On the effectiveness of higher-order terms in layer-wise shell models

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ABSTRACT: Refined layer-wise theories for shell and plate models are presented and discussed in this paper. Higher-order models are obtained through the Unified Formulation (UF) developed by the first author over the last decade. The effectiveness of each higher-order term is evaluated through the so-called mixed axiomatic/asympthotic approach (MAAA). MAAA has been recently developed and can be seen as a tool to build reduced refined models against full expansion theories. The development of reduced models is carried out through the investigation of the effectiveness of each unknown variable on the solution for a given problem. Reduced models are then built for various structural cases, such as thin and thick shells, layered shells and sandwiches. Results show the enhanced capabilities of MAAA and UF to develop refined layer-wise models with reduced computational costs.

1 INTRODUCTION

Composite panels are frequently exploited in mechanical and aerospace applications. The structural analysis of these components requires appropriate models able to deal with high transverse shear and normal deformability and the discontinuity of physical properties through the thickness of a panel. An accurate stress and strain field evaluation can be often obtained only by means of refined models.

Classical and well-known plate/shell theories are those by Kirchhoff (1850), Love (1927), Reissner (1945), Mindlin (1950), Vlasov (1957), Koiter (1970) and Naghdi (1972). These theories are based on axiomatic hypotheses leading to simplified models of the three-dimensional deformed configuration of the structure. Typical examples of these hypotheses are: sections remain plane, the thickness deformation can be neglected, shear strains are negligible. For a complete review of this topic, including laminated composite structures, the readers can refer to the many available survey articles on plates and shells (Librescu & Reddy 1986, Reddy 2004).

Approximated structural models can be also built through asymptotic approaches. In this case, expansions of the unknown variables are developed by evaluating the order of magnitude of significant terms referring to a geometrical parameter (thickness-to-length in the case of plates and shells). The asymptotic approach provides consistent approximations since all the terms retained have the same order of magnitude of the perturbation parameter when the latter vanishes. Articles on the application of asymptotic methods to shell structures can be found in Cicala (1959), Fettahlioglu & Steele (1974), Berdichevsky (1979, Berdichevsky & Misyura (1992), Widera & Logan (1980), Widera & Fan (1988), Spencer, Watson, & Rogers (1990), and in the monographes by Cicala (1965) and Gol'denweizer (1961).

Axiomatic and asymptotic methods have been historically motivated by the need to work with simplified theories capable of leading to simple formulas and equations to be solved by hand calculations. Up to five decades ago, in fact, it was quite prohibitive to solve problems with many unknowns (more than 5, 6); nowadays, this limitation no longer holds. The approach exploited in this paper adopts a condensed notation technique introduced by the first author during the last decade and referred to as the Carrera Unified Formulation (CUF) for beams, plates and shell structures (Carrera 2003, Carrera & Petrolo 2012). Through CUF, governing equations are given in terms of a few 'fundamental nuclei' whose form does not depend on either the order of the introduced approximations or on the type base functions in the thickness direction. In order to obtain more general conclusions and to draw general guidelines and recommendations to build bidimensional theories for metallic and composite plates and shells, CUF can be also used to evaluate the effectiveness of each term of a refined theory. This has been done in the present paper. In CUF, in fact, the role of each displacement variable in the solution is investigated by measuring the loss of accuracy due to its neglect. A term is considered ineffective, i.e. negligible, if it does not affect the accuracy of the solution with respect to a reference

3D solution. Reduced kinematics models, based on a set of retained displacement variables, are then obtained for each configuration considered . Full and reduced models are then compared in order to highlight the sensitivity of a kinematics model to variations in the structural problem. This method can somehow be considered as a *mixed axiomatic/asymptotic approach* since it provides asymptotic-like results starting from a preliminary axiomatic choice of the base functions. Companion investigations - related to equivalent single layer and layer-wise plate models - have been proposed in Carrera & Petrolo (2010), Carrera, Miglioretti, & Petrolo (2011a), Carrera, Miglioretti, & Petrolo (2011b) and Carrera, Lamberti, & Petrolo (2011).
In this paper, layer-wise refined plate and shell models are investigated. The effectiveness of each displacement variable has been established varying a number of parameters (e.g. the thickness ratio, the orthotropic ratio and the stacking sequence of the lay-out).

