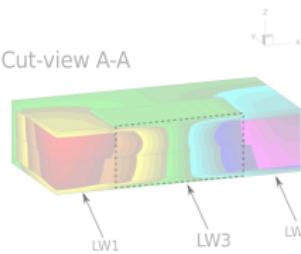
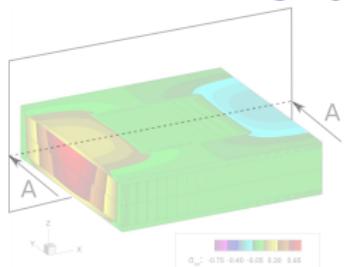
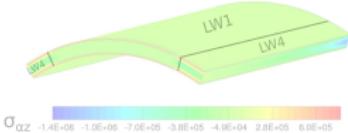
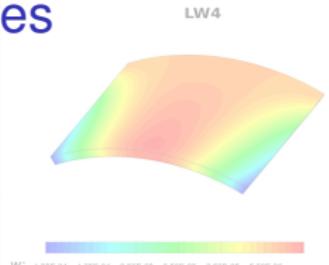


# Node-dependent kinematic shell elements for the analysis of smart structures



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Moscow, 21-24 May 2017

# Contents

- Unified Formulation for plates and shells
- Finite Element Method, and shear locking problem overcoming (MITC)
- Governing Equations in weak form
- Global/local method: Node dependent variable kinematics
- Results: Composite Plate with piezoelectric layers subjected to mechanical and electrical loads
- Results: Composite Shell with piezoelectric layers subjected to electrical load
- Results: Sandwich plate with piezoelectric patch subjected to electrical load
- Conclusions

## Unified Formulation

2D approximation of **mechanical displacements** and **electric potential**  
using the *thickness functions*

$$\left\{ \begin{array}{l} u^k(\alpha, \beta, z) = F_0(z) u_0^k(\alpha, \beta) + F_1(z) u_1^k(\alpha, \beta) + \dots + F_N(z) u_N^k(\alpha, \beta) \\ v^k(\alpha, \beta, z) = F_0(z) v_0^k(\alpha, \beta) + F_1(z) v_1^k(\alpha, \beta) + \dots + F_N(z) v_N^k(\alpha, \beta) \\ w^k(\alpha, \beta, z) = F_0(z) w_0^k(\alpha, \beta) + F_1(z) w_1^k(\alpha, \beta) + \dots + F_N(z) w_N^k(\alpha, \beta) \\ \Phi^k(\alpha, \beta, z) = F_0(z) \Phi_0^k(\alpha, \beta) + F_1(z) \Phi_1^k(\alpha, \beta) + \dots + F_N(z) \Phi_N^k(\alpha, \beta) \end{array} \right.$$

in compact form:

$$\mathbf{u}^k(\alpha, \beta, z) = F_\tau(z)\mathbf{u}_\tau^k(\alpha, \beta) \quad ; \quad \delta\mathbf{u}^k(\alpha, \beta, z) = F_s(z)\delta\mathbf{u}_s^k(\alpha, \beta) \quad ; \quad \tau, s = 0, 1, \dots, N$$

$$\Phi^k(\alpha, \beta, z) = F_\tau(z)\Phi_\tau^k(\alpha, \beta) \quad ; \quad \delta\Phi^k(\alpha, \beta, z) = F_s(z)\delta\Phi_s^k(\alpha, \beta) \quad ; \quad \tau, s = 0, 1, \dots, N$$

## Taylor Polynomials

$$\mathbf{u}^k = F_0 \mathbf{u}_0^k + F_1 \mathbf{u}_1^k + \dots + F_N \mathbf{u}_N^k = F_\tau \mathbf{u}_\tau^k$$

$$\Phi^k = F_0 \Phi_0^k + F_1 \Phi_1^k + \dots + F_N \Phi_N^k = F_\tau \Phi_\tau^k$$

$$\tau = 0, 1, \dots, N$$

$$F_0 = (z)^0 = 1; F_1 = (z)^1 = z; \dots; F_N = (z)^N$$

## Legendre Polynomials

$$\mathbf{u}^k = F_t \mathbf{u}_t^k + F_b \mathbf{u}_b^k + F_r \mathbf{u}_r^k = F_\tau \mathbf{u}_\tau^k$$

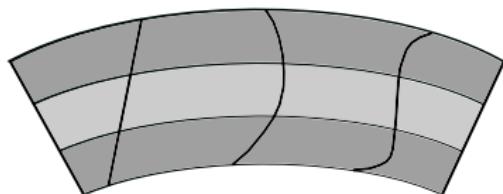
$$\Phi^k = F_t \Phi_t^k + F_b \Phi_b^k + F_r \Phi_r^k = F_\tau \Phi_\tau^k$$

$$\tau = t, b, r ; r = 2, \dots, N$$

$$F_t = \frac{P_0 + P_1}{2}; F_b = \frac{P_0 - P_1}{2}; F_r = P_r - P_{r-2}$$

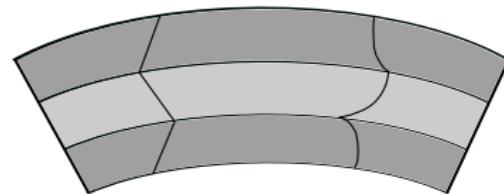
### Equivalent Single Layer Approach (ESL)

$$N = 1 \quad N = 2 \quad N = 3$$



### Layer Wise Approach (LW)

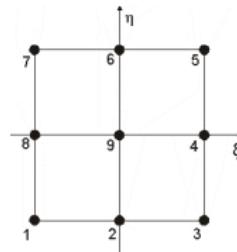
$$N = 1 \quad N = 2$$



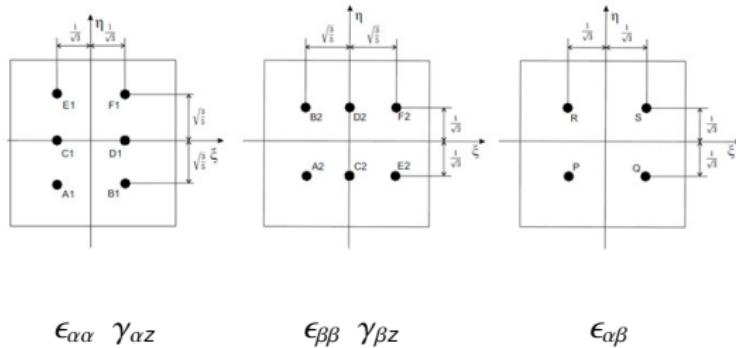
# Finite Element Method

Approximation of **variables** in the reference midplane surface using the *Langrangian* shape functions:

$$\mathbf{u}_\tau = N_i(\xi, \eta) \mathbf{u}_{\tau i}$$



To overcome the problem of the **membrane and shear locking**, the strain components are calculated using a specific interpolation strategy:



