

FINITE ELEMENT MODELS WITH NODE-DEPENDENT KINEMATICS ADOPTING LEGENDRE POLYNOMIAL EXPANSIONS FOR THE ANALYSIS OF LAMINATED PLATES

G. Li, A.G. de Miguel, E. Zappino, A. Pagani and E. Carrera

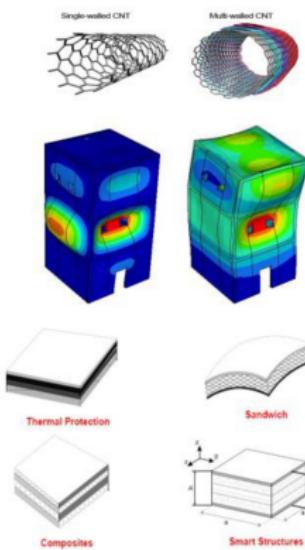
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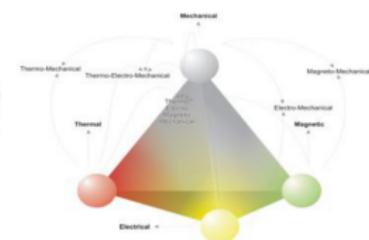
21st International Conference on Composite Materials
ICCM21
25 August 2017, Xi'an

MUL² - MULtilayered structures & MULtifield effects

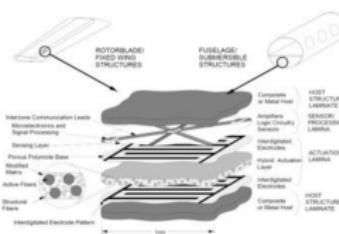
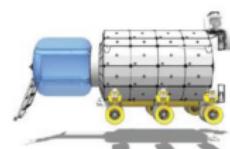
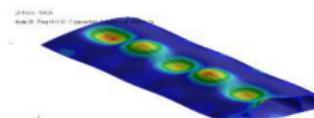
MUL_{tilayered structures}



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MUL_{tifield interaction}



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Overview

- 1 Carrera Unified Formulation (CUF) for refined 2D models;
- 2 Node-dependent kinematic 2D FEM models;
- 3 Hierarchical Legendre Expansions as shape functions for 2D elements;
- 4 Numerical cases;
- 5 Conclusions.

An Example: A Higher-order Deformation Theory for Plate Written in CUF

– Displacement description

$$\begin{cases} u = u_0(x, y) & +z \cdot u_1(x, y) & + \dots & +z^N u_N(x, y) \\ v = v_0(x, y) & +z \cdot v_1(x, y) & + \dots & +z^N v_N(x, y) \\ w = w_0(x, y) & +z \cdot w_1(x, y) & + \dots & +z^N w_N(x, y) \end{cases}$$

$$\boldsymbol{u}^T = \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} \mathbf{F}_\tau u_\tau & \mathbf{F}_\tau v_\tau & \mathbf{F}_\tau w_\tau \end{bmatrix} = \mathbf{F}_\tau \boldsymbol{u}_\tau^T$$

$$\mathbf{F}_\tau(z) = z^{\tau-1} \quad \tau = 1, 2, \dots, N+1$$

– FEM discretization

$$\boldsymbol{u}(x, y, z) = N_i(x, y) \boldsymbol{u}_i(z) = N_i(x, y) \mathbf{F}_\tau(z) \boldsymbol{U}_{i\tau}$$

– PVD

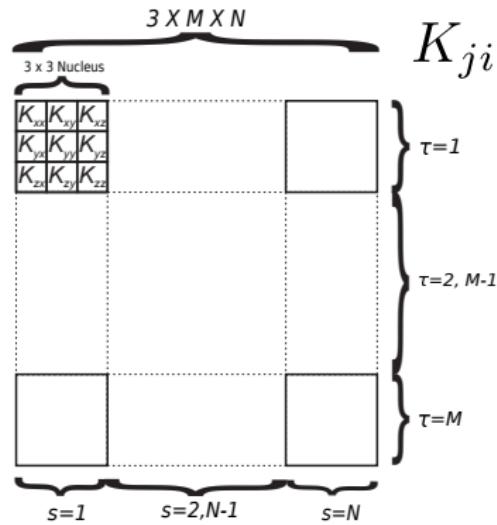
$$\boldsymbol{u}(x, y, z) = N_i(x, y) \mathbf{F}_\tau(z) \boldsymbol{U}_{i\tau} \quad \delta \boldsymbol{u}(x, y, z) = N_j(x, y) \mathbf{F}_\tau(z) \delta \boldsymbol{U}_{j\tau}$$

