Advanced layer-wise shells theories based on trigonometric functions expansion

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Advanced layer-wise shells theories based on trigonometric functions expansions are considered to evaluate the static behavior of multi-layered, orthotropic shells. The aim of the present work is to extend the basis functions used for layer-wise formulation to a trigonometric basis functions properly defined. Carrera Unified Formulation for the modeling of composite shell structures is adopted. Via this approach, higher order, zig-zag, layer-wise and mixed theories can be easily formulated. The governing differential equations of the problem are presented in a compact general form. These equations are solved via a Navier-type, closed form solution. As assessment, results are compared with available exact solutions present in literature.

Nomenclature

A_k, B_k	=	coefficients of the first fundamental form of Ω_k
α, β	=	shell in-plane coordinates
$d\Omega_k$	=	area of an infinitesimal rectangle on Ω_k
ds^2_k	=	square of line element
dV	=	infinitesimal volume
F_τ	=	generic thickness function
Γ_k	=	boundary of Ω_k
h_k	=	thickness of the k-layer
k	=	integer indicating layer number starting from the shell-bottom
N	=	expansion order
N_l	=	number of layers of the laminate
Ω_k	=	k-layer reference shell surface
$\boldsymbol{p}$	=	pressure loading vector
R^k_α ,R^k_β	=	radii of curvature along α and β directions respectively
$\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$	=	stress and strain vectors
$\boldsymbol{u}$	=	displacements vector
u_α ,u_β ,u_z	=	components of the displacements vector along α, β and z directions respectively
ζ_k	=	non dimensioned layer coordinate
z	=	shell thickness coordinate

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I. Introduction

Classical theories (Classical Lamination Theory, CLT) developed for thin elastic shells are based on Love-Kirchhoff's assumptions[1]. Nowadays curved shell structures made of composite laminae have gained widespread acceptance for primary structural components due to high value of strength- and stiffness-to-weight ratios. Love-Kirchhoff's kinematic assumptions applied to layered anisotropic composite shells may not yield a correct prediction of displacements and stresses fields. These materials exhibit high transverse shear deformation and discontinuous material properties in the thickness direction.
These features require the development of refined theories [2,3].
According to published research, various theories in mechanics for composite structures have been developed. They can be classified as:

- Equivalent Single Layer (ESL): the number of unknowns is independent from the number of layers, but the shear stress continuity on the interfaces of layers is often violated.
- Layer-wise approach (LW): this theory aims at overcoming the restriction of the ESL about the discontinuity of in-plane displacement on the interface layers.

A review of equivalent-single-layer and layer-wise laminate theories is presented by Reddy [4]. Concerning trigonometric theories for structural analysis the following literature is found.
Shimpi and Ghugal [5] have used trigonometric terms in the displacement field for the analysis of two layers composite plate. An ESL model is developed by Arya et al.[6] using a sine term to represent the non-linear displacement field across the thickness in symmetric laminated beams. An extension of Ref. 6 to composite plates is presented by Ferreira et al.[7]. A trigonometric shear deformation theory is used to model symmetric composite plates discretized by a meshless method based on global multiquadric radial basis functions. A specialized version of this theory with a layer-wise approach is proposed by the same authors in Ref. 8. Vidal and Polit [9] designed a new three-noded beam finite element for the analysis of laminated beams, based on a sinus distribution with layer refinement. A recent work from the same authors[10] deals with the influence of the Murakami's zig-zag function in the sine model for static and vibration analysis of laminated beams. Static and free vibration analysis of laminated shells is performed by radial basis functions collocation, according to a sinusoidal shear deformation theory in Ferreira et al.[11]. It accounts for through-the-thickness deformation, by considering a sinusoidal evolution of all displacements with the thickness coordinate.

Over the last decade the second author has developed a Unified Formulation[12] (CUF) that allows formulating several two-dimensional models on the basis of the choice of the a-priori main unknowns (displacements or mixed models), the approximation level (laminate or lamina level) and the through-the-thickness polynomial approximation order. As a result, an exhaustive variable kinematic model has been obtained: models that account for the transverse normal and shear deformability, the continuity of the transverse stress components and the zig-zag variation along the thickness of displacements and transverse normal stresses can be formulated straightforwardly.
In the framework of the Unified Formulation, to *postulate* the displacements distribution, we present a new set of trigonometric functions of the shell thickness coordinate z. This approach is adopted in both case: ESL and LW models. In the next subsections we present the unified and compact form of resulting equations.

II. Geometry

Shells are bi-dimensional structures with one dimension, in general the thickness along z direction, negligible with respect to the others two on the reference surface directions. The main features of shell geometry are shown in Fig. 1. A laminated shell composed of N_l layers is considered. The integer k, used as superscript or subscript, indicates each layer starting from the shell bottom. The layer geometry is denoted by the same symbols as those used for the whole multilayered shell and vice-versa. α_k and β_k are the curvilinear orthogonal co-ordinates (coinciding with lines of principal curvature) on the layer reference surface k (middle surface of the k-layer). z_k denotes the rectilinear co-ordinate measured along the normal direction to k. The following relations hold in the orthogonal system of coordinates above described:

$$\begin{aligned} ds_k^2 &= H_\alpha^k d\alpha^2 + H_\beta^k d\beta^2 + H_z^k dz^2 \\ d\Omega_k &= H_\alpha^k H_\beta^k d\alpha_k d\beta_k \\ dV &= H_\alpha^k H_\beta^k H_z^k d\alpha_k d\beta_k dz_k \end{aligned} \tag{1}$$

where ds_k^2 is the square of line element, $d\Omega_k$ is the area of an infinitesimal rectangle on Ω_k and dV is an infinitesimal volume. Here:

$$\begin{aligned} H_\alpha^k &= A^k(1+z_k/R_\alpha^k) \\ H_\beta^k &= B^k(1+z_k/R_\beta^k) \\ H_z^k &= 1 \end{aligned} \tag{2}$$

R_α^k and R_β^k are the radii of curvature along the two in plane directions α_k and β_k respectively. A_k and B_k are the coefficients of the first fundamental form of Ω_k. For shells with constant curvature these coefficients are equal to unity.

