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Buckling of Beams by Refined Theories and Dynamic Stiffness Method

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Abstract

This paper is concerned with linearized elastic buckling analysis of columns. The Carrera unified formulation (CUF) is used to formulate variable kinematics beam theories. According to the CUF, Taylor-like polynomials of order N are used to interpolate the cross-sectional displacement field. By using the strong form of the principle of virtual displacements, governing equations and natural boundary conditions are formulated in terms of fundamental nuclei, whose formal expressions do not depend on the order of the theory N. The dynamic stiffness matrix is straightforwardly developed and the algorithm of Wittrick and Williams is used as solution technique to compute critical buckling loads for a number of solid and thin-walled metallic and composite beam-columns. The results computed using the proposed method are compared with those available in the literature. The accuracy and efficiency of the current approach as well as its capability to deal with bending-torsion and coupled buckling modes are demonstrated.

Keywords: buckling analysis, Carrera unified formulation, beams, dynamic stiffness method.

1 Introduction

Linearized buckling analysis of axially loaded beam-columns plays an important role in the design of aerospace, civil and other engineering structures. Several methodologies to solve the problem have been developed during the years and there are excellent texts on the subject, see for example Timoshenko [1] and Matsunaga [2].

In most of the classical works on column buckling, it has been assumed that when the equilibrium of the column is disturbed, it becomes unstable due to bending in a plane of symmetry of the cross-section. There are cases of practical interest where the column may buckle due to twisting or due to a combination of both twisting and bending. Such types of torsion or bending-torsion buckling are particularly prevalent in columns of thin-walled cross-sections, which generally exhibit low torsional rigidity. Some noteworthy contributions on instability of thin-walled columns are due to Wagner [3], Goodier [4] and Vlasov [5], amongst others. More recent papers on this topic can be found in Vo and Lee [6, 7] and Kim et al. [8]. In essence, Vo and Lee [6, 7] developed an analytical model based on the shear deformable beam theory whereas Kim et al. [8] proposed a formulation based on the displacement parameters defined at an arbitrarily chosen axis, including second order terms of finite semitangential rotations. Furthermore, the developments on the generalized beam theory [9, 10] deserve some special mention. Other contributions on the subject include Zhang and Tong [11], Mohri et al. [12] and Beale et al. [13].

In the present work, a general formulation for buckling analysis of both solid and thin-walled columns is proposed. The methodology can deal with pure bending or torsional buckling modes as well as with coupled bending-torsion instability phenomena. In the formulation presented, both metallic and composite columns can be analysed with no restrictions on the cross-sectional geometry. This is achieved by exploiting the Carrera Unified Formulation (CUF) [14], which has received wide attention in recent years [15, 16, 17]. CUF enables the development of 1D displacement fields in an arbitrary, but kinematically enriched manner. The governing differential equations can, in fact, be written in terms of the fundamental nuclei that depend nether on the order of the theory nor on the cross-sectional geometry. In recent works, CUF has already been applied to buckling analysis of columns by using both Finite Element Method (FEM) [18] and a Navier type solution [19]. As it is known, FEM is a widely used numerical method in solid mechanics which transforms the governing differential equations into a system of algebraic equations. However, only approximate solutions are given by FEM. On the other hand, if the Navier solution is used, no numerical approximations are made, but of course, only simply supported boundary conditions can be addressed.

A more powerful, but elegant approach for CUF theories can be achieved through the application of the Dynamic Stiffness Method (DSM), which was recently applied by Pagani et al. for free vibration analysis of metallic [20] and composite [21] beams. The DSM is appealing in free vibration and buckling analyses because unlike the FEM, it provides exact solution of the governing equations of a structure for any boundary conditions, once the initial assumptions on the displacements field have been made. The uncompromising accuracy of the DSM when dealing with buckling analysis has been demonstrated by Banerjee and Williams [22], Banerjee [23], Eisenberger and Reich [24], Eisenberger [25] and Abramovich et al. [26], amongst others.