$$\begin{aligned} \delta L_{int} &= \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV = \int_V \delta \boldsymbol{u}^T \mathbf{b}^T \mathbf{C} \mathbf{b} \boldsymbol{u} dV \\ &= \delta \boldsymbol{U}_{js}^T \cdot \int_V \mathbf{F}_s N_j \mathbf{b}^T \mathbf{C} \mathbf{b} N_i \mathbf{F}_\tau dV \cdot \boldsymbol{U}_{i\tau} = \delta \boldsymbol{U}_{js}^T \cdot \mathbf{K}_{ij\tau s} \cdot \boldsymbol{U}_{i\tau} \end{aligned}$$

$$\delta L_{ext} = \int_V \delta \boldsymbol{u}^T \mathbf{p} dV = \delta \boldsymbol{U}_{js}^T \cdot \int_V N_j \mathbf{F}_s \mathbf{p} dV = \delta \boldsymbol{U}_{js}^T \cdot \mathbf{P}_{js}$$

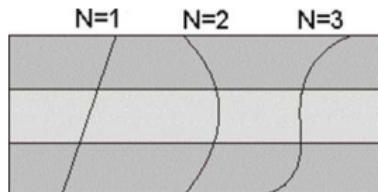
$$\mathbf{K}_{ij\tau s} = \int_V N_j \mathbf{F}_s \mathbf{b}^T \mathbf{C} \mathbf{b} \mathbf{F}_\tau N_i dV$$

$\mathbf{K}_{ij\tau s}$: Fundamental Nucleus

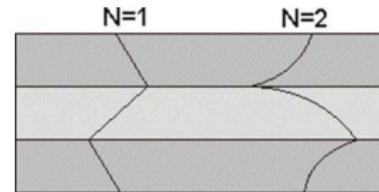


CUF for Two Major Frameworks of Refined 2D Models

Equivalent Single-Layer models (ESL)



Layer-Wise models (LW)



$$\boldsymbol{u} = F_0 \boldsymbol{u}_0 + F_1 \boldsymbol{u}_1 + \cdots + F_N \boldsymbol{u}_N$$

Note:

- F_τ defined on the whole through-thickness domain;
- Applicable to series expansion;
- Taylor, Fourier, Exponential, Hyperbolic, et al.;
- Taylor Expansions (TE) are most commonly used.

$$\boldsymbol{u}^k = F_t \boldsymbol{u}_t^k + F_b \boldsymbol{u}_b^k + F_r \boldsymbol{u}_r^k$$

Note:

- F_τ defined on the thickness domain of layer k ;
- Continuity constraints at layer interfaces: $\boldsymbol{u}_t^k = \boldsymbol{u}_b^{k+1}$;
- Can adopt interpolation polynomials;
- Lagrange, Legendre, Chebyshev, et al.;
- Lagrange Expansions (LE) are the most widely used.



Carrera, E., Cinefra, M., Li, G. and Kulikov, G.M.

MITC9 shell finite elements with miscellaneous through-the-thickness functions for the analysis of laminated structures.

Composite Structures, 154(2016), pp.360–373.

3D Constitutive Equations

Stress-strain relations for an orthotropic material:

$$\sigma_p^k = C_{pp}^k \epsilon_p^k + C_{pn}^k \epsilon_n^k$$

$$\sigma_n^k = C_{np}^k \epsilon_p^k + C_{nn}^k \epsilon_n^k$$

where

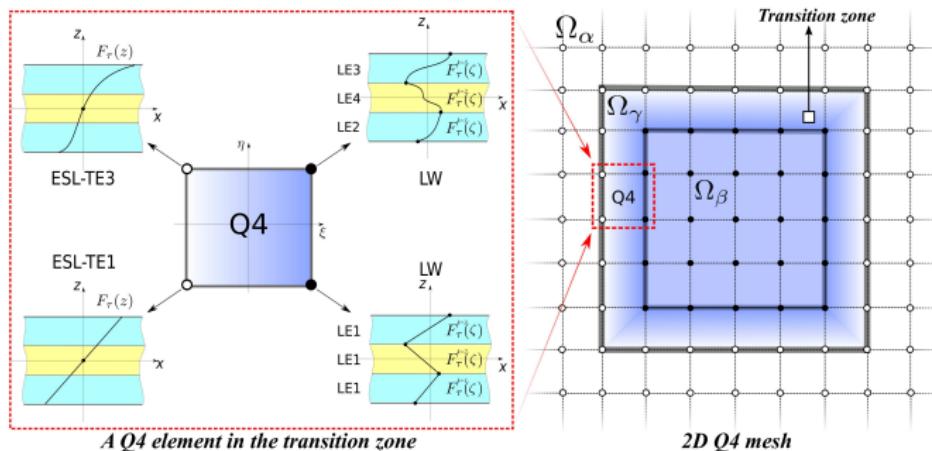
$$C_{pp}^k = \begin{bmatrix} C_{11}^k & C_{12}^k & C_{16}^k \\ C_{12}^k & C_{22}^k & C_{26}^k \\ C_{16}^k & C_{26}^k & C_{66}^k \end{bmatrix} \quad C_{pn}^k = \begin{bmatrix} 0 & 0 & C_{13}^k \\ 0 & 0 & C_{23}^k \\ 0 & 0 & C_{36}^k \end{bmatrix}$$