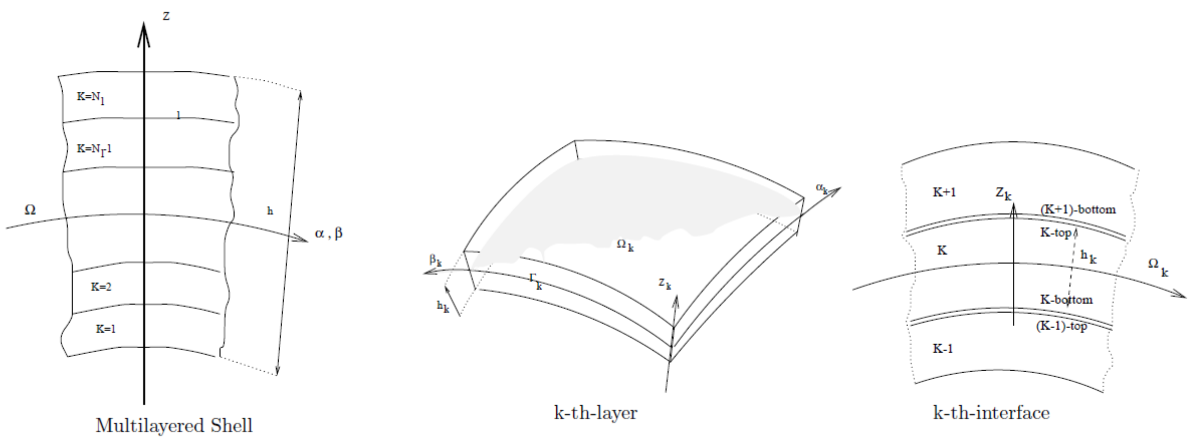

Figure 1. Geometry and notation used for multilayered shell

III. Carrera Unified Formulation

Carrera's Unified Formulation (CUF) permits several two dimensional models to be obtained for shells, thanks to the separation of the unknown variables into a set of thickness functions only depending on the thickness coordinate z, and the correspondent unknowns depending on the in-plane coordinates (α, β). In force of that various shell theories can be unified considering that CLT and FSDT (First-Order Shear Deformation Theory) are a peculiar case of ESL higher order models. These latter models can be regarded as a particular case of Layer-Wise (LW) models in which the number of layers is equal to one and the through-the-thickness polynomial approximation is performed via the classical base $\{z^\tau : \tau = 0, 1, ..., N\}$. In case only displacement assumption are introduced, the following expansion in the thickness coordinate z can be written:

$$\boldsymbol{u}^k = F_\tau(z)\boldsymbol{u}_\tau^k(\alpha,\beta) \qquad k = 1,2,\ldots,N_l \tag{3}$$

where:

- $\boldsymbol{u}=(u_\alpha , u_\beta , u_z)$ are the three displacement components of the generic shell point $P(\alpha, \beta, z)$ measured in a cartesian reference system.
- $\boldsymbol{u}_\tau=(u_{\tau\alpha} , u_{\tau\beta} , u_{\tau z})$ are the introduced displacement variables of $P_\Omega(\alpha, \beta, z)$ which lie on reference shell surface Ω.
- F_τ are the introduced functions of the thickness coordinates z.
- N_l is the number of layers of the laminate

Einstein convention for repeated indexes is referred to. The order of the expansion as well as the choice of the base functions used to build the thickness function F_τ is completely free.

The governing equations are derived according to the chosen variational statement (PVD approach) in a general way that does not depend upon the variable description (ESL or LW) and the expansion order.

IV. Overview of the considered Shell Theories

Since a large variety of two-dimensional theories can be formulated on the basis of different kinematic assumptions, it may be useful to recall some details about shell's theories.

The introduced assumptions for displacements (Eq. (3)) can be made at layer or at multilayer level. Layer-Wise (LW) description is obtained in the first case whereas Equivalent Single Layer (ESL) description is acquired in the latter one. If LW description is employed than $\boldsymbol{u}_\tau$ are layer variables. These are different in each layer. If ESL description is referred to then $\boldsymbol{u}_\tau$ are shell variables. These are the same for the whole multilayer. Examples of ESL and LW assumption are given in Fig. 2.

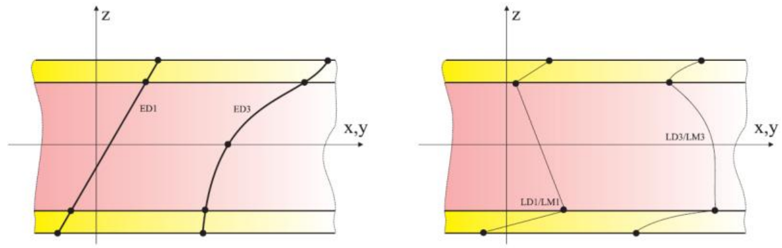

Figure 2. Examples of ESL (left) and LW (right) assumptions. Linear and cubic cases

A. Equivalent single layer models

1. ESL with Lagrange polynomial basis functions

Higher Order Theories (HOTs) can be formulated adopting the following expansion for displacements variables $\boldsymbol{u}=\{u_\alpha, u_\beta, u_z\}$:

$$\boldsymbol{u}(\alpha,\beta,z) = z^\tau \boldsymbol{u}_\tau \qquad \tau = 0,1,2,\ldots,N \tag{4}$$

A N-order theory based upon Eq. (4) is addressed as 'EDN'. Letter 'E' denotes that the kinematic is preserved for the whole layers of the shell, as in Equivalent Single Layer (ESL) approach. 'D' indicates that only displacement unknowns are used. 'N' stands for the expansion order of the through-the thickness polynomial approximation. For instance, the displacement field of an ED3 model is:

$$\begin{aligned} u_\alpha &= u_{\alpha 0} + z\,u_{\alpha 1} + z^2 u_{\alpha 2} + z^3 u_{\alpha 3} \\ u_\beta &= u_{\beta 0} + z\,u_{\beta 1} + z^2 u_{\beta 2} + z^3 u_{\beta 3} \\ u_z &= u_{z0} + z\,u_{z1} + z^2 u_{z2} + z^3 u_{z3} \end{aligned} \tag{5}$$

This theory accounts for a parabolic and cubic variation along the thickness of transverse normal and shear strains, respectively.

2. ESL with trigonometric basis functions

In the present work we extended the basis functions used for the equivalent single layer formulation to the trigonometric basis functions defined as:

$$\{1, z, \cos(\pi z/h), \sin(\pi z/h), \cos(2\pi z/h), \sin(2\pi z/h), \cos(3\pi z/h), \ldots, \cos(n\pi z/h), \sin(n\pi z/h)\} \tag{6}$$

then the displacements field can be written as:

$$\begin{aligned} &\boldsymbol{u}(\alpha,\beta,z) = \boldsymbol{u}_0 + z\boldsymbol{u}_1 + \cos(\pi z/h)\boldsymbol{u}_2 + \sin(\pi z/h)\boldsymbol{u}_3 + \cos(2\pi z/h)\boldsymbol{u}_4 + \\ &+ \sin(2\pi z/h)\boldsymbol{u}_5 + \ldots + \cos(N\pi z/2h)\boldsymbol{u}_N + \sin(N\pi z/2h)\boldsymbol{u}_{N+1} \end{aligned} \tag{7}$$

We mention this theory with the acronym EDTN, where 'T' refers to the trigonometric basis functions adopted to write the assumed displacements field. The kinematics of the proposed theory (Eq. (7)) is useful, because if the trigonometric term (involving thickness coordinate z) is expanded in power series, the kinematics of higher order theories (which are usually obtained by power series in thickness coordinate z) are implicitly taken into account.

