FREE VIBRATION ANALYSIS OF ROTATING STRUCTURES BY ONE-DIMENSIONAL, VARIABLE KINEMATIC THEORIES.

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ABSTRACT

In this paper, Carrera’s Unified Formulation (CUF) is extended to perform free-vibrational analyses of rotating structures. CUF is a hierarchical formulation which offers a procedure to obtain refined structural theories that account for variable kinematic description. These theories are obtained by expanding the unknown displacement variables over the beam section axes by adopting Taylor’s polynomials of N-order, in which N is a free parameter. The linear case (N=1) permits us to obtain classical beam theories while higher order expansions could lead to three-dimensional description of dynamic response of both rotors and centrifugally stiffened beams. The Finite Element method is used to derive the weak form of the three-dimensional differential equations of motion in term of fundamental nuclei, whose forms do not depend on the approximation used (N). The present formulations include gyroscopic effects and stiffening due to centrifugal stresses. In order to verify the accuracy of the new theories, several analyses are carried out and the results are compared with solutions presented in the literature in graphical and numerical form. The advantages of the variable kinematic models are evident especially when shafts with deformable discs and thin-walled rotating beams made up with composite materials are studied.

Introduction

Since shafts and blades constitute the fundamental parts of many machines, a thorough understanding of their dynamic features serves as a starting point for the study of fatigue effects, forced-response and flutter instability, which occur in airplane engines, helicopters and turbomachinery. The modelling of the rotating structures is usually made with beam elements to which some rigid discs may be connected. Many researchers adopted Euler-Bernoulli theory to investigate the vibrational behaviour of spinning shafts “[1-3]” and centrifugally stiffened beams “[4-7]” by adopting both analytical and numerical approaches to solve the equations of motion (EoM). Unfortunately, these solutions are inadequate for short and stubby bodies in which rotary inertia and shear deformations play an important role. In order to take into account these effects, in the open literature, there are many papers devoted to the development of theories based on the Timoshenko beam model “[8-13]”. Although this formulation introduces evident improvements, it is not able to detect some effects such as the torsion and the warping of the transversal cross-section. In addition, the design of advanced rotating structures has been strongly affected by the advent of composite material by introducing further difficulties in the modelling. Its high specific strength and stiffness combined with an ease formability allow one to produce more efficient and lighter thin-walled structures. For instance, in ”[14-16]” refined beam formulation are proposed in order to study the dynamic behaviour of shafts constituted by composite and functionally graded materials. As far as the ro-
tating blades are concerned, interesting theories are presented in "[17-19]". It was previously pointed out that, in the simplified models, the discs are generally considered rigid. This assumption is too restrictive when the disc is thin and, hence, highly deformable. The complex shapes of real rotors and blades have led the researchers to develop suitable theories for overcoming these problems. Among the several examples, in [20] and [21], the authors proposed that the displacements of a disc can be written as a superimposition of a rigid motion and a deflection relative to the rigid body configuration. In accordance with this assumption, the last contribution was approximated with a truncated Fourier series in the tangential direction whereas, for the radial direction, polynomial functions were used. In [22], the disc was modelled by using the Kirchhoff plate theory, while in contrast, in [23] and [24] the more computational expensive two- and three dimensional finite elements were used for studying the dynamic features of the spinning shafts. The dynamic of the rotating structures clearly represents a complex and interesting topic (see "[25], [26]"), but it seems that a reliable and general method for its complete analysis is not yet available. With the aim to overcome the limitations of ad hoc theories, in this paper Carrera Unified Formulation (CUF) is used for the study of the dynamics of the rotating structures [27], [28]. CUF provides a procedure to obtain refined structural models merely by enriching the displacement field components. CUF was first developed for plate and shell models [29], [30] and was later extended to the beam model. In the open literature, there are a considerable number of published papers in which the refined beam elements were tested [31], [32], but for a thorough and clear description, see [33].

