Node-dependent kinematics, refined zig-zag and multi-line beam theories for the analysis of composite structures

Erasmo Carrera,* Matteo Filippi,† Alfonso Pagani‡ and Enrico Zappino§

Mul ^{2} Group

Department of Mechanical and Aerospace Engineering, Politecnico di Torino, Turin, 10029, Italy

Three different approaches to the analysis of composite structures are assessed in the present paper. These three methodologies are based on the refined one-dimensional models developed in the framework of the Carrera Unified Formulation (CUF). The first approach, in order to address the C_z^0 requirements of laminates, allows the kinematics of the model to be enriched by using zig-zag functions. The second method is referred to as Multi-Line and it makes use of Lagrange multipliers in the domain of higher-order and hierarchical theories to develop variable kinematic models with layer-wise capabilities. Finally, as the main novelty of this work, one-dimensional models with node-dependent kinematics are presented. This new approach enables the automatic implementation of variable-kinematic models by using hybrid/transition elements. The three models have been used for the analysis of a laminated beam benchmark and the results have been compared in terms of computational costs and accuracy. The proposed assessment highlights the capability of the present models to deal efficiently with the analysis of composite structures and, in particular, underlines the global/local features of the node-dependent kinematics beam theories.

I. Introduction

The design of composite structures requires the use of accurate computational tools. The complex behaviour of the material and the anisotropy of the mechanical properties may cause complex stress fields within structures, which have to be described accurately in order to predict the arising of cracks and therefore the failure of a given component. Classical one-dimensional structural models[1,2] are widely used in the design of structures but their employability is limited by their fundamental assumptions. Nevertheless, the use of refined one-dimensional models allows the limitations introduced by the fundamental assumptions of the classical models to be overcome and, eventually, the stress singularities due to local effects[3,4,5] to be predicted. Many refined one-dimensional models have been proposed over the last few decades, see for example the use of warping functions as proposed by Vlasov.[6] Schardt[7] proposed a one-dimensional model for the thin-walled structures analysis where the displacement field was considered as an expansion around the mid-plane of the thin-walled cross-section. The Variation Asymptotic Method, VAM, proposed by Berdichevsky,[8] uses a characteristic cross-section parameter to build an asymptotic expansion of the solution. Volovoi[9] and Yu,[10,11] in particular, extended this method to composite materials and beams with an arbitrary cross-sections.

When dealing with the modelling of laminated composite structures, one of the main issue is the fulfilment of the so-called C_z^0 *Requirements*, i.e. the through-the-thickness continuity of the transverse shear and normal stresses and the zig-zag behaviour of the displacement components. Many efforts have been done over the

*Professor, Dept. of Mech. and Aero. Eng., Politecnico di Torino, erasmo.carrera@polit.it, AIAA Member.

†Research Fellow, Dept. of Mech. and Aero. Eng., Politecnico di Torino, matteo.filippi@polit.it.

‡Assistant Professor, Dept. of Mech. and Aero. Eng., Politecnico di Torino, alfonso.pagani@polit.it.

§Assistant Professor, Dept. of Mech. and Aero. Eng., Politecnico di Torino, enrico.zappino@polit.it.

years in order to extend the use of classical structural models to account for these requirements. On the other hand, the use of Layer-Wise (LW)[12,13] models allows each layer of the laminate to be described independently and in detail. However, the C_z^0 Requirements can be automatically satisfied a-priori by using Mixed Formulations, which makes use of the Reissner Mixed Variational Theorem[14] for displacement and stresses.

An important step in the analysis of laminated structures was the introduction of the zig-zag function as proposed by Murakami,[15] which allows the Equivalent-Single-Layer (ESL) models to be used efficaciously in the analysis of composite structures. This method was used by Carrera *et. al.*[16] in the case of beam models. A compromise between ESL and LW models has been recently proposed by Carrera and Pagani[17] with the multi-line approach that uses the Lagrange multipliers to connect different ESL models in a global/local and variable kinematic sense.

The use of refined structural models increases the accuracy of the analyses but also the computational costs. The use of *ad-hoc* or global/local formulations allows refined models to be used only in small parts of the structure, where accurate results are required. These class of models reduces the computational costs preserving the accuracy of the solution where accurate description of the stress/strain fields are of interest. Examples of this approach are those based on the Arlequin method, formerly proposed by.[18,19] In this case, the compatibility in the overlapping region (a region where higher-order and lower-order formulations co-exist) is imposed using Lagrange multipliers. This approach has also been used in the domain of Carrera Unified Formulation (CUF), in particular in the work by Biscani *et al.*[20]

As an alternative method to those addressed above, the use of node-dependent kinematics in one-dimensional models as presented by Carrera and Zappino,[21] make it possible to vary the kinematics of the structural theory in any beam node that is, refined beam model can be used only were an accurate solution is requested.

