# 1D multi-line refined models for the analysis of aerospace and civil engineering structures

Erasmo Carrera[1], Alfonso Pagani[1], Marco Petrolo[1]
[1]*Department of Mechanical and Aerospace Engineering, Politecnico di Torino, Italy*
*E-mail: erasmo.carrera@polito.it, alfonso.pagani@polito.it, marco.petrolo@polito.it*

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Finite element models are often composed by various sub-models over the problem domain based on various formulations. This approach can lead to reduced computational costs and acceptable accuracy losses. Locally refined 1D structural models are presented in this paper. Refined models are obtained through the Carrera Unified Formulation (CUF). CUF is a hierarchical formulation able to deal with any-order structural model with no need of ad hoc formulations. The order and the type of the expansions (e.g. Taylor, Lagrange, Legendre) are free parameters of the analysis. Local refinements are obtained through a multi-line approach. 1D refined finite elements with different order of expansion over the cross-sectional plane are employed in different regions of the domain of the structure. Lagrange multipliers are used to couple different order elements. Constraints are imposed on displacement variables at a number of points (connection points) whose location over the interface boundaries is a parameter of the method. Various structural problems are considered with particular attention paid to those problems where local refinements are of particular interest, such as locally loaded thin-walled structures or composite structures. The accuracy and the computational costs of the multi-line models are evaluated through comparisons with 2D and 3D models from commercial codes.

## 1 INTRODUCTION

This paper is devoted to the analysis of isotropic and laminated composite beams for aerospace and civil engineering applications. Beam models are widely used to analyze the mechanical behavior of slender bodies, such as columns, aircraft wings, towers and bridges, amongst others. The ease of application of one-dimensional (1D) theories and their computational efficiency are some of the main reasons why structural analysts prefer them to two-dimensional (2D) and three-dimensional (3D) models.

The classical and best-known beam theories are those by Euler [1] (hereinafter referred to as EBBM) and Timoshenko [2] (hereinafter referred to as TBM). These models yield reasonably good results when slender, solid section, homogeneous structures are subjected to flexure. Conversely, the analysis of deep, thin-walled, open section beams may require more sophisticated theories to achieve sufficiently accurate results (see [3]).

Over the last century, many refined beam theories have been proposed to overcome the limitation of classical beam modelling. Different approaches have been used to improve the beam models, which include the introduction of shear correction factors, the use of warping functions based on de Saint-Venant's solution, the variational asymptotic solution (VABS), the generalized beam theory (GBT), and others. Some selected references and noteworthy contributions are briefly discussed below.

Early investigators have focused on the use of appropriate shear corrections factors to increase the accuracy of classical 1D formulations, such as Timoshenko and Goodier [4], Sokolniko [5]. Another important class of refinement methods in the literature is based on the use of warping functions. The contributions by El Fatmi [6] and Ladevéze et al. [7] are some excellent examples.

Asymptotic type expansion in conjunction with variational methods has also been proposed; see for example Berdichevsky et al. [8], which also includes a commendable review of previous works on beam theory development.

GBT probably originated from the work of Schardt [9]. GBT improves classical theories by using piece-wise beam description of thin-walled sections.

Higher-order theories are generally obtained by using refined displacement fields of the beam cross-sections. Washizu [10] ascertained how the use of an arbitrarily chosen rich displacement fields can lead to closed form exact 3D solutions. However, when complex cross-sections are considered, the solution becomes increasingly inaccurate as the distance from the reference axis of the beam increases.

To overcome this limitation of higher-order models, Multi-Line (ML) [11, 12, 13] beam models are introduced in this paper. In the ML beam modelling approach, a slender body is discretized by means of multiple beam axes which are placed in different regions over the problem domain. 1D higher-order finite elements are developed within the framework of the Carrera Unified Formulation (CUF), which has been well established in the literature for over a decade [14, 15, 16]. CUF is a hierarchical formulation that considers the order of the model $N$ as a free-parameter (i.e. as input) of the analysis. In the present work, beam theories using CUF are obtained on the basis of Taylor-type expansions (TE). EBBM and TBM can be obtained as particular or special cases.