The Kinetic and Potential Energies
Let us consider a structure that is free to rotate about its longitudinal or transverse axis, as shown in Fig.1. The absolute velocity of the point \( P \) is the sum of the relative velocity and the transfer velocity.

\[
v_{abs} = v_{rel} + vr = \dot{u} + \Omega \times r_{tot}\tag{1}\]

\[
\Omega = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & 0 \\ -\Omega_y & 0 & 0 \end{bmatrix}\tag{2}\]

where \( u = \{u_x, u_y, u_z\}^T \) is the displacement vector and \( r_{tot} = r + u \) is the distance of \( P \) from the neutral axis. The kinetic energy of the whole structure can be written as the following expression:

\[
T = \frac{1}{2} \int_V \rho (\dot{u}^T \dot{u} + 2\dot{u}^T \Omega T \dot{u} \Omega + 2u^T \Omega T \Omega u + 2u^T \Omega T \Omega r + r^T \Omega T \Omega r) dV \tag{3}\]

In the linear analysis, the potential energy of a rotating structure is given by the sum of the elastic term and a geometric contribution:

\[
U = \frac{1}{2} \int_V (\varepsilon_l^T C \varepsilon_l) dV + \int_V (\sigma_0^T \varepsilon_{nl}) dV \tag{4}\]

where \( C \) is the matrix of material coefficients and \( \varepsilon_l \) and \( \varepsilon_{nl} \) are the linear and nonlinear components of the strain field. Nevertheless in the following analyses, the geometric potential energy is disregarded when a spinning shaft is studied (\( \Omega = 0 \) in Eq.2), while in contrast, for a centrifugally stiffened structure the tension \( \sigma_0 \) is assumed to be:

\[
\sigma_0 = \Omega^2 \rho \left[ r_2 L + \frac{1}{2} L^2 - r_h y - \frac{1}{2} y^2 \right]\tag{5}\]

where \( L \) and \( r_h \) are the length of the beam and the dimension of the hub.

Carrera Unified Formulation
The CUF states that the displacement field, \( u(x,y,z,t) \), is an expansion of generic functions, \( F_\tau(x,z) \) for the vector displacement, \( u_\tau(y) \):

\[
u(x,y,z,t) = F_\tau(x,z) u_\tau(y,t) \quad \tau = 1,2,\ldots,T\tag{6}\]
where $T$ is the number of terms of the expansion and, in accordance to the generalized Einstein’s notation, $\tau$ indicates summation. In this work, Eq.6 consists of polynomials, that are functions of the coordinates of the cross-section. For example, the second-order displacement field is:

\[
\begin{align*}
  u_1 &= u_1 + x u_2 + z u_3 + x^2 u_4 + xz u_5 + z^2 u_6 \\
  u_2 &= u_2 + x u_3 + z u_4 + x^2 u_5 + xz u_6 + z^2 u_6 \\
  u_3 &= u_3 + x u_4 + z u_5 + x^2 u_6 + xz u_6 + z^2 u_6 \\
\end{align*}
\]

(7)

The classical beam theories are obtainable as particular cases of linear expansion. It should be noted that classical theories require reduced material stiffness coefficients to contrast Poisson’s locking. Unless otherwise specified, for classical and first-order models Poisson’s locking is corrected according to [33]. If a classical Finite Element technique is adopted with the purpose of easily dealing with arbitrary shaped cross-sections, the generalized displacement vector becomes:

\[
\begin{align*}
  \mathbf{u}_g(y) &= N_i(y) q_{i2}(t) \\
\end{align*}
\]

(8)

where $N_i(y)$ are the shape functions and $q_{i2}(t)$ is the nodal displacement vector:

\[
\begin{align*}
  q_{i2}(t) &= \{ q_{u_1}, q_{u_2}, q_{u_3} \}^T \\
\end{align*}
\]

(9)

**Equations of motion in CUF form**

The EoM are obtained by using the Hamilton Principle:

\[
\delta \int_{t_0}^{t_f} (T - U) \, dt = 0
\]

(10)

where $\delta$ is the virtual variation of the functional. The homogeneous EoM for the spinning shaft and for the rotating blade are:

\[
M\ddot{q} + G_{\Omega} \dot{q} + (K - K_{\Omega}) q = 0
\]

(11)

\[
M\ddot{q} + G_{\Omega} \dot{q} + (K - K_{\Omega} + K_{\Omega 0}) q = 0
\]

(12)

where the matrices in terms of “fundamental nuclei” are:

\[
M^{ij \Omega} = I_i^j \langle F_0 \rho F_3 \rangle
\]

\[
G^{ij \Omega} = I_i^j \langle F_0 \rho F_3 \rangle > 2\Omega
\]

\[
K^{ij \Omega} = I_i^j \langle F_0 \rho F_3 \rangle > \Omega^2 \Omega
\]

\[
K^{i j \Omega 0} = I_i^j \langle F_0 \rho F_3 \rangle > \Omega^2 \Omega
\]