The present paper presents a comparison between three diverse CUF-based approaches in the analysis of composite structures. The first approach proposes the use of the zig-zag functions.[16] The second approach proposes a multi-line model[17] that uses the Lagrange multipliers to join different beam theories with, eventually, incompatible kinematics. The last approach uses a node-dependent kinematic beam model that allows the displacement field and, thus, the theory of structure to be varied in each node of the beam model. Those three methodologies have been developed in the framework of the CUF and their formulation can be expressed in a general, but still unified, compact form. A standard benchmark has been used to compare the performances of the three models addressed. The computational costs and the accuracy of the solution have been used in the comparison. Moreover, a brief theoretical introduction of three approaches has been reported in the next sections.

II. Theoretical model

A. One-dimensional model with variable-kinematics assumptions

Classical beam models are based on fundamental assumptions that limit the use of these models to a small number of applications. The Euler-Bernoulli beam theory, EBBT, does not consider, for example, the shear effects and the warping of the cross-section, which is considered rigid. The displacement field of EBBT can be written as:

$$\begin{aligned} u_x &= u_{x_1} \\ u_y &= u_{y_1} + x\,\frac{\partial u_{z_1}}{\partial y} + z\,\frac{\partial u_{x_1}}{\partial y} \\ u_z &= u_{z_1} \end{aligned} \tag{1}$$

On the other hand, the Timoshenko beam theory or First-order Shear Deformation Theory, FSDT, includes the effects of the shear but it is considered constant over the cross-section, therefore, it violates the homogeneous conditions at the free edges. In this case the displacement field can be expressed as:

$$\begin{aligned} u_x &= u_{x_1} \\ u_y &= u_{y_1} + x\,u_{y_2} + z\,u_{y_3} \\ u_z &= u_{z_1} \end{aligned} \tag{2}$$

The use of a still inconsistent and non-physical shear correction factor may reduce the error due to the constant shear assumption.

Refined models allows the range of applicability of these one-dimensional models to be extended to a large number of applications and problems. In this work, CUF is used to formulate higher-order beam theories with node-dependent kinematics. According to CUF, the one-dimensional approximation requires to assume an axiomatic/asymptotic approximation over the cross-section. The three-dimensional displacement field can be, hence, written as and arbitrary expansion of the generalized displacement unknowns $\mathbf{u}_\tau(y)$:

$$\mathbf{u} = \mathbf{u}_\tau(y) F_\tau(x,z), \qquad \tau = 1 \dots M. \tag{3}$$

Where $F_\tau(x,z)$ are arbitrary function which determine the expansion and approximation over the cross-section. M is the number of terms in the functions expansion $F_\tau(x,z)$. The choice of these functions allows the kinematic of the model to be modified automatically. In the present work, Taylor- and Lagrange-like expanding functions are considered and more details are given in the next sections.

A Finite Element (FE) approximation is used to describe the axial unknowns $\mathbf{u}_\tau(y)$. Using the one-dimensional shape functions $N_i(y)$ the generalized displacement field can be assumed as the linear combination of thereof to have:

$$\mathbf{u} = \mathbf{u}_{i\tau} N_i(y) F_\tau(x,z), \qquad \tau = 1 \dots M; \qquad i = 1 \dots N_n. \tag{4}$$

Where N_i are the shape functions introduced by the FE model and N_n is the number of nodes of the element. $\mathbf{u}_{i\tau}$ are the nodal generalized unknowns. Similarly, the virtual variation of the displacement field can be written as:

$$\delta\mathbf{u} = \delta\mathbf{u}_{js} N_j(y) F_s(x,z), \qquad s = 1 \dots M; \qquad j = 1 \dots N_n. \tag{5}$$

1. Taylor-Expansion models (TE)

The TE 1D model consists of an expansion that uses 2D polynomials $x^m \; z^n$, as F_τ, where m and n are positive integers. For instance, the second-order TE beam model according to CUF is:

$$\begin{aligned}
u_x = &\quad u_{x_1} + x\, u_{x_2} + z\, u_{x_3} + x^2\, u_{x_4} + xz\, u_{x_5} + z^2\, u_{x_6} \\
u_y = &\quad u_{y_1} + x\, u_{y_2} + z\, u_{y_3} + x^2\, u_{y_4} + xz\, u_{y_5} + z^2\, u_{y_6} \\
u_z = &\quad u_{z_1} + x\, u_{z_2} + z\, u_{z_3} + x^2\, u_{z_4} + xz\, u_{z_5} + z^2\, u_{z_6}
\end{aligned} \tag{6}$$

It is interesting to note that classical beam models, such as EBBT abd FSDT, can be derived as degenerated cases of the linear TE-CUF model.[22]

2. Lagrange-Expansion models (LE)

In the case of LE models, Lagrange polynomials are used to build 1D higher-order theories. Two types of cross-section polynomial set are adopted in this paper: nine-point LE 9, and four-point LE 4. The isoparametric formulation is exploited to deal with arbitrary cross-sections and complex shaped geometries. For instance, the LE 4 interpolation functions are:

$$\begin{aligned}
&F_1 = \tfrac{1}{4}\,(1-\xi)\,(1-\eta); \quad F_2 = \tfrac{1}{4}\,(1+\xi)\,(1-z); \\[1em]
&F_3 = \tfrac{1}{4}\,(1+\xi)\,(1+\eta); \quad F_4 = \tfrac{1}{4}\,(1-\xi)\,(1+\eta)
\end{aligned} \tag{7}$$

