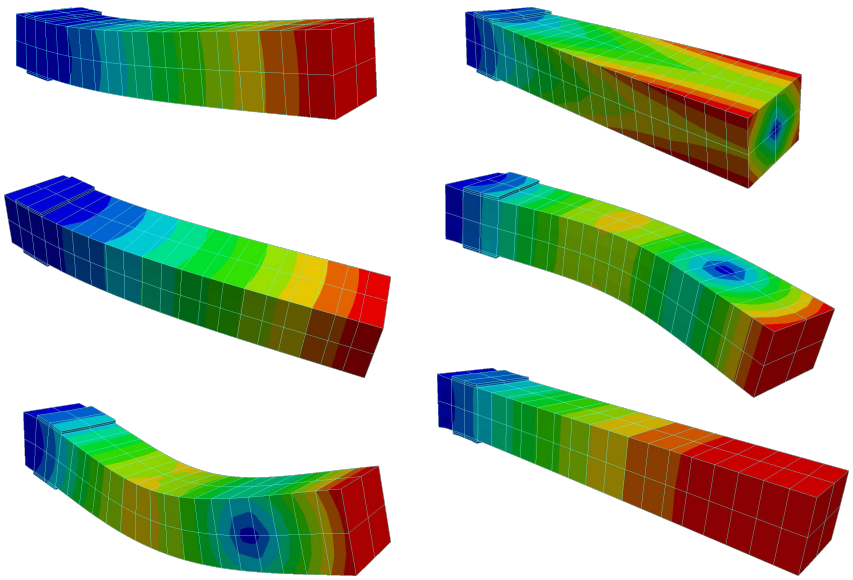
*Mechanics of Advanced Materials and Structures*, DOI:10.1080/15376494.2017.1285466. In press.

# Analysis of a beam with piezo-patches

## Free Vibration analysis

The free-vibration analysis has been performed to compare the performances of the two models, the Full LE and the mixed LE/TE.