In this paper, DSM is applied to refined CUF beam models and then extended to the linearized buckling analysis of columns, i.e. axially loaded beams. The investigation is carried out in the following steps: (i) first CUF is introduced and higher-order models are formulated, (ii) then, the Principle of Virtual Displacements (PVD) is used to derive the differential governing equations and the associated natural boundary conditions for the generic N-order model, (iii) next, the Dynamic Stiffness (DS) matrix

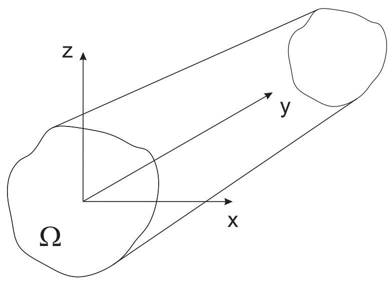

Figure 1: Beam structure.

is formulated and (iv) finally the algorithm by Wittrick and Williams [27] is used to compute the critical buckling loads of both isotropic and composite laminated beam-columns.

2 Carrera Unified Formulation

2.1 Preliminaries

Figure 1 shows the adopted rectangular Cartesian right-handed coordinate system, together with the geometry of a generic beam structure. The cross-section of the beam lies on the xz-plane and it is denoted by Ω, whereas the boundaries over y are $0 \leq y \leq L$. Let us introduce the transposed displacement vector,

$$\mathbf{u}(x,y,z) = \left\{ \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\}^T \tag{1}$$

The linear part of the strain vector, $\boldsymbol{\epsilon}$, can be derived as follows:

$$\boldsymbol{\epsilon} = \mathbf{D}\mathbf{u} \tag{2}$$

where $\mathbf{D}$ is a linear differential operator. In this work, geometric non-linearities are introduced in the axial strain in a Green-Lagrange manner.

$$\epsilon_{yy}^{\mathrm{nl}} = \frac{1}{2}(u_{x,y}^2 + u_{y,y}^2 + u_{z,y}^2) \tag{3}$$

The suffix after the comma in Eq. (3) denotes the derivatives. Constitutive laws are then exploited to obtain stress components to give

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\epsilon} \tag{4}$$

The linear differential matrix $\mathbf{D}$ and the components of matrix $\tilde{\mathbf{C}}$ are not given here for the sake of brevity, but they can be found in [21].

Within the framework of the CUF, the displacement field $\mathbf{u}(x,y,z)$ can be expressed as

$$\mathbf{u}(x,y,z) = F_\tau(x,z)\mathbf{u}_\tau(y), \qquad \tau = 1,2,....,M \tag{5}$$

where F_τ are the functions of the coordinates x and z on the cross-section. $\mathbf{u}_\tau$ is the vector of the *generalized* displacements, M stands for the number of the terms used in the expansion, and the repeated subscript, τ, indicates summation. The choice of F_τ determines the class of the 1D CUF model. According to Eq. (5), TE (Taylor expansion) 1D CUF models consist of a MacLaurin series that uses the 2D polynomials x^i z^j as F_τ functions, where i and j are positive integers. For instance, the displacement field of the second-order ($N = 2$) TE model can be expressed as

$$
\begin{aligned}
u_x &= u_{x_1} + x\ u_{x_2} + z\ u_{x_3} + x^2\ u_{x_4} + xz\ u_{x_5} + z^2\ u_{x_6} \\
u_y &= u_{y_1} + x\ u_{y_2} + z\ u_{y_3} + x^2\ u_{y_4} + xz\ u_{y_5} + z^2\ u_{y_6} \\
u_z &= u_{z_1} + x\ u_{z_2} + z\ u_{z_3} + x^2\ u_{z_4} + xz\ u_{z_5} + z^2\ u_{z_6}
\end{aligned}
\tag{6}
$$

The order N of the expansion is set as an input option of the analysis; the integer N is arbitrary and it defines the order the beam theory. Classical Euler-Bernoulli Beam Model (EBBM) and Timoshenko Beam Model (TBM) can be realised by using a suitable F_τ expansions as shown in [14].