Different-order refined beam elements can be adopted for each beam-line in ML models. Then, once each beam axis has been discretized with 1D elements, Lagrange multipliers are used to impose constraints on displacement variables at a number of *connecting points* at the interface boundaries between each beam-line. The number of beam-lines and the order of the beam elements used to discretize each beam-line, as well as the number and the location of *connecting points* at the boundary interfaces, are all parameters of the ML model. In the case of laminated composite structures, ML modelling approach represents a step forward from the classical *equivalent single layer* (ESL) and *layer-wise* (LW) approaches since each layer (or group of layers) of the structure is modelled by one higher-order beam.

In the next section a brief overview of CUF is provided. Subsequently, the use of Lagrange multipliers for the development of ML models is described. Then, numerical results concerning reinforced thin-walled isotropic and composite structures are presented and the main conclusions are outlined.

## 2 1D REFINED ELEMENTS AND MULTI-LINE MODELS

The adopted rectangular cartesian coordinate system is shown in Figure 1, together with the geometry of the beam. The cross-sectional plane of the structure is denoted by $\Omega$, and the beam boundaries over $y$ are $0 \leq y \leq L$. Let us introduce the transposed displacement vector,

$$\mathbf{u}(x, y, z) = \left\{ \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\}^T \tag{1}$$

Within the framework of the CUF, the displacement field can be expressed as

$$\mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y), \qquad \tau = 1, 2, ...., M \tag{2}$$

where $F_\tau$ are the functions of the coordinates $x$ and $z$ on the cross-section. $\mathbf{u}_\tau$ is the vector of the generalized displacements, $M$ stands for the number of terms used in the expansion, and the repeated

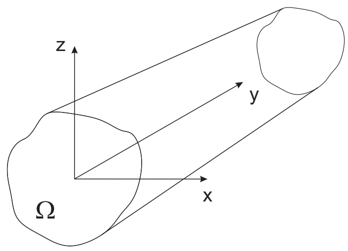



Figure 1: Coordinate frame of the beam model.

subscript, $\tau$ , indicates summation. The choice of $F_\tau$ determines the class of the 1D CUF model. The refined beam elements adopted in the present paper, consist of a Taylor series that uses the 2D polynomials $x^i\, z^j$ as base - where $i$ and $j$ are positive integers - and they are referred to be Taylor Expansion (TE) models. For instance, the displacement field of the second-order ($N = 2$) TE model can be expressed as

$$\begin{aligned} u_x &= u_{x_1} + x\, u_{x_2} + z\, u_{x_3} + x^2\, u_{x_4} + xz\, u_{x_5} + z^2\, u_{x_6} \\ u_y &= u_{y_1} + x\, u_{y_2} + z\, u_{y_3} + x^2\, u_{y_4} + xz\, u_{y_5} + z^2\, u_{y_6} \\ u_z &= u_{z_1} + x\, u_{z_2} + z\, u_{z_3} + x^2\, u_{z_4} + xz\, u_{z_5} + z^2\, u_{z_6} \end{aligned} \tag{3}$$

The order $N$ of the expansion is set as an input option of the analysis; the integer $N$ is arbitrary and defines the order the beam theory. Classical TBM and EBBM theories are also captured from the formulation as degenerate cases.

The FE approach was adopted to discretize the structure along the $y$-axis. This process is conducted via a classical finite element technique, where the displacement vector is given by

$$\mathbf{u}(x, y, z) = F_\tau(x, z) N_i(y) \mathbf{q}_{\tau i} \tag{4}$$

$N_i$ stands for the shape functions and $\mathbf{q}_{\tau i}$ for the nodal displacement vector. For the sake of brevity, the shape functions are not reported here. They can be found in many books, for instance in [17]. Elements with four nodes (B4) were adopted in this work, that is, a cubic approximation along the $y$ axis was assumed. The stiffness matrix of the elements and the external loadings vector were obtained via the *principle of virtual displacements*

$$\delta L_{\text{int}} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} \mathrm{d}\, V = \delta L_{\text{ext}} \tag{5}$$

where $L_{\text{int}}$ stands for the strain energy, $L_{\text{ext}}$ is the work of the external loadings and $\delta$ stands for the virtual variation. $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the vectors of the strains and stresses, respectively. For the sake of brevity, the fundamental nuclei of the stiffness matrix and the loading vector is not reported here. They can be found in [16], together with a more comprehensive discussion on refined 1D CUF models.

