1D Higher-Order Finite Element Models with only Displacement Variables for the Analysis of Fiber-Reinforced Composite Structures

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Abstract

This paper presents 1D higher-order structural models for the finite element analysis of fiber-reinforced composite structures. The formulation proposed is embedded in the framework of the Carrera Unified Formulation (CUF) for plates, shells (2D) and beams (1D). CUF 1D models are based on a hierarchical formulation which permits us to deal with any-order models in a unified manner. This implies that the order of the formulation is an input of the analysis. The unified formulation key feature is based on the so-called fundamental nucleus structure of the FE matrices which is independent of the order of the structural model. Equivalent Single Layer (ESL) and Layer Wise (LW) descriptions are accountable. The former is obtained by means of Taylor-like polynomial expansions (hereafter referred to as TE) of the cross-section displacement field. The latter is obtained by exploiting Lagrange-like polynomial models (hereafter referred to as LE) which are defined on subdomains of the cross-section. LE models have solely displacement unknowns. Results from the present formulation are compared with those from solid models. The results obtained suggest that CUF 1D models provide accurate 3D-like results with low computational costs.

Introduction

Composite structures are well-known for their high strength-to-weight ratio, high stiffness-to-weight ratio, ease of formability, wide range of operating temperatures and their capability to be tailored according to a given requirement, see the book by Tsai [1]. The structural analysis of a composite structure requires tools which have to account for some important capabilities. Typical examples are the proper detection of the shear effects due to the typically low shear moduli of composites and the possibility of modeling the different components of the structure (i.e. fibers and matrices).

This paper is devoted to 1D structural models for the analysis of composite structures. Many structures can be analyzed by means of 1D models, typical examples are given by aircraft wings, rotor and wind blades. These structures are, nowadays, widely designed by means of anisotropic layered composite materials. 1D models are advantageous since they require relatively lower computational costs if compared to 2D, plate and shell, or 3D, solid models.

The analysis of composites via 1D models requires the adoption of refined models to overcome the well-known limits of classical 'beam' theories, e.g. the Euler-Bernoulli and the Timoshenko models. These models work properly when slender, compact, homogeneous structures are considered under bending. Conversely, the analysis of deep, thin-walled, open beams requires more complex advanced models, see [2]. A considerable amount of work has been done to refine 1D models as stated in the excellent review of Kapania and Raciti [3]. Some of the most relevant contributions are discussed below, with particular attention being given to the static analysis of composite beam structures, which is the main task of the present paper. Different approaches were considered such as: 1. the use of appropriate shear correction factors [4]; 2. the use of warping functions [5]; 3. the exploitation of asymptotic methods [6, 7]; 4. the development of the Generalized Beam Theory (GBT) [8, 9]; 5. the adoption of higher-order theories [10, 11].

The present work is embedded in the framework of the Carrera Unified Formulation (CUF) as a tool to develop refined 1D formulations. CUF 1D models have been introduced in recent years for different applications including closed-form solutions [12], finite element formulations for static [13] and free-vibration [14] analysis, joined wings [15], bridge-like cross-sections [16, 17], thin-walled cross-sections made of isotropic materials [18], composite structures [19] and asymptotic-like analyses [20]. A comprehensive overview of the most recent developments of the proposed refined models can be found in the book chapter by Carrera et al. [21], moreover, a book by the same authors [22] provides an extended theoretical description of CUF.

A novel extension of 1D CUF models is presented in this paper where composite structures are analyzed by means of different approaches. The structure can be modeled, in fact, at different scale levels. 1D CUF allows us to consider composites as homogenized laminates (Equivalent Single Layer approach, ESL), or as being composed by a set of homogenized laminae (Layer-Wise approach, LW), or as being composed by fibers and matrices. This latter modeling option represents the novelty of this paper. 1D CUF models are herein obtained by means of Taylor and Lagrange expansions of the displacement field above the cross-section (hereafter referred to as TE and LE, respectively). Arbitrary refined models are obtained by exploiting the hierarchical capabilities of CUF, that is, without changing the formal expressions of the problem matrices. A set of numerical examples is carried out and comparisons with models from commercial codes are provided.