2.2 Governing equations of the N-order TE model

The principle of virtual displacements is used to derive the equations of motion.

$$\delta L_{\text{int}} - \delta L_{\sigma^0_{\text{yy}}} = 0 \tag{7}$$

where L_{int} stands for the strain energy and $L_{\sigma^0_{\text{yy}}}$ is the work done by the axial pre-stress σ^0_{yy} on the corresponding non-linear strain $\epsilon^{\text{nl}}_{yy}$. δ stands for the usual virtual variation operator. The virtual variation of the strain energy is

$$\delta L_{\text{int}} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, \mathrm{d}V \tag{8}$$

Equation (8) is rewritten using Eq.s (2), (4) and (5). After integrations by part, it reads

$$\delta L_{\text{int}} = \int_L \delta \mathbf{u}_\tau^T \mathbf{K}^{\tau s} \mathbf{u}_s \, \mathrm{d}y + \left[\delta \mathbf{u}_\tau^T \boldsymbol{\Pi}^{\tau s} \mathbf{u}_s \right]_{y=0}^{y=L} \tag{9}$$

where $\mathbf{K}^{\tau s}$ is the differential linear stiffness matrix and $\boldsymbol{\Pi}^{\tau s}$ is the matrix of the natural boundary conditions in the form of 3×3 fundamental nuclei. The explicit expressions of the components of $\mathbf{K}^{\tau s}$ and $\boldsymbol{\Pi}^{\tau s}$ can be found in [21].

The virtual variation of the axial pre-stress is

$$\delta L_{\sigma^0_{\text{yy}}} = \int_L \left(\int_\Omega \sigma^0_{yy} \delta \epsilon^{\text{nl}}_{yy} \, \mathrm{d}\Omega \right) \mathrm{d}y \tag{10}$$

After substituting Eqs. (5) and (3) into Eq. (10) and after integration by parts, one has

$$\delta L_{\sigma^0_{\text{yy}}} = -\sigma^0_{yy} \int_L \delta \mathbf{u}_\tau^T \mathbf{K}^{\tau s}_{\sigma^0_{yy}} \mathbf{u}_s \, \mathrm{d}y + \sigma^0_{yy} \left[\delta \mathbf{u}_\tau^T \boldsymbol{\Pi}^{\tau s}_{\sigma^0_{yy}} \mathbf{u}_s \right]_{y=0}^{y=L} \tag{11}$$

where $\mathbf{K}^{\tau s}_{\sigma^0_{yy}}$ is the fundamental nucleus of the differential geometric stiffness matrix.

$$\mathbf{K}^{\tau s}_{\sigma^0_{yy}} = \begin{bmatrix} E_{\tau s}\frac{\partial^2}{\partial y^2} & 0 & 0 \\ 0 & E_{\tau s}\frac{\partial^2}{\partial y^2} & 0 \\ 0 & 0 & E_{\tau s}\frac{\partial^2}{\partial y^2} \end{bmatrix} \tag{12}$$

where

$$E_{\tau s} = \int_{\Omega} F_\tau F_s \, \mathrm{d}\Omega \tag{13}$$

The components of $\mathbf{\Pi}^{\tau s}_{\sigma^0_{yy}}$ are

$$\mathbf{\Pi}^{\tau s}_{\sigma^0_{yy}} = \begin{bmatrix} E_{\tau s}\frac{\partial}{\partial y} & 0 & 0 \\ 0 & E_{\tau s}\frac{\partial}{\partial y} & 0 \\ 0 & 0 & E_{\tau s}\frac{\partial}{\partial y} \end{bmatrix} \tag{14}$$