### 2.1 *The Multi-Line Approach*

In the present paper, Lagrange multipliers are used to implement ML models. In Fig. 2 a slender structure discretized by two different beam axes is shown. Higher-order elements of arbitrary order are placed on each beam-line, which separately describes a given sub-region of the whole structure.

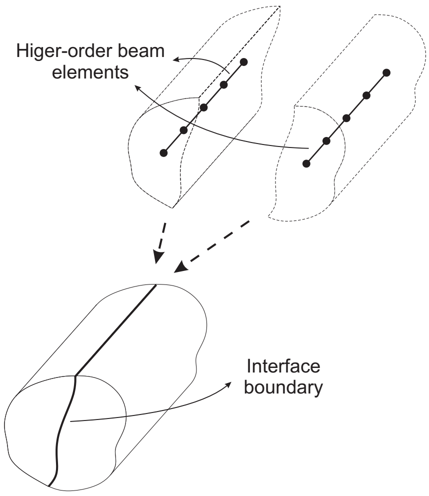



Figure 2: Multi-line approach

Lagrange multipliers are then used to impose compatibility on displacement variables at a number of *connecting points* at the interface boundary between beam-lines.

A comprehensive discussion about the extension of Lagrange multipliers to 1D refined CUF elements can be found in [18]. The final system that has to be solved is the following:

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{B}^T \\ \boldsymbol{B} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{B}$ is the matrix containing the coefficients of the constraints imposed at the interfaces, $\boldsymbol{\lambda}$ is the vector of the Lagrange multipliers, and $\boldsymbol{F}$ is the vector of the generalised loads.

## 3 NUMERICAL RESULTS

Two different structural problems were considered and the results are discussed below. In the first example, an I-shaped beam is addressed in order to highlight the advantages of the present ML approach when applied to the analysis of thin-walled structures. The results from the analysis of a cross-ply laminated beam are subsequently shown. The comparisons with the solutions from the literature and 3D solid model by MSC Nastran clearly show the capability of the present ML models to deal with LW results.

### 3.1 I-section beam

A cantilever beam with a I-shaped cross-section such as the one shown in Fig. 3 is considered first. It is assumed that the beam has a height $h = 100$ mm and a width $w = 96$ mm. The length to height ratio, $l/h$, is 10. The thickness of the flanges is $t_1 = 8$ mm, whereas the thickness of the web is $t_2 = 5$ mm. The material data are: elastic modulus $E = 200$ GPa and Poisson ratio, $\nu$, equal to $0.29$. A vertical force $F_z = -2 \times 10^3$ N is applied at point B (see Fig. 3) at the free end of the beam.

Table 1 shows the vertical displacements, $u_z$, at the tip of the beam, at points A and B. The