(13)

where:

\[
I_{\Omega y} = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

\[(14)\]

\[
\int \left( \begin{array}{c}
I_i^j, I_i^{j y}, I_i^{j y y}, I_i^{j y y y}, I_i^{j y y y y}
\end{array} \right) \left( \begin{array}{c}
N_i, N_j, N_{i j}, N_{i y}, N_{i y y}, N_{i y y y}, N_{i y y y y}, N_{i y y y y y}
\end{array} \right) dy
\]

(15)

In addition to the Mass Matrix $M^{ij \Omega 0}$ and the Stiffness Matrix $K^{ij \Omega 0}$, the other three terms are:

the Coriolis Matrix $G^{ij \Omega 0}$

the Softening Matrix $K^{ij \Omega 0}$

the Stiffening Matrix $K^{i j \Omega 0}$

**Results and Discussion**

**Spinning Shafts**

Several illustrative examples are presented in which the boundary conditions and the ratio between the cross-section dimensions are assumed to be problem parameters. With the purpose of enabling a general application of results, they are presented in non-dimensional form by adopting the following non-dimensional natural frequency and spinning speed parameters:

\[
\omega^* = \frac{\omega}{\omega_0}, \quad \Omega^* = \frac{\Omega}{\omega_0}, \quad \omega_0 = \sqrt{\frac{EJ_0 L^4}{\rho AL^4}},
\]

(16)

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where \( J_{xx}, J_{zz} \) are the moments of inertia in the two principal planes, \( E \) is the Young’s Modulus and \( A \) the area of the cross-section. In the first example, the effects of shear deformation and rotary inertia are considered to be small and, for this reason, the study is conducted using the Euler-Bernoulli theory. The reference solutions are obtained by Eq.17, in which \( \omega_0 \) and \( \omega_r \) are the natural frequencies at standstill.

\[
\omega_{h,z} = \sqrt{\frac{1}{2} \left\{ -(\omega_0^2 + \omega_r^2 + 2\Omega^2) \pm \sqrt{(\omega_0^2 - \omega_r^2)^2 + 8\Omega^2(\omega_0^2 + \omega_r^2)} \right\}}
\]

Figure 2 shows how the dimensionless frequencies change with the rotational speed parameter for different aspect-ratios and, as can be seen, the results are in strong agreement with the reference solution.

In the following test case, a moderate thick cylindrical beam simply supported is analyzed. The length and the radius are assumed to be equal to 10m and 0.25m, respectively. The material has the Young’s modulus equal to 210 GPa and the density, \( \rho \), 7860 kg/m³. Figure 3 shows the dependency of angular frequencies (rad/sec) with the angular speed (rad/sec) when Euler-Bernoulli model (continuous line) and the forth-order expansion (empty dots) are used. It is evident that the refined model is able to detect a greater number of frequencies and related modal shapes than the classical beam theory, similarly to a 3-D model. As the \( Q \) increases above the critical speeds, the modes of whirling are stable until the point ‘C’. This point is the beginning of the self-excited range where the frequencies are complex conjugate and the rotor system is moving in an ellipse at the frequency \( \omega \). As can be seen, with the TE4 expansion, the instability region starts at a lower value of rotating speed. Anyway, as said before, in these analyses the stiffening contribution due to the centrifugal forces has not been considered despite of it becomes important when a deformable structure is rotating. For this reason in the future works, the stiffening term will be included so as to study correctly the deformable discs. Nevertheless, it must be highlighted that the possibility to study more complex spinning shafts with simple 1-D finite elements already appears a remarkable result.

### The Rotating Blade

**Euler-Bernoulli Beam Model.** A slender cantilever rotating beam is considered in order to evaluate its natural frequencies as function of rotating speed. The results are compared with those shown in [4] and [34] and presented in dimensionless form (see Eq.16):

\[
\omega_0 = \sqrt{\frac{\rho AL^4}{EJ_{xx}}} \quad \delta = \frac{r_h}{L} \quad S = \sqrt{\frac{AL^2}{J_{xx}}} \quad (18)
\]

With the purpose of ensuring the validity of the Euler-Bernoulli model assumptions, the parameter ‘S’ is assumed to be equal