In a matrix form, the equilibrium equations can be therefore written as follows:

$$\delta \mathbf{u}_\tau : \quad \mathbf{L}^{\tau s}\, \tilde{\mathbf{u}}_s = 0 \tag{15}$$

where

$$\tilde{\mathbf{u}}_s = \left\{ \begin{array}{ccccccccc} u_{xs} & u_{xs,y} & u_{xs,yy} & u_{ys} & u_{ys,y} & u_{ys,yy} & u_{zs} & u_{zs,y} & u_{zs,yy} \end{array} \right\}^T \tag{16}$$

and $\mathbf{L}^{\tau s}$ is the 3×9 fundamental nucleus which contains the coefficients of the ordinary differential equations. The components of $\mathbf{L}^{\tau s}$ are provided below and they are referred to as $L^{\tau s}_{(ij)}$, where i is the row number ($i = 1, 2, 3$) and j is the column number ($j = 1, 2, ..., 9$)

$$\begin{aligned}
&L^{\tau s}_{(11)} = E^{22}_{\tau_{,x}s_{,x}} + E^{44}_{\tau_{,z}s_{,z}}, \quad L^{\tau s}_{(12)} = E^{26}_{\tau_{,x}s} - E^{26}_{\tau s_{,x}}, \quad L^{\tau s}_{(13)} = \sigma^0_{yy}E_{\tau s} - E^{66}_{\tau s} \\
&L^{\tau s}_{(14)} = E^{26}_{\tau_{,x}s_{,x}} + E^{45}_{\tau_{,z}s_{,z}}, \quad L^{\tau s}_{(15)} = E^{23}_{\tau s_{,x}} - E^{66}_{\tau s_{,x}}, \quad L^{\tau s}_{(16)} = -E^{36}_{\tau s} \\
&L^{\tau s}_{(17)} = E^{12}_{\tau_{,x}s_{,z}} + E^{44}_{\tau_{,z}s_{,x}}, \quad L^{\tau s}_{(18)} = E^{45}_{\tau_{,z}s} - E^{16}_{\tau s_{,z}}, \quad L^{\tau s}_{(19)} = 0 \\
&L^{\tau s}_{(21)} = E^{26}_{\tau_{,x}s_{,x}} + E^{45}_{\tau_{,z}s_{,z}}, \quad L^{\tau s}_{(22)} = E^{66}_{\tau_{,x}s} - E^{23}_{\tau s_{,x}}, \quad L^{\tau s}_{(23)} = -E^{36}_{\tau s} \\
&L^{\tau s}_{(24)} = E^{66}_{\tau_{,x}s_{,x}} + E^{55}_{\tau_{,z}s_{,z}}, \quad L^{\tau s}_{(25)} = E^{36}_{\tau_{,x}s} - E^{36}_{\tau s_{,x}}, \quad L^{\tau s}_{(26)} = \sigma^0_{yy}E_{\tau s} - E^{33}_{\tau s} \\
&L^{\tau s}_{(27)} = E^{16}_{\tau_{,x}s_{,z}} + E^{45}_{\tau_{,z}s_{,x}}, \quad L^{\tau s}_{(28)} = E^{55}_{\tau_{,z}s} - E^{13}_{\tau s_{,z}}, \quad L^{\tau s}_{(29)} = 0 \\
&L^{\tau s}_{(31)} = E^{44}_{\tau_{,x}s_{,z}} + E^{12}_{\tau_{,z}s_{,x}}, \quad L^{\tau s}_{(32)} = E^{16}_{\tau_{,z}s_{,x}} - E^{45}_{\tau s_{,z}}, \quad L^{\tau s}_{(33)} = 0 \\
&L^{\tau s}_{(34)} = E^{45}_{\tau_{,x}s_{,z}} + E^{16}_{\tau_{,z}s_{,x}}, \quad L^{\tau s}_{(35)} = E^{13}_{\tau_{,z}s} - E^{55}_{\tau s_{,z}}, \quad L^{\tau s}_{(36)} = 0 \\
&L^{\tau s}_{(37)} = E^{44}_{\tau_{,x}s_{,x}} + E^{11}_{\tau_{,z}s_{,z}}, \quad L^{\tau s}_{(38)} = E^{45}_{\tau_{,x}s} - E^{45}_{\tau s_{,x}}, \quad L^{\tau s}_{(39)} = \sigma^0_{yy}E_{\tau s} - E^{55}_{\tau s}
\end{aligned} \tag{17}$$