For example:

$$\epsilon_{\alpha\alpha} = N_{A1}\epsilon_{\alpha\alpha A1} + N_{B1}\epsilon_{\alpha\alpha B1} + N_{C1}\epsilon_{\alpha\alpha C1} + N_{D1}\epsilon_{\alpha\alpha D1} + N_{E1}\epsilon_{\alpha\alpha E1} + N_{F1}\epsilon_{\alpha\alpha F1}$$

# PVD for electro-mechanical problems

## Static Analysis

### Principle of Virtual Displacements

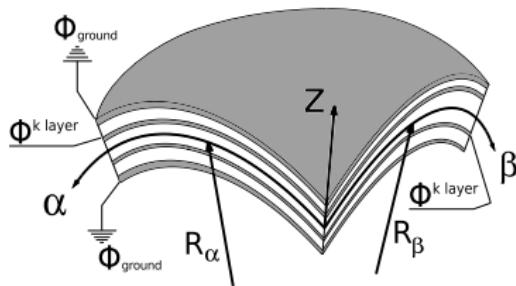
$$\int_V \left\{ \delta \epsilon^k \cdot \sigma^k - \delta \mathcal{E}^k \cdot \mathcal{D}^k \right\} dV = \delta L_e$$

$$\sigma^k = \mathbf{C}^k \epsilon^k - \mathbf{e}^k \cdot \mathcal{E}^k$$

$$\mathcal{D}^k = \mathbf{e}^k \epsilon^k + \epsilon^k \mathcal{E}^k$$

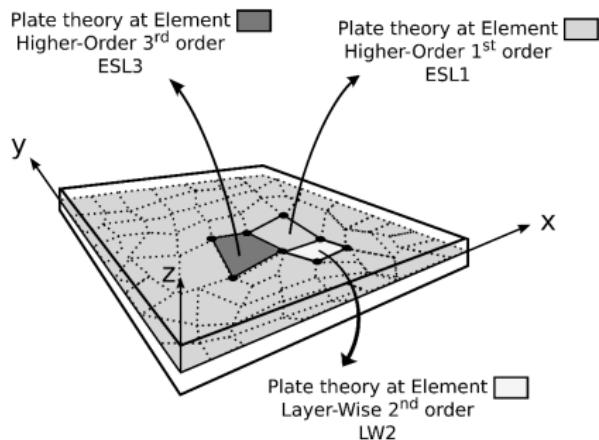
In compact form:

$$(4 \times 4) \quad \begin{aligned} \delta \mathbf{u}_{\tau i}^k : & \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\phi} \\ \mathbf{K}_{\phi u} & \mathbf{K}_{\phi\phi} \end{bmatrix}^{k\tau sij} \begin{Bmatrix} \mathbf{u}_{sj}^k \\ \Phi_{sj}^k \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}_u \\ \mathbf{P}_\phi \end{Bmatrix}^{k\tau i} \\ \delta \Phi_{\tau i}^k : & \begin{bmatrix} \mathbf{K}_{\phi u} & \mathbf{K}_{\phi\phi} \end{bmatrix}^{k\tau sij} \end{aligned}$$



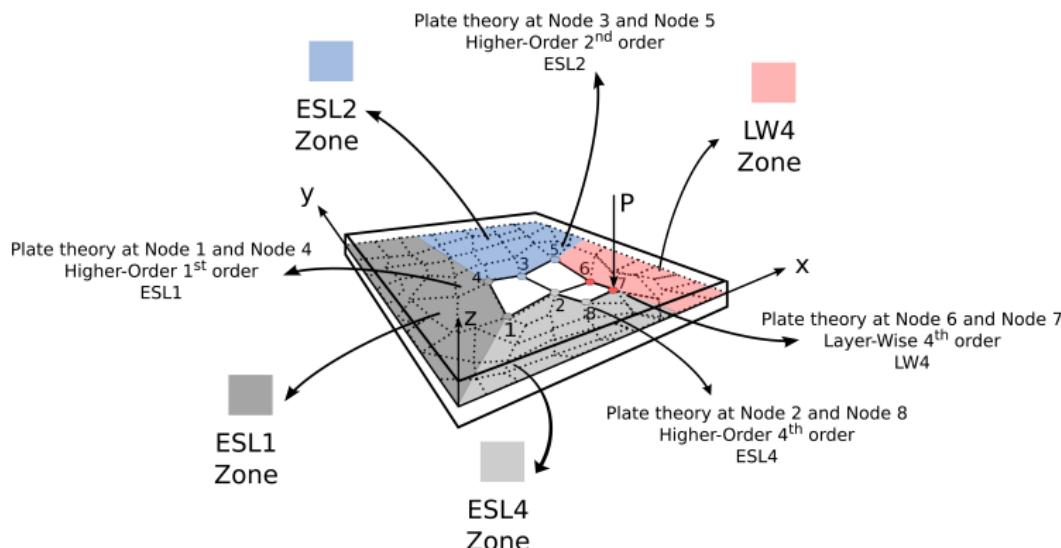
$$\mathbf{K}^{k\tau sij} = \begin{bmatrix} K_{\alpha\alpha} & K_{\alpha\beta} & K_{\alpha z} & K_{\alpha\phi} \\ K_{\beta\alpha} & K_{\beta\beta} & K_{\beta z} & K_{\beta\phi} \\ K_{z\alpha} & K_{z\beta} & K_{zz} & K_{z\phi} \\ K_{\phi\alpha} & K_{\phi\beta} & K_{\phi z} & K_{\phi\phi} \end{bmatrix}^{k\tau sij}$$

## Coupling of Different Kinematics in Literature



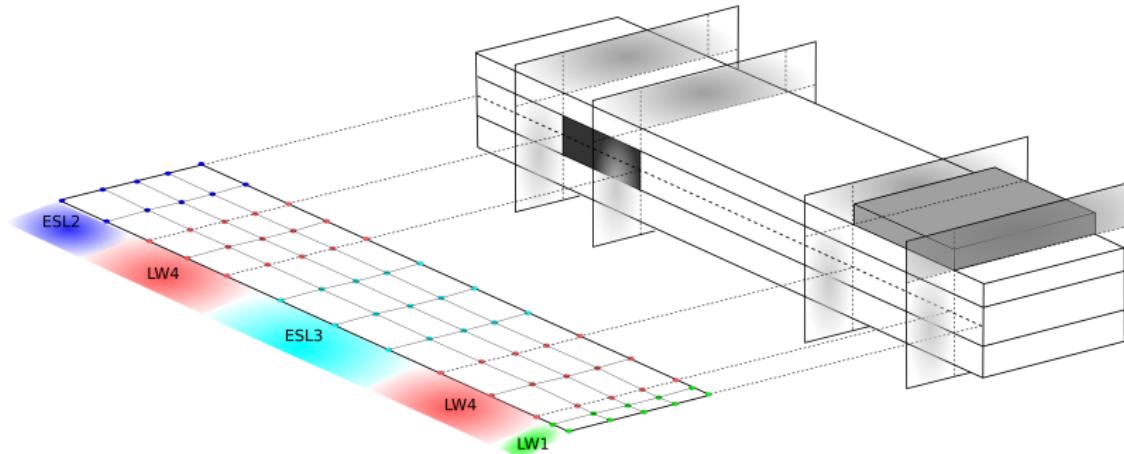
In literature different kinematics can be coupled between elements with additional equations or mathematical artifices. The most common method to couple different kinematics are: the Arlequin Method, and the Lagrange Multipliers Method.

## Node-Dependent Variable Kinematic Finite Element for Global/Local analysis



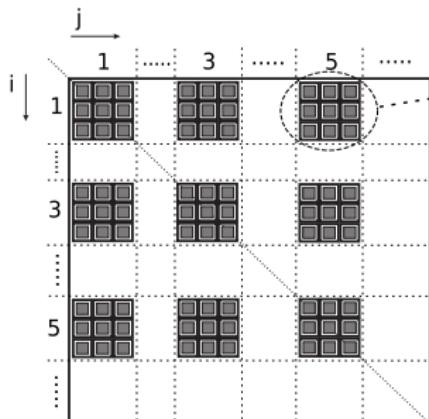
Different Kinematics are defined in the **Global Nodes**. **Shared nodes have the same kinematics**. The coupling of different kinematics is naturally obtained inside the finite element **without** any mathematical artifice or additional equations.

## Node-Dependent Variable Kinematic Finite Element for Global/Local analysis

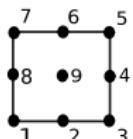
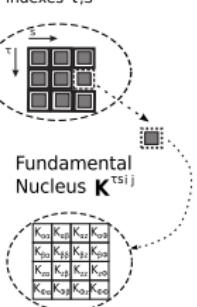


# Finite Element assembling scheme

Assembling on the nodes  $i,j$



Assembling on expansion order indexes  $\tau, S$



$$\mathbf{u}(x,y,z) = F_S(z) N_j(\alpha,\beta) \mathbf{u}_S$$

$$\delta\mathbf{u}(\alpha,\beta,z) = F_T(z) N_j(\alpha,\beta) \delta\mathbf{u}_T$$

$$\Phi(x,y,z) = F_S(z) N_j(\alpha,\beta) \Phi_S$$

$$\delta\Phi(\alpha,\beta,z) = F_T(z) N_j(\alpha,\beta) \delta\Phi_T$$

$$S = 0, 1, \dots, N \quad \tau = 0, 1, \dots, N \quad i, j = 1, \dots, 9$$

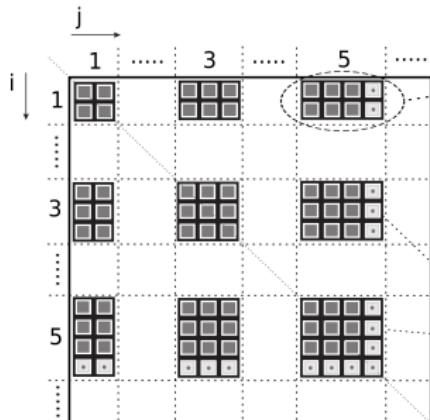
ESL2 model for the entire finite element

$$\mathbf{u}(\alpha,\beta,z) = (u_{01} + z u_{11} + z^2 u_{21}) N_1(\alpha,\beta) + \dots + (u_{03} + z u_{13} + z^2 u_{23}) N_3(\alpha,\beta) + \dots + (u_{05} + z u_{15} + z^2 u_{25}) N_5(\alpha,\beta) + \dots$$

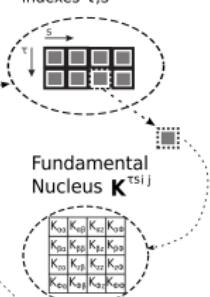
$$\Phi(\alpha,\beta,z) = (\Phi_{01} + z \Phi_{11} + z^2 \Phi_{21}) N_1(\alpha,\beta) + \dots + (\Phi_{03} + z \Phi_{13} + z^2 \Phi_{23}) N_3(\alpha,\beta) + \dots + (\Phi_{05} + z \Phi_{15} + z^2 \Phi_{25}) N_5(\alpha,\beta) + \dots$$

# Node-Dependent Variable assembling example

Assembling on the nodes  $i,j$

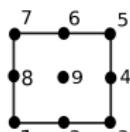


Assembling on expansion order indexes  $\tau, s$



Fundamental Nucleus  $K^{tsij}$

Cubic term influence



$$\mathbf{u}(\alpha, \gamma, z) = F_S(z) N_j(\alpha, \beta) \mathbf{u}_S$$

$$\delta \mathbf{u}(\alpha, \beta, z) = F_\tau^i(z) N_i(\alpha, \beta) \delta \mathbf{u}_\tau$$

$$\Phi(\alpha, \gamma, z) = F_S(z) N_j(\alpha, \beta) \Phi_S$$

$$\delta \Phi(\alpha, \beta, z) = F_\tau^i(z) N_i(\alpha, \beta) \delta \Phi_\tau$$

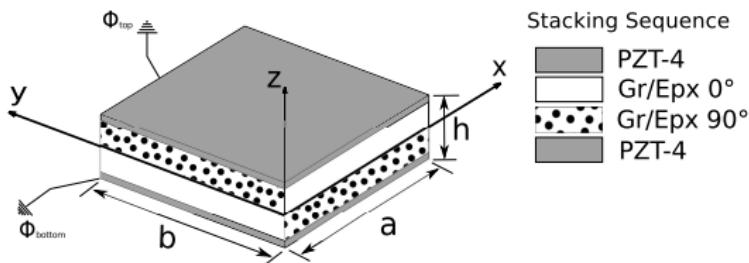
$$s = 0, 1, \dots, N^j \quad \tau = 0, 1, \dots, N^i \quad i, j = 1, \dots, 9$$

$$N^{(node\ 1)} = 1 \dots N^{(node\ 3)} = 2 \dots N^{(node\ 5)} = 3$$

$$\begin{aligned} \mathbf{u}(\alpha, \beta, z) = & [(1 + \zeta_k/2) U_{01} + (1 - \zeta_k/2) U_{11}] N_1(\alpha, \beta) + \dots + \\ & + (U_{03} + z U_{13} + z^2 U_{23}) N_3(\alpha, \beta) + \dots + \\ & + (U_{05} + z U_{15} + z^2 U_{25} + z^3 U_{35}) N_5(\alpha, \beta) + \dots \end{aligned}$$

$$\begin{aligned} \Phi(\alpha, \beta, z) = & [(1 + \zeta_k/2) \Phi_{01} + (1 - \zeta_k/2) \Phi_{11}] N_1(\alpha, \beta) + \dots + \\ & + (\Phi_{03} + z \Phi_{13} + z^2 \Phi_{23}) N_3(\alpha, \beta) + \dots + \\ & + (\Phi_{05} + z \Phi_{15} + z^2 \Phi_{25} + z^3 \Phi_{35}) N_5(\alpha, \beta) + \dots \end{aligned}$$

# Composite Plate with Piezoelectric Skins



## Simply-Supported

Geometry:  
 $a=b=4$  ;  $h=1$   
 $h_{\text{composite}} = 0.4 * h_{\text{total}}$   
 $h_{\text{skin}} = 0.1 * h_{\text{total}}$

## Sensor Case

$$p(x, y, z_{top}) = \hat{p}_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

$$\hat{\Phi}(z = \text{top}) = 0$$

$$\hat{\Phi}(z = \text{bottom}) = 0$$

## Actuator Case

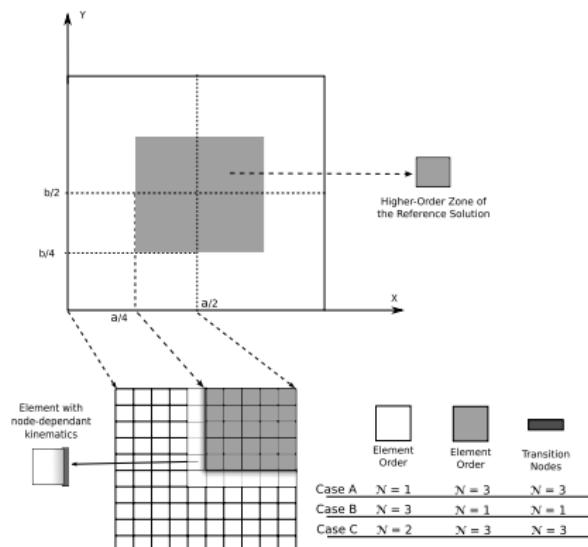
$$\Phi(x, y, z_{top}) = \hat{\Phi}_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right)$$

$$\hat{\Phi}(z = \text{bottom}) = 0$$

# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models

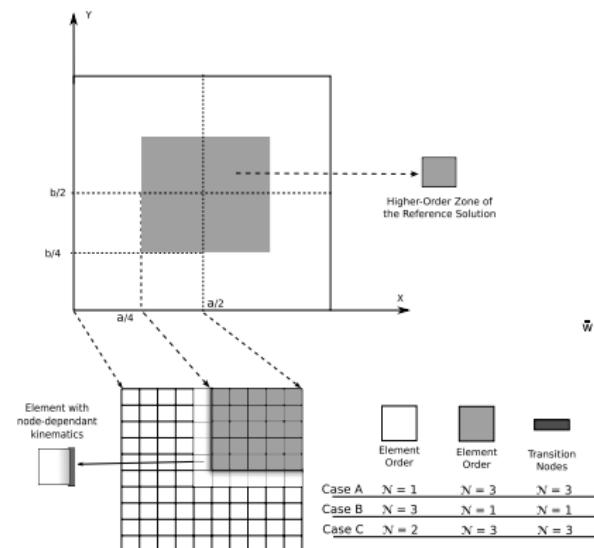
### Actuator Case



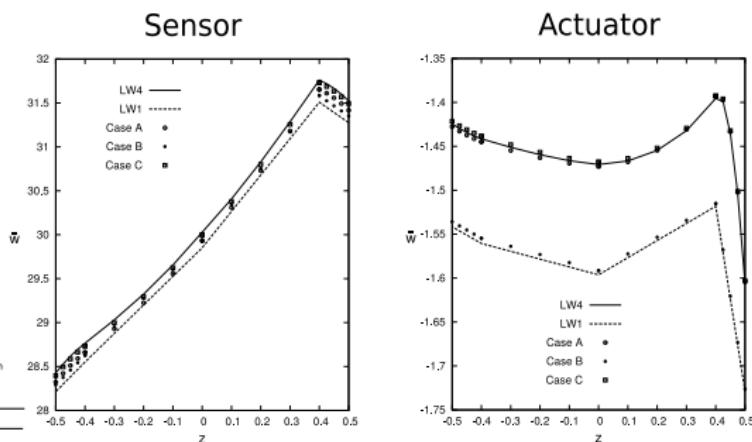
( $x, y, z$ )	$\hat{w}$ $(\frac{a}{2}, \frac{b}{2}, 0)$	$\hat{\sigma}_{xx}$ $(\frac{a}{2}, \frac{b}{2}, +\frac{b}{2})$	$\hat{\sigma}_{xz}$ $(0, \frac{b}{2}, 0)$	$\hat{\sigma}_{zz}$ $(\frac{a}{2}, \frac{b}{2}, 0)$	$\hat{\phi}$ $(\frac{a}{2}, \frac{b}{2}, 0)$	$\hat{D}_z$ $(\frac{a}{2}, \frac{b}{2}, +\frac{b}{2})$	DOFs
Reference solutions							
<b>3D Exact Analytical solutions</b>	-1.471	1.1181	-0.2387	-14.612	0.4476	-	
LW4 <sub>a</sub>	-1.4707	1.1180	-0.239	-	0.4477	-2.4184	
LW1 <sub>a</sub>	-1.5962	3.3433	-	-	0.4468	-1.3814	
Arlequin solutions							
$(LW1 - LWM3)^A$	-1.420	1.119	-	-	-	-	
$(LW2 - LWM3)^C$	-1.410	-	-	-	-	-	
Present single- and multi-theory models							
LW4	-1.4707	1.1248	-0.2411 <sup>+</sup>	-14.732 <sup>+</sup>	0.4477	-2.4186	29988
LW3	-1.4707	1.1261	-0.2270 <sup>+</sup>	-14.660 <sup>+</sup>	0.4477	-2.4183	22932
LW2	-1.4662	1.1311	-0.3592 <sup>+</sup>	-13.541 <sup>+</sup>	0.4477	-2.4167	15876
LW1	-1.5962	3.3531	-0.0293 <sup>+</sup>	-15.343 <sup>-</sup>	0.4468	-1.3816	8820
Case A	-1.4729	1.1315	-0.0302 <sup>+</sup>	-12.729 <sup>+</sup>	0.4479	-2.4183	12692
Case B	-1.5916	3.3590	-0.2210 <sup>+</sup>	-14.026 <sup>+</sup>	0.4467	-1.3818	19060
Case C	-1.4679	1.1250	-0.3599 <sup>+</sup>	-13.641 <sup>+</sup>	0.4477	-2.4183	17812

# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models

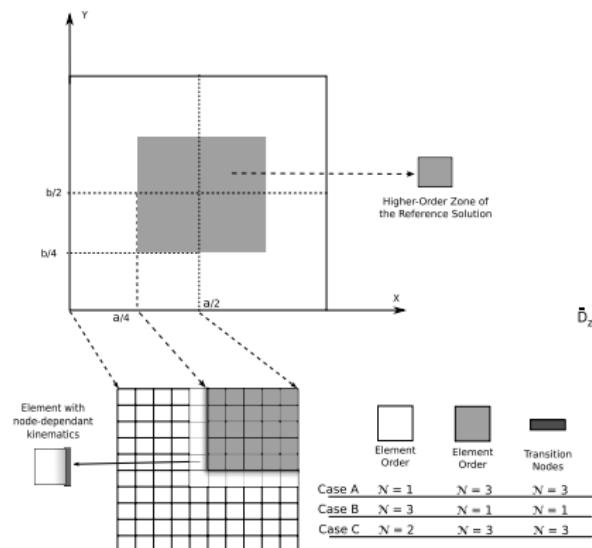


Transverse displacement  
 $w(x, y) : (a/2, b/2)$



# Node-Dependent Variable Kinematics

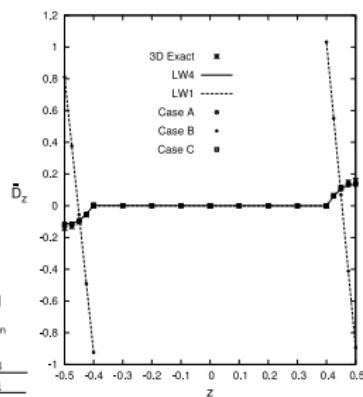
## Cases with Layer-Wise models



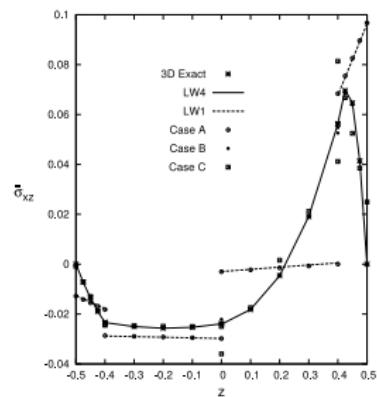
Transverse electric displacement and shear stress

$$\mathcal{D}_z(x, y) : \sigma_{xz}(x, y) : (a/2, b/2)$$

Sensor

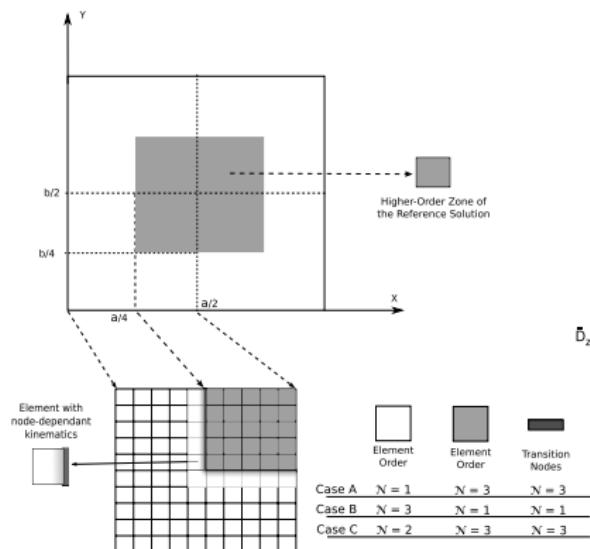


Actuator



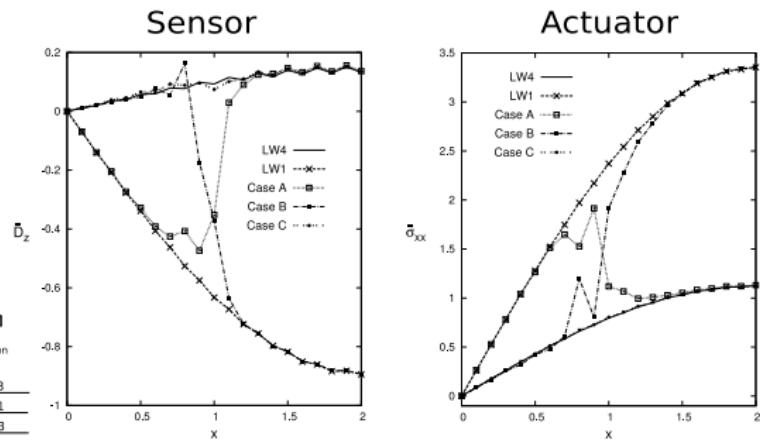
# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models



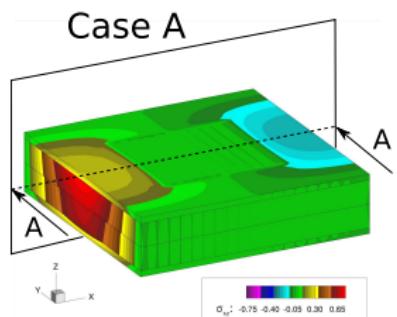
### Transverse electric displacement and in-plane stress along X-axis

$$\mathcal{D}_z(y, z) : \sigma_{xx}(y, z) : (b/2, +h/2)$$

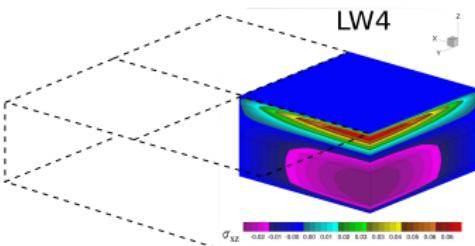


# Transverse Shear Stress $\sigma_{xz}$

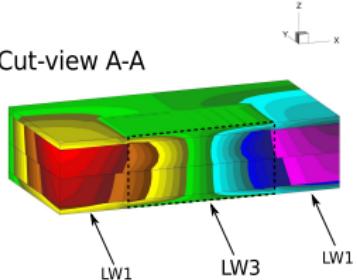
## Sensor



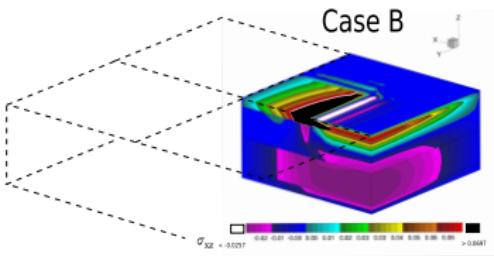
## Actuator



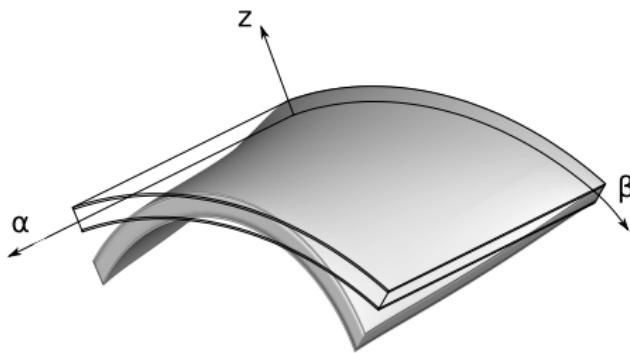
Cut-view A-A



Case B



# Cylindrical shell with Piezoelectric Skins



Stacking Sequence

- [Grey square] PZT-5H
- [White square] Aluminum
- [Grey square] PZT-5H

## Cantilevered

Geometry:  
 $a=b=300$  mm  
 $h_{\text{Aluminum}} = 2.5$  mm  
 $h_{\text{skin}} = 0.25$  mm  
 $R/b=1$

## Actuator Case

Upper Skin

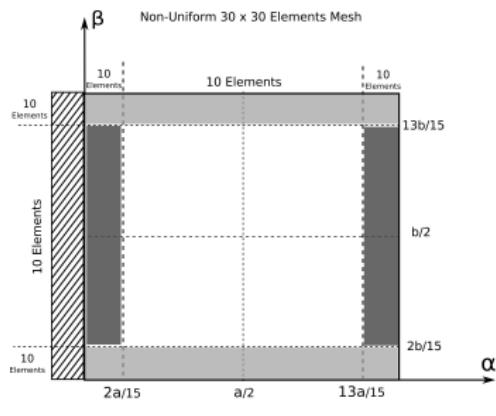
$$\Phi_{top} = 50 \text{ V}, \Phi_{bottom} = 0 \text{ V}$$

Lower Skin

$$\Phi_{top} = 0 \text{ V}, \Phi_{bottom} = 50 \text{ V}$$

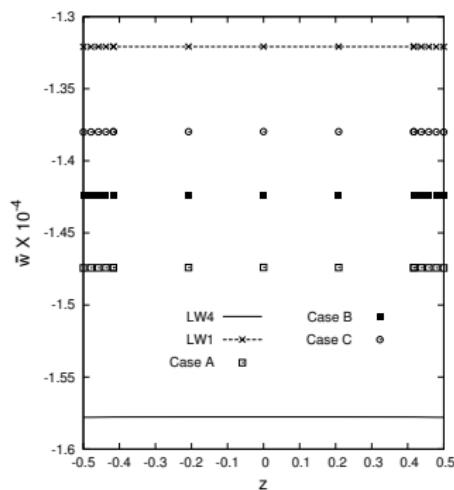
# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models



	Element Order	Element Order	Element Order
Case A	$N = 1$	$N = 4$	$N = 4$
Case B	$N = 4$	$N = 1$	$N = 1$
Case C	$N = 1$	$N = 4$	$N = 1$

Transverse displacement  
 $w(\alpha, \beta) : (a, 0)$

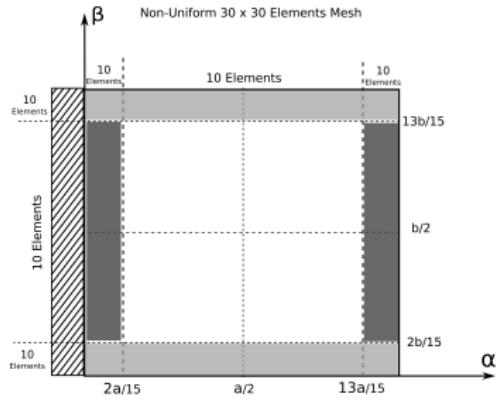


# Node-Dependent Variable Kinematics

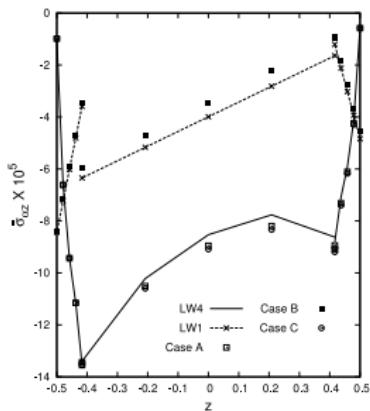
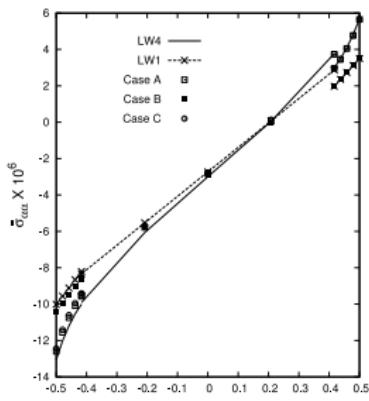
## Cases with Layer-Wise models

In-plane and transverse shear stresses

$$\sigma_{\alpha\alpha}(\alpha, \beta) : \sigma_{az}(\alpha, \beta) : (0, 0)$$



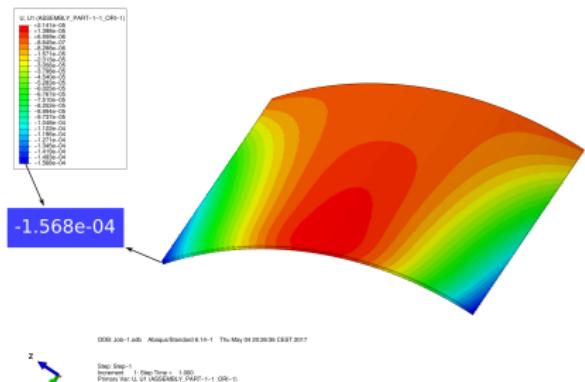
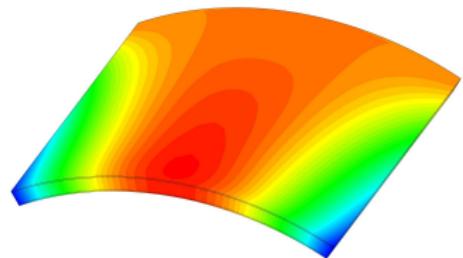
	Case A	$N = 1$	$N = 4$	$N = 4$
Case B		$N = 4$	$N = 1$	$N = 1$
Case C		$N = 1$	$N = 4$	$N = 1$



# Transverse displacement $w$

Present shell model **LW4**

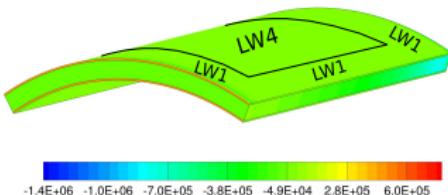
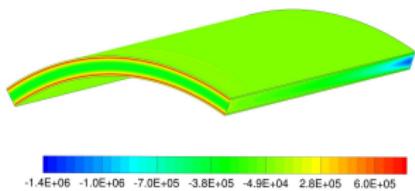
Abaqus: 3D finite element C3D20RE



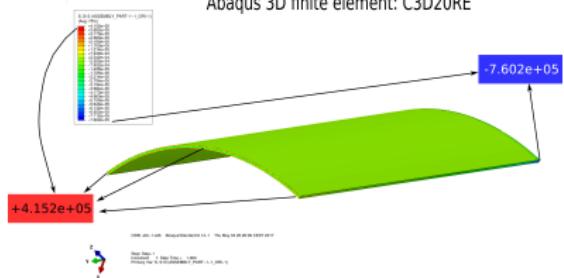
# Transverse shear stress $\sigma_{\alpha z}$

Case B

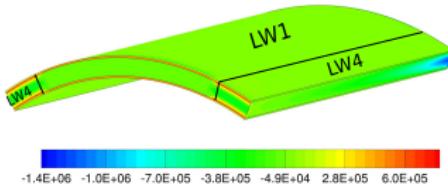
Present shell model LW4



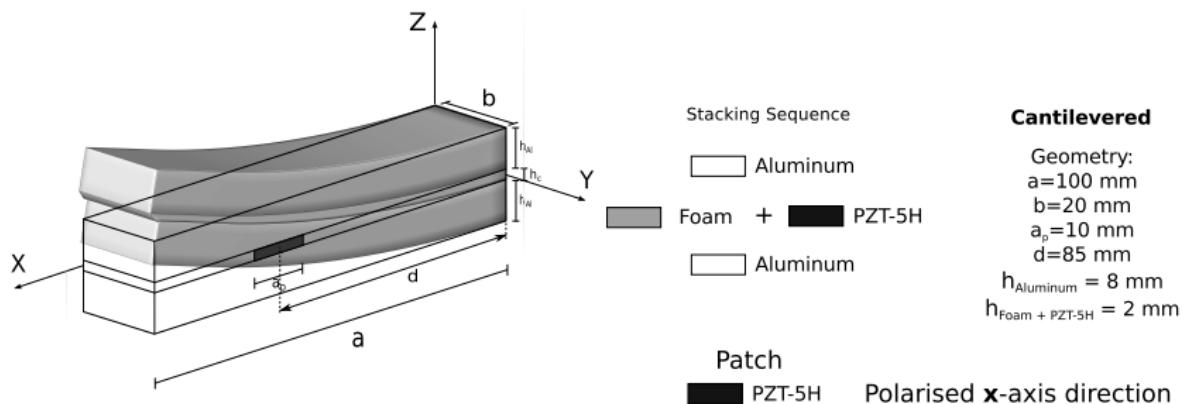
Abaqus 3D finite element: C3D20RE



Case C



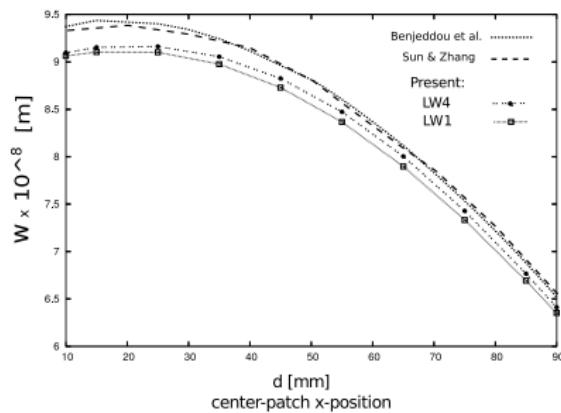
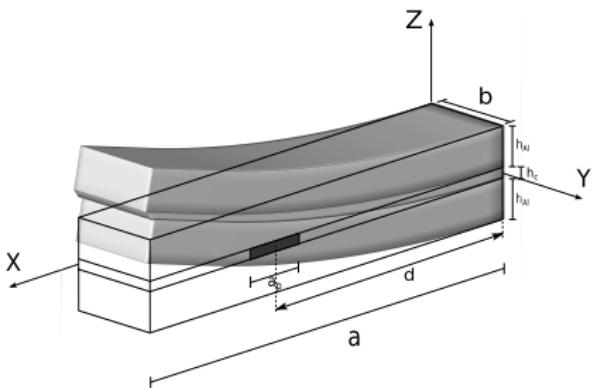
## Sandwich plate with piezoelectric patch shear actuated mode-15



Piezoelectric patch used in **Actuator configuration**

$$\Phi_{\text{patch top}} = -10 \text{ V} \quad \Phi_{\text{patch bottom}} = +10 \text{ V}$$

## Model validation varying the patch position along the x-axis



### Reference solutions:



C T Sun and X D Zhang.

Use of thickness-shear mode in adaptive sandwich structures.  
*Smart Materials and Structures*, 4:202–206, 1995.



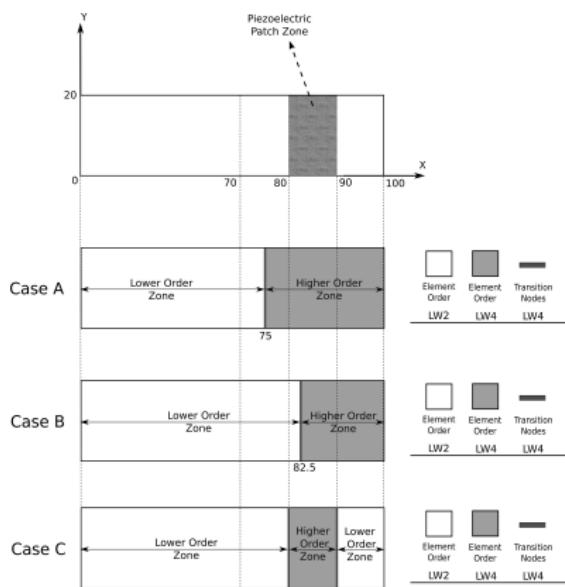
A Benjeddu, M Trindade, and R Ohayon.

A unified beam finite element model for extension and shear piezoelectric actuation mechanisms.

*Journal of Intelligent Material Systems and Structures*, 8:1012–1025, 1997.