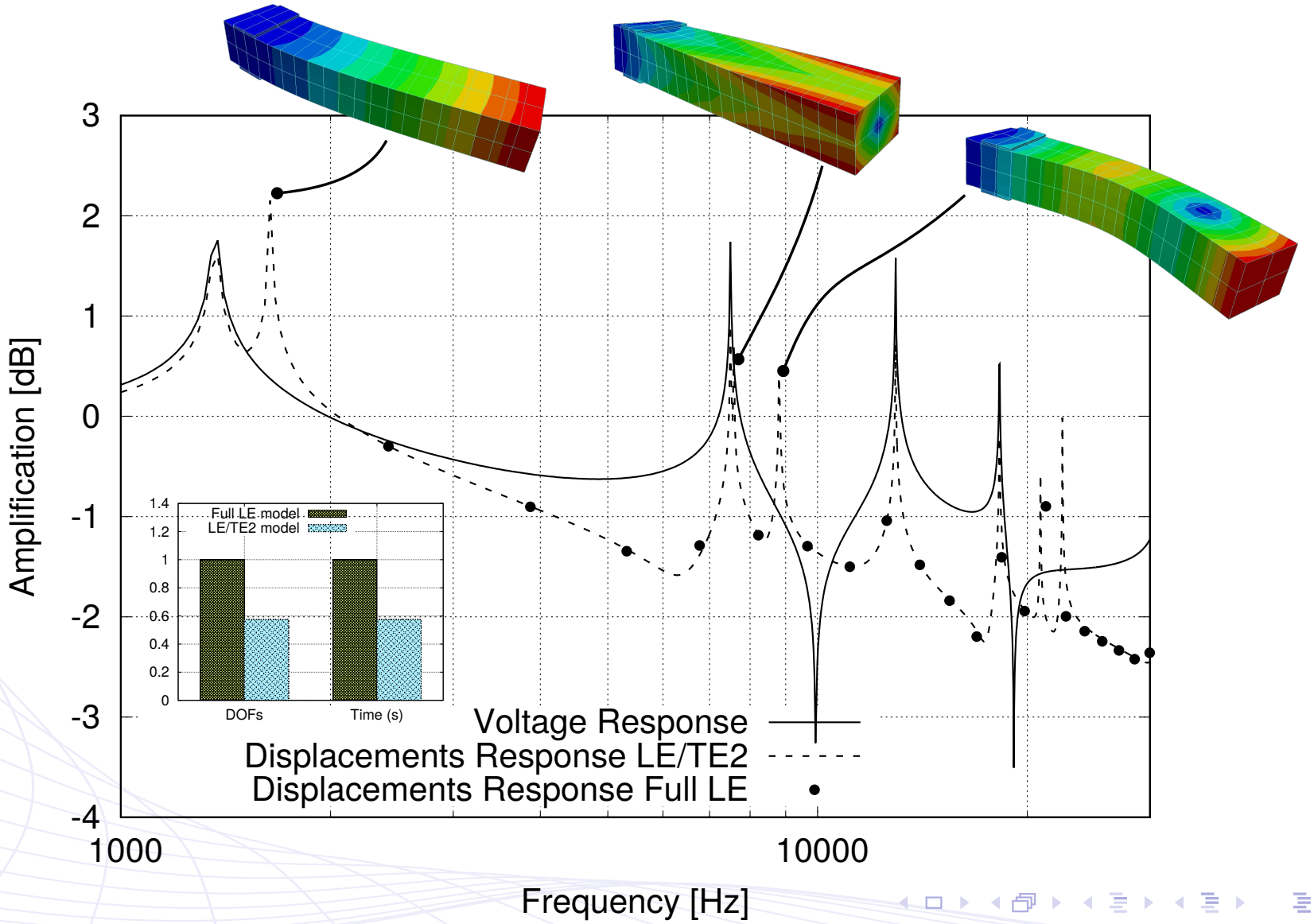
| | Full LE | LE-TE2 | % Error |
|---|---|---|---|
| 1 | 1363.1 | 1365.1 | 0.1% |
| 2 | 1637.2 | 1640.7 | 0.2% |
| 3 | 7214.3 | 7500.8 | 4.0% |
| 4 | 7460.0 | 7586.2 | 1.7% |
| 5 | 8744.9 | 8810.0 | 0.7% |
| 6 | 12941.5 | 12941.6 | 0.0% |
| 7 | 18080.8 | 18240.9 | 0.9% |
| 8 | 20658.1 | 20885.9 | 1.1% |
| 9 | 21308.1 | 22464.8 | 5.4% |
| DOFs | 5765 | 3317 | −42.5% |
| TIME(s) | 14.55 | 10.08 | −30.7% |



The results show that the use of a variable kinematic model may reduce the computational cost of the 30% in term of computational time.