where

$$E^{\alpha\beta}_{\tau_{,\theta}s_{,\zeta}} = \int_{\Omega} \tilde{C}_{\alpha\beta} F_{\tau_{,\theta}} F_{s_{,\zeta}} \, \mathrm{d}\Omega \tag{18}$$

For a given expansion order, N, the equilibrium equations can be obtained in the form of Eq. (19) as given below by expanding $\mathbf{L}^{\tau s}$ for $\tau = 1, 2, ..., (N+1)(N+2)/2$ and $s = 1, 2, ..., (N+1)(N+2)/2$. It reads:

$$\mathbf{L}\,\tilde{\mathbf{u}} = 0 \tag{19}$$

In a similar way, the boundary conditions can be written in a matrix form as

$$\delta \mathbf{u}_{\tau} : \quad \mathbf{P}_s = \mathbf{B}^{\tau s}\,\hat{\mathbf{u}}_s \tag{20}$$

where

$$\hat{\mathbf{u}}_s = \left\{ \begin{array}{cccccc} u_{xs} & u_{xs,y} & u_{ys} & u_{ys,y} & u_{zs} & u_{zs,y} \end{array} \right\}^T \tag{21}$$

and $\mathbf{B}^{\tau s}$ is the 3×6 fundamental nucleus which contains the coefficients of the natural boundary conditions.

$$\mathbf{B}^{\tau s} = \left[\begin{array}{cccccc} E^{26}_{\tau s_{,x}} & (E^{66}_{\tau s} - \sigma^0_{yy} E_{\tau s}) & E^{66}_{\tau s_{,x}} & E^{36}_{\tau s} & E^{16}_{\tau s_{,z}} & 0 \\ E^{23}_{\tau s_{,x}} & E^{36}_{\tau s} & E^{36}_{\tau s_{,x}} & (E^{33}_{\tau s} - \sigma^0_{yy} E_{\tau s}) & E^{13}_{\tau s_{,z}} & 0 \\ E^{45}_{\tau s_{,z}} & 0 & E^{55}_{\tau s_{,z}} & 0 & E^{45}_{\tau s_{,x}} & (E^{55}_{\tau s} - \sigma^0_{yy} E_{\tau s}) \end{array} \right] \tag{22}$$

For a given expansion order, N, the natural boundary conditions can be obtained in the form of Eq. (23) by expanding $\mathbf{B}^{\tau s}$ in the same way as $\mathbf{L}^{\tau s}$ to finally give

$$\mathbf{P} = \mathbf{B}\,\hat{\mathbf{u}} \tag{23}$$

In the case of laminated structures, matrices $\mathbf{L}$ and $\mathbf{B}$ are evaluated for each layer; global matrices are then obtained by summing the contribution of each lamina.

3 Dynamic Stiffness Method

Equation (19) is a system of ordinary differential equations (ODEs) of second order in y with constant coefficients. A change of variables is used to reduce the second order system of ODEs to a first order system,

$$\mathbf{Z} = \left\{ \begin{array}{cccc} Z_1 & Z_2 & \dots & Z_n \end{array} \right\}^T = \hat{\mathbf{u}} \tag{24}$$