2 CARRERA UNIFIED FORMULATION

The Carrera Unified Formulation (CUF) can deal with a wide variety of beam/plate/shell structures. More details on CUF can be found in many papers (Carrera 2002, Carrera 2003). According to CUF, the governing equations are written in terms of a few *fundamental nuclei* which do not formally depend on both the order of the expansion N and the approximating functions in the thickness direction. The displacement field is modeled as follows:

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \qquad \tau = 0, 1, ..., N \tag{1}$$

where F_τ are functions of z. $\mathbf{u}_\tau$ is the displacements vector and N stands for the order of the expansion. According to Einstein's notation, the repeated subscript τ indicates summation. In this work, layer-wise models are considered based on Legendre polynomials. For the sake of brevity, these polynomials are not given here, they can be found in Carrera (2003).
The Principle of Virtual Displacements (PVD) is used to obtain the governing equations and boundary conditions. In the general case of multi-layered plates/shells subjected to mechanical loads, the governing equations are

$$\delta \mathbf{u}_s^{k^T} : \qquad \mathbf{K}_{uu}^{k\tau s}\, \mathbf{u}_\tau^k = \mathbf{P}_{u\tau}^k \tag{2}$$

where T indicates the transpose and k the layer. $\mathbf{K}_{uu}^{k\tau s}$ and $\mathbf{P}_{u\tau}^k$ are the fundamental nuclei for the stiffness and load terms, respectively, and they are assembled through the depicted indexes, τ and s, which consider the order of the expansion in z for the displacements.

The corresponding Neumann-type boundary conditions are:

$$\mathbf{\Pi}_d^{k\tau s}\, \mathbf{u}_\tau^k = \mathbf{\Pi}_d^{k\tau s}\, \bar{\mathbf{u}}_\tau^k\,, \tag{3}$$

where $\mathbf{\Pi}_d^{k\tau s}$ is the fundamental nucleus for the boundary conditions and the over-line indicates an assigned condition.

For the explicit form of fundamental nuclei for the Navier-type closed-form solution and more details about the constitutive equations and geometrical relations for laminated plates and shells in the framework of CUF, one can refer to (Carrera 2002). A simply supported plate has been considered and a bi-sinusoidal transverse distributed load is applied to the top surface:

$$p_z = \bar{p}_z\; sin(\frac{\pi x}{a})\; sin(\frac{\pi y}{b}) \tag{4}$$

where a and b are the side length of the plate.

2.1 Mixed Axiomatic/Asymptotic approach

Significant advantages are offered by refined plate/shell theories in terms of accuracy of the solution, but a higher computational effort is necessary because of the presence of a larger number of displacement variables. This work is an effort to understand the convenience of using a fully refined model rather than a reduced one. The effectiveness of each term, as well as the terms that have to be retained in the formulation, are investigated as follows.

1. The problem data are fixed (geometry, boundary conditions, loadings, materials and layer layouts).

2. A set of output variables is chosen (maximum displacements, stress/displacement component at a given point, etc.).

3. A theory is fixed, that is, the terms that have to be considered in the expansion of u_x, u_y, and u_z are established.

4. A reference solution is used to establish the accuracy (the $N = 4$ case is assumed as the best-reference solution since it offers an excellent agreement with the 3D solutions).

5. The effectiveness of each term is numerically established measuring the error produced compared to the reference solution.

6. Any term which does not give any contribution to the mechanical response is not considered as effective in the kinematics model.

7. The most suitable kinematics model is then detected for a given structural lay-out.

A graphical notation has been introduced to make the representation of results more readable. This consists of a table with three lines for the displacement components and a number of columns that depends on the number of displacement variables which are used in

Table 1: Symbols to indicate the status of a displacement variable.