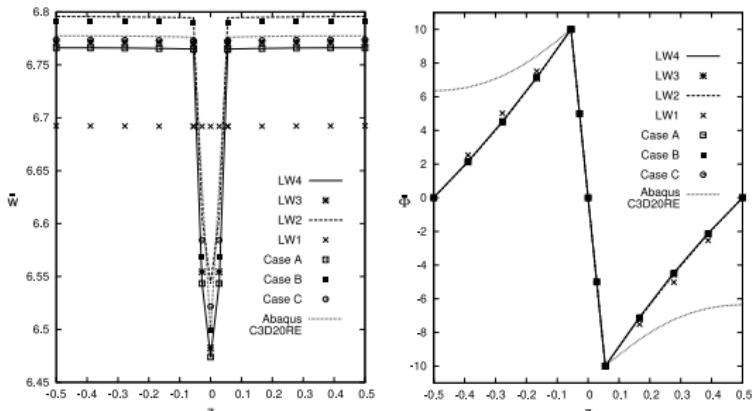
# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models



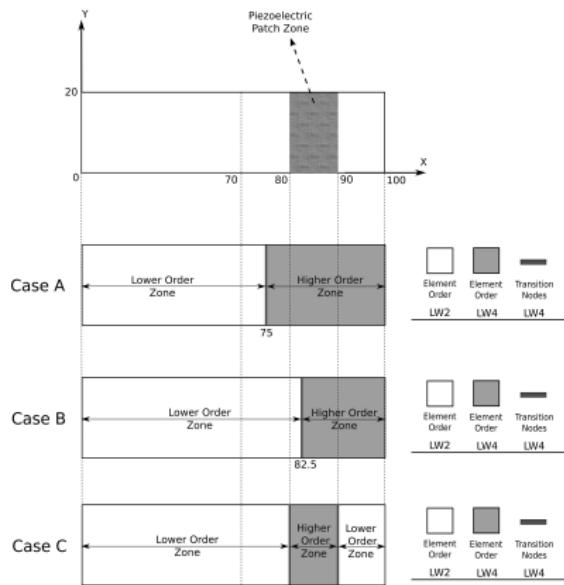
Transverse displacement and electric potential

$$w(x, y) : (a, b/2) \quad \Phi(x, y) : (d = 85, b/2)$$



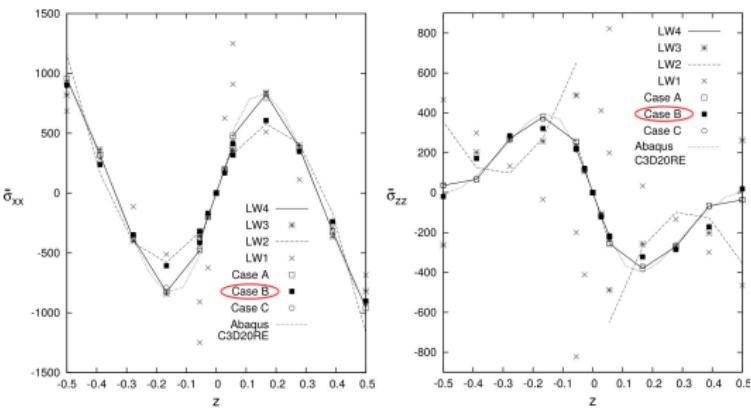
# Node-Dependent Variable Kinematics

## Cases with Layer-Wise models



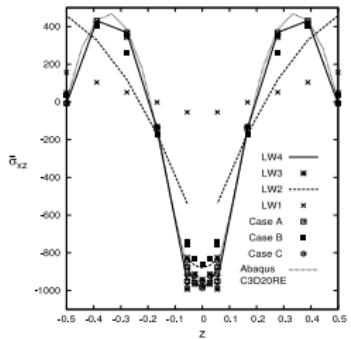
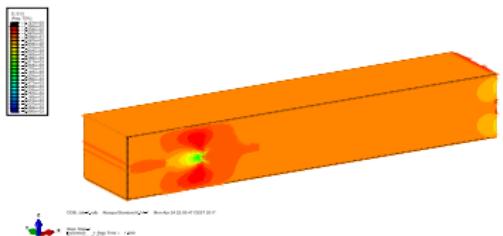
In-plane and Transverse stresses

$$\sigma_{xx}(x, y) : \sigma_{zz}(x, y) : (d = 85, b/2)$$

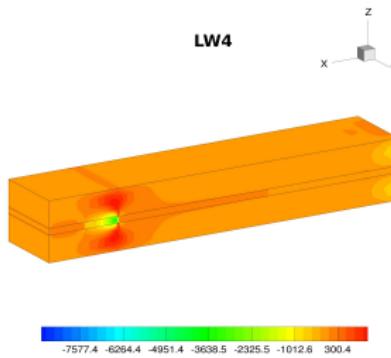


# Transverse shear stress $\sigma_{\alpha z}$

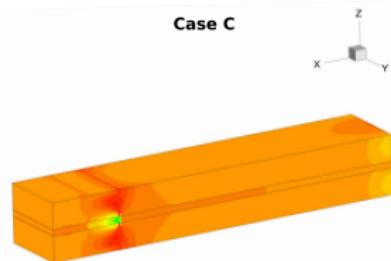
3D finite element **Abaqus C3D20RE**



LW4



Case C



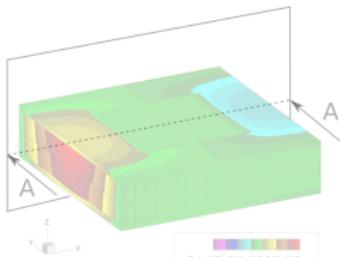
## Conclusions

- Unified Formulation is the ideal tool for the implementation of **node-dependent** variable kinematic theories. In fact, the theory approximation order and the modelling technique (ESL, LW) are free parameters of the FEM arrays, which are written in a compact and very general form.
- The present node-dependent variable kinematic model allows to locally improve the solution. Two main aspects can be highlighted: a reduction of computational costs with respect to Layer-Wise single-model solutions, and a simultaneous multi-models global-local analysis can be performed in one-single analysis step.
- The **Node-Dependent Variable Kinematic** method permits to enrich locally the theory approximation accuracy by enforcing the same kinematics at the interface nodes between kinematically incompatible plate/shell elements. The resulting global/local approach is very efficient because it does not employ any mathematical artifice to enforce the displacement and stress continuity, such as those methods based on Lagrange multipliers or overlapping regions.

# Conclusions

- An accurate representation of secondary variables (mechanical stresses and electric displacements) in localized zones is possible with DOFs reduction if an accurate distribution of the higher-order kinematic capabilities is performed. On the contrary, the accuracy of the solution in terms of primary variables (mechanical displacements and electric potential) values depends on the global approximation over the whole structure. The efficacy of the node-dependent variable kinematic and global/local models, thus, depends on the characteristics of the problem under consideration as well as on the required analysis type.
- The **Node-Dependent Variable Kinematic** method is very promising. Further investigations will be carried out on thermo-mechanical problems with localized loadings, plate/shell with non-uniform thickness problems etc. etc.

# Thanks for the attention



Cut-view A-A