Where ξ and η are the coordinates in the natural reference system. Equation (7) coincides with the linear Lagrange polynomials in two dimensions. Using LE approximations, higher-order models make use of only pure displacement variables. Moreover, by discretizing the cross-section with a number of sub-domains and by utilizing step-wise continuous LE expansions, models with layer-wise capabilities can be easily implemented.[23]

3. Zig-Zag functions

Murakami[24] introduced a zig-zag function in the first order shear deformation theory with the purpose of reproducing the zig-zag form for the displacements through the thickness of laminated plates. Since it is possible to use the zig-zag function in CUF framework, hereafter the theories which contain the term are identified with the exponent $(^{zz})$. For example, $TE2^{zz}$:

$$
\begin{aligned}
u_x &= u_{x_1} + x\, u_{x_2} + z\, u_{x_3} + x^2\, u_{x_4} + xz\, u_{x_5} + z^2\, u_{x_6} + (-1)^k \zeta_k u_{x_{7Z}} \\
u_y &= u_{y_1} + x\, u_{y_2} + z\, u_{y_3} + x^2\, u_{y_4} + xz\, u_{y_5} + z^2\, u_{y_6} + (-1)^k \zeta_k u_{y_{7Z}} \\
u_z &= u_{z_1} + x\, u_{z_2} + z\, u_{z_3} + x^2\, u_{z_4} + xz\, u_{z_5} + z^2\, u_{z_6} + (-1)^k \zeta_k u_{z_{7Z}}
\end{aligned}
$$

Where $\zeta_k = 2z_k/h_k$ is a non-dimensional layer coordinate and h_k the thickness of the $k-layer$. The exponent k changes the sign of the zig-zag term in each layer.

4. Multi-line beam

The use of the Lagrange Multipliers allows the displacement equivalence to be imposed in a finite number of points. This approach can be used to join two or more beams with different kinematics as shown by Carrera and Pagani.[17] The model obtained, which is called multi-line model, allows the beam to be divided in many sub-domain, e.g. the layers of a composite beam. Each of the sub-domain can be modelled with different kinematics and finally the displacement congruence can be imposed in a finite number of point via the Lagrange multipliers. This approach allow to have a local approximation of the displacement field, that is it is able to provide an accurate description of the laminated structures.

5. Node-dependent kinematics

In most of the cases, a refined kinematics is required just in some parts of the domain, where local and complex effects are present, while, classical models could be used elsewhere.

A generic configuration and example is shown in Fig. 1. Depending on the geometry and the boundary conditions, some sections of a beam can be just involved in pure bending phenomena. Thus, in this case, a classical model could be used. Other sections may be slightly deformed, therefore, a low-order but still refined beam model can be used there. Finally, some sections could show a large warping in- and out-of-plane and, in these parts of structure, a higher order model is required.

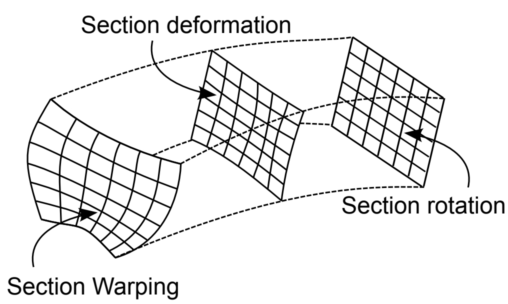

Figure 1. Beam undergoing a general deformed configuration.

In this work a new class of node-dependent kinematics elements is introduced in order to refine the kinematics only where it is strictly required. The approach introduced in this section can be easily included in the CUF formulation and extended to any order beam model. The node-dependent kinematics in one-dimensional model displacement field can be written including two main novelties:

$$F_\tau(x,z) \longrightarrow F_\tau^i(x,z) \tag{8}$$

$$M \longrightarrow M^i \tag{9}$$

The first equation, Eq. (8), says that the functions expansion is not a property of the finite element but is a property of the node. For this reason the index i is included in the notation. Equation (9), moreover, remarks that the number of the terms in the expansion, M, can be different in each node and the notation M^i is used to underline this aspect. Hence, the generic displacement field can be written as:

$$\mathbf{u} = \mathbf{u}_{i\tau} N_i(y) F^i_\tau(x,z), \qquad \tau = 1 \dots M^i; \qquad i = 1 \dots N_n. \tag{10}$$

B. Governing equations

The governing equations can be derived using the Principle of Virtual Displacements (PVD) that in the case of static response assumes the following form:

$$\delta L_{int} = \delta L_{ext} \tag{11}$$

Where L_{int} stands for the strain energy and L_{ext} is the work of the external loadings. δ denotes the virtual variation. The virtual variation of the internal work can be written as:

$$\delta L_{int} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV \tag{12}$$

By introducing the constitutive equations for elastic materials and the linear geometrical relations and introducing the displacement field given in Eq. (10), the variation of the internal work becomes:

$$\delta L_{int} = \delta \mathbf{u}^T_{js} \int_V \left[N_j(y) F^j_s(x,z) \;\; \boldsymbol{D}^T \;\; \boldsymbol{C} \;\; \boldsymbol{D} \;\; F^i_\tau(x,z) N_i(y) \right] dV \mathbf{u}_{i\tau} \tag{13}$$

$$= \delta \mathbf{u}^T_{js} \;\; \boldsymbol{K}^{ij\tau s} \;\; \mathbf{u}_{i\tau} \tag{14}$$

Where $\boldsymbol{K}^{ij\tau s}$ is the stiffness matrix expressed in form of *fundamental nucleus* that is a 3×3 array.