B. Layer-Wise theory

3. Legendre expansion

The ESL theories mentioned above have the number of unknown variables that are independent from the number of constitutive layers N_l. If detailed response of individual layers is required and if significant variations in

displacements gradients between layers exist, as it is the case of local phenomena description, a layer-wise approach is necessary. That is, each layer is seen as an independent plate/shell and compatibility of displacement components with correspondence to each interface is then imposed as a constraint.
Legendre Polynomials base function is usually adopted. The following expansion is, therefore, adopted:

$$
\begin{aligned}
u_\alpha^k &= F_t u_{\alpha t}^k + F_b u_{\alpha b}^k + F_r u_{\alpha r}^k \\
u_\beta^k &= F_t u_{\beta t}^k + F_b u_{\beta b}^k + F_r u_{\beta r}^k \qquad r = 2,3,..,N \\
u_z^k &= F_t u_{z t}^k + F_b u_{z b}^k + F_r u_{z r}^k
\end{aligned}
\tag{8}
$$

where $k=1, 2, \ldots, N_l$ counts the laminae and N_l represents the total number of layers. Subscripts t and b denote values evaluated at top and bottom surface of a k-layer, respectively. The thickness functions F_t , F_b , and F_r, depend on the non dimensional thickness coordinate $\zeta_k = 2z_k/h_k$. ζ_k is defined such that $-1 \leq \zeta_k \leq 1$, being z_k a k-layer local coordinate and h_k stands for the thickness of a k-layer. F_t , F_b , and F_r are defined as follows:

$$
F_t = \frac{P_0 + P_1}{2} \quad F_b = \frac{P_0 - P_1}{2} \quad F_r = P_r - P_{r-2}
\tag{9}
$$

$P_j = P_j(\zeta_k)$ is a Legendre's polynomials of order j. The first four Legendre's polynomials are:

$$
\begin{aligned}
&P_0 = 1 \quad P_1 = \zeta_k \quad P_2 = \frac{3\zeta_k^2 - 1}{2} \\
&P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2} \quad P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8}
\end{aligned}
\tag{10}
$$

The following properties hold:

$$
\begin{aligned}
\zeta_k = 1: &\quad F_t = 1, F_b = 0, F_r = 0 \\
\zeta_k = -1: &\quad F_t = 0, F_b = 1, F_r = 0
\end{aligned}
\tag{11}
$$

Top and bottom displacements of each lamina are assumed as unknown variable. Interlaminar compatibility of displacements can be easily linked:

$$
\begin{aligned}
u_{\alpha t}^k &= u_{\alpha b}^{(k+1)} \\
u_{\beta t}^k &= u_{\beta b}^{(k+1)} \qquad k = 1,2,\ldots,N_l - 1 \\
u_{z t}^k &= u_{z b}^{(k+1)}
\end{aligned}
\tag{12}
$$

The acronym used for these theories is ‘LDN’, where ‘L’ stands for the LW approach.

4. *Trigonometric functions expansion*

In the framework of trigonometric functions expansions, the main purpose of the paper is to present and validate a new set of thickness functions for layer-wise approach. We use functions containing sine and cosine terms instead of Legendre Polynomials. This new set of trigonometric functions meets conditions (11). The thickness functions F_t and F_b are defined as previously:

$$
F_t = \frac{P_0 + P_1}{2} \quad F_b = \frac{P_0 - P_1}{2}
\tag{13}
$$

whereas F_r , $r=2,3,\ldots N$, are redefined in the following way:

$$
\begin{aligned}
&F_2 = \cos\left(\frac{\pi}{2}\zeta_k\right), F_3 = \sin\left(\frac{\pi}{2}\zeta_k\right) - \zeta_k, F_4 = \cos(\pi\zeta_k) + 1, F_5 = \sin(\pi\zeta_k), F_6 = \cos\left(\frac{3}{2}\pi\zeta_k\right), \\
&F_7 = \sin\left(\frac{3}{2}\pi\zeta_k\right) + \zeta_k, F_8 = \cos(2\pi\zeta_k) - 1, F_9 = \sin(2\pi\zeta_k), F_{10} = \cos\left(\frac{5}{2}\pi\zeta_k\right), F_{11} = \sin\left(\frac{5}{2}\pi\zeta_k\right) - \zeta_k
\end{aligned}
\tag{14}
$$

The maximum expansion order introduced is 11. Besides to the sine and cosine functions, in some case we added a unit or a linear term, so as to satisfy the conditions (11) seen before. The displacements expansion (8) is still valid, but F_r are now defined by Eqs. (14). We refer to this theories with the acronym 'LDTN', where 'T' stands for trigonometric thickness functions basis.
For all the models that have been previously described, the governing equations and the boundary conditions are derived via the Principle of Virtual Displacement (PVD) in Sec. V.
Carrera Unified Formulation (CUF) is employed to derive shell equations that are solved for the case of simply supported boundary conditions and doubly curved shells with constant curvatures. Navier-type closed form solution are obtained. Results for multi-layer cross-ply shells are presented to validate the proposed theory, and further are compared with exact solutions present in literature.

V. Governing equations

C. Equations for the N_l layers

The displacement approach is formulated in terms of $\boldsymbol{u}_k$ by variational imposing the equilibrium via PVD:

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{h_k} (\delta\boldsymbol{\varepsilon}_p^{kT} \boldsymbol{\sigma}_p^k + \delta\boldsymbol{\varepsilon}_n^{kT} \boldsymbol{\sigma}_n^k)\, dz_k \, d\Omega_k = \sum_{k=1}^{N_l} \int_{\Sigma_k^b \cup \Sigma_k^t} \delta\boldsymbol{u}^{kT} \boldsymbol{p}^k d\Omega_k \tag{15}$$

where

$$\boldsymbol{\varepsilon}_p = \begin{Bmatrix} \varepsilon_{\alpha\alpha} \\ \varepsilon_{\beta\beta} \\ \varepsilon_{\alpha\beta} \end{Bmatrix}, \quad \boldsymbol{\varepsilon}_n = \begin{Bmatrix} \varepsilon_{\alpha z} \\ \varepsilon_{\beta z} \\ \varepsilon_{zz} \end{Bmatrix}, \quad \boldsymbol{\sigma}_p = \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{Bmatrix}, \quad \boldsymbol{\sigma}_n = \begin{Bmatrix} \sigma_{\alpha z} \\ \sigma_{\beta z} \\ \sigma_{zz} \end{Bmatrix} \tag{16}$$