CUF 1D Formulation

The transposed displacement vector is defined as

$$\mathbf{u}(x,y,z) = \left\{ \begin{array}{ccc} u_x & u_y & u_z \end{array} \right\}^T \tag{1}$$

where x, y, and z are orthonormal axes as shown in Fig. 1. The cross-section of the structure is Ω, the

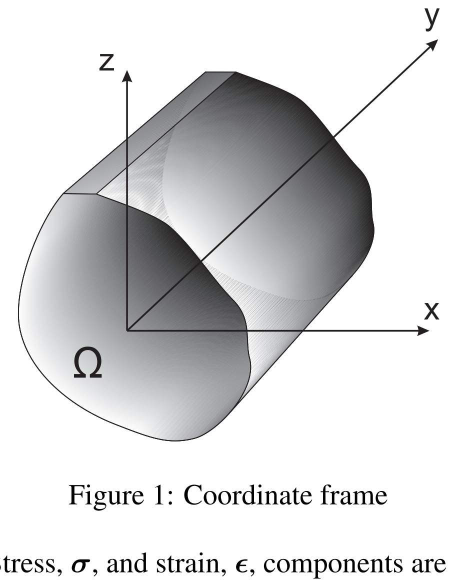

Figure 1: Coordinate frame

longitudinal axis is $0 \leq y \leq L$. Stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\epsilon}$, components are grouped as

$$\begin{array}{ll} \sigma_p = \left\{ \begin{array}{ccc} \sigma_{zz} & \sigma_{xx} & \sigma_{zx} \end{array} \right\}^T, & \varepsilon_p = \left\{ \begin{array}{ccc} \varepsilon_{zz} & \varepsilon_{xx} & \varepsilon_{zx} \end{array} \right\}^T \\ \sigma_n = \left\{ \begin{array}{ccc} \sigma_{zy} & \sigma_{xy} & \sigma_{yy} \end{array} \right\}^T, & \varepsilon_n = \left\{ \begin{array}{ccc} \varepsilon_{zy} & \varepsilon_{xy} & \varepsilon_{yy} \end{array} \right\}^T \end{array} \tag{2}$$

The subscript "n" stands for terms lying on the cross-section, while "p" stands for terms lying on planes which are orthogonal to Ω.

Strains are obtained as

$$\begin{aligned}\varepsilon_p &= D_p\mathbf{u}\\ \varepsilon_n &= D_n\mathbf{u} = (D_{np}+D_{ny})\mathbf{u}\end{aligned} \tag{3}$$

where D_p and D_n are differential operators whose explicit expressions is not reported here for the sake of brevity, it can be found in [16]. Constitutive laws are introduced to obtain stress components:

$$\sigma = \tilde{C}\varepsilon \tag{4}$$

According to Eq.s 2, the previous equation becomes:

$$\begin{aligned}\sigma_p &= \tilde{C}_{pp}\varepsilon_p + \tilde{C}_{pn}\varepsilon_n\\ \sigma_n &= \tilde{C}_{np}\varepsilon_p + \tilde{C}_{nn}\varepsilon_n\end{aligned} \tag{5}$$

where $\tilde{C}_{pp}$, $\tilde{C}_{pn}$, $\tilde{C}_{np}$, and $\tilde{C}_{nn}$ are the material coefficient matrices whose explicit expressions is not reported here for the sake of brevity, it can be found in [16].

Hierarchical Variable Kinematics: TE and LE models

In the CUF framework, the displacement field is the expansion of generic functions, F_τ:

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \qquad \tau = 1,2,....,M \tag{6}$$

where F_τ vary above the cross-section. $\mathbf{u}_\tau$ is the displacement vector and M stands for the number of terms of the expansion. According to the Einstein notation, the repeated subscript, τ, indicates summation. The choice of F_τ determines the class of 1D CUF model to adopt. Taylor-like polynomial expansions, $x^i\ z^j$, of the displacement field above the cross-section of the structure are adopted in this paper (i and j are positive integers). One of the three displacement components of generic N-order is then expressed by:

$$u_x = \sum_{N_i=0}^{N}\left(\sum_{M=0}^{N_i} x^{N-M}\ z^M\ u_{x_{\frac{N(N+1)+M+1}{2}}}\right) \tag{7}$$

For example, the second-order model, $N = 2$, has the following kinematic model:

$$u_x = u_{x_1} + x\,u_{x_2} + z\,u_{x_3} + