Free Vibrations of Beams accounting for Refined Theories and Radial Basis Functions

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The free vibration response analysis of beams with arbitrary cross-sections and geometrical boundary conditions consists of one of the most classical problem of dynamics of continuum systems. Depending on the beam length as well as on the geometrical characteristics of the cross-section, classical beam theories neglecting (Euler-Bernoulli) and accounting (Timoshenko) for transverse shear deformation can lead to inaccurate results, especially in those cases in which vibration modes involving significant cross-section deformations are of interest. The use of refined theories, which are able to accurately detect in-plane deformations and warping, is therefore mandatory in many practical problems related to the high-frequency response analysis and thin-walled beams. In this work, one-dimensional refined theories are developed on the basis of the Unified Formulation (UF), which was recently introduced by the authors [1].
Closed-form exact solutions of governing equations related to free vibration analysis are available only for a few cases of boundary conditions. Therefore, to obtain the solutions for a wide range of problems, adequate approximated/numerical methods have to be developed, see [2]. Numerical methods can be classified as meshless and mesh-based methods. Among the latter, the finite element method (FEM) is probably the most popular. FEM, as well as any other method introducing a finite discretization of the problem domain, reveals by definition deficiencies in the evaluation of the high-frequency response, which plays a fundamental role in many practical problems such as wave propagation and vibro-acoustics. The use of meshless methods is mandatory in these cases. Among the many available meshless methods, the radial basis functions (RBF) method is considered in this work to investigate the free vibration characteristics of beams by accounting for refined theories developed on the basis of UF. In the recent past, authors applied RBF and UF to plate and shell problems and encouraging results were obtained [3]. A few details on the technique adopted are outlined below.

Beam governing equations based on UF. Within the framework of the UF, the displacement field of a *N-order* beam theory can be expressed as

$$\mathbf{u}(x, y, z; t) = F_\tau(x, z)\mathbf{u}_\tau(y; t), \qquad \tau = 1, 2, ..., M \tag{1}$$

where **u** is the displacement vector, xyz is a rectangular coordinate system with y lying along the beam axis, and F_τ are the functions of the coordinates x and z on the cross-section plane of the beam, which is denoted as Ω. $\mathbf{u}_\tau$ is the vector of the *generalized* displacements, M stands for the number of terms used in the expansion, and the repeated subscript, τ,

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indicates summation. The principle of virtual displacements is used to derive the equations of motion.

$$\delta L_{\text{int}} + \delta L_{\text{ine}} = 0, \qquad \text{with } \delta L_{\text{int}} = \int_V \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma}\, \mathrm{d}V \tag{2}$$

where L_{int} stands for the strain energy and δL_{ine} is the work done by the inertial loadings. δ stands for the usual virtual variation operator and V is the volume of the beam. $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are the strain and stress vectors respectively. The virtual variation of the strain energy is rewritten using strain - displacement relations, constitutive laws, and Eq. (1). After integrations by part, Eq. (2) becomes

$$\delta L_{\text{int}} = \int_L \delta \mathbf{u}_\tau^T \mathbf{K}^{\tau s} \mathbf{u}_s \,\mathrm{d}y + \left[\delta \mathbf{u}_\tau^T \boldsymbol{\Pi}^{\tau s} \mathbf{u}_s\right]_{y=0}^{y=L} \tag{3}$$

where $\mathbf{K}^{\tau s}$ is the differential linear stiffness matrix and $\boldsymbol{\Pi}^{\tau s}$ is the matrix of the natural boundary conditions in the form of 3×3 fundamental nuclei. The virtual variation of the inertial work is given by $\delta L_{\text{ine}} = \int_V \delta \mathbf{u} \rho \ddot{\mathbf{u}}\, \mathrm{d}V$. The explicit form of the governing equations was obtained in a recent work [4]. In the following, the component along the x-axis of the equations of motion of the refined beam in free vibration is reported

$$\begin{aligned} \delta u_{x\tau}: \quad & -E^{66}_{\tau s} u_{xs,yy} + \left(E^{26}_{\tau_{,x} s} - E^{26}_{\tau s_{,x}}\right) u_{xs,y} + \left(E^{22}_{\tau_{,x} s_{,x}} + E^{44}_{\tau_{,z} s_{,z}}\right) u_{xs} \\ & -E^{36}_{\tau s} u_{ys,yy} + \left(E^{23}_{\tau_{,x} s} - E^{66}_{\tau s_{,x}}\right) u_{ys,y} + \left(E^{26}_{\tau_{,x} s_{,x}} + E^{45}_{\tau_{,z} s_{,z}}\right) u_{ys} \\ & + \left(E^{45}_{\tau_{,z} s} - E^{16}_{\tau s_{,z}}\right) u_{zs,y} + \left(E^{44}_{\tau_{,z} s_{,x}} + E^{12}_{\tau_{,x} s_{,z}}\right) u_{zs} = -E^{\rho}_{\tau s} \ddot{u}_{xs} \end{aligned} \tag{4}$$

