ENHANCED FREE VIBRATION ANALYSIS OF COMPOSITE WING-BOX STRUCTURES BY ONE-DIMENSIONAL COMPONENT-WISE AND DYNAMIC STIFFNESS FORMULATIONS

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Abstract: Free vibration analysis is an important aspect of aeronautical engineering. The availability of enhanced models for accurate modal analysis is therefore of primary interest. This work deals with advanced 1D formulations able to foresee higher-order phenomena, such as elastic bending/shear coupling, restrained torsional warping and 3D strain effects. The proposed beam models are developed in the framework of the Carrera Unified Formulation (CUF), whose hierarchical capability allows the analyst to automatically implement refined models with arbitrarily rich kinematics. The capabilities of the resulting 1D theories are assessed by both weak- and strong-form solutions and the results from free vibration analyses of wing-like composite structures are compared to those from the literature, experiments and commercial FEM software.

1 INTRODUCTION

The use of composite materials in various weight sensitive structures (e.g. high-speed aircraft, rocket, launchers, etc.) is quite popular due to their well-known attractive properties. The wide use of laminated composite materials has aroused considerable interest in the related theoretical models and numerical simulation methods, including one-dimensional (1D) structural theories. A considerable number of theories have been devised in order to overcome the limitations of the first beam models introduced by Euler and Bernoulli [1] (hereinafter referred to as EBBM, i.e. Euler–Bernoulli beam model) and by Timoshenko [2] (hereinafter referred to as TBM, i.e. Timoshenko beam model) (see Kapania and Raciti [3,4]). Some recent noteworthy contributions about refined composite beam analysis are mentioned in the following. The attention is focused on free vibration analysis, which is the main topic of the present work.

A family of sinus models was presented by Vidal and Polit [5] for the vibration analysis of laminated beams. In [6], a trigonometric shear deformation theory was developed and the closed form solution was provided. Subramanian [7] presented two different one-dimensional (1D) Finite Elements (FEs) for laminated beams, in which a 5th order expansion was used to expand the axial displacement and a 4th order power series was used for the transverse displacement. In the work by Marur and Kant [8], Taylor's series expansions were used for axial displacement in order to describe the warping of cross-sections of sandwich and composite beams. Interesting mixed formulations were presented in [9], where through-the-thickness continuity of transverse stress and displacement fields was enforced. The beam model presented by Heyliger and Reddy [10] accounted for the stress-free conditions on the upper and lower surfaces of the beam while retaining a parabolic shear strain distribution. Other noteworthy contributions are those by Chen et al. [11], Hodges et al. [12], Stemple and Lee [13], Mitra et al. [14], Chandrashekhara et al. [15] and Chandrashekhara and Bangera [16].

The present paper deals with 1D higher-order theories able to accurately capture the mechanical behavior of laminated orthotropic and anisotropic structures. Refined beam models are developed within the framework of Carrera Unified Formulation (CUF) [17]. Two classes of CUF 1D models were formulated in recent works: the Taylor-expansion class, hereafter referred to as TE, and the Lagrange-expansion class, hereafter referred to as LE. TE models exploit N-order Taylor-like polynomials to define the displacement field above the cross-section with N as a free parameter of the formulation. The strength of CUF TE beam models in dealing with arbitrary geometries, thin-walled structures, and local effects were evident in static [18] and free vibration analysis [19]. Recently, CUF TE theories were applied with reference to Dynamic Stiffness Method (DSM) to investigate the free vibration characteristics of thin-walled structures [20]. On the other hand, the LE class is based on Lagrange-like polynomials to discretize the cross-section displacement field and they have only pure displacement variables. Recently, static analyses (see for example [21]) have revealed the strength of LE models in dealing with open cross-sections, arbitrary boundary conditions, and obtaining layer-wise descriptions of the 1D model. Moreover, LE models have been successfully used for the component-wise analyses of aeronautical metallic structures [22].

In this work, 1D refined CUF models of composite wing-like structures are implemented. Various laminations are considered and complex geometries are assessed, including wing structures with sweep angle and anisotropic laminated box structures. The paper is organized as follows: (i) first, the unified formulation of structures is briefly introduced; (ii) then, both weak- and strong-form governing equations are obtained in terms of fundamental nuclei; next, numerical results are presented and discussed; finally, the main conclusions are outlined.