'T' as superscript stands for the transposition operator. δ signifies virtual variations. $\boldsymbol{p}^{kT} = \{p_\alpha^k, p_\beta^k, p_z^k\}$ is a generic pressure loading acting on the top, Σ_k^t, and on the bottom, Σ_k^b, of each lamina. The variation of the internal work has been split into in-plane and out-of-plane parts and involves the stress obtained from Hooke's Law and the strain from the geometrical relations. Geometrical relations link strains $\boldsymbol{\varepsilon}$ and displacements $\boldsymbol{u}$. Strains are conveniently grouped into in-plane and normal components denoted by the subscripts p and n, respectively. The geometric relations are:

$$\begin{aligned} \boldsymbol{\varepsilon}_p^k &= \boldsymbol{D}_p \boldsymbol{u}^k + \boldsymbol{A}_p \boldsymbol{u}^k \\ \boldsymbol{\varepsilon}_n^k &= \boldsymbol{D}_{n\Omega} \boldsymbol{u}^k + \boldsymbol{A}_n \boldsymbol{u}^k + \boldsymbol{D}_{nz} \boldsymbol{u}^k \end{aligned} \tag{17}$$

in which $\boldsymbol{D}_p$, $\boldsymbol{D}_{n\Omega}$ and $\boldsymbol{D}_{nz}$ are differential matrix operators and $\boldsymbol{A}_p$ and $\boldsymbol{A}_n$ are geometrical terms accounting for the through-the-thickness variation of the curvature:

$$\boldsymbol{D}_p = \begin{bmatrix} \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} & 0 & 0 \\ 0 & \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} & 0 \\ \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} & \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} & 0 \end{bmatrix}, \quad \boldsymbol{A}_p = \begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k R_\alpha^k} \\ 0 & 0 & \frac{1}{H_\beta^k R_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}$$

$$
\boldsymbol{D}_{n\Omega}=\begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} \\ 0 & 0 & \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{A}_n=\begin{bmatrix} -\frac{1}{H_\alpha^k R_\alpha^k} & 0 & 0 \\ 0 & -\frac{1}{H_\beta^k R_\beta^k} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{D}_{nz}=\begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \tag{18}
$$

H_α^k and H_β^k account for the change in length of a k-layer segment due to the curvature. Curvature terms have been entirely retained in the following developments.
In the case of linear elastic material, stresses and strains are related via Hooke's generalized law:

$$
\begin{aligned}
\boldsymbol{\sigma}_p^k &= \tilde{\boldsymbol{C}}_{pp}^k \boldsymbol{\varepsilon}_p^k + \tilde{\boldsymbol{C}}_{pn}^k \boldsymbol{\varepsilon}_n^k \\
\boldsymbol{\sigma}_n^k &= \tilde{\boldsymbol{C}}_{np}^k \boldsymbol{\varepsilon}_p^k + \tilde{\boldsymbol{C}}_{nn}^k \boldsymbol{\varepsilon}_n^k
\end{aligned} \tag{19}
$$

Terms $\tilde{\boldsymbol{C}}_{pp}^k$, $\tilde{\boldsymbol{C}}_{pn}^k$, $\tilde{\boldsymbol{C}}_{np}^k$ and $\tilde{\boldsymbol{C}}_{nn}^k$ are the material stiffness matrices for a k-layer in the global reference system, see Carrera[12]. By replacing Eqs. (17), (19) and the unified displacement field in Eq. (3). into Eq. (15), PVD reads:

$$
\begin{aligned}
&\sum_{k=1}^{N_l} \int_{\Omega_k} \delta\boldsymbol{u}_\tau^{kT} \int_{h_k} \{(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T)[\tilde{\boldsymbol{C}}_{pp}^k (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pn}^k (F_s \boldsymbol{D}_{n\Omega} + F_s \boldsymbol{A}_n + F_{s,z})] + \\
&+ (-F_\tau \boldsymbol{D}_{n\Omega}^T + F_\tau \boldsymbol{A}_n^T + F_{\tau,z})[\tilde{\boldsymbol{C}}_{np}^k (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn}^k (F_s \boldsymbol{D}_{n\Omega} + F_s \boldsymbol{A}_n + F_{s,z})]\} H_\alpha^k H_\beta^k dz_k \boldsymbol{u}_s^k \, d\Omega_k + \\
&+ \sum_{k=1}^{N_l} \int_{\Gamma_k} \delta\boldsymbol{u}_\tau^{kT} \int_{h_k} \{F_\tau \boldsymbol{I}_p^T [\tilde{\boldsymbol{C}}_{pp}^k (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pn}^k (F_s \boldsymbol{D}_{n\Omega} + F_s \boldsymbol{A}_n + F_{s,z})] + \\
&+ F_\tau \boldsymbol{I}_{n\Omega}^T [\tilde{\boldsymbol{C}}_{np}^k (F_s \boldsymbol{D}_p + F_s \boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn}^k (F_s \boldsymbol{D}_{n\Omega} + F_s \boldsymbol{A}_n + F_{s,z})]\} H_\alpha^k H_\beta^k dz_k \boldsymbol{u}_s^k \, d\Gamma_k = \sum_{k=1}^{N_l} \int_{\Sigma_k^b \cup \Sigma_k^t} \delta\boldsymbol{u}_\tau^{kT} \boldsymbol{p}_\tau^{k^k} d\Omega_k
\end{aligned} \tag{20}
$$

being:

$$
\boldsymbol{I}_p=\begin{bmatrix} \frac{1}{H_\alpha^k} & 0 & 0 \\ 0 & \frac{1}{H_\beta^k} & 0 \\ \frac{1}{H_\beta^k} & \frac{1}{H_\alpha^k} & 0 \end{bmatrix}, \quad \boldsymbol{I}_{n\Omega}=\begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k} \\ 0 & 0 & \frac{1}{H_\beta^k} \\ 0 & 0 & 0 \end{bmatrix} \tag{21}
$$

By imposing the definition of virtual variations for the unknown displacement variables, the differential system of governing equations and related boundary conditions for the N_l k-layers in each Ω_k domain are found. The governing equations are:

$$
\boldsymbol{K}_d^{k\tau s} \boldsymbol{u}_s^k = \boldsymbol{p}_\tau^k \tag{22}
$$

with boundary conditions

$$
\begin{aligned}
\boldsymbol{u}_\tau^k &= \overline{\boldsymbol{u}}_\tau^k && \text{geometrical on } \Gamma_k^g \\
\boldsymbol{\Pi}_d^{k\tau s} \boldsymbol{u}_s^k &= \boldsymbol{\Pi}_d^{k\tau s} \overline{\boldsymbol{u}}_s^k && \text{mechanical on } \Gamma_k^m
\end{aligned} \tag{23}
$$