Active term	Inactive term	Fixed term
▲	△	▼

the expansion. In an equivalent single layer approach, the number of variables does not depend on the number of layers. A fourth-order models, for instance, has 15 unknowns (generalized displacement variables), as shown in figure 1. In a LW approach, the number of unknowns depends on the number of layers. Figure 2 shows the unknowns for a generic layer. In this case, the top and bottom variables are pure displacement variables. Table 1 shows the symbols adopted to indicate the status of a variables. The top and bottom terms of LW are considered as fixed since they cannot be neglected, this would imply the constraining of the structure.

$N=0$	$N=1$	$N=2$	$N=3$	$N=4$
u_1	$u_2\ z$	$u_3\ z^2$	$u_4\ z^3$	$u_5\ z^4$
v_1	$v_2\ z$	$v_3\ z^2$	$v_4\ z^3$	$v_5\ z^4$
w_1	$w_2\ z$	$w_3\ z^2$	$w_4\ z^3$	$w_5\ z^4$

Figure 1: Locations of the displacement variables within the tables layout

$u^k_{x_t} \cdot F^k_t$	$u^k_{x_2} \cdot F^k_2$	$u^k_{x_3} \cdot F^k_3$	$u^k_{x_4} \cdot F^k_4$	$u^k_{x_b} \cdot F^k_b$
$u^k_{y_t} \cdot F^k_t$	$u^k_{y_2} \cdot F^k_2$	$u^k_{y_3} \cdot F^k_3$	$u^k_{y_4} \cdot F^k_4$	$u^k_{y_b} \cdot F^k_b$
$u^k_{z_t} \cdot F^k_t$	$u^k_{z_2} \cdot F^k_2$	$u^k_{z_3} \cdot F_{3^k}$	$u^k_{z_4} \cdot F^k_4$	$u^k_{z_b} \cdot F^k_b$

Figure 2: Locations of the displacement variables within the tables layout. LW approach for a generic *k* layer.

3 RESULTS AND DISCUSSION

As first assessment, composite shells were analyzed to assess the shell theory accuracy against the stacking sequence. A three-layered symmetrical $(90°/0°/90°)$ shell and a two-layered asymmetric $(90°/0°)$ shell were considered. In all the cases, the layers are of equal thickness. The orthotropic ratio E_L/E_T equal to 25 and the thickness ratio $R_\beta/h = 100$.

Results were obtained through equivalent single layer models, results from LW will be given during the conference. Table 2 shows the shell model for each stacking sequence in order to obtain a fourth-order model accuracy. The stacking sequence influences the construction of adequate models to a great extent and an asymmetric lamination sequence requires a higher number of displacement variables than a symmetric one.

Layer-wise models were considered for plates and results are given in Tables 3-5 for different stacking sequences and thickness ratios (E_L = 40 GPa, E_T = E_z = 1 GPa, G_{LT} = 0.5 GPa, G_z = 0.6 GPa, ν = 0.25.). Reduced models equivalent to full fourth-order models require significantly less unknown variables. High thickness and asymmetric stacking lead to increments of the number of variables required.

Table 2: Comparison of the sets of effective terms for composite shells with different stacking sequences.

u_z					σ_{yy}					σ_{yz}				
							$90°/0°/90°$							
$M_e=8$					$M_e=9$					$M_e=6$				
▲	▲	△	△	△	▲	▲	△	△	△	▲	▲	△	△	△
▲	▲	△	▲	△	▲	▲	△	▲	△	▲	▲	△	▲	△
▲	▲	▲	△	△	▲	▲	▲	▲	△	▲	△	△	△	△
							$90°/0°$							
$M_e=6$					$M_e=9$					$M_e=10$				
▲	▲	△	△	△	▲	▲	△	△	△	▲	▲	△	△	△
▲	▲	△	△	△	▲	▲	△	△	△	▲	▲	▲	▲	▲
▲	△	▲	△	△	▲	▲	▲	▲	▲	▲	▲	▲	△	△

Table 3: Summary of the effective terms for a symmetric (0°/90°/0°) composite plate, $a/h = 100$.