2 CARRERA UNIFIED FORMULATION

The adopted rectangular Cartesian coordinate system is shown in Fig. 1. The cross-section of the beam lies on the xz-plane and it is denoted by Ω, whereas the boundaries over y are $0 \leq y \leq L$. Let us introduce the transposed displacement vector,

$$u(x,y,z;t) = \{u_x \ \ u_y \ \ u_z\}^T \tag{1}$$

Within the framework of CUF, the 3D displacement field of Eq. (1) is expressed as

$$u(x,y,z;t) = F_\tau(x,z) u_\tau(y;t), \qquad \tau = 1,2,\dots,M \tag{2}$$

where F_τ are the functions of the coordinates x and z on the cross-section; u_τ is the vector of the generalized displacements; M stands for the number of the terms used in the expansion; and the repeated subscript, τ, indicates summation. TE (Taylor Expansion) 1D CUF models consist of McLaurin series that uses the 2D polynomials $x^i \ z^j$ as F_τ functions, where i and j are positive integers. For instance, the displacement field of the second-order ($N = 2$) TE model can be expressed as

$$u_x = u_{x_1} + x\,u_{x_2} + z\,u_{x_3} + x^2\,u_{x_4} + xz\,u_{x_5} + z^2\,u_{x_6}$$
$$u_y = u_{y_1} + x\,u_{y_2} + z\,u_{y_3} + x^2\,u_{y_4} + xz\,u_{y_5} + z^2\,u_{y_6} \quad (3)$$
$$u_z = u_{z_1} + x\,u_{z_2} + z\,u_{z_3} + x^2\,u_{z_4} + xz\,u\_(z\_5\ ) + z^2\,u_{z_6}$$

The order N of the expansion is set as an input of the analysis; the integer N is arbitrary and it defines the order of the beam theory. EBBM and TBM can be realized as degenerated cases of the linear ($N = 1$) TE model. For further information about TE models see [17].

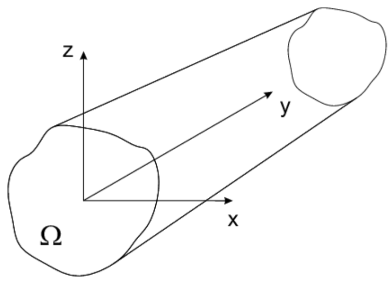


Figure 1: Coordinate frame of the beam model

The LE class exploits Lagrange-like polynomials on the cross-section to build 1D higher order models. The isoparametric formulation is exploited to deal with arbitrary shape geometries. In this paper, the nine-point (L9) cross-sectional cubic polynomial set was adopted. For the L9 element (Figure 2) and other orders, the interpolation functions can be found in [23]. However, the displacement field of an L9 beam is given in the following for illustrative purpose:

$$\begin{aligned} u_x &= F_1 u_{x1} + F_2 u_{x2} + \cdots + F_9 u_{x9} \\ u_y &= F_1 u_{y1} + F_2 u_{y2} + \cdots + F_9 u_{y9} \\ u_z &= F_1 u_{z1} + F_2 u_{z2} + \cdots + F_9 u_{z9} \end{aligned} \quad (4)$$

Where $u_{x1}(y;t), \dots, u_{z9}(y;t)$ are the displacement variables of the problem and they represent the translational displacement components of each of the nine points of the L9 element; $F_1(x,z), \dots, F_9(x,z)$ are the Lagrange polynomials. According to Carrera and Petrolo [23], the beam cross-section can be discretized by using several L-elements for further refinements.