where $\hat{\mathbf{u}}$ is the expansion of $\hat{\mathbf{u}}_s$ for a given theory order and $n = 6 \times M$ is the dimension of the unknown vector as well as the number of differential equations. In [20], an automatic algorithm to transform the $\mathbf{L}$ matrix of Eq. (19) into the matrix $\mathbf{S}$ of the following linear differential system was described:

$$\mathbf{Z}'(y) = \mathbf{S}\mathbf{Z}(y) \tag{25}$$

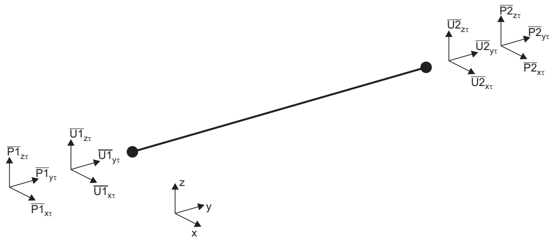

Figure 2: Boundary conditions of the beam element and sign conventions.

Once the differential problem is described in terms of Eq. (25), the solution can be written as follows:

$$\mathbf{Z} = \boldsymbol{\delta}\mathrm{C}\mathrm{e}^{\boldsymbol{\lambda} y} \tag{26}$$

where $\boldsymbol{\lambda}$ is the vector of the eigenvalues of $\mathbf{S}$. The element δ_{ij} of matrix δ is the j-th component of the i-th eigenvector of matrix $\mathbf{S}$ and the vector $\mathbf{C}$ contains the integration constants that need to be determined by using the boundary conditions.

Once the closed form analytical solution has been found, the generic boundary conditions for the generalized displacements and forces need to be applied (see Fig. 2). It should be noted that the vector $\mathbf{Z}$ of Eq. (26) does not only contain the displacements but also their first derivatives. If only displacements are needed, by recalling Eq. (26), only the lines $1, 3, 5, ..., n-1$ should be taken into account. Therefore, by evaluating Eq. (26) in 0 and L and applying the boundary conditions as shown in Fig. 2, the following matrix relation for the nodal displacements is obtained:

$$\overline{\mathbf{U}} = \mathbf{AC} \tag{27}$$

Similarly, boundary conditions for generalized nodal forces are written as follows:

$$\overline{\mathbf{P}} = \mathbf{RC} \tag{28}$$

The constants vector $\mathbf{C}$ from Eq.s (27) and (28) can now be eliminated to give the DS matrix of one beam element as follows:

$$\overline{\mathbf{P}} = \boldsymbol{\mathcal{K}}\overline{\mathbf{U}} \tag{29}$$

where

$$\boldsymbol{\mathcal{K}} = \mathbf{RA}^{-1} \tag{30}$$

is the required DS matrix.

The DS matrix given above is the basic building block to compute the exact critical buckling loads of a higher-order beam. The DSM has also many of the general features of the FEM. In particular, it is possible to assemble elemental DS matrices to form the overall DS matrix of any complex structures consisting of beam elements. Once the global DS matrix of the final structure is obtained, the boundary conditions can be applied by using the well-known penalty method (often used in FEM) or by simply removing rows and columns of the dynamic stiffness matrix corresponding to the degrees of freedom that are zeroes.

3.1 The Wittrick-Williams algorithm

For linearized buckling analysis of structures, FEM generally leads to a linear eigenvalue problem. By contrast, the DSM leads to a transcendental (non-linear) eigenvalue problem for which the Wittrick-Williams algorithm [28] is recognisably the best available solution technique at present. The basic working principle of the algorithm can be briefly summarised in the following steps:

(i) A trial critical load $-\sigma^0_{yy} = \lambda^*$ is chosen to compute the dynamic stiffness matrix $\mathcal{K}^*$ of the final structure;

(ii) $\mathcal{K}^*$ is reduced to its upper triangular form by the usual form of Gauss elimination to obtain $\mathcal{K}^{*\triangle}$ and the number of negative terms on the leading diagonal of $\mathcal{K}^{*\triangle}$ is counted; this is known as the sign count $s(\mathcal{K}^*)$ of the algorithm;

(iii) The number, j, of critical loads (λ) of the structure which lie below the trial buckling load (λ^*) is given by:

$$j = j_0 + s(\mathcal{K}^*) \tag{31}$$

where j_0 is the number of critical buckling loads of all individual elements with clamped-clamped (CC) boundary conditions on their opposite sides which still lie below the trial critical buckling load λ^*.