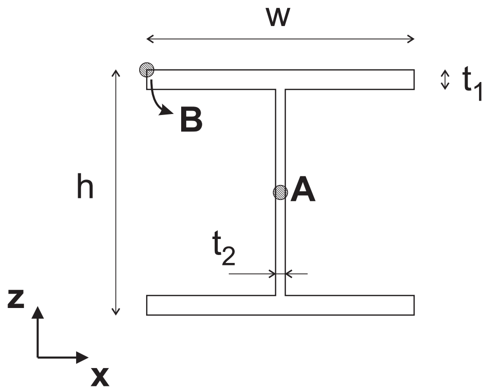



Figure 3: I-section beam geometry

number of the degrees of freedom (DOFs) is also given for each model in Table 1. The results are compared with 3D (Solid) and 2D (Shell) FEM solutions obtained using commercial code MSC Nastran. The analytical result achieved through Euler-Bernoulli beam theory is also given for comparison purposes, $u_{z_b} = \frac{F_z L^3}{3EI}$, where $I$ is the cross-section moment of inertia. The results by classical and refined CUF single-line models are shown in rows 6 to 15, where up to eight-order ($N = 8$) beam models are considered. The last rows of Table 1 give the results by the ML models of the I-section beam. The ML models of the I-section beam are referred to as $\text{ML}_{N_f, N_w}$, where $N_f$ is the expansion order of the beam elements placed on beam-lines of the flanges and $N_w$ is the expansion order of the beam elements placed on the beam-line of the web.

Based on these results, it is clear that classical and higher-order single-line models cannot detect the warping phenomena; lower-order elements can be effective when used in a ML approach; the number of the degrees of freedom of ML models is extremely reduced if compared to MSC Nastran and single-line refined models.

### 3.2 Symmetric laminated beam

The analysis of a symmetric cross-ply $[0\,°/90\,°/0\,°]$ laminated cantilever beam was carried out next. The three layers have the same thickness. The non-dimensional proprieties of the adopted orthotropic material are

$$E_L/E_T = 25 \qquad G_{LT}/G_{TT} = 2.5 \qquad \nu_{LT} = \nu_{TT} = 0.33$$

where $L$ refers to the direction of the fibers and $T$ stands for the direction normal to the fibers. For the sake of convenience, the results are given in non-dimensional form.

$$u_z^* = 100\frac{bh^3 E_T}{q_0 l^4} u_z, \qquad \sigma_{ij}^* = \frac{\sigma_{ij}}{q_0}, \qquad \text{with } i, j = x, y, z \tag{7}$$

here $l$ is the length of the beam, whereas $b$ and $h$ are the dimensions of the rectangular cross-section. $q_0$ is the intensity of the load uniformly distributed over the lower face of the beam. The beam is considered to be very short ($l/h = 4$), with clamped-free boundary conditions.

The ML scheme adopted is depicted in Fig. 4, together with the loading condition. The ML models addressed make use of three beam-lines. Two ML configurations are considered: (1) in the

| | | $-u_{z_A}$, mm | $-u_{z_B}$, mm | DOFs |
|---|---|---|---|---|
| $-u_{z_b} = \frac{F_z L^3}{3EI} = 0.951$, mm | | | | |
| MSC Nastran models | | | | |
| Solid | | 0.956 | 2.316 | 355800 |
| Shell | | 1.006 | 2.437 | 61000 |
| Classical and refined single-line models | | | | |
| EBBM | | 0.951 | 0.951 | 93 |
| $N = 2$ | | 0.956 | 0.978 | 558 |
| $N = 4$ | | 0.989 | 1.287 | 1395 |
| $N = 6$ | | 0.992 | 1.462 | 2604 |
| $N = 8$ | | 0.997 | 1.851 | 4185 |
| Present $\text{ML}_{N_f,N_w}$ | | | | |
| $N_f$ | $N_w$ | | | |
| 1 | 1 | 1.016 | 1.016 | 837 |
| 2 | 1 | 0.990 | 1.028 | 1116 |
| 2 | 2 | 0.951 | 1.940 | 1674 |
| 3 | 2 | 0.950 | 1.984 | 2046 |
| 3 | 3 | 0.952 | 2.186 | 2790 |
| 4 | 3 | 0.954 | 2.197 | 3255 |
| 4 | 4 | 0.952 | 2.230 | 4185 |

Table 1: Vertical displacement at points A and B on the free end of the cantilever I-section beam

first case a second-order ($N = 2$) expansion is used in the three layers; (2) in the second case a third-order ($N = 3$) expansion is used in the top/bottom layers, whereas a parabolic distribution on the cross-section is assumed in the central layer. ML models of the symmetric laminated beam are referred to as ML $\alpha/\beta/\gamma$, where $\alpha$ is the expansion order of the beam discretizing the bottom layer, $\beta$ is the expansion order of the central layer, and $\gamma$ is the expansion order of the top layer.