Differential stiffness and mechanical boundary conditions matrices are:

$$
\begin{aligned}
\boldsymbol{K}_d^{k\tau s} &= \int_{h_k} \{(-F_\tau \boldsymbol{D}_p^T + F_\tau \boldsymbol{A}_p^T)[\tilde{\boldsymbol{C}}_{pp}^k(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pn}^k(F_s\boldsymbol{D}_{n\Omega} + F_s\boldsymbol{A}_n + F_{s,z})] + \\
&(-F_\tau \boldsymbol{D}_{n\Omega}^T + F_\tau \boldsymbol{A}_n^T + F_{\tau,z})[\tilde{\boldsymbol{C}}_{np}^k(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn}^k(F_s\boldsymbol{D}_{n\Omega} + F_s\boldsymbol{A}_n + F_{s,z})]\} H_\alpha^k H_\beta^k dz_k \\
\boldsymbol{\Pi}_d^{k\tau s} &= \int_{h_k} \{F_\tau \boldsymbol{I}_p^T[\tilde{\boldsymbol{C}}_{pp}^k(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{pn}^k(F_s\boldsymbol{D}_{n\Omega} + F_s\boldsymbol{A}_n + F_{s,z})] + \\
&+ F_\tau \boldsymbol{I}_{n\Omega}^T[\tilde{\boldsymbol{C}}_{np}^k(F_s\boldsymbol{D}_p + F_s\boldsymbol{A}_p) + \tilde{\boldsymbol{C}}_{nn}^k(F_s\boldsymbol{D}_{n\Omega} + F_s\boldsymbol{A}_n + F_{s,z})]\} H_\alpha^k H_\beta^k dz_k
\end{aligned} \tag{24}
$$

$\boldsymbol{I}$ is the unit array.
Previous equations consist of 3×3 fundamental nuclei. The following sub/super–scripts apply: τ, s and k. Explicit forms of the governing equations for each layer can be written by expanding the introduced subscripts and superscripts in the previous arrays as follows

$$
k = 1, 2, \ldots, N_l; \quad \tau = t, r, b; \quad s = t, r, b; \quad r = 2, \ldots N \tag{25}
$$

D. Assemblage and Multilayer Equations

PVD has been written for the N_l independent layers. C_z^0 requirements must be imposed to drive equations from layer to multilayer level. Multilayered equations can be written according to the usual variational statements: stiffness related to the same variables is accumulated in this process. Interlaminar continuity conditions are imposed at this stage. An example is shown in Fig. 3. Details on this procedure can be found in the mentioned author's papers. Multilayered arrays are obtained at the very end of the assemblage. The equilibrium and boundary conditions for the displacement formulation take on the following form

$$
\begin{aligned}
&\boldsymbol{K}_d \boldsymbol{u} = \boldsymbol{p} \\
&\boldsymbol{u} = \bar{\boldsymbol{u}} \quad \text{or} \quad \boldsymbol{\Pi}_d = \boldsymbol{\Pi}_d \boldsymbol{u}
\end{aligned} \tag{26}
$$

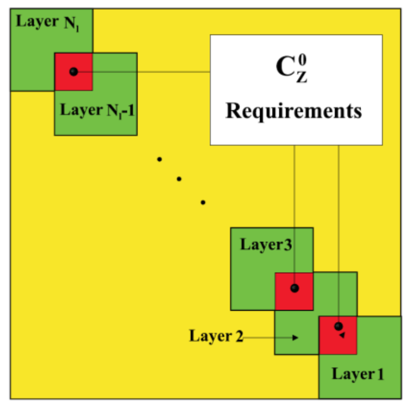

Figure 3. Example: how C_z^0 requirements are imposed from layer-to multilayer level

VI. Closed form solution

In order to assess the proposed models, equations (26) are herein solved for a special case in which closed form solutions are given. The particular case in which the material has the following properties (as it is the case of cross-ply shells) $\tilde{\boldsymbol{C}}_{16} = \tilde{\boldsymbol{C}}_{26} = \tilde{\boldsymbol{C}}_{36} = \tilde{\boldsymbol{C}}_{45} = 0$ has been considered, for which Navier-type closed form solutions can be found by assuming the following harmonic forms for the applied loadings $\boldsymbol{p}_k = \{p^k_{\alpha\tau}, p^k_{\beta\tau}, p^k_{z\tau}\}$ and unknown displacement $\boldsymbol{u}_k = \{u^k_{\alpha\tau}, u^k_{\beta\tau}, u^k_{z\tau}\}$ variables in each k–layer,

$$
\begin{aligned}
(u_{\alpha\tau}^k, p_{\alpha\tau}^k) &= \sum_{m=1}^{\bar{m}} \sum_{n=1}^{\bar{n}} (U_{\alpha\tau}^k(z), P_{\alpha\tau}^k(z)) \cos\left(\frac{m\pi}{a_k}\alpha_k\right) \sin\left(\frac{n\pi}{b_k}\beta_k\right) && k = 1, \ldots, N_l \\
(u_{\beta\tau}^k, p_{\beta\tau}^k) &= \sum_{m=1}^{\bar{m}} \sum_{n=1}^{\bar{n}} (U_{\beta\tau}^k(z), P_{\beta\tau}^k(z)) \cos\left(\frac{m\pi}{a_k}\alpha_k\right) \sin\left(\frac{n\pi}{b_k}\beta_k\right) && \tau = t, b, r \\
(u_{z\tau}^k, p_{z\tau}^k) &= \sum_{m=1}^{\bar{m}} \sum_{n=1}^{\bar{n}} (U_{z\tau}^k(z), P_{z\tau}^k(z)) \cos\left(\frac{m\pi}{a_k}\alpha_k\right) \sin\left(\frac{n\pi}{b_k}\beta_k\right) && r = 2, \ldots, N
\end{aligned} \tag{27}
$$

which correspond to simply-supported boundary conditions. a_k and b_k are the lengths of the shell along the two coordinates α_k and β_k. m and n represent the number of half-waves in α_k and β_k direction, respectively. Capital letters indicate maximal amplitudes. These assumptions correspond to the simply-supported boundary conditions. Upon substitution of Eq. (25), the governing equations assume the form of a linear system of ordinary differential equations.

VII. Results

The higher-order theories described above have been applied to the static analysis of multilayered composite plates and shell structures. The mechanical material properties of the lamina are: $E_L = 25E_T$, $G_{LT} = 0.5E_T$, $G_{TT} = 0.2E_T$, $\nu_{LT} = \nu_{TT} = 0.25$. Subscript 'L' stands for direction parallel to the fibers, 'T' identifies the transverse direction, ν_{LT} is the major Poisson ratio. Thin and moderately thick plates and shells are considered. The total thickness of the laminate is h. In this paper, we present results for ESL and LW trigonometric theories. A comparison with available exact solutions is given in the following paragraphs.

