Multi-Line-Beam with Variable Kinematic Models for the Analysis of Composite Structures

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ABSTRACT

In the finite element analysis of composite structures, it is a common practice to use different formulations in different sub-regions of the problem domain. In the present work, the Carrera Unified Formulation (CUF) is used to develop variable kinematic structural models. CUF is a higher-order, one-dimensional (1D), finite element formulation which was recently introduced by the first author. By exploiting the hierarchical characteristics of the CUF, a multi-line approach is developed straightforwardly and used for the analysis of a multilayered structure. In the multi-line method, 1D refined finite elements with different order of expansion over the cross-sectional plane are employed in different regions of the domain of the structure. Lagrange multipliers are used to ”mix” different order elements. Constraints are imposed on displacement variables at a number of points (”connection points”) whose location over the interface boundaries is a parameter of the method. The accuracy of the proposed method is verified both through published literature and through finite element solutions using the commercial code MSC/NASTRAN.

INTRODUCTION

Beam models are widely used to analyze the mechanical behavior of slender bodies, such as columns, rotor-blades, aircraft wings, towers and bridges, amongst others. The simplicity of one-dimensional (1D) theories, their ease of application and their computational efficiency are some of the main reasons why structural analysts prefer them to two-dimensional (2D) and three-dimensional (3D) models. The classical and best-known beam theories are those by Euler [1] - hereinafter referred to as EBBM - and Timoshenko [2] - hereinafter referred to as TBM. The former does not account for transverse shear deformations, whereas the latter assumes a uniform shear distribution along the cross-section of the beam.

This paper is devoted to the analysis of laminated composite beams. The advantages of composite materials are known and, amongst these advantages, the most

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relevant are: high strength-to-weight ratio, high stiffness-to-weight ratio, ease of formability, wide range of operating temperatures, and their capability to be tailored according to a given requirement, see the book by Tsai [3]. One of the main issues related to the proper modeling of a composite structure is related to its low transverse shear moduli compared to the axial tensile moduli, see the excellent review of Kapania and Raciti [4] which includes a comprehensive overview on composite beam works. Moreover, the characterization of anisotropic layered composite structures requires models able to reproduce piecewise continuous displacement and transverse stress fields in the thickness direction. These two effects, which were summarized by the acronym C_z^0 requirements in [5], are not automatically satisfied by those models that were originally devoted to the analysis of single-layered structures. For these reasons, a number of refined *Equivalent Single Layer* (ESL) and *Layer-Wise* (LW) beam theories have been developed over the years.

A great deal of literature exists on classical and refined beam theories for the analysis of multilayered composite structures. A brief, though not exhaustive review, is given hereafter. Reddy [6] presented a plate theory which provides a parabolic distribution of the transverse shear strains ensuring that the transverse shear stresses are null on the top and bottom surfaces. By using this model, exact closed-form solutions for static analyses of cross-ply laminated beams with arbitrary boundary conditions were presented in [7]. In [8], Surana and Nguyen presented an interesting two dimensional hierarchical curved beam element. In Matsunaga's paper [9], the displacement components were expanded into power series of the z-thickness coordinate. Mantari et al. [10] expressed the displacement components of laminated plates by adopting a combination of exponential and trigonometric functions. Recently, Vidal et al. [11] proposed the approximation of the displacement field as a sum of separated functions of axial and transverse coordinates by adopting the Proper Generalized Decomposition procedure.

All the aforementioned theories are based on the ESL approach and, although the results agree very well with the three-dimensional solutions for several structural problems, the continuity of shear strains (hence the discontinuity of the shear stresses if they are computed through the constitutive equations) at interfaces represents the major drawback. To enhance this shortcoming, many researchers have adopted the LW approach and a few examples are given in the following. Shimpi and Ghugal [12] presented a new LW trigonometric model for two-layered cross-ply beams. The main feature of this theory is that the shear stresses were derived directly from the constitutive equations satisfying both the shear-stress-free condition at the free surfaces of the beam and the condition of continuity of the shear stresses at the interface. On the same topic, Tahani [13] proposed two theories to analyze the static and dynamic behaviour of the laminated beams. Unfortunately, when the number of layers increases, the LW approach becomes unfavorable because it is too expensive in terms of computational cost. To overcome this problem, many researchers have introduced layer independent theories in which zig-zag or Heaviside's functions are widely used.

Murakami [14] was the first to introduce a zig-zag function into Reissner's new mixed variational principle to develop a plate theory (for a complete review of Murakami's zig-zag method, see Carrera [15]). Vidal and Polit [16] presented a refined sine model by exploiting a Heaviside function for each layer to satisfy the continuity conditions for both displacements and transverse shear stress and the free conditions of the upper and lower surfaces.

It is clear that many attempts have been made in order to provide a general and reliable theory able to capture every aspect of the complex nature of the composite materials. In the present paper, a new method for the analysis of laminated composite structures is proposed. This method, which is called *Multi-Line* (ML) [17], represents a step forward to the classical ESL and LW approaches. In a ML model, each layer (or group of layers) of the structure is modelled by a higher-order beam. Subsequently, higher-order beams are assembled at the layer interfaces through Lagrange multipliers. In this work, refined beam elements are formulated using the Carrera Unified Formulation (CUF). According to CUF, Taylor-like polynomials are used on the cross-section of each beam to expand generalised displacement variables in a neighborhood of the beam axis. CUF was originally devoted to the analysis of plate and shell structures [18] and recently it has been expanded to 1D theories by the first author and his co-workers [19]. Several papers are available on the analysis of composite structures via CUF models, see for example [20, 21, 22].

In the next section a brief overview on CUF and ML approach is provided. Numerical results concerning a laminated beam are then presented. Finally, the main conclusions are outlined.

1D REFINED ELEMENTS AND MULTI-LINE MODELS

The adopted rectangular cartesian coordinate system is shown in Figure 1, together with the geometry of a beam which can be considered as a single layer, a group of layers, as well as a whole multilayer. The cross-sectional plane of the struc-

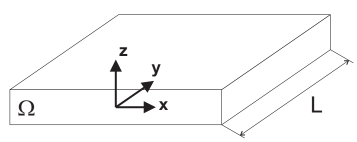


Figure 1. Coordinate frame of the beam model.

ture is denoted by Ω, and the beam boundaries over y are $0 \leq y \leq L$. Subscript k, which is usually used to denote variables and parameters related to the k^{th} layer, is neglected in this work for the sake of simplicity. Let us introduce the transposed displacement vector,

$$\mathbf{u}(x, y, z) = \left\{ \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\}^T \tag{1}$$

Within the framework of the CUF, the displacement field can be expressed as

$$\mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y), \qquad \tau = 1, 2, ..., M \tag{2}$$

where F_τ are the functions of the coordinates x and z on the cross-section. $\mathbf{u}_\tau$ is the vector of the generalized displacements, M stands for the number of terms used in the expansion, and the repeated subscript, τ , indicates summation. The choice of F_τ determines the class of the 1D CUF model. The refined beam elements adopted in the present paper, consist of a Taylor series that uses the 2D polynomials $x^i\, z^j$ as base - where i and j are positive integers - and they are referred to be Taylor Expansion (TE) models. For instance, the displacement field of the second-order ($N = 2$) TE model can be expressed as

$$\begin{aligned} u_x &= u_{x_1} + x\, u_{x_2} + z\, u_{x_3} + x^2\, u_{x_4} + xz\, u_{x_5} + z^2\, u_{x_6} \\ u_y &= u_{y_1} + x\, u_{y_2} + z\, u_{y_3} + x^2\, u_{y_4} + xz\, u_{y_5} + z^2\, u_{y_6} \\ u_z &= u_{z_1} + x\, u_{z_2} + z\, u_{z_3} + x^2\, u_{z_4} + xz\, u_{z_5} + z^2\, u_{z_6} \end{aligned} \tag{3}$$

The order N of the expansion is set as an input option of the analysis; the integer N is arbitrary and defines the order the beam theory. Classical TBM and EBBM theories are also captured from the formulation as degenerate cases.