$$C_{np}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13}^k & C_{23}^k & C_{36}^k \end{bmatrix} \quad C_{nn}^k = \begin{bmatrix} C_{55}^k & C_{45}^k & 0 \\ C_{45}^k & C_{44}^k & 0 \\ 0 & 0 & C_{33}^k \end{bmatrix}$$

Plate Elements with Node-Dependent Kinematics (NDK)

- Introduction of the dependency of kinematics on FEM nodes

$$\boldsymbol{u} = N_i \boldsymbol{F}_\tau \boldsymbol{u}_{i\tau} \quad \Rightarrow \quad \boldsymbol{u} = N_i \boldsymbol{F}_\tau^i \boldsymbol{u}_{i\tau}$$



- Variable LW/ESL nodal capabilities;
- Modeling of patches;
- Global-local analysis.

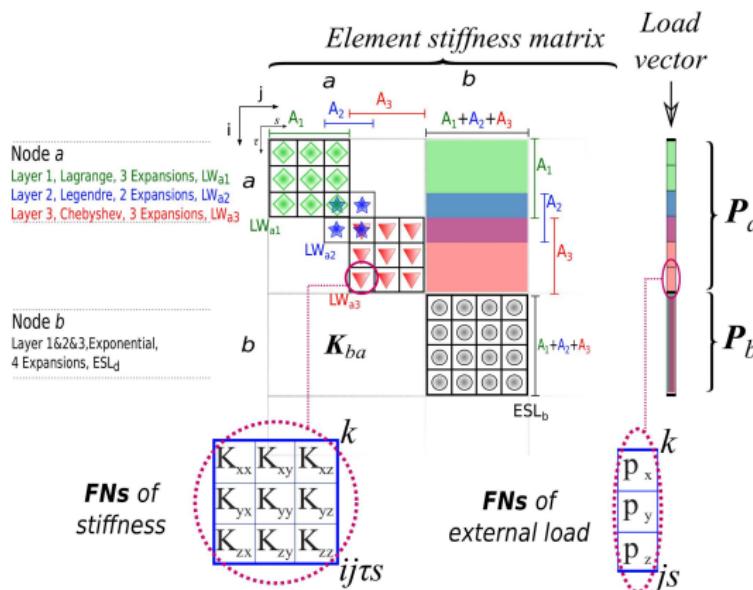
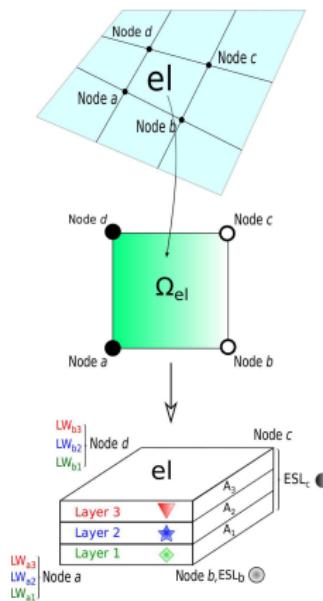


Zappino, E., Li, G., Pagani, A. and Carrera, E.

Global-local analysis of laminated plates by node-dependent kinematic finite elements with variable ESL/LW capabilities.
Composite Structures, 172(2017), pp.1–14.