E. Elastic behavior of multilayered Plates

In this section, we investigate the bending of square bidirectional plates consisting of several layers, loaded via a bisinusoidal loading p_z. The elastic solution for this problem was given by Pagano and Hatfield[13]. Symmetric laminations with respect to the central plane are considered, with fiber oriented alterning between 0° and 90°, with respect to the α-axis. The outer layers are 0° oriented and the total thickness of the 0° and 90° layers is the same. Moreover each layer with same orientation has same thickness. Results present the following dimensionless quantities:

$$\bar{u}_z = \frac{\pi^4 Q}{12\, p_0 h (a/h)^4} u_z$$

$$\bar{\sigma}_{\alpha\alpha} = \frac{1}{p_0 (a/h)^2} \sigma_{\alpha\alpha} \qquad (28)$$

$$\bar{\sigma}_{\alpha z} = \frac{1}{p_0 (a/h)} \sigma_{\alpha z}$$

$Q = 4G_{LT} + (E_L + E_T(1 + 2\upsilon_{TT}))/(1 - \upsilon_{LT}\upsilon_{TL})$, p_0 is a constant representing the load's amplitude and a is the edge length. In terms of normalized functions (28), the CLT solution is independent of the ratio *a/h*.

Table 1 reports the displacement u_z for a 9-ply laminate. When the plate is thin (*a/h=100*) we have agreement with the reference solution for all the considered theories and expansion orders. Instead, when *a/h* decreases we notice the difference between ESL and LW theories. In the case *a/h=10* we have an error of 7*%* and it reaches 15*%* for *a/h*=4. For this case, the trigonometric theories do not give improvements respect to the classic ones. This behavior is highlighted in Figure 4, which represents the displacement u_z in function of

Table 1. Maximum transverse deflection $\bar{u}_z$ evaluated in *z=0* for 9-ply square plate.

	ED*N*	EDT*N*	LD*N*	LDT*N*
a/h=100				
Ref. 13		1.005		
N=4	1.004	1.004	1.005	1.005
N=3	1.004	1.004	1.005	1.005
N=2	1.003	1.003	1.005	1.005
a/h=10				
Ref. 13		1.512		
N=4	1.413	1.412	1.512	1.512
N=3	1.413	1.412	1.512	1.512
N=2	1.342	1.342	1.512	1.512
a/h=4				
Ref. 13		4.079		
N=4	3.488	3.479	4.079	4.079
N=3	3.493	3.482	4.079	4.078
N=2	3.074	3.072	4.078	4.078
CLT		1		

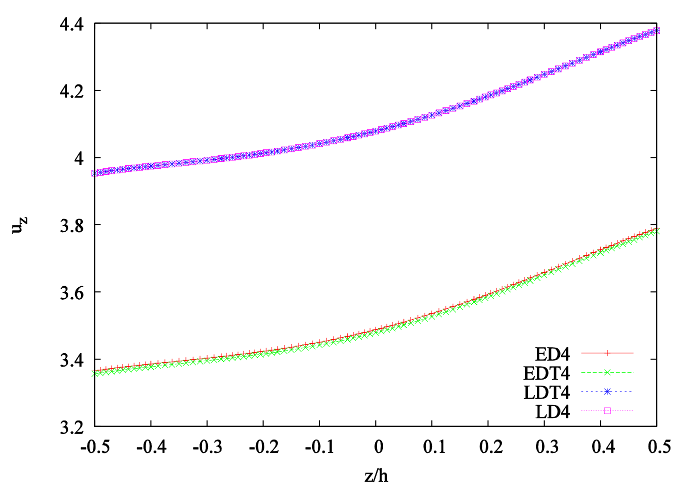

Figure 4: Distribution of displacement u_z for 9-ply laminate, *a/h*=4, expansion order *N=4*. Results for classical and trigonometric ESL and LW models.

the plate's thickness. The most thick plate case has been considered (*a/h=4*). The ED4 and EDT4 curves are overlapped as well as those related to LD4 and LDT4.

Figures 5 and 6 show the distributions of displacement u_z and stresses $\bar{\sigma}_{\alpha\alpha}$ and $\bar{\sigma}_{\alpha z}$ for a 3-ply laminate, in the a/h=10 case. Results for classical and trigonometric theories are comparable.

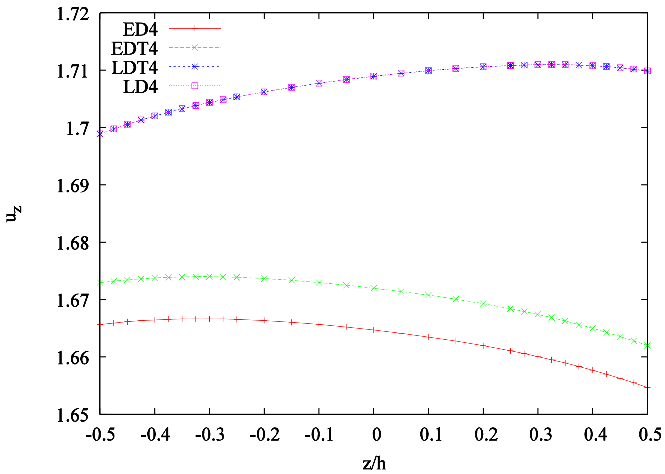

Figure 5: Distribution of displacement u_z for 3-ply laminate, a/h=10.

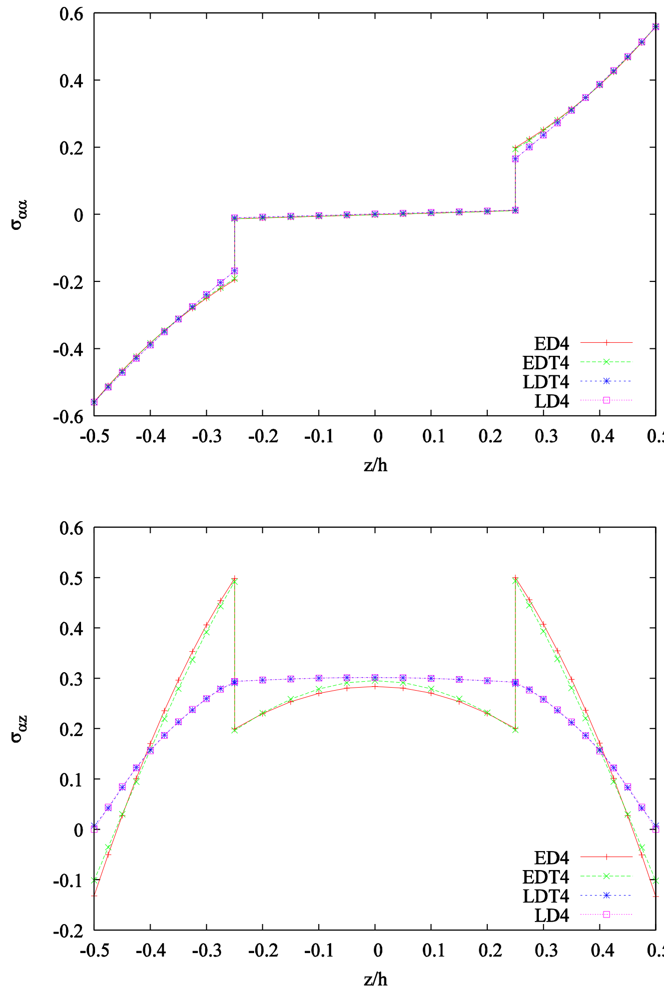

Figure 6: Distribution of stresses $\bar{\sigma}_{\alpha\alpha}$ and $\bar{\sigma}_{\alpha z}$ for 3-ply laminate, a/h=10.