Note that j_0 is required because the DSM allows for an infinite number of critical buckling loads to be accounted for when all the nodes of the structure are fully clamped so that one or more individual elements of the structure can still buckle on their own between the nodes. Assuming that j_0 is known, and $s(\mathcal{K}^*)$ can be obtained by counting the number of negative terms in $\mathcal{K}^{*\triangle}$, a suitable procedure can be devised (e.g. the bi-section method) to bracket any critical buckling load between an upper and lower bound of the trial load λ^* to any desired accuracy.

4 Numerical Results

4.1 Metallic rectangular cross-section beam

A cantilever metallic beam is considered as the first illustrative example to demonstrate the theory. The same structure was addressed in [2, 18], whose results are quoted in this paper for comparison purposes. The beam has a rectangular cross-section and length-to-height ratio, L/h, equal to 20. The material is aluminium alloy with elastic modulus $E = 71.7$ GPa and Poisson's ratio $\nu = 0.3$.

Table 1 shows the first three non-dimensional critical buckling loads. The second column shows the n-th critical buckling load from the Euler buckling formula give by

$$P_{cr}^{Euler} = n^2\pi^2\frac{EI}{L^2}, \qquad \text{with } I = \frac{bh^3}{12} \tag{32}$$

Mode	Euler	Matsunaga [2]	TBM		N=1		N=2		N=3	
			FEM [18]	DSM	FEM [18]	DSM	FEM [18]	DSM	FEM [18]	DSM
1	1	0.992	0.990	0.993	0.993	0.993	0.993	0.993	0.992	0.992
2	4	3.873	3.875	3.886	3.884	3.886	3.885	3.887	3.873	3.874
3	9	8.387	8.422	8.444	8.437	8.444	8.444	8.451	8.387	8.391

Table 1: First three non-dimensional buckling loads ($P^*_{cr} = \frac{P_{cr}L^2}{\pi^2 EI}$) of the metallic beam, $L/h = 20$.

In column 3 the results by Matsunaga [2] are given whereas columns 4 to 7 report the results by classical and refined models based on TE CUF models of the present paper. The exact solution by the present DSM are compared to those from FEM, which was used in [18]. The following comments arise from the analysis:

- Euler buckling formula overestimates the critical loads of the beam addressed, even though a high length-to-side ratio is considered.
- Higher-order CUF theories are effective in refining the solution and the results are in good agreement with those available in the literature.
- The critical buckling load becomes lower as the expansion order for TE CUF models increases. This is significant because other theories give unconservative estimates of critical buckling loads.
- The exact solutions provided with the DSM is slightly higher then those by FEM. This is unusual and may be due to numerical problems inherent in FEM.

4.2 Channel section beam

The cantilever C-shaped section beam of Fig. 3 is now addressed. The main dimensions of the cross-section are $a_1 = 4$ cm, $a_2 = 2$ cm, $h = 10$ cm and $t = 0.5$ cm. The beam has a length $L = 2$ m and is made of homogeneous isotropic material with elastic modulus $E = 3 \times 10^4$ N/cm^2 and shear modulus $G = 1.15 \times 10^4$ N/cm^2.

Table 2 shows the first three critical buckling loads by higher-order ($N = 7$ and $N = 8$) beam models by the present CUF-DSM methodology. The results are compared with those given by Vo and Lee [6], who developed an analytical model based on the shear deformable beam theory. Figure 4 shows the second buckling mode by the seventh-order ($N = 7$) CUF-DSM model. The figure clearly shows that the present method can predict the flexural-torsional buckling load accurately. The analysis highlights that

- Relatively higher-order kinematics are needed to detect flexural-torsional buckling modes of axially loaded thin-walled structures accurately.
- The results by the proposed CUF-DSM models are in good agreement with the results found in the literature.