The work done by the external loads can be written, in the case of a point load, as:

$$\delta L_{ext} = \int_V \delta \boldsymbol{u}^T \boldsymbol{P} dV \tag{15}$$

Where $\boldsymbol{u}$ is the displacement vector in one point and $\boldsymbol{P}$ is the load applied in that point. By substituting CUF one has:

$$\delta L_{ext} = \delta \mathbf{u}^T_{js} \int_V N_j(y) F^j_s(x,z) \boldsymbol{P} dV = \delta \mathbf{u}^T_{js} \boldsymbol{P}^{js} \tag{16}$$

Other loading conditions can be treated similarly.

C. The matrix assembly

The formulation and the use of the fundamental nucleus as introduced in the last section is widely discussed in Carrera *et. al.*.[25] It can be used as a fundamental brick to build the matrix of the complete structure for any arbitrarily refined theory. Figure 2 shows the procedure used to build the stiffness matrix starting from the fundamental nucleus.

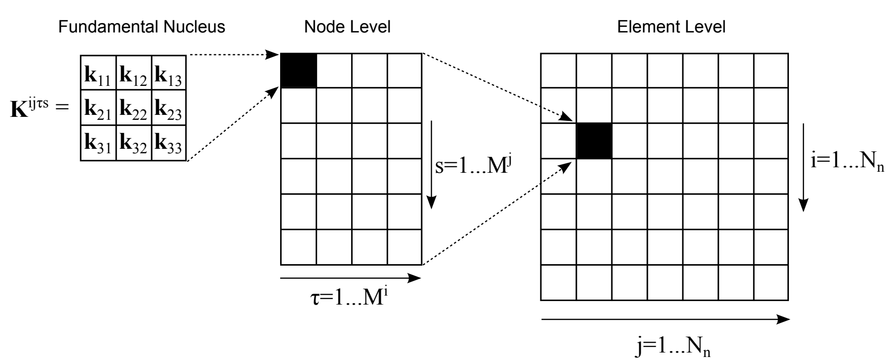

Figure 2. Stiffness matrix assembly.

This approach has been introduced in many works by the authors[25] considering the structural model constant in each beam element. In the present paper the approximation used in each beam node can be

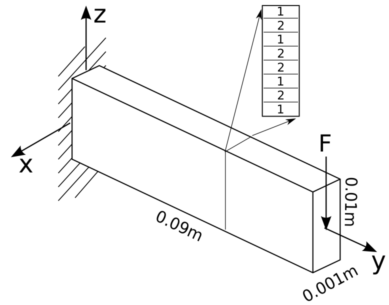

Figure 3. laminated beam geomentry.

	Material 1	Material 2
E_L [GPa]	30	5
E_T [GPa]	1	1
ν_{LT}	0.25	0.25
G_{LT} [GPa]	0.5	0.5

Table 1. Materials properties

different. This aspect can be seen at the node level, that is, the loop on the indices τ and s can have a different number of terms therefore the matrix may appear rectangular at this level. The dimensions of this block depends by the number of terms in the expansions in the nodes i and j. When $i = j$ the matrix at the node level is always square because $M^i = M^j$.

III. Numerical Results

The following section aims at presenting a number of results obtained with different beam models developed in the CUF framework. The considered structure is an 8-layer cantilevered beam, whose dimensions and lamination scheme are shown in Figure 3. The properties of the materials '1' and '2' are listed in Table 1. The beam is subjected to a point load of 0.2 N that is applied at its tip. The results have been reported in terms of transverse displacement (at y = 0.09 m) and through-the-thickness distributions of normal and shear stress, which are evaluated at the mid-span.

A. One-dimensional model assessment

The capabilities of the Taylor- and Lagrange-like expansions have been first evaluated. For the equivalent single layer approach, different kinematic theories based on the Taylor-like expansions (TEN) of arbitrary orders, N, have been considered. These models also encompassed the classical beam theories, namely the Euler-Bernoulli model (EBBT) and the first-order shear deformation theory (FSDT). As far as the layer-wise approach is concerned, eight 9-node Lagrange elements (L9) have been used to build the mathematical model.

Table 2 reports the numerical results obtained using the considered theories. The comparisons with other solutions taken from the literature have revealed that all models predicted the normal stress with a significant accuracy. However, it should be noted that the classical theories underestimated the transverse displacement.

Figures 4 and 5 show the through-the-thickness distributions of shear and normal stresses. The results are compared with the analytical solution proposed by Lekhnitskii.[30] As far as the normal stress distribution is concerned, all kinematic theories have provided results in strong agreement with the reference solution. However, the TE models did not provide an accurate description of the shear stress distribution due to the significant anisotropy of the structure. On the contrary, the layer-wise CUF model is able to accurately reproduce the stress profile.

Model	$\sigma^{*}_{yy} \times 10^3$ $[MPa]$	$\sigma^{**}_{yz} \times 10^3$ $[MPa]$	$u^{***}_{z} \times 10^{-2}$ $[mm]$	DOFs
Surana and Nguyen[26]	720		-3.03	
Davalos and KimBarbero[27]	700		-3.03	
Xiaoshan Lin[28]	750		-3.06	
Vo and Thai[29]			-3.02	
EBBT	730	0.00	-2.62	93
FSDT	730	-2.00	-2.98	155
TE1	730	-2.00	-2.98	279
TE3	730	-2.82	-3.03	930
TE5	730	-2.75	-3.03	1953
TE7	729	-2.86	-3.04	3348
TE9	730	-2.88	-3.04	5115
LE	730	-2.79	-3.05	4743

* evaluated at y=0.045, x=0.0 , z=0.005

** evaluated at y=0.045, x=0.0 , z=0.0

*** evaluated at y=0.09, x=0.0 , z=0.00

Table 2. Refined one-dimensional models results

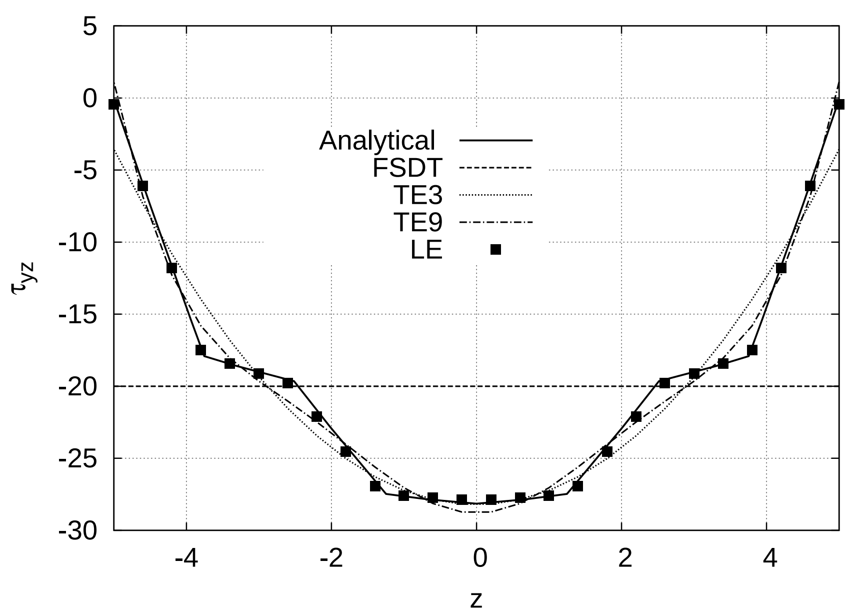

Figure 4. Shear stress evaluated at y=0.045 and x=0

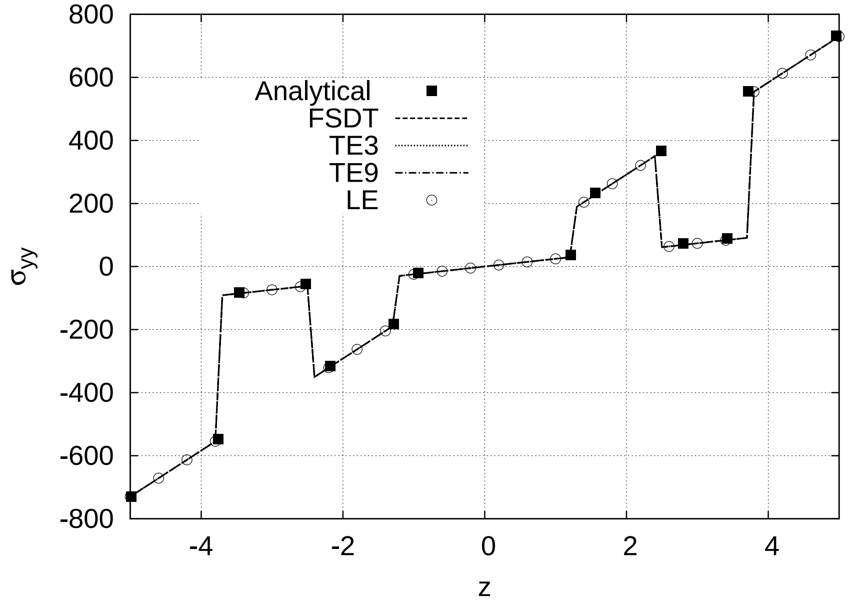

Figure 5. Normal stress evaluated at y=0.045 and x=0

B. CUF-based models for composite material analysis

The previous section showed how the use of refined structural models can lead to accurate results. However, these theories require a higher computational effort with respect to the beam classical models. In order to reduce the number of degrees of freedom (DOFs), the node-dependent kinematic models have been developed. The main idea of this approach is the use of refined structural theories only in few structural sub-domains, where an accurate solution is required, while simpler mathematical models can be adopted elsewhere. The capabilities of different node-dependent kinematic models have been compared with other two approaches, which are the zig-zag higher-order theories and the multi-line method. The results have been obtained considering the previous structural problem. Different models have been conceived in order to refine the solution at the mid-span of the beam. Figure 6 shows these models. The FEM model consists of ten 4-node beam elements. The TE3 model has been used for the external parts of the beam (from node 1 to 12 and from node 20 to 31), while the central sub-domain (from node 13 to 19) has been modeled using different kinematic combinations:

- in models $TE^{3/5}_{M1}$, $TE^{3/7}_{M1}$, and $TE^{3/9}_{M1}$, higher order TE models have been adopted in all seven central nodes;
- in models $TE^{3/5}_{M2}$, $TE^{3/7}_{M2}$, and $TE^{3/9}_{M2}$, higher order TE models have been adopted only in the central node (node 16);
- in model TE/LE_{M1} the Lagrange discretization is used in the seven central nodes;
- in model TE/LE_{M2} the Lagrange discretization is only used in the central node.

The related results are shown in Table 3 and they are compared with the zig-zag and multi-line solutions. The transverse displacement and the stress values are in good agreement. As far as the shear stress distributions are concerned, Figure 7 shows that the model $TE^{3/9}_{M1}$ reproduced the results obtained with the complete TE9 model, but with a lower computational effort. On the other hand, the $TE^{3/9}_{M2}$ solution is in close agreement with the TE3 distribution.

A further improvement in the solution can be obtained using the layer-wise approach. In fact, Figure 8 shows that the TE/LE_{M1} model provided a solution comparable to the full LE model. If the LE model is used only in the central node, model TE/LEM2, slight differences are observed with respect to the reference solutions. Moreover, it must be noted that the computational cost has been strongly reduced using the node

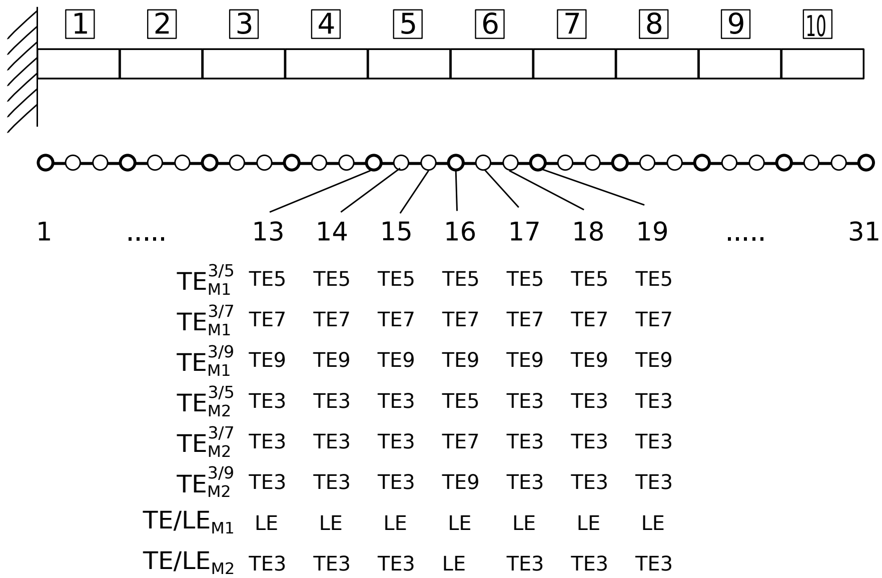

Figure 6. Node-dependent kinematic models.

Model	$\sigma_{yy}^{*} \times 10^3$ $[MPa]$	$\sigma_{yz}^{**} \times 10^3$ $[MPa]$	$u_z^{***} \times 10^{-2}$ $[mm]$	DOFs
$N9^{zz}$ [16]	661	-	-3.04	5208
$N3^{zz}$ [16]	729	-	-3.04	1023
Multi-Line [17]	731	-	3.026	6696
$TE_{M1}^{3/5}$	730	-2.77	-3.03	1161
$TE_{M1}^{3/7}$	731	-2.87	-3.03	1476
$TE_{M1}^{3/9}$	730	-2.89	-3.03	1875
$TE_{M2}^{3/5}$	736	-2.80	-3.03	963
$TE_{M2}^{3/7}$	755	-2.86	-3.03	1008
$TE_{M2}^{3/9}$	760	-2.84	-3.04	1065
TE/LE_{M1}	731	-2.80	-3.04	1791
TE/LE_{M2}	750	-2.80	-3.05	1053

* evaluated at y=0.045, x=0.0 , z=0.005

** evaluated at y=0.045, x=0.0 , z=0.0

*** evaluated at y=0.09, x=0.0 , z=0.00

Table 3. Refined one-dimensional models results

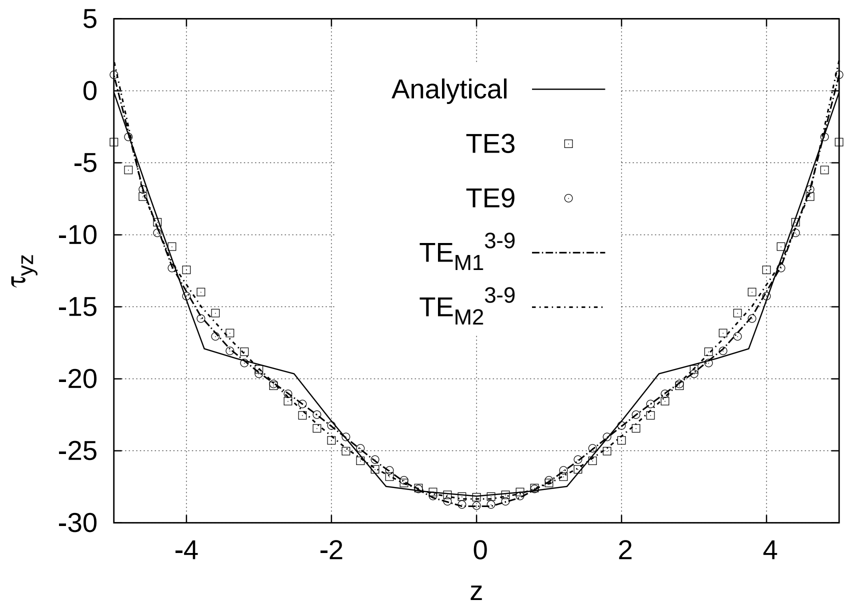

Figure 7. Shear stress evaluated at y=0.045 and x=0

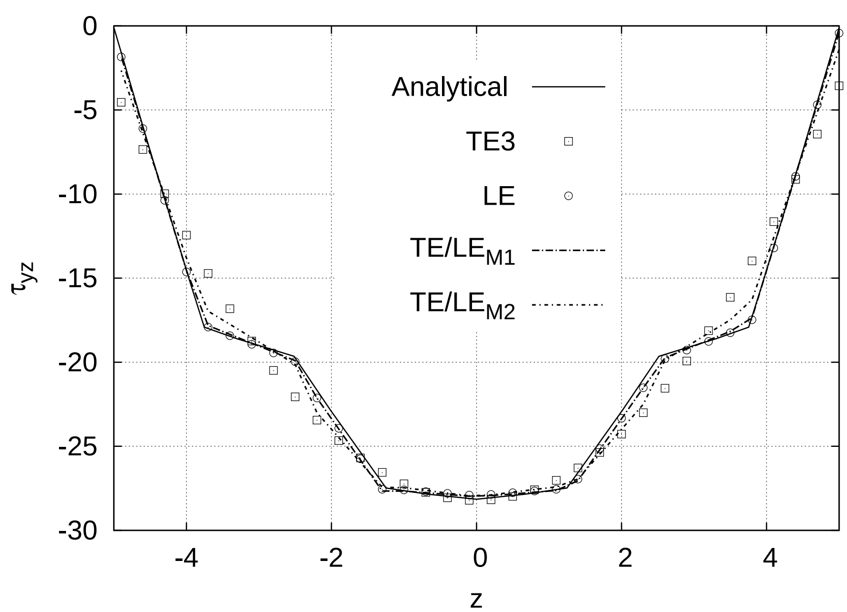

Figure 8. Normal stress evaluated at y=0.045 and x=0

dependent kinematic. In fact, the full LE model requires almost 5000 DOFs, while the TE/LE_{M1} has less than 2000 DOFs.

IV. Conclusions

The present paper compared the node-dependent kinematic one-dimensional models with multi-line models and higher-order zig-zag theories for the stress analysis of laminated structures. All considered structural theories have been derived using the Carrera Unified Formulation. Different combined models have been presented in order to investigate the capabilities of the node-dependent structural models. In the light of the obtained results, it is possible to drawn the following remarks:

- Refined one-dimensional models are able to provide accurate results when composite structures are considered.
- The Carrera Unified Formulation can be used to derive node-dependent kinematic models without any 'ad hoc' assumptions.
- The use of node-dependent kinematic allows refined models to be used only where an accurate solution is required.
- The computational cost can be significantly reduced.

In conclusion, the node-dependent kinematics method is a valuable trade-off between accuracy and computational cost in the study of complex structures. In particular, the use of the higher-order zig-zag theories and the multi-line method within the node-dependent kinematics approach can represent an effective solution for the analysis of laminated structures.

References

[1]Euler, L., "De curvis elasticis. Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis iso-perimetrici lattissimo sensu accepti." 1744.

[2]Timoshenko, S. P., "On the corrections for shear of the differential equation for transverse vibrations of prismatic bars," *Philosophical Magazine*, Vol. 41, 1921, pp. 744–746.

[3]Horgan, C. and Simmonds, J., "Saint-Venant end effects in composite structures," *Composites Engineering*, Vol. 4, No. 3, 1994, pp. 279–286.

[4]Bar-Yoseph, P. and Ben-David, D., "Free-edge effects in unsymmetrically laminated composite plates," *Composite Structures*, Vol. 30, No. 1, 1995, pp. 13–23.