# Analysis of a beam with piezo-patches

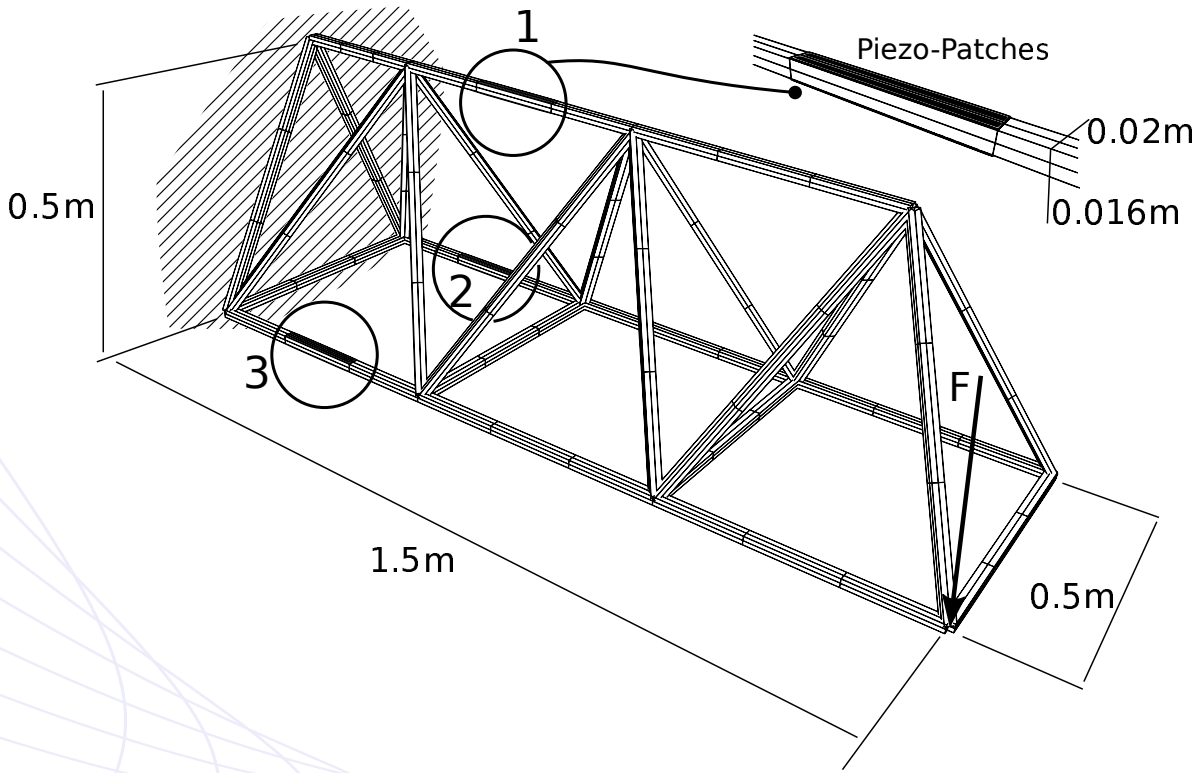
## Frequency response analysis



# Complex structure analysis

## Model description

Since the previous cases showed the accuracy and the efficiency provided by the present approach, a truss structure has been here considered to show how, the present model, can be applied to complex geometries.