The FE approach was adopted to discretize the structure along the y-axis. This process is conducted via a classical finite element technique, where the displacement vector is given by

$$\mathbf{u}(x, y, z) = F_\tau(x, z) N_i(y) \mathbf{q}_{\tau i} \tag{4}$$

N_i stands for the shape functions and $\mathbf{q}_{\tau i}$ for the nodal displacement vector. For the sake of brevity, the shape functions are not reported here. They can be found in many books, for instance in [23]. Elements with four nodes (B4) were adopted in this work, that is, a cubic approximation along the y axis was assumed. The stiffness matrix of the elements and the external loadings vector were obtained via the *principle of virtual displacements*

$$\delta L_{\text{int}} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \mathrm{d}\, V = \delta L_{\text{ext}} \tag{5}$$

where L_{int} stands for the strain energy, L_{ext} is the work of the external loadings and δ stands for the virtual variation. $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the vectors of the strains and stresses, respectively. For the sake of brevity, the fundamental nuclei of the stiffness matrix and the loading vector is not reported here. They can be found in [19], together with a more comprehensive discussion on refined 1D CUF models.

The Multi-Line Approach

In Figure 2 a multilayered structure is considered. The structure is composed by

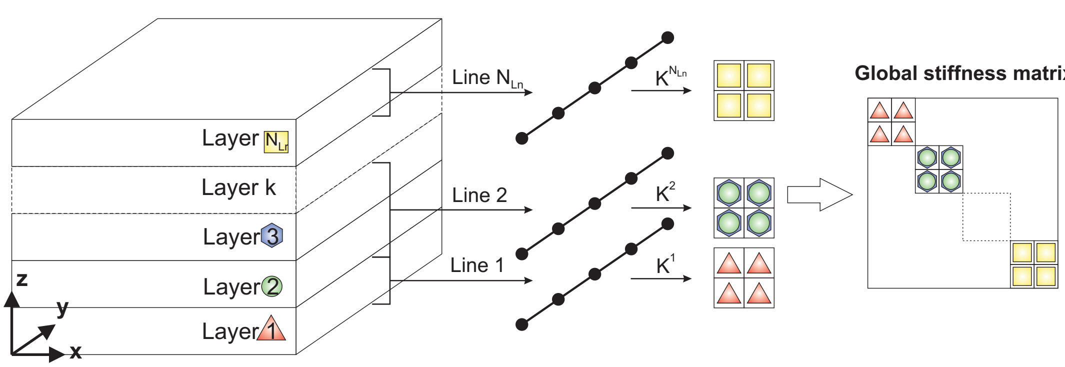


Figure 2. Multilayered structures and Multi-Line approach.

$\mathrm{N_{Lr}}$ layers. Each layer has its own geometrical and material properties. In the ML modelling approach, each layer (or group of layers) of the structure is modelled by a higher-order beam. Choice of the analyst is the number of beam lines ($\mathrm{N_{Ln}}$) to be used in order to find the right balance between accuracy and efficiency. In the present paper, the single beam line is discretized by means of refined CUF beam elements, according to the previous section. Once the stiffness matrices of each beam line have been computed, the global stiffness matrix is assembled according to Figure 2. Subsequently, boundary constraints are imposed at a number of points on the interfaces ("connection points") between each beam line. These constraints are imposed through Lagrange multipliers. A comprehensive discussion about the extension of Lagrange multipliers to 1D refined CUF elements can be found in [24]. The final system that has to be solved is the following:

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{B}$ is the matrix containing the coefficients of the constraints imposed at the interfaces, $\boldsymbol{\lambda}$ is the vector of the Lagrange multipliers, and $\boldsymbol{F}$ is the vector of the generalised loads.

NUMERICAL RESULTS

A symmetric cross-ply $(0^o/90^o/0^o)$ laminated cantilever beam is considered. All the layers have the same thickness. The non-dimensional properties of the orthotropic material considered are: $E_L/E_T = 25$, $G_{LT}/G_{TT} = 2.5$, $\nu_{LT} = \nu_{TT} = 0.33$. L refers to the fiber direction, whereas T stands for a direction normal to the fiber. For the sake of convenience, the results are given in non-dimensional form:

$$\overline{u}_z = 100\,\frac{bh^3 E_T}{q_0 L^4}\,u_z, \qquad \overline{\sigma}_{ij} = \frac{\sigma_{ij}}{q_0}, \qquad \text{with } \; i,\, j = x,\, y,\, z$$

where L is the length of the beam, b and h are the dimensions of the rectangular cross-section, and q_0 is the intensity of the load uniformly distributed over the upper face of the beam. In the proposed example, the length-to-thickness ratio, L/h, is equal to 4.