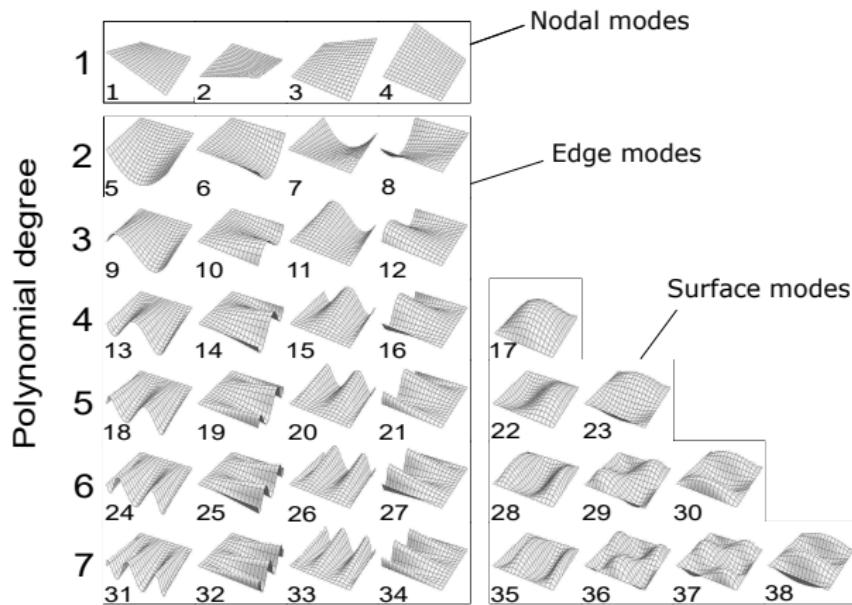
Plate Elements with Node-Dependent Kinematics (NDK)

– Assembly technique



Assembly technique of stiffness matrix for 2D elements adopting NDK.

Hierarchical Legendre Expansions as shape functions of 2D elements



Szabó, B. and Babuška, I.
Finite Element Analysis.
John Wiley & Sons, 1991.



Szabó, B., Düster, A. and
Rank, E.
The p -version of the finite
element method.
Encyclopedia of computational
mechanics (2004).



Pagani, A., De Miguel, A.G.,
Petrolo, M. and Carrera, E.
Analysis of laminated beams
via Unified Formulation and
Legendre polynomial
expansions.
Composite Structures,
156(2016), pp.78-92.

HLE as shape functions for 2D p -version elements.

Hierarchical Legendre Expansions as shape functions of 2D elements

- Nodal modes:

$$N_i(\xi, \eta) = \frac{1}{4} (1 - \xi_i \xi)(1 - \eta_i \eta) \quad i = 1, 2, 3, 4$$

- Edge modes:

$$N_i(\xi, \eta) = \frac{1}{2} (1 - \eta) \phi_p(\xi) \quad i = 5, 9, 13, 18, \dots$$

$$N_i(\xi, \eta) = \frac{1}{2} (1 + \xi) \phi_p(\eta) \quad i = 6, 10, 14, 19, \dots$$

$$N_i(\xi, \eta) = \frac{1}{2} (1 + \eta) \phi_p(\xi) \quad i = 7, 11, 15, 20, \dots$$

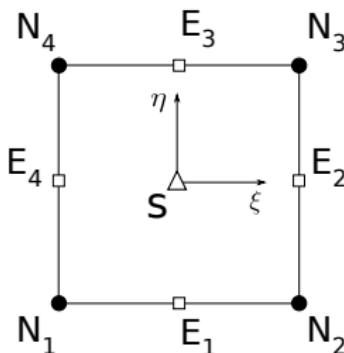
$$N_i(\xi, \eta) = \frac{1}{2} (1 - \xi) \phi_p(\eta) \quad i = 8, 14, 16, 21, \dots$$

- Surface modes:

$$N_i(\xi, \eta) = \phi_p(\xi) \phi_q(\eta) \quad p, q \geq 4$$

- *Basis functions:

$$\phi_p(\xi) = \sqrt{\frac{2p-1}{2}} \int_{-1}^{\xi} L_{p-1}(x) dx = \frac{L_p(\xi) - L_{p-2}(\xi)}{\sqrt{4p-2}} \quad p = 2, 3, \dots$$