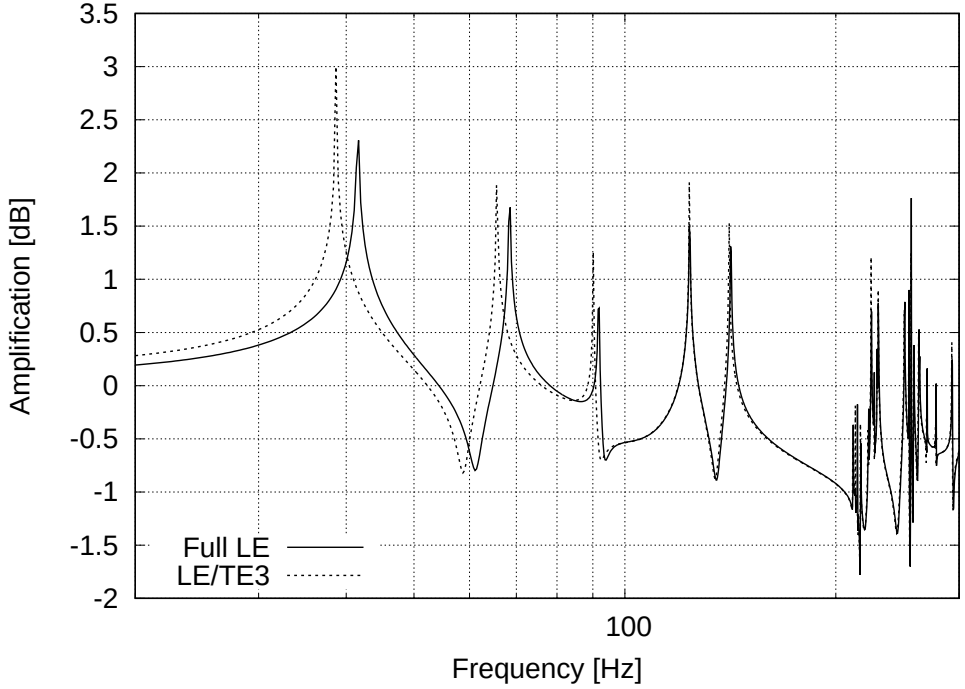


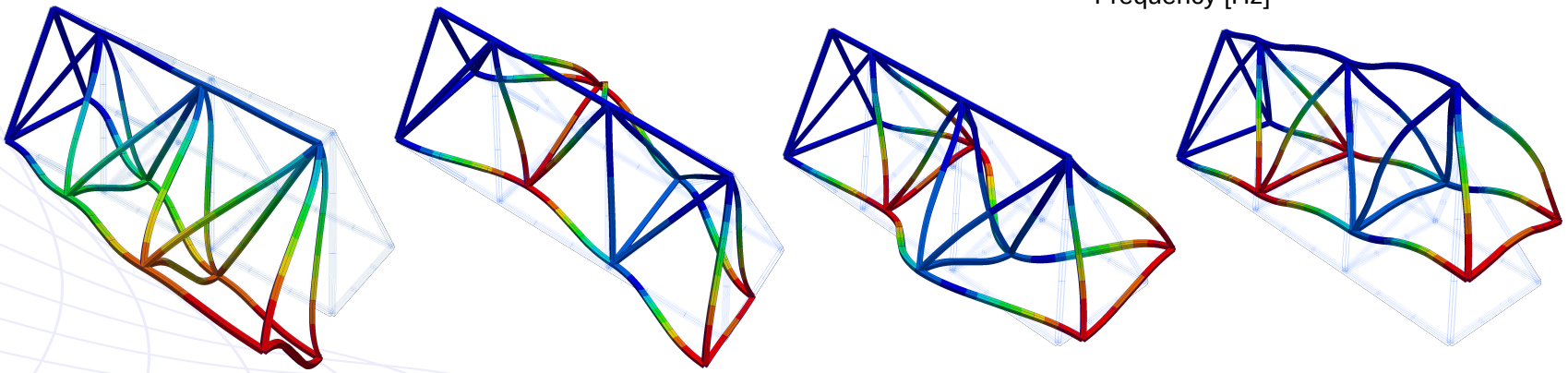

The structure is reported in the Figure. Three piezo-patches have been placed in three different positions. At first a free-vibration analysis and a frequency response analysis have been performed, then a time dependent force, F=1000N, has been considered. Full LE and midex LE/TE3 models have been used in the analysis and compared each others.

# Complex structure analysis

## Free Vibration analysis and Frequency Response

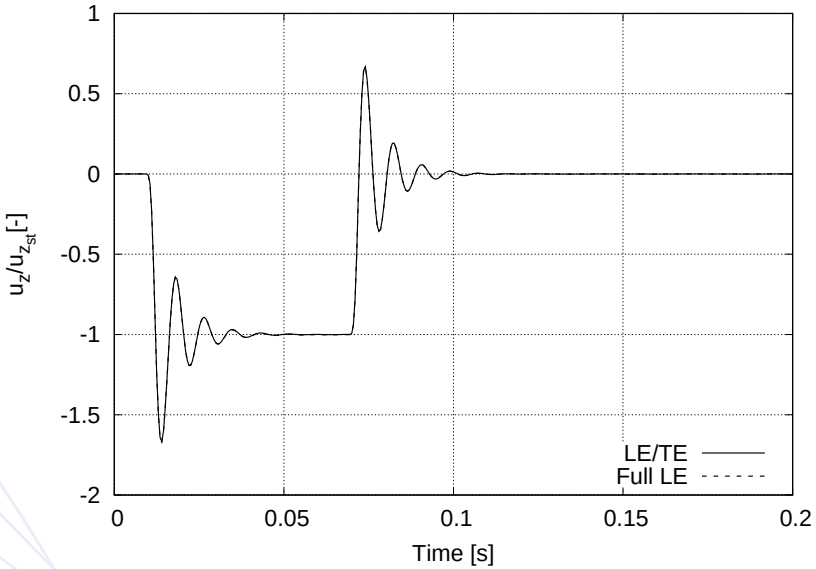
| | LE-TE model | Full LE model |
|---|---|---|
| DOFs | 8733 | 13143 |
| 1 | 41.55 | 38.69 |
| 2 | 68.44 | 65.66 |
| 3 | 91.75 | 90.15 |
| 4 | 123.72 | 123.64 |
| 5 | 141.66 | 140.85 |
| 6 | 211.72 | 211.58 |
| 7 | 213.42 | 213.28 |
| 8 | 214.79 | 214.65 |
| 9 | 215.28 | 215.13 |
| 10 | 216.35 | 216.21 |