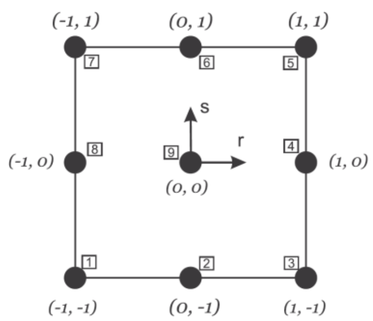


Figure 2: L9 element in the natural coordinate system

3 WEAK AND STRONG FORM GOVERNING EQUATIONS

In this work, the governing equations of the refined beam in free vibration are formulated by using the principle of virtual displacements

$$\delta L_{int} = \int_V \delta \boldsymbol{\epsilon}\, \boldsymbol{\sigma}\, dV = -\delta L_{ine} \quad (5)$$

Where L_{int} is the work of internal strains; L_{ine} is the work of inertial loadings; $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the vectors of strain and stress components, respectively; and δ is the virtual variation.

If finite elements are used along the beam axis, the generalized displacements can be approximated through generic 1D shape functions

$$u_\tau\,(y;t) = N_i\,(y) q_\tau(t), \qquad i = 1, \dots, p+1 \quad (6)$$

In this work, classical cubic ($p = 3$) shape-functions are used. By substituting FEM approximation (6), CUF (2), constitutive and linear strain-displacement relations, the virtual variation of the internal work can be expressed as

$$\delta L_{int} = \delta \boldsymbol{q}_{\tau i}^T\, \boldsymbol{K}^{ij\tau s}\, \boldsymbol{q}_{sj} \quad (7)$$

Where $\boldsymbol{K}^{ij\tau s}$ is the 3x3 fundamental nucleus of the algebraic stiffness FE matrix. The components of this matrix and those of the mass matrix coming from δL_{ine} are not given here for the sake of brevity, but they can be found in [17]. The most important characteristic of CUF, however, is that the elemental stiffness matrix of the generic beam element can be found by automatically expanding $\boldsymbol{K}^{ij\tau s}$ versus the four indices. In fact, the formal expression of the coefficients of the fundamental nucleus does not depend on the order and class (F_τ) adopted.

If finite elements are not employed, by only using CUF and elastica fundamental equations, the expression of the internal work reads as follows:

$$\delta L_{int} = \int_L \delta \boldsymbol{u}_\tau^T\, \boldsymbol{K}^{\tau s}\, \boldsymbol{u}_s\, dy + [\delta \boldsymbol{u}_\tau^T\, \boldsymbol{\Pi}^{\tau s}\, \boldsymbol{u}_s]_{y=0}^{y=L} \quad (8)$$

Where $\boldsymbol{K}^{\tau s}$ is the differential linear stiffness matrix and $\boldsymbol{\Pi}^{\tau s}$ is the matrix of the natural boundary conditions in the form of 3x3 fundamental nuclei. As in FEM formulation, the expansion of these fundamental kernels allow one to automatically find the governing equations of arbitrary order 1D theories. The components of the differential nuclei are not given in the present work. They can be found in [20], where also a detailed DSM procedure is devised for the exact solution of the system of differential equations described by Eq. (5).

Model	f_1	f_2	f_3	f_4	f_5
N = 2	7.4	46.1*	59.1	129.5*	182.7
N = 3	7.2	45.1*	59.1	126.5*	182.4
N = 4	7.2	45.0*	59.1	126.4*	182.3
CLT [24]	7.3	45.4*	59.1	127.7*	182.3

*Torsional mode

Table 1: Natural frequencies (Hz) of the eight-layer straight wing

4 NUMERICAL RESULTS

In this section, the proposed beam theories are evaluated for the free vibration analysis of wing structures. First, plate-like composite wings are considered, and the effect of sweep angle is evaluated. Then, complex composite box structure are addressed. The analyses are carried out by both TE and component-wise LE models, whereas DSM and classical FEM procedure are indistinctly employed for the solution of the free vibration problem. The results are compared with those from the literature and commercial software.