F. Laminated shells in cylindrical bending

5. Ren's cylindrical panel

Results presented in Tables 2-4 refer to a [90°/0°/90°] cylindrical shell with layers of equal thickness, loaded via a bisinusoidal loading p_z applied at the top layer. Exact solutions were considered by Ren[14] for cross-ply cylindrical panels in cylindrical bending. The geometrical data (see fig. 7) are: number of half-waves in β_k direction n=1 and $b/R_\beta=\pi/3$. Layers are numbered, starting from the shell bottom - internal surface. The fiber *L*-orientation coincided with the α_k -layers direction.

Transverse displacement, in-plane and transverse shear stresses values are normalized by:

$$\bar{u}_z = \frac{10E_T}{p_0 h(R_\beta / h)^4} u_z$$
$$\bar{\sigma}_{\beta\beta} = \frac{1}{p_0 (R_\beta / h)^2} \sigma_{\beta\beta} \qquad (29)$$
$$\bar{\sigma}_{\beta z} = \frac{1}{p_0 (R_\beta / h)} \sigma_{\beta z}$$

We consider the new trigonometric set of thickness functions to carry out the analyses. In tables 2 - 4 we present results also for the classical ESL and LW models in order to provide a comparison with the new trigonometric basis functions.

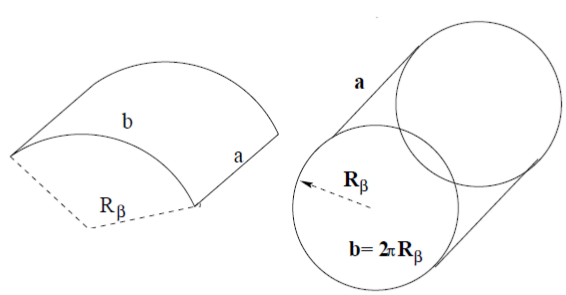

Figure 7: Geometrical notations used for the investigated cylindrical panels and cylindrical shells.

Table 2 reports the displacement $\bar{u}_z$. Trigonometric LW theories provide results in agreement with the reference solution for an expansion order as low as 3 and for both thin and thick shells. Therefore higher expansion orders are not necessary for this case. Instead, if we consider the EDTN models we note that for the displacement u_z there is a regular trend. Adding terms in the expansion improves the result, especially considering thick shells (R_β/h=4), but at least we get to an error of 11%. When the shell is thin (R_β/h=100) higher-order theories do not give improvement in the solution, respect the CLT results. In fact the difference between CLT and LDT11 solutions is about 1%.

Table 2. Maximum transverse deflection $\bar{u}_z$ evaluated in *z=0.*

R_β/h	100	10	4
Ref. 14	0.0787	0.144	0.457
LDT11	0.0786	0.144	0.458
LDT10	0.0786	0.144	0.458
LDT9	0.0786	0.144	0.458
LDT8	0.0786	0.144	0.458
LDT7	0.0786	0.144	0.458
LDT6	0.0786	0.144	0.458
LDT5	0.0786	0.144	0.458
LDT4	0.0786	0.144	0.458
LDT3	0.0786	0.144	0.458
LDT2	0.0786	0.144	0.454
LD4	0.0786	0.144	0.458
LD3	0.0786	0.144	0.458
LD2	0.0786	0.144	0.454
LD1	0.0785	0.141	0.441
EDT11	0.0782	0.135	0.404
EDT10	0.0782	0.135	0.403
EDT9	0.0782	0.135	0.396
EDT8	0.0782	0.135	0.395
EDT7	0.0782	0.135	0.397
EDT6	0.0781	0.133	0.388
EDT5	0.0781	0.133	0.387
EDT4	0.0781	0.131	0.380
EDT3	0.0781	0.131	0.380
EDT2	0.0779	0.113	0.292
ED4	0.0785	0.136	0.428
ED3	0.0785	0.136	0.428
ED2	0.0783	0.119	0.331
ED1	0.0780	0.119	0.333
CLT	0.0776	0.078	0.078

Tables 3 and 4 report stresses $\bar{\sigma}_{\beta\beta}$ and $\bar{\sigma}_{\beta z}$, for ESL and LW results.

Table 3. Maximum stresses $\bar{\sigma}_{\beta\beta}$ evaluated in *z=h/2*

R_β/h	100	10	4
Ref. 14	0.781	0.897	1.367
LDT4-11	0.779	0.897	1.367
LDT3	0.779	0.896	1.363
LDT2	0.779	0.895	1.344
LD4	0.779	0.897	1.367
LD3	0.779	0.897	1.367
LD2	0.779	0.896	1.347
LD1	0.779	0.866	1.213
EDT11	0.775	0.831	0.910
EDT10	0.775	0.830	0.895
EDT9	0.774	0.846	1.140
EDT8	0.775	0.845	1.136
EDT7	0.775	0.849	1.187
EDT6	0.775	0.845	1.180
EDT5	0.775	0.846	1.169
EDT4	0.775	0.842	1.179
EDT3	0.776	0.844	1.196
EDT2	0.774	0.739	0.710
ED4	0.779	0.883	1.339
ED3	0.779	0.884	1.354
ED2	0.777	0.777	0.813
ED1	0.778	0.778	0.777
CLT	0.776	0.759	0.732

In table 3, considering trigonometric ESL models we notice that in some cases adding a term in the expansion leads to major errors. The thick shell case (R_β/h=4) presents an irregular trend. For example the solution that we obtain with an expansion order of 10 is worse than that relative to N=3. Therefore it becomes interesting to study the contribution of each term in the expansion. LDTN results convergence to the reference solution already for N=4 and thus we do not need to add other terms to increase the accuracy of the solution. The transverse shear component $\bar{\sigma}_{\beta z}$ is presented in table 4. In this case there is an unexpected behavior. The solutions that we obtain with the trigonometric ESL models are better than the LW ones, when we consider high expansion orders. In fact, the LW solutions do not change from N=3 onwards, but the error remains high especially in the case of thin plate where it reaches 30%. The EDTN models instead have a regular trend and as we add terms to the expansion we obtain more accurate solutions.

6. *Varadan and Bhaskar's cylindrical shell*

Varadan and Bhaskar[15] considered exact solutions for cross-ply laminated, cylindrical shells, subjected to transverse pressure p_z at the bottom (internal) surface. The geometrical data (see fig. 7) are: a/R_β=4, m=1, n=8. The two layers [0°/90°] lamination schemes have been considered. The following dimensionless quantities are considered:

$$\bar{u}_z = \frac{10E_L}{p_0 h(R_\beta / h)^4} u_z$$
$$\bar{\sigma}_{zz} = \frac{1}{p_0}\sigma_{zz} \qquad (30)$$
$$\bar{\sigma}_{\beta z} = \frac{1}{p_0 (R_\beta / h)}\sigma_{\beta z}$$

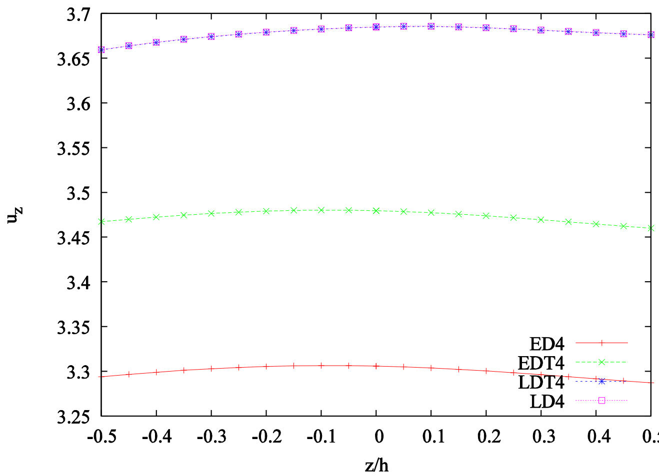

Figure 8: Distribution of displacement u_z for [0°/90°] cylindrical shell, *a/h*=10. 3D solution in Ref.15

Table 4. Maximum stresses $\bar{\sigma}_{\beta z}$ evaluated in *z=0*

R_β/h	100	10	4
Ref. 14	0.523	0.525	0.476
LDT4-11	0.367	0.497	0.442
LDT3	0.367	0.497	0.442
LDT2	0.366	0.496	0.438
LD3-4	0.367	0.497	0.442
LD2	0.366	0.496	0.437
LD1	0.366	0.498	0.446
EDT11	0.416	0.524	0.436
EDT10	0.430	0.537	0.454
EDT9	0.430	0.537	0.451
EDT8	0.408	0.515	0.426
EDT7	0.408	0.515	0.427
EDT6	0.337	0.446	0.365
EDT5	0.337	0.446	0.365
EDT4	0.233	0.356	0.317
EDT3	0.233	0.355	0.317
EDT2	0.033	0.166	0.164
ED4	0.217	0.357	0.348
ED3	0.217	0.357	0.347
ED2	0.033	0.174	0.186
ED1	0.033	0.174	0.187

For the sake of brevity, results quoted in figures 8 and 9 are referred to the R_β/h=10 case and present the behavior of the investigated quantities (30) along the shell's thickness. We compared the solutions obtained for N= 4 for each theory that we have considered in this work, that is, the classical theories and the trigonometric ESL and LW. Being equal the approach adopted, the trends do not change, with or without the use of trigonometric functions. This applies especially to stresses, whereas the displacement u_z undergoes an improvement when we consider the EDT model, compared to the ED one.

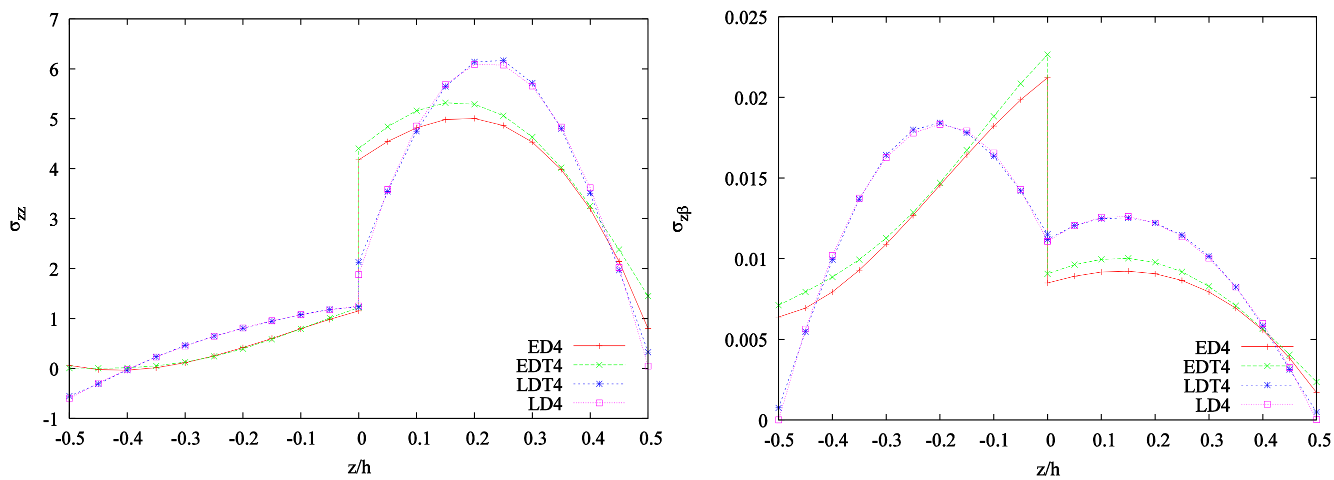

Figure 9: Distributions of stresses $\bar{\sigma}_{zz}$ and $\bar{\sigma}_{\beta z}$ for [0°/90°] cylindrical shell, *a/h*=10. 3D solution in Ref. 15.

VIII. Conclusion

In the framework of axiomatic approaches which can be developed on the basis of variational statements, new trigonometric displacement distributions in the thickness of the shell *z* have been *postulated*. In this work we presented higher-order shell theories based on Equivalent Single Layer and Layer-Wise approaches, formulated on the basis of new kinematic assumptions. A unified approach to formulate two-dimensional shell theories has been here addressed to evaluate the static response of cylindrical multilayered shells made of composite materials. Results have been compared with exact solutions available in literature. Some considerations about the effectiveness of the formulations can be made. In general higher expansion orders result necessary when we consider tick shells, but in some case adding more terms in the expansion do not give an improvement of the solution. Then it is evident that theories developed from a new basis of trigonometric thickness functions are effective depending on the terms that are adopted in the expansion. It becomes interesting to evaluate the importance of higher-order terms (see Ref. 16).

Acknowledgments

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