The results are shown in Table 2, where the non-dimensional vertical displacement at the tip is given, together with the number of degrees of freedom for each model. The results by ML models are compared to classical and refined ESL theories from [19] and to a solid FE model by MSC/Nastran.

Stress distributions shown in Fig. 5 confirm the advantages of the ML approach over one-line beam solutions. In fact, less DOFs than ESL cases are used (both with and without Zig-Zag func-

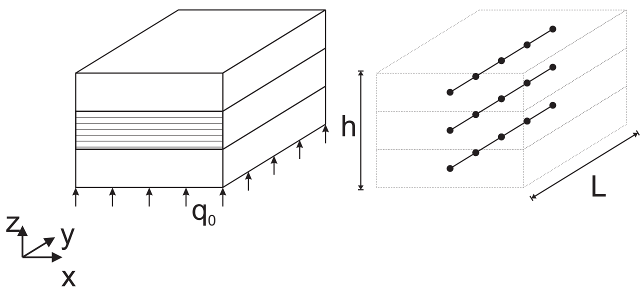
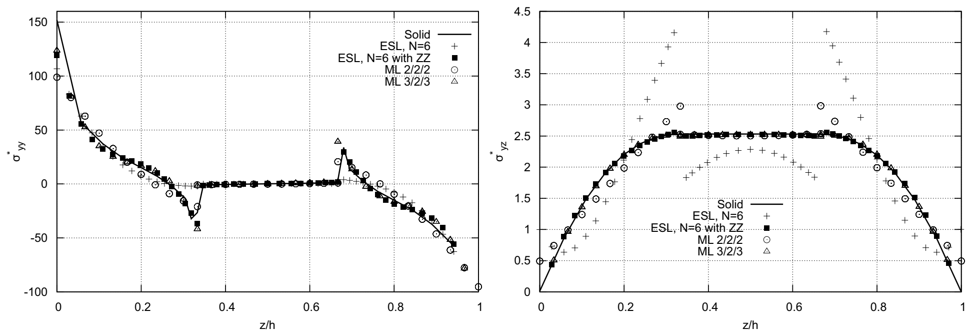



Figure 4: Symmetric laminated $[0/90/0]$ beam

| Model | $u_z^*$ | DOFs |
|---|---|---|
| Present ML | | |
| ML $2/2/2$ | $17.69$ | $1188$ |
| ML $3/2/3$ | $17.83$ | $1716$ |
| MSC Nastran, [19] | | |
| Solid | $17.98$ | $103920$ |
| Classical and higher-order ESL, [19] | | |
| $N = 6$ with ZZ | $17.84$ | $1914$ |
| $N = 6$ | $17.14$ | $1848$ |
| $N = 3$ with ZZ | $17.83$ | $726$ |
| $N = 3$ | $16.76$ | $660$ |
| TBM | $14.02$ | $110$ |
| EBBM | $6.22$ | $66$ |

Table 2: Non-dimensional displacement at the tip, symmetric cross-ply beam

tions) and LW solutions are obtained.




(a) $\sigma_{yy}$ at $y = 0$

(b) $\sigma_{yz}$ at $y = \frac{L}{2}$

Figure 5: Distribution of axial, $\sigma_{yy}^*$, and transverse, $\sigma_{yz}^*$, stresses for the symmetric laminated beam

## 4 CONCLUSIONS

A higher-order multi-line approach to the analysis of isotropic and composite slender bodies has been presented in this paper. Refined beam elements have been formulated using CUF, which allows for the formulation of any-order beam theories by setting the expansion order as input of the analysis. Different-order beam elements have been formulated and subsequently used over different beam-lines discretizing various structures. Beam-lines have then been coupled by imposing the compatibility of displacements at boundary interface using Lagrange multipliers. Although the efficiency of the present approach is problem dependent, Multi-line modelling undoubtedly represents a step forward in the analysis of thin-walled and laminated composite structures and it is able to deal with shell- and solid-like solutions with a significant reduction of the computational costs.

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