where the generic term $E^{\alpha\beta}_{\tau_{,\theta} s_{,\zeta}} = \int_\Omega \tilde{C}_{\alpha\beta} F_{\tau_{,\theta}} F_{s_{,\zeta}}\, \mathrm{d}\Omega$ is a cross-sectional moment parameter. Double over dots stand as second derivative with respect to time (t). Letting $\mathbf{P}_\tau = \left\{ \begin{array}{ccc} P_{x\tau} & P_{y\tau} & P_{z\tau} \end{array} \right\}^T$ to be the vector of the generalized forces, the component along the x-axis of natural boundary conditions is

$$\delta u_{x\tau}: \quad P_{xs} = E^{66}_{\tau s} u_{xs,y} + E^{26}_{\tau s_{,x}} u_{xs} + E^{36}_{\tau s} u_{ys,y} + E^{66}_{\tau s_{,x}} u_{ys} + E^{16}_{\tau s_{,z}} u_{zs} \tag{5}$$

The complete form of governing equations and natural boundary conditions, as well as the components of the fundamental nuclei can be found in [4].

Radial basis functions for one-dimensional refined theories. The solution is assumed to be harmonic, $\mathbf{u}_\tau(y;t) = \mathbf{U}_\tau(y) e^{i\omega t}$. The radial basis function (ϕ) approximation of the generalised displacement field can be defined as

$$\mathbf{U}_\tau(y) = \sum_{i=1}^{N_c} \alpha_i \phi(\|y - y_i\|_2), \qquad 0 \leq y \leq L \tag{6}$$

where y_i, $i = 1, ..., N_c$ is a finite set of distinct points (centers) on the beam axis whose length is referred to as L. The coefficients α_i are chosen so that $\mathbf{u}_\tau$ satisfies some variationally-consistent boundary conditions. Some examples of RBFs are given below

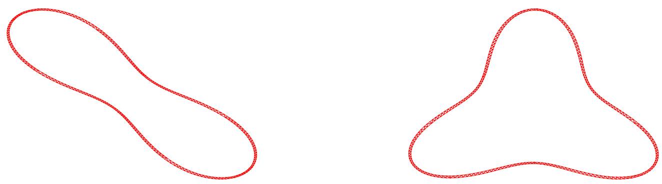
$$\begin{array}{ll} \phi(r) = r^3, & \textit{cubic} \\ \phi(r) = e^{-cr^2}, & \textit{Gaussian} \\ \phi(r) = \sqrt{c^2 + r^2}, & \textit{multiquadrics} \end{array} \tag{7}$$

By substituting Eq. (6) into the governing equations and by appliying the boundary conditions at the collocations points that lie at the beam ends, the following eigenproblem can be formulated

$$\left[\mathcal{L} - \omega^2 \mathcal{G}\right] \mathbf{X} = \mathbf{0} \tag{8}$$

with $\mathcal{L}$ collecting stiffness terms and $\mathcal{G}$ collecting inertial terms. In Eq. (8), $\mathbf{X}$ are the modal shapes associated with the natural frequencies which are referred to as ω.

Preliminary results Several results concerning free vibration analysis of beam-like structures, obtained through the combination of UF and FEM, are available in the literature. Among those examples, the free vibration characteristics of a thin-walled cylinder are described in this section. As shown in [1], lower-order beam models are able to detect 'global' modes such as bending and torsional ones. However, it was shown that higher-order beam theories are necessary when local *shell-like* modes involving the presence of lobes along the circumferential direction of the cylinder have to be characterized. Figures 1a and 1b show a two- and three-lobe mode of the cylinder, respectively. The frequency values of the first three-lobe mode are given in Table 1 for both classical and refined beam theories and the results are compared to shell and solid FEM solutions. In Table 1, N is the expansion-order of the refined beam model. More insight on the capability of the present higher-order beam models of dealing with shell- and solid-like solutions will be given during the presentation.



(a) Two lobes. (b) Three lobes.

Figure 1: Circumferential natural modes, 2D view, [1].

Theory	DOFs	f [Hz]
Euler-Bernoulli	93	–
Timoshenko	155	–
N = 2	558	–
N = 4	1395	75.690
N = 6	2604	52.386
N = 8	4185	40.102
Shell	49500	40.427
Solid	174000	46.444

Table 1: First three-lobe frequency [1].

The use of RBF method as opposed to FEM, in conjunction with UF, represents a powerful tool for the high-frequency response analysis of both solid and thin-walled structures and the results will be disclosed during the 9^{th} International Symposium on Vibrations of Continuous Systems.

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