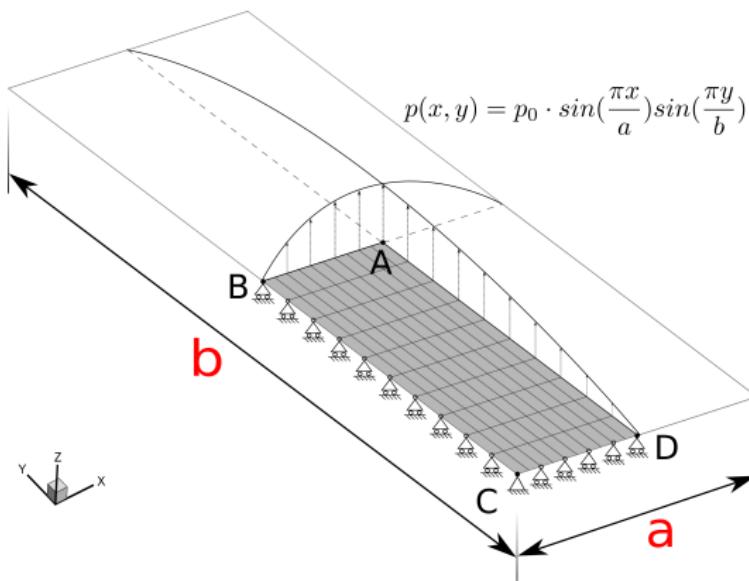
- Nodal modes
- Edge modes
- △ Surface modes



Szabó, B., Düster, A. and Rank, E.

The p -version of the finite element method.
Encyclopedia of computational mechanics (2004).

Three-layered cross-ply plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 2$ and 100, loaded on top surface



2D FEM 1/4 model with symmetry for the composite plate.



Pagano, N.J.

Exact solutions for rectangular bidirectional composites and sandwich plates.

Journal of Composite Materials, Vol 4, pp 20-34 (January 1970). Composites 1.4(1970):257-257.

Three-layered cross-ply plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 2$ Thick plate ($\frac{a}{h} = 2$), with LE4 as thickness functions.

Element	Mesh	\bar{w} ($\frac{a}{2}, \frac{b}{2}, 0$)	$\bar{\sigma}_{yy}$ ($\frac{a}{2}, \frac{b}{2}, \frac{h}{6}$)	$10\bar{\sigma}_{yz}$ ($\frac{a}{2}, 0, 0$)	$\bar{\sigma}_{zz}$ ($\frac{a}{2}, \frac{b}{2}, \frac{h}{2}$)	DOFs
Q9	1x1	3.196	1.006	2.668	0.5103	351
Q9	2x2	8.167	2.356	7.525	1.010	975
Q9	4x4	7.882	2.315	6.904	0.9693	3159
Q9	5x5	8.165	2.308	6.825	1.004	4719
HLE2	1x1	8.137	2.671	9.853	1.038	312
HLE3	1x1	8.005	2.866	6.961	0.9598	468
HLE4	1x1	8.120	2.333	6.600	0.9918	663
HLE5	1x1	8.167	2.279	6.622	1.004	897
HLE6	1x1	8.166	2.293	6.688	1.004	1170
HLE7	1x1	8.165	2.295	6.686	1.004	1482
Pagano		8.17	2.30	6.68	1.000	
Kulikov and Plotnikova		8.1659	2.6772	6.6778	1.0001	
Carrera et al.		8.166	2.296	6.690	1.000	



Carrera, E., Cinefra, M., Li, G. and Kulikov, G.M.

MITC9 shell finite elements with miscellaneous through-the-thickness functions for the analysis of laminated structures.

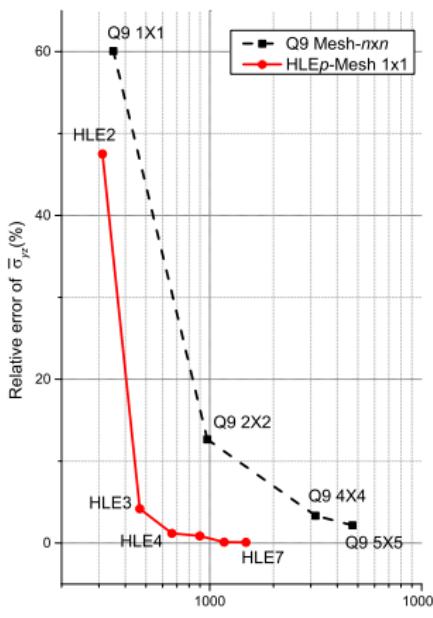
Composite Structures, 154(2016), pp.360–373.



Kulikov, GM and Plotnikova, SV

Exact 3D stress analysis of laminated composite plates by sampling surfaces method.