[5]Tullini, N., "Decay rate of Saint-Venant end effects for multilayered orthotropic strips," *International Journal of Solids and Structures*, Vol. 34, No. 33-34, nov 1997, pp. 4263–4280.

[6]Vlasov, V. Z., *Thin walled elastic beams*, 1961.

[7]Schardt, R., "Eine Erweiterung der Technischen Biegetheorie zur Berech- nung Prismatischer Faltwerke." *Der Stahlbau*, Vol. 35, 1966, pp. 161–171.

[8]Berdichevsky, V. L., "Equations of the theory of anisotropic inhomogeneous rods," *Dokl. Akad. Nauk*, Vol. 228, 1976, pp. 558–561.

[9]Volovoi, V., "Asymptotic theory for static behavior of elastic anisotropic I-beams," *International Journal of Solids and Structures*, Vol. 36, No. 7, mar 1999, pp. 1017–1043.

[10]Yu, W., Volovoi, V. V., Hodges, D. H., and Hong, X., "Validation of the variational asymptotic beam sectional analysis," *AIAA Journal*, Vol. 40, 2002, pp. 2105–2113.

[11]Yu, W. and Hodges, D. H., "Asymptotic Approach for Thermoelastic Analysis of Laminated Composite Plates," *Journal of Engineering Mechanics*, Vol. 130, No. 5, 2004, pp. 531.

[12]Carrera, E., "Multilayered Shell Theories that Account for a Layer-Wise Mixed Description. Part I. Governing Equations," *AIAA Journal*, Vol. 37, 1999, pp. 1107–1116.

[13]Shimpi, R., "A beam finite element based on layerwise trigonometric shear deformation theory," *Composite Structures*, Vol. 53, No. 2, aug 2001, pp. 153–162.

[14]Carrera, E., "Developments, ideas and evaluations based upon the Reissner's mixed variational theorem in the modeling of multilayered plates and shells," *Applied Mechanics Reviews*, Vol. 54, 2001, pp. 301–329.

[15]Carrera, E., "A historical review of zig-zag theories for multilayered plates and shells," *Applied Mechanics Reviews*, Vol. 56, No. 3, 2003, pp. 287–308.

[16]Carrera, E., Filippi, M., and Zappino, E., "Free vibration analysis of laminated beam by polynomial, trigonometric, exponential and zig-zag theories," *Journal of Composite Materials*, Vol. 48, No. 19, 2014, pp. 2299–2316.

[17]Carrera, E. and Pagani, A., "Multi-line Enhanced Beam Model for the Analysis of Laminated Composite Structures," *Composites Part B*, Vol. 57, 2014, pp. 112–119.

[18]Ben Dhia, H. and Rateau, H., "The arlequin method as a flexible engineering tool," *International journal for numerical methods in engineering*, Vol. 62, No. 11, 2005, pp. 1442–1462.

[19]Ben Dhia, H., "Multiscale mechanical problems: the arlequin method," *Comptes Rendus de l'Academie des Sciences Series IIB Mechanics Physics Astronomy*, Vol. 326, No. 12, 1998, pp. 899–904.

[20]Biscani, F., Giunta, G., Belouettar, S., Carrera, E., and Hu, H., "Variable kinematic beam elements coupled via Arlequin method," *Composite Structures*, Vol. 93, 2011, pp. 697–708.

[21]Carrera, E. and Zappino, E., "Node-dependent kinematic one dimensional models for the analysis of composite structures," To be Submitted.

[22]Carrera, E., Giunta, G., and Petrolo, M., *Beam Structures: Classical and Advanced Theories.*, John Wiley & Sons Ltd., 2011.

[23]Carrera, E. and Petrolo, M., "Refined One-Dimensional Formulations for Laminated Structure Analysis," *AIAA Journal*, Vol. 50, No. 1, 2012, pp. 176–189.

[24]Murakami, H., "Laminated composite plate theory with improved in-plane responses," *Journal Applied Mechanics*, Vol. 53, 1986, pp. 661–666.

[25]Carrera, E., Cinefra M. Petrolo, M., and Zappino, E., *Finite Element Analysis of Structures Through Unified Formulation*, John Wiley & Sons, 2014.

[26]Surana, K. and Nguyen, S., "Two-dimensional curved beam element with higher-order hierarchical transverse approximation for laminated composites," *Computer and Structures*, Vol. 36, 1990, pp. 499–511.

[27]Davalos, J., Y.Kim, and Barbero, E., "Analysis of laminated beams with a layerwise constant shear theory," *Composite Structures*, Vol. 28, 1994, pp. 241–253.

[28]Xiaoshan Lin, Y. Z., "A novel one-dimensional two-node shear-exible layered composite beam element," *Finite Elements in Analysis and Design*, Vol. 47, 2011, pp. 676–682.

[29]Vo, T. P. and Thai, H.-T., "Static behavior of composite beams using various refined shear deformation theories," *Composites Structures*, 2012, In press.

[30]Lekhnitskii, S. G., *Anisotropic Plates*, Gordon and Breach Science Publishers.