In Table I the vertical displacement at the free end of the beam is given, together with $\overline{\sigma}_{yy}$ and $\overline{\sigma}_{zz}$ stress components at the clamped end and at the mid-span, respectively. In the last column of Table I, the number of the degrees of freedom is reported for each model considered. In the second row, a reference solution by a solid model by MSC/NASTRAN is given. Rows 3 and 4 show the results by classical beam models (EBBM and TBM). In rows 5 and 6, the results by a third-order and a sixth-order ESL TE models are shown. Row 7 shows the results obtained by including a zig-zag function in the $TE\,6$ model [25]. Finally, in rows 9 and 10 the results by the Multi-Line approach are given. For the structure considered, the Multi-Line models are obtained by modelling each layer with a higher-order beam and they are referred to as $ML\,\alpha/\beta/\gamma$, where α is the expansion order of the bottom layer, β is the expansion order of the central layer, and γ is the expansion order of the top layer.

Figure 3 shows the distributions of the normal and transverse stress components versus the z axis. The following comments arise from the analysis:

TABLE I. NON-DIMENSIONAL DISPLACEMENTS AND STRESSES. $\overline{u}_z$ AT $(\frac{b}{2}, L, \frac{h}{2})$, $\overline{\sigma}_{yy}$ AT $(\frac{b}{2}, 0, \frac{2}{15}h)$, AND $\overline{\sigma}_{zz}$ AT $(\frac{b}{2}, \frac{L}{2}, \frac{h}{2})$.

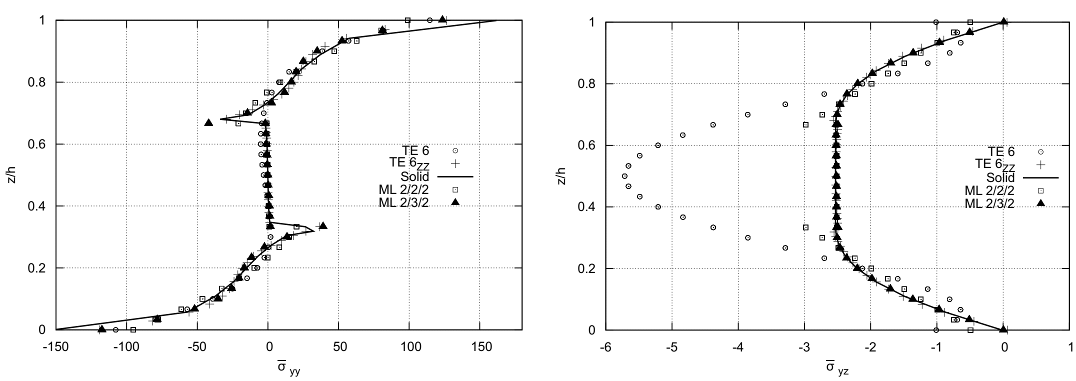
Model	$-\overline{u}_z$	$-\overline{\sigma}_{yy}$	$-\overline{\sigma}_{zz}$	DOFs
Solid [25]	17.97	30.58	1.028	103920
EBBT [25]	6.225	36.52	0.000	66
TBM [25]	14.02	36.52	0.000	110
$TE\,3$ [25]	16.76	28.39	1.044	660
$TE\,6$ [25]	17.14	25.29	0.995	1848
$TE\,6_{\mathrm{ZZ}}$ [25]	17.84	27.36	1.032	1914
Present Models				
$ML\,2/2/2$	17.70	32.70	1.004	1188
$ML\,3/2/3$	17.83	25.70	0.980	1716

1. As it is known from previous literature, classical theories cannot be used for the analysis of laminated structures.

2. ESL models, and in particular the $TE\,3$ and $TE\,6$ models, are not able to correctly detect transverse stress distributions.

3. Both $TE\,6_{ZZ}$ and ML models are able to deal with solid-like analysis.

4. The ML models have the best accuracy-to-DOFs ratio and they appears to the authors as the most convenient way to analyze composite laminated structures.



(a) (b)

Figure 3. Through-the-thickness distributions of $\overline{\sigma}_{yy}$ (a) and $\overline{\sigma}_{yz}$ (b) stress components.

CONCLUSIONS

This paper presents a Multi-Line approach based on higher-order theories to analyze composite structures. Refined 1D models were obtained using the Carrera Unified Formulation, CUF, which allows us to obtain any order theory in a hierarchical manner. Multi-Line approach represents a step forward to the analysis of laminated structure and it is able to deal with solid layer-wise solutions with a significant reduction of the computational costs.

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