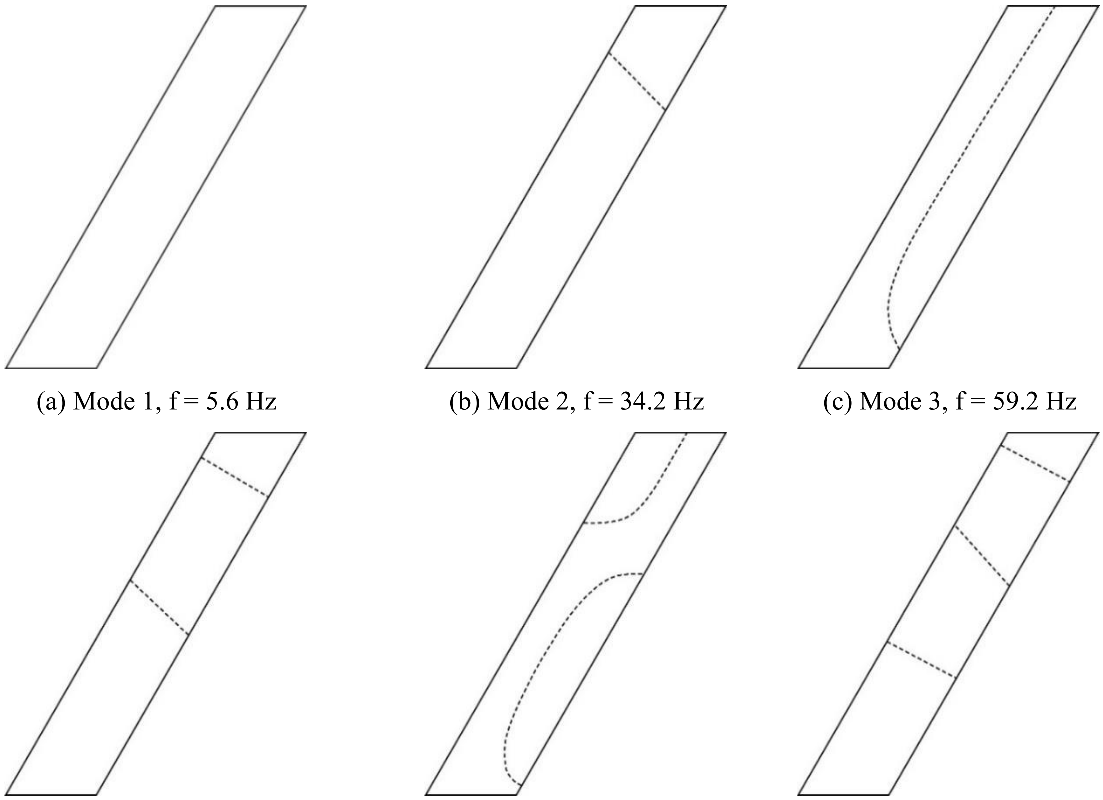
Model	f_1	f_2	f_3	f_4	f_5
N = 2	5.6	34.7	76.5*	97.5	193.3*
N = 3	5.6	34.4	60.1*	95.9	187.0*
N = 4	5.6	34.2	59.2*	95.3	180.1*
CLT [24]	5.6	34.4	60.0*	95.4	182.0*

*Torsional mode

Table 2: Natural frequencies (Hz) of the eight-layer back-swept (Λ = 30°) wing

4.1 Plate-like composite wing

In the first analysis case, composite wing structures were considered. Composite plate wing models were retrieved from [24] and [25]. A graphite/epoxy composite material with the following characteristics was used: $E_L = 98.0$ GPa, $E_T = 7.90$ GPa, $G_{LT} = 5.60$ GPa, Poisson ratio $\nu = 0.28$ and $\rho = 1520$ Kg/m^3, where L denotes the fibres direction and T a direction perpendicular to the fibres. The length of the wing (L) is equal to 305 mm and the chord (c) is equal to 76.2 mm. The total thickness of the laminate is 0.804 mm.



(a) Mode 1, f = 5.6 Hz
(b) Mode 2, f = 34.2 Hz
(c) Mode 3, f = 59.2 Hz
(d) Mode 4, 95.3 Hz
(e) Mode 5, f = 180.1 Hz
(f) Mode 6, 185.7 Hz

Figure 3: Modal shapes of the 8-layer swept plate wing, N = 4 DSM model

An eight-layer symmetric stacking sequence was considered. The stacking sequence was equal to [-22.5/67.5/22.5/-67.5]s, whereas the thickness sequence was [0.09/0.12/0.16/0.63]s, where each term indicates the thickness ratio of each ply with respect to the half of the thickness of the laminate. For instance, the thickness of the first layer is the 9% of the half thickness of the laminate. Two sweep angles were considered, $\Lambda = 0°$ and $\Lambda = 30°$. The natural frequencies are given in Tables 1 and 2, in which the results from the present TE variable order 1D DSM models are compared with those from a 2D solution. Finally, the nodal lines of the first sixth mode shapes of the swept wing ($\Lambda = 30°$) via the N = 4 DSM beam model are shown in Fig. 3. More results about metallic and composite plate-like wings from the present TE DSM approach can be found in [26], where flutter stability analyses were also addressed.