---

**TABLE 1.** Forward Critical Speeds Predicted by CUF for the Test Case of a Simply Supported Shaft

<table>
<thead>
<tr>
<th>Mode Number</th>
<th>[10] Forward Speed</th>
<th>EBBM Forward Speed</th>
<th>FSDT Forward Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>63.869</td>
<td>63.699</td>
<td>63.550</td>
</tr>
<tr>
<td>2</td>
<td>253.769</td>
<td>255.001</td>
<td>252.449</td>
</tr>
<tr>
<td>3</td>
<td>564.741</td>
<td>573.379</td>
<td>564.299</td>
</tr>
<tr>
<td>4</td>
<td>989.050</td>
<td>1019.405</td>
<td>988.400</td>
</tr>
<tr>
<td>5</td>
<td>1516.866</td>
<td>1599.400</td>
<td>1519.259</td>
</tr>
</tbody>
</table>
to 70. Table 2 shows the first three flapwise frequencies for different boundary conditions (Clamped-Free, Clamped-Pinched and Pinched-Pinched) and hub-ratios, whereas Tab.3 shows the first dimensionless frequency in chordwise direction as function of rotating speed parameter and hub-ratio. The results strongly agree with the reference solutions in both cases.

In the following example, the structure has been taken from [25]. The dimensions of the beam are $10 \times 5 \times 200 \text{ mm}$ and it is made of aluminium ($E = 73.1 \text{ GPa}$, $\rho = 2770 \text{ kg/m}^3$ and $\nu = 0.33$). In [25], the results were obtained via finite element method by using brick elements and by adopting theoretical approaches. The variation of natural frequencies (Hz) as function of rotating speed (rad/sec) is shown in Fig.4. Also in this case, both solutions are in agreement for all values of rotating speed.

### Table 2. Dependency of $\omega^*_{1}$, $\omega^*_{2}$ and $\omega^*_{3}$ on the Variation of $\Omega^*$, Hub Dimension and Boundary Conditions for the Flapwise Motion.

<table>
<thead>
<tr>
<th>B.C.</th>
<th>$\omega^*$</th>
<th>Theory</th>
<th>$\Omega^* = 1$</th>
<th>$\Omega^* = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\delta = 0$</td>
<td>$\delta = 1$</td>
<td>$\delta = 2$</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>Present</td>
<td>10.862</td>
<td>12.483</td>
</tr>
<tr>
<td>2</td>
<td>C – F</td>
<td>Present</td>
<td>32.764</td>
<td>35.827</td>
</tr>
<tr>
<td>3</td>
<td>Present</td>
<td>Present</td>
<td>73.984</td>
<td>77.935</td>
</tr>
<tr>
<td>1</td>
<td>15.513</td>
<td>15.650</td>
<td>15.512</td>
<td>15.649</td>
</tr>
<tr>
<td>2</td>
<td>C – P</td>
<td>Present</td>
<td>60.906</td>
<td>64.382</td>
</tr>
<tr>
<td>3</td>
<td>Present</td>
<td>Present</td>
<td>116.99</td>
<td>121.30</td>
</tr>
<tr>
<td></td>
<td>[4] 104.42</td>
<td>104.62</td>
<td>104.42</td>
<td>104.62</td>
</tr>
<tr>
<td>1</td>
<td>10.022</td>
<td>10.264</td>
<td>19.684</td>
<td>22.078</td>
</tr>
<tr>
<td>2</td>
<td>P – P</td>
<td>Present</td>
<td>53.132</td>
<td>57.235</td>
</tr>
<tr>
<td>3</td>
<td>Present</td>
<td>Present</td>
<td>103.93</td>
<td>108.93</td>
</tr>
<tr>
<td></td>
<td>[4] 88.991</td>
<td>89.241</td>
<td>89.003</td>
<td>89.253</td>
</tr>
</tbody>
</table>

**First-order Shear Deformation Theory.** When the structure is no longer so thin ($S = 30$) as to consider the approximations of Euler-Bernoulli model valid, at least, the first order shear deformation theory is needed. The variation of the non-dimensional frequency of a cantilever Timoshenko beam with the rotating speed is shown in Tab.4 and it is compared with [35]. The agreement between the two sets of results is evident, in fact, the relative error remains below 1%.

CUF makes it possible to consider the non-linear term, $u_{y y}$, within the geometric stiffness matrix. Considering that the extensional modes and the chordwise modal shapes are coupled by the Coriolis matrix, this term may have significant influence on the natural frequencies, especially when the spinning speed is large. In or-
TABLE 3. DEPENDENCY OF \( \omega_1^* \) ON THE VARIATION OF \( \Omega^* \) AND HUB DIMENSION FOR THE CHORDWISE MOTION.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( \Omega^* ) Present [34]</th>
<th>Diff. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.6196 3.6173</td>
<td>0.06</td>
</tr>
<tr>
<td>0.1</td>
<td>7.3337 7.4537</td>
<td>1.63</td>
</tr>
<tr>
<td>0.2</td>
<td>4.3978 4.3960</td>
<td>0.04</td>
</tr>
<tr>
<td>0.1</td>
<td>13.048 13.047</td>
<td>0.01</td>
</tr>
<tr>
<td>0.1</td>
<td>41.227 41.346</td>
<td>0.28</td>
</tr>
<tr>
<td>0.5</td>
<td>2.6430 6.6421</td>
<td>0.01</td>
</tr>
<tr>
<td>0.5</td>
<td>27.266 27.276</td>
<td>0.03</td>
</tr>
<tr>
<td>0.5</td>
<td>74.003 74.178</td>
<td>0.23</td>
</tr>
</tbody>
</table>

TABLE 4. NON-DIMENSIONAL FUNDAMENTAL FREQUENCY OF A CANTILEVER TIMOSHENKO BEAM AS A FUNCTION OF THE ROTATING SPEED.

<table>
<thead>
<tr>
<th>( \Omega^* ) [35]</th>
<th>Present</th>
<th>Diff. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.4798 3.4831</td>
<td>0.09</td>
</tr>
<tr>
<td>1</td>
<td>3.6452 3.6494</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>4.0994 4.1064</td>
<td>0.17</td>
</tr>
<tr>
<td>3</td>
<td>4.7558 4.7667</td>
<td>0.22</td>
</tr>
<tr>
<td>5</td>
<td>6.5375 6.5530</td>
<td>0.27</td>
</tr>
<tr>
<td>5</td>
<td>6.3934 6.4144</td>
<td>0.32</td>
</tr>
<tr>
<td>6</td>
<td>7.2929 7.3205</td>
<td>0.37</td>
</tr>
<tr>
<td>7</td>
<td>8.6184 8.5538</td>
<td>0.43</td>
</tr>
<tr>
<td>8</td>
<td>8.2596 9.2039</td>
<td>0.48</td>
</tr>
<tr>
<td>9</td>
<td>10.109 10.164</td>
<td>0.54</td>
</tr>
</tbody>
</table>

FIGURE 5. THE VARIATION OF THE FIRST TWO FREQUENCY PARAMETERS OF A TIMOSHENKO ROTATING BEAM.

1. Where possible, the solutions obtained via CUF are compared with those found in [36], in which an exact power-series solution was adopted providing for the extensional deformation. Figure 5 shows the variations of the first two frequencies parameter with respect to the dimensionless rotating speed \( \Omega^* / S \). In accordance with [36], both terms appear very important in the analysis of dynamic of rotating beam. In general, the frequency values are lessened by the Coriolis force but, they increase when the axial non-linear contribution is included. Therefore, the error in the computation of natural frequencies of the blades may be considerable if any one of the both factors are not considered, especially at high rotating speed.

Higher-Order Theories. So far, the finite elements based on the classical theories have been tested by introducing suitable assumptions on the considered structures. The aim of this section is to evaluate the capabilities of the higher-order models and, to this end, a laminated thin-walled box is studied. The geometrical features of the structure are fixed and shown in Fig.6. Each edge is constituted by four layers with the same thickness \( t_e \). The length of the beam is equal to 4 m and the ratios \( L / h, h / b, b / t_e \) are assumed to be 6.667, 2 and 10, respectively. The material properties are:

\[
E_L = 144 GPa \quad E_T = 9.65 GPa \quad G_{LT} = 4.14 GPa \quad (19)
\]

\[
G_{TT} = 3.45 GPa \quad v_{LT} = v_{TT} = 0.3, \quad \rho = 1389 K g / m^3 \quad (20)
\]