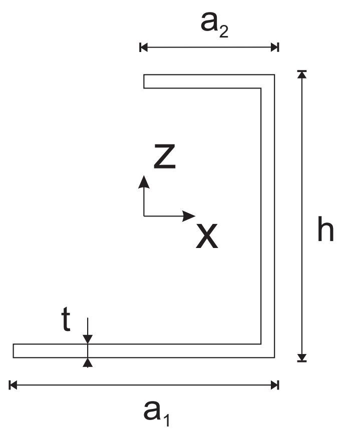

Figure 3: Cross-section of the C-shaped beam.

Mode	Present CUF-DSM		Vo and Lee [6]
	$N = 7$	$N = 8$	
1	14.111	13.875	12.977
2	119.034	117.375	113.440
3	201.510	199.125	190.567

Table 2: Flexural-torsional buckling loads (N) for the axially compressed C-section beam.

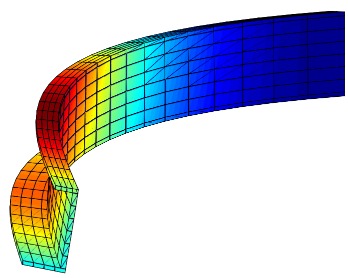

Figure 4: Second flexural-torsional buckling mode of the C-shaped section beam by the seventh-order ($N = 7$) CUF model.

L/h	Present CUF-DSM				Vo and Thai [29]	Aydogdu [30]
	$N = 1$	$N = 2$	$N = 3$	$N = 4$		
5	4.810	4.992	4.668	4.653	4.709	4.726
10	6.929	6.937	6.751	6.750	6.778	–
20	7.663	7.677	7.618	7.618	7.620	7.666
50	7.903	7.917	7.904	7.904	7.896	–

Table 3: Effect of L/h on the non-dimensional critical load, ($P^*_{cr} = \frac{P_{cr}L^2}{\pi^2 E_2 bh^3}$), of the symmetric cross-ply beam.

4.3 Cross-ply laminated beams

A simply-supported composite rectangular beam is now analysed. The lay-up is a symmetric cross-ply $[0°/90°/0°]$. Each layer has the same thickness and is made of an orthotropic material with following elastic constants:

$$E_1/E_2 = 10,\ G_{12} = G_{13} = 0.6E_2,\ G_{23} = 0.5E_2,\ \nu_{12} = 0.25$$

where 1 stands for the fibre direction and $(2,\ 3)$ are two directions orthogonal to 1.

Table 3 shows the first buckling load for different length-to-side ratios, L/h. In columns 2 to 5 the results from the present CUF-DSM method with higher-order beam models are given and compared with those from Vo and Thai [29] and Aydogdu [30]. The following comments are worth noting:

- Higher-order kinematics is needed as the length-to-side ratio decreases.
- According to the fourth-order ($N = 4$) CUF model, reference beam theories [29, 30] overestimates the first critical load if short beams are considered.
- The present method is effective even though composite cross-ply beams are analysed.
- The DSM is an effective and powerful means for the solution of CUF models.

5 Conclusions

Using the Carrera Unified Formulation, the governing equations of axially loaded higher-order beams have been formulated in a compact and concise notation. The exact dynamic stiffness matrix has been subsequently developed and the algorithm of Wittrick and Williams has been used to determine the critical buckling loads. Different structures have been analysed and the results from the present methodology have been compared to those available in the literature. The results from the theory clearly shows the strength of the CUF-DSM method, which can be successfully applied to linearized buckling analysis of both compact and thin-walled metallic beams as well as composite laminates.

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