	$M_e=12/39$												
	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
u_z	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	$M_e=14/39$												
	▼	△	△	△	▼	△	△	△	▼	▲	△	△	▼
σ_{xx}	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	△	△	△	▼	△	△	△	▼	▲	△	△	▼
	$M_e=13/39$												
	▼	△	△	△	▼	△	▲	△	▼	△	△	△	▼
σ_{xz}	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	$M_e=15/39$												
	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
σ_{zz}	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	▲	▲	▲	▼	△	△	△	▼	△	△	△	▼
	$M_e=20/39$												
	▼	△	△	△	▼	△	▲	△	▼	▲	△	△	▼
COMBINED	▼	△	△	△	▼	△	▲	△	▼	△	△	△	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	△	△	▼

Table 4: Summary of the effective terms for a symmetric composite plate (0°/90°/0°), $a/h = 2$.

	$M_e=24/39$												
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	▲	△	▼
u_z	▼	▲	△	△	▼	△	▲	△	▼	▲	△	△	▼
	▼	▲	△	△	▼	▲	△	△	▼	▲	△	△	▼
	$M_e=26/39$												
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	▲	△	▼
σ_{xx}	▼	▲	△	△	▼	△	▲	△	▼	▲	△	△	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	△	△	▼
	$M_e=24/39$												
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	▲	△	▼
σ_{xz}	▼	▲	△	△	▼	△	▲	△	▼	▲	△	△	▼
	▼	▲	△	△	▼	▲	△	△	▼	▲	△	△	▼
	$M_e=17/39$												
	▼	▲	△	▲	▼	△	△	△	▼	△	△	△	▼
σ_{zz}	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	▲	▲	▲	▼	△	△	△	▼	△	△	△	▼
	$M_e=29/39$												
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	▲	▲	▼
COMBINED	▼	▲	△	△	▼	▲	▲	▲	▼	▲	△	△	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	△	△	▼

4 CONCLUSIONS

This paper has dealt with the development of reduced higher-order models for shell structures. Results have been obtained through the Carrera Unified Formulation (CUF) and the so-called mixed axiomatic/asymptotic approach (MAAA). CUF is a hi-

Table 5: Summary of the effective terms for an asymmetric composite plate (0°/0°/90°), $a/h = 2$.

	$M_e = 24/39$												
u_z	▼	▲	△	△	▼	▲	▲	△	▼	▲	▲	△	▼
	▼	▲	▲	△	▼	△	△	△	▼	▲	△	△	▼
	▼	▲	△	△	▼	▲	△	△	▼	▲	△	△	▼
	$M_e = 28/39$												
σ_{xx}	▼	▲	△	△	▼	▲	▲	△	▼	▲	▲	▲	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	△	△	▼
	▼	▲	△	△	▼	▲	△	△	▼	▲	▲	▲	▼
	$M_e = 23/39$												
σ_{xz}	▼	▲	△	△	▼	△	▲	△	▼	▲	▲	△	▼
	▼	▲	▲	△	▼	▲	△	△	▼	▲	△	△	▼
	▼	▲	△	△	▼	▲	△	△	▼	▲	△	△	▼
	$M_e = 17/39$												
σ_{zz}	▼	△	△	△	▼	△	△	△	▼	△	△	△	▼
	▼	▲	△	▲	▼	△	△	△	▼	△	△	△	▼
	▼	▲	▲	▲	▼	△	△	△	▼	△	△	△	▼
	$M_e = 30/39$												
COMBINED	▼	▲	△	△	▼	▲	▲	△	▼	▲	▲	▲	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	△	△	▼
	▼	▲	▲	▲	▼	▲	△	△	▼	▲	▲	▲	▼

erarchical formulation allowing for the automatic implementation of any-order theory with no need of ad hoc formulations. MAAA stems from CUF and provides reduced models againt full expansion theories with reduced computational costs. Analyses have been carried out on various composite plate and shells structures. The following main conclusions can be drawn:

1. MAAA provides reduced models as accurate as full expansion models.

2. Reduced models by MAAA can need significantly less unknown variables than full models.

3. In particular, for LW models, the reduction of unknowns is generally larger than in ESL.

4. As expected, higher thicknesses or anisotropy make the reduced models more cumbersome.

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