To check the accuracy of the 1-D elements proposed in this study, the first ten natural frequencies at standstill are compared with those obtained with the shell element QUAD4 of MSC.
TABLE 5. FIRST TEN FREQUENCIES (Hz) OF THIN-WALLED CANTILEVER BOX.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode I</th>
<th>Mode II</th>
<th>Mode III</th>
<th>Mode IV</th>
<th>Mode V</th>
<th>Mode VI</th>
<th>Mode VII</th>
<th>Mode VIII</th>
<th>Mode IX</th>
<th>Mode X</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shell</td>
<td>35.447</td>
<td>58.533</td>
<td>76.011</td>
<td>114.23</td>
<td>130.12</td>
<td>143.15</td>
<td>148.54</td>
<td>163.65</td>
<td>165.24</td>
<td>170.34</td>
</tr>
<tr>
<td>TE4</td>
<td>38.345</td>
<td>60.171</td>
<td>89.811</td>
<td>144.93</td>
<td>851.27</td>
<td>1211.7</td>
<td>1215.8</td>
<td>-</td>
<td>292.80</td>
<td></td>
</tr>
<tr>
<td>TE7</td>
<td>36.849</td>
<td>59.290</td>
<td>85.739</td>
<td>134.58</td>
<td>223.95</td>
<td>177.53</td>
<td>191.41</td>
<td>220.26</td>
<td>203.65</td>
<td>259.63</td>
</tr>
<tr>
<td>TE11</td>
<td>36.413</td>
<td>59.043</td>
<td>81.861</td>
<td>126.82</td>
<td>158.22</td>
<td>156.38</td>
<td>164.12</td>
<td>183.16</td>
<td>179.62</td>
<td>214.26</td>
</tr>
</tbody>
</table>

(a): bending mode in x direction; (b): bending mode in z direction; (c): torsional mode; (d): shell-like mode; (e): axial mode.

FIGURE 6. CROSS-SECTION OF THE LAMINATED BEAM.

NASTRAN®, when the lamination scheme is assumed to be (0°/0°/0°/0°).

Despite the structure being very complex, the variable kinematic models are able to describe its dynamic behaviour quite accurately. Indeed, by enriching the displacement fields, the Taylor-like expansions used predict flexural, torsional, coupled and shell-like modes with increasing accuracy for all lamination schemes. Examples of shell-like modal shapes are shown in Fig.7.

In order to underline the effects of ply orientation on the dynamics of rotating structures, a number of analyses are carried out by using different displacement theories. Figures 8 and 9 show the variation of natural frequencies with respect to rotating speed (rad/sec) computed with the second, fourth and sixth order polynomial, when the angle of fibers is set to 0° or 90° for all layers. For both cases, the curves tend to lower values of frequencies when the model is enriched assuring the convergence to the correct solutions and, as expected, for the second lamination, the modes occur to lower values of frequencies.

In order to evaluate the influence of the lamination scheme, the evolutions of the first three frequencies with the rotating speed are considered and shown in Fig.10 for the following stacking sequences: (0°/0°/0°/0°) (Case A), (90°/90°/90°/90°) (Case B) and (45°/−45°/−45°/45°) (Case C). For all examples, the first two frequencies are related to the chordwise and flapwise motion, while in contrast, the third mode is torsional for the cases A and B and again flexural for the remaining stacking sequence. For this reason, the tendencies of the first two curves are quite similar for the whole speed interval. The main difference is observable considering the third modal shape, indeed, contrary to cases A and B, for the (45°/−45°/−45°/45°) lamination scheme, the third curve rapidly increases with the rotational speed.

As far as the damping of the structures is concerned, it is always null because all lamination schemes considered are symmetric. Indeed, when a generic non-symmetric stacking sequence is used, the damping value becomes a function of the rotating speed.
A possible explanation of this phenomena could be that the modal shapes involve both torsional and flexural deformations and, these modes can be excited or damped during the rotation. Naturally, only the refined models are able to identify the flexural/torsional coupling and in order to present a qualitative example, the damping of the first three modes of a non-symmetric laminated box is shown in Fig.11. In the future works, this aspect will be properly studied.

**Conclusion**

In the present paper, Carrera’s Unified Formulation has been developed and extended to the free vibration of rotating beams. The equations of motion were derived through Hamilton’s Principle and they have been solved with the Finite Element Method. Many rotating beams were considered and, in order to assess the new theory, the results were compared with those published in the literature, where possible. The analyses have concerned isotropic, composite and non-uniform beams as well as thin-walled structures. The hierarchical property of CUF has enabled refined displacement fields with Taylor-type expansions to be obtained. The equations are written in a fully three dimensional form. In the light of the results it is possible to make the following remarks:

- the use of finite elements based on the classical beam model leads to results that agree very well with reference solutions for all boundary conditions and cross-sections considered;
- the refined models detect new modes of deformation and their relative frequencies with good accuracy with respect to classical theories;
- in the case of laminated beams and thin-walled boxes, the refined elements have described the evolution of the natural frequencies and relating modes with increasing accuracy with respect to the classical theories, also yielding remark-
able results for shell-like deformations.

The research reported in this paper is expected to stimulate further research on dynamic behaviour of more complex rotating structural systems and, among these, the thin-walled shafts and discs appear of particular interest to the authors.

REFERENCES

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