4.2 Composite box structure

A hollow rectangular cross-section laminated box beam was subsequently considered for verification. Clamped-free boundary conditions were addressed. The same structure was used for experimental [27] and analytical [28] investigations in previous works. The dimensions of the beam are as follows: length $L = 844.55$ mm, height $h = 13.6$ mm, width $b = 24.2$ mm, and thickness $t = 0.762$ mm. The box beam was made of six layers with the following orthotropic material properties: $E_1 = 141.96$ GPa, $E_2 = E_3 = 9.79$ GPa, $\nu_{12} = \nu_{13} = 0.42$, $\nu_{23} = 0.5$, $G_{12} = G_{13} = 6.0$ GPa, $G_{23} = 4.83$ GPa, and $\rho = 1445.0$ kg/m^3. The six layers had the same thickness. Different lamination schemes are considered for the box beam under consideration. Both circumferentially asymmetric stiffness (CAS) and circumferentially uniform stiffness (CUS) stacking sequences are addressed and they are detailed in Table 3.

	Flanges		Webs	
Lay-up	Top	Bottom	Left	Right
CAS2	$[30]_6$	$[30]_6$	$[30/-30]_3$	$[30/-30]_3$
CAS3	$[45]_6$	$[45]_6$	$[45/-45]_3$	$[45/-45]_3$
CUS1	$[15]_6$	$-[15]_6$	$[15]_6$	$-[15]_6$
CUS2	$[0/30]_3$	$[0/-30]_3$	$[0/30]_3$	$[0/-30]_3$
CUS3	$[0/45]_3$	$[0/-45]_3$	$[0/45]_3$	$[0/-45]_3$

Table 3: Various stacking sequences of the box beam

The values of the natural frequencies obtained from these box beam configurations are listed in Table 4, where the results from the present LE and TE FEM models are compared to those from the literature. In particular, TBM, the seventh-order TE models as well as a LE model made with 24 L9 elements are compared to experimental data [27] analytical solutions [28] and a 2D FE model by ANSYS [29]. Further results of composite box structures by the present beam model and related FEM formulations can be found in [30].

5 CONCLUSIONS

Advanced beam models have been used in this work for the free vibration analysis of composite wing structures. Refined 1D theories have been implemented by using CUF, whose hierarchical characteristics allow us to formulate weak and strong form governing equations in terms of fundamental nuclei, which are invariant with respect to the beam model. First, plate-like composite wings have been addressed and the effect of sweep angle has been evaluated. In the second analysis, a complex composite box structure has been considered and the natural frequencies have been evaluated for various stacking sequences. The results highlight the accuracy of the present beam models, which are able to foresee complex structural behavior (twisting, warping and shear/bending couplings) and provide solutions that are very close to those from the literature.

Lay-up	Mode	CUF LE	CUF TE		Exp. [26]	Anlt. [27]	2D FEM [28]
		24 L9	TBM	$N = 7$			
CAS2	1	20.06	20.96	20.60	20.96	19.92	19.73
	2	38.21	41.76	39.42	38.06	-	37.53
CAS3	1	14.75	15.00	14.69	16.67	14.69	14.58
	2	25.41	26.38	25.44	29.48	-	25.01
CUS1	1	29.51	32.36	29.19	28.66	28.67	28.37
CUS2	1	34.69	35.09	34.61	30.66	34.23	34.29
CUS3	1	33.03	33.11	33.01	30.00	32.75	32.35

Table 6. Natural frequencies (Hz) for different stacking sequences of the laminated box beam.

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