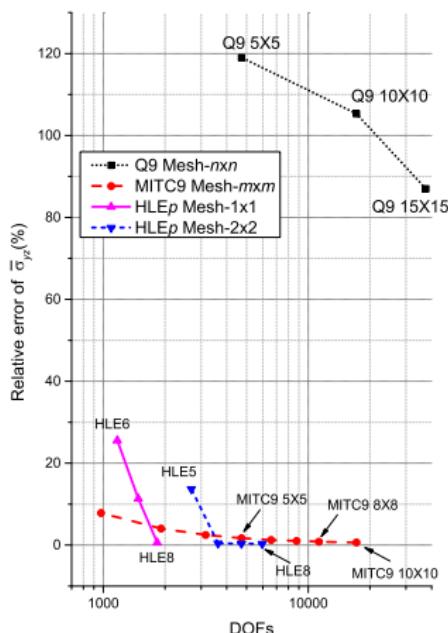
Composite Structures, 94(12), pp.3654-3663.



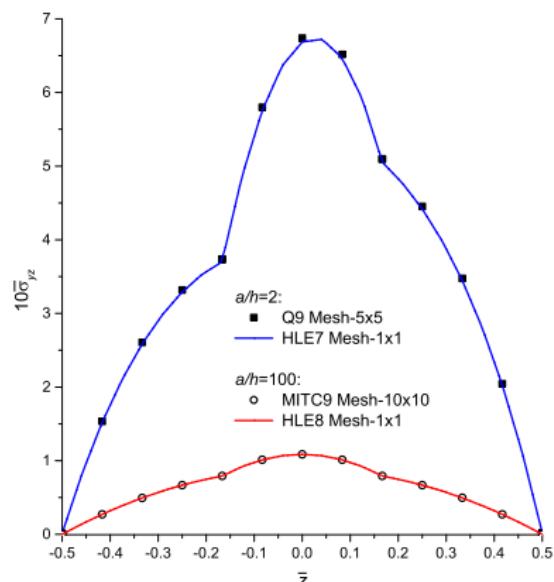
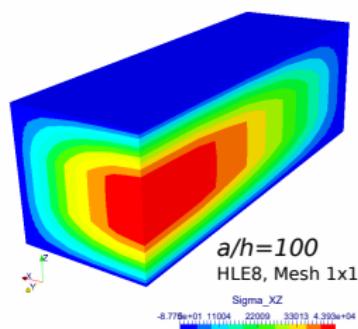
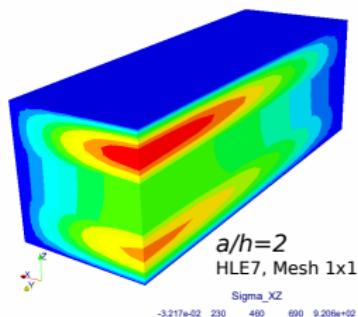
DOFs vs Accuracy

Three-layered cross-ply plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 100$ Thin plate ($\frac{a}{h} = 100$), with LE4 as thickness kinematics.

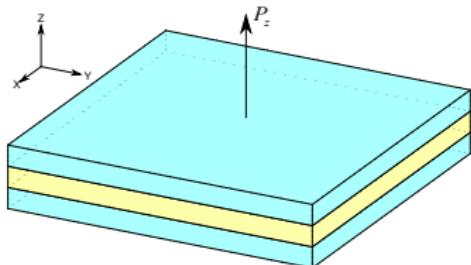
Element	Mesh	\bar{w} $(\frac{a}{2}, \frac{b}{2}, 0)$	$\bar{\sigma}_{yy}$ $(\frac{a}{2}, \frac{b}{2}, \frac{h}{6})$	$10\bar{\sigma}_{yz}$ $(\frac{a}{2}, 0, 0)$	$\bar{\sigma}_{zz}$ $(\frac{a}{2}, \frac{b}{2}, \frac{h}{2})$	DOFs
Q9	5x5	0.5069	0.2511	2.365	1.813	4719
Q9	10x10	0.5076	0.2528	2.217	1.128	17199
Q9	15x15	0.5076	0.2530	2.020	1.041	37479
MITC9	5x5	0.5077	0.2551	1.098	1.018	4719
MITC9	8x8	0.5077	0.2539	1.089	1.003	11271
MITC9	10x10	0.5077	0.2536	1.087	1.001	17199
HLE3	1x1	0.4487	0.5085	105.1	5.404	468
HLE4	1x1	0.5054	0.3479	42.23	-1.431	663
HLE5	1x1	0.5087	0.2830	-7.019	0.4908	897
HLE6	1x1	0.5077	0.2486	0.8042	1.164	1170
HLE7	1x1	0.5077	0.2531	1.202	1.011	1482
HLE8	1x1	0.5077	0.2532	1.087	0.9971	1833
HLE3	2x2	0.5075	0.3472	8.386	2.541	1287
HLE4	2x2	0.5075	0.2629	3.799	0.8828	1911
HLE5	2x2	0.5077	0.2540	0.9318	0.9758	2691
HLE6	2x2	0.5077	0.2530	1.075	1.003	3627
HLE7	2x2	0.5077	0.2531	1.084	1.000	4719
HLE8	2x2	0.5077	0.2531	1.084	1.000	5967
Pagano Kulikov and Plotnikova Carrera et al.		0.508 0.50766 0.5077	0.253 0.25236 0.2533	1.08 1.0836 1.085	1.000 1.000 1.000	