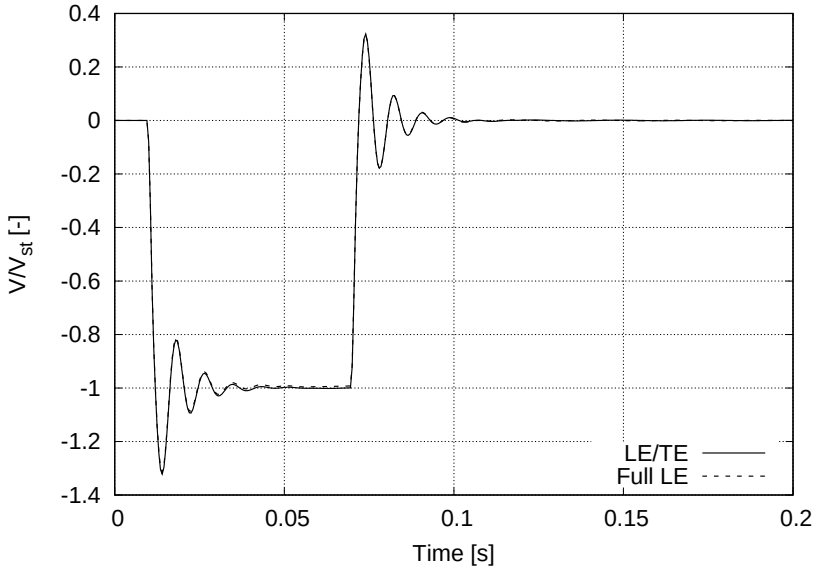


# Time response analysis

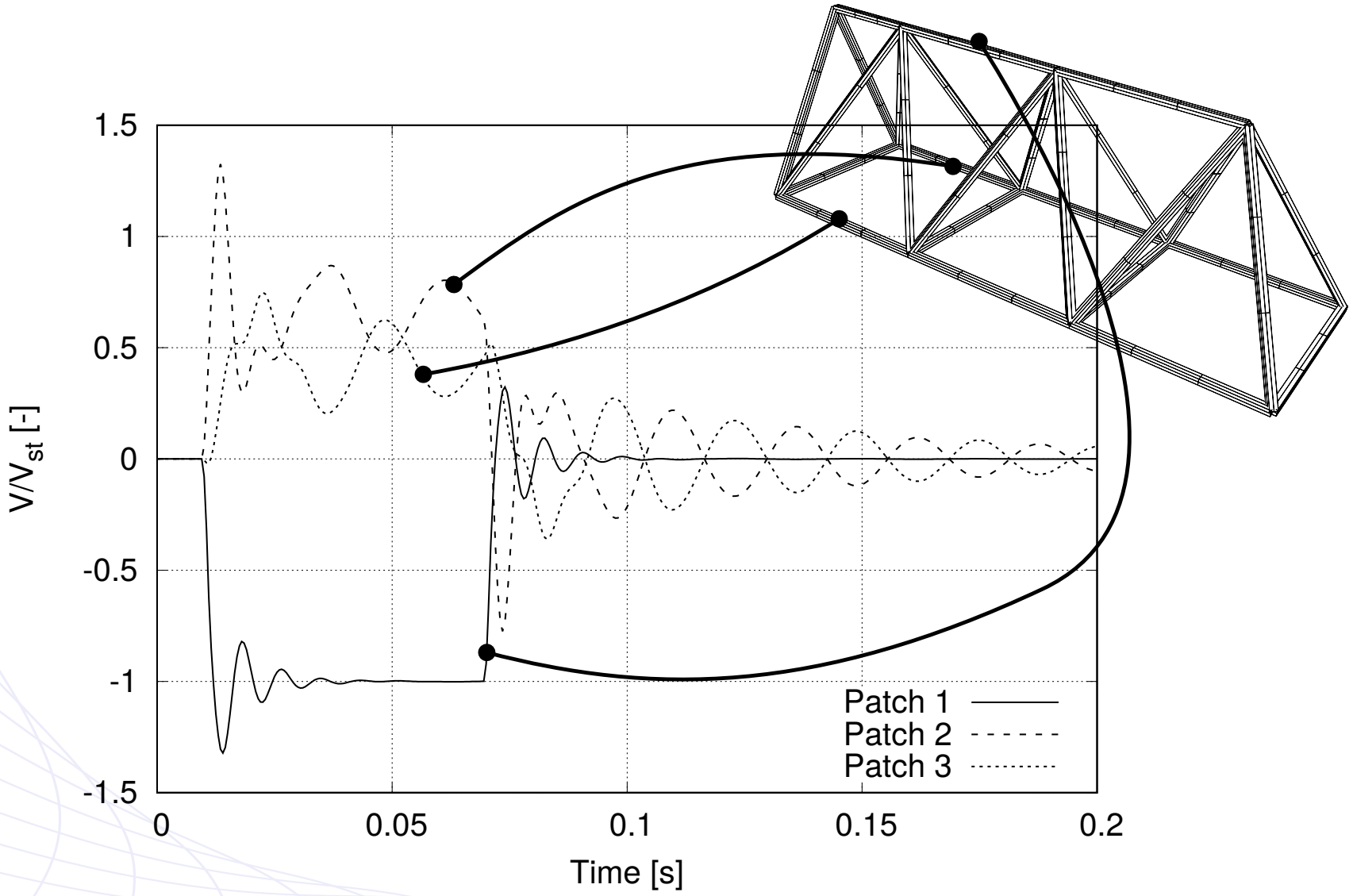
A time repose analysis has been performed. A positive step load has been placed at 0.01s and a negative one , with the same magnitude, at 0.07s. The simulation covers 0.2s and 400 time steps have been considered. The damping as been assumed as a linear combination of K and M.




The figures show the time evolution of the displacements and voltage in the patch number 1. It can be seen that the node-dependent kinematic model provide results close to the full LE model but it take 51.91s to run instead to 114.80s.

# Complex structure analysis



# Conclusions

The following considerations can be drawn from the obtained results:

- The node-dependent kinematic elements allow a different kinematic model to be used at each node of the structure. These models can be derived in the CUF framework, without the use of a specific approach.
- The node-dependent kinematic formulation can be easily extended to the electro-mechanical problem.
- The use of node-dependent kinematic elements allows to obtain accurate results where complex phenomenons appear (e.g. piezo-patches) using refined models while low order or classical models can be used elsewhere.
- The variation in the kinematics allows the number of unknowns and the computational time to be reduced because refined models are only used where they are really needed.

In short, the present node-dependent kinematic model can be considered a breakthrough with respect to uniform kinematic elements. The use of these elements could lead to benefits in several applications. Their accuracy a and their low computational cost make them suitable for optimization problem, health monitoring applications, virtual testing, etc.

POLITECNICO
DI TORINO
Department of MEchanical
and AeroSpace Engineering

Thank you for your attention!