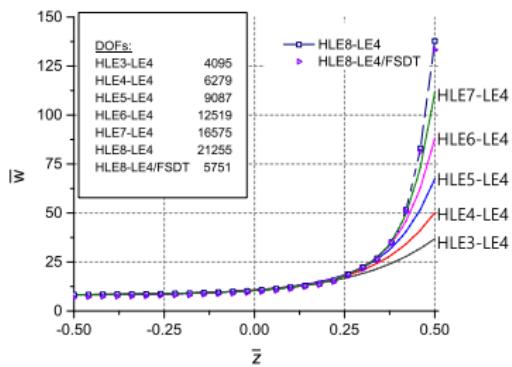
DOFs vs Accuracy

Three-layered cross-ply plate, $[0^\circ/90^\circ/0^\circ]$, σ_{xz} and σ_{yz} Variation of $10\bar{\sigma}_{yz}$ along \bar{z} Contour plot of σ_{xz}

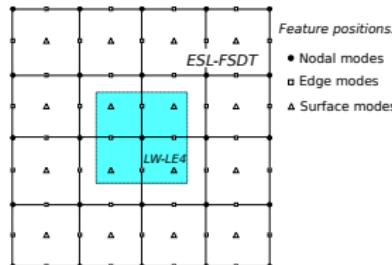
Cross-ply plate subjected to point load on the top surface



Geometry and loading.



Through-the-thickness variation of w



Assignment of nodal kinematics

$$\bar{w} \text{ at } (\frac{a}{2}, \frac{b}{2}, 0)$$

Element	Kinematics	w
HLE8	LE4	10.46
HLE8	LE4/FSDT	9.512
ABAQUS 3D		11.76
Carrera (closed-form)		13.188

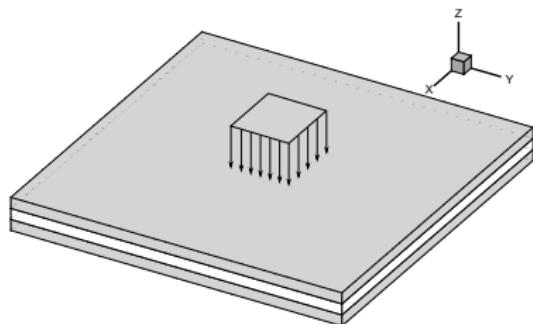


Carrera, E. and Ciuffreda, A.

Bending of composites and sandwich plates subjected to localized lateral loadings: a comparison of various theories.

Composite Structures, 68(2), pp.185-202.

Three-layered cross-ply composite plate under local pressure, $a/h = 10$



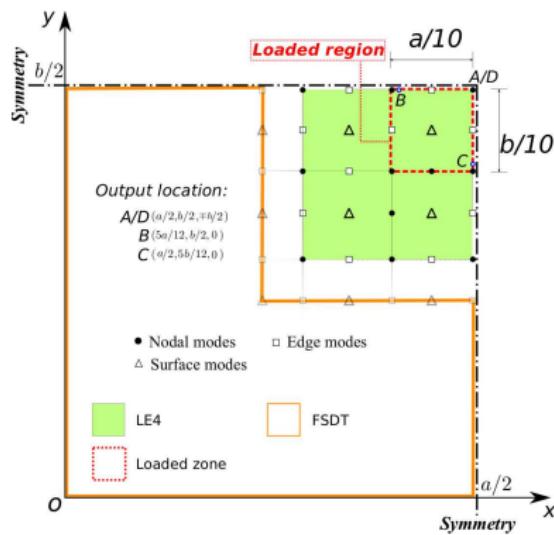
Geometry and loading, $a/h = 10$, $(0^\circ/90^\circ/0^\circ)$



Biscani, F., Giunta, G., Belouettar, S., Carrera, E. and Hu, H.
Variable kinematic plate elements coupled via Arlequin method.
International Journal for Numerical Methods in Engineering, 91(12),
pp.1264-1290.



Zappino, E., Li, G., Pagani, A. and Carrera, E.
Global-local analysis of laminated plates by node-dependent kinematic
finite elements with variable ESL/LW capabilities.
Composite Structures, 172(2017), pp.1–14.

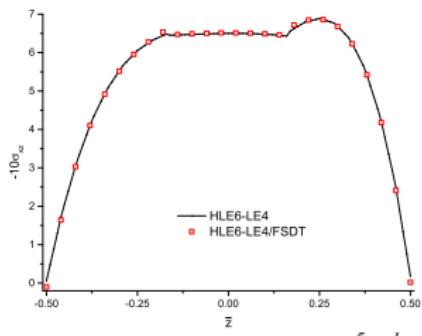


Assignment of nodal kinematics.

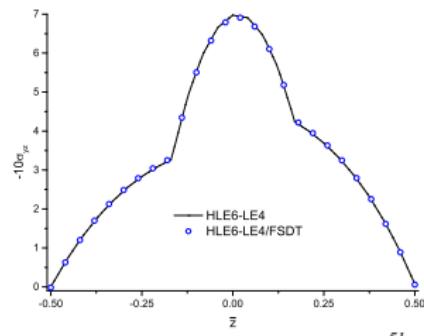
Three-layered cross-ply composite plate under local pressure, $\frac{a}{h} = 10$

Displacement and stress evaluation on the three-layered plate under local pressure

Element	Mesh	Kinematics	$w[10^{-5}\text{m}]$ ($\frac{a}{2}, \frac{b}{2}, -\frac{h}{2}$)	$\sigma_{xx}[\text{MPa}]$ ($\frac{a}{2}, \frac{b}{2}, -\frac{h}{2}$)	$\sigma_{yy}[\text{MPa}]$ ($\frac{a}{2}, \frac{b}{2}, -\frac{h}{2}$)	$-10\sigma_{xz}[\text{MPa}]$ ($\frac{5a}{12}, \frac{b}{2}, 0$)	$-10\sigma_{yz}[\text{MPa}]$ ($\frac{a}{2}, \frac{5b}{12}, 0$)	$-\sigma_{zz}[\text{MPa}]$ ($\frac{a}{2}, \frac{b}{2}, \frac{h}{2}$)	DOFs
HLE3	5x5	LE4	1.681	12.08	2.061	6.596	7.094	1.241	6084
HLE4	5x5	LE4	1.682	11.92	2.023	6.501	6.965	0.881	9399
HLE5	5x5	LE4	1.682	11.97	2.039	6.497	6.972	1.045	13689
HLE6	5x5	LE4	1.682	11.98	2.037	6.499	6.974	1.003	18954
HLE6	5x5	FSDT	1.610	10.44	1.850	3.829	4.647	0.938	2430
HLE6	5x5	LE4/FSDT	1.702	12.50	2.036	6.516	6.941	1.008	5592
Zappino et al.			1.675	11.99	2.033	6.463	6.902	0.993	37479
Biscani et al.			1.674	11.94	2.019	6.524	—	—	—



Variation of $-10\sigma_{xz}$ through \bar{z} at $B(\frac{5a}{12}, \frac{b}{2})$



Variation of $-10\sigma_{yz}$ through \bar{z} at $C(\frac{a}{2}, \frac{5b}{12})$

Conclusions

With node-dependent kinematic plate elements adopting HLE as shape functions:

- ① FEM models with variable LW/ESL nodal capabilities can be conveniently constructed;
- ② Local kinematic refinement can be carried out without modifying the mesh neither using any *ad hoc* coupling method;
- ③ With sufficiently high polynomial degree p , 2D elements adopting HLE can achieve comparable accuracy with fewer computational resources compared with pure h -version refinement;
- ④ Node-dependent kinematics can help to further reduce the computational consumption when the structure undergoes strong local effects;
- ⑤ Such an approach is ideal for global-local analysis.

Acknowledgment

- FULLComp: **FULLy** integrated analysis, design, manufacturing and health-monitoring of **COMPosite** structures
- Funded by the European Commission under a *Marie Skłodowska-Curie* Innovative Training Networks grant for European Training Networks (grant agreement No. 642121).
- The full spectrum of the design of composite structures will be dealt with, such as **manufacturing, health-monitoring, failure, modeling, multiscale approaches, testing, prognosis, and prognostic.**
- Research activities are aimed at engineering fields such as aeronautics, automotive, mechanical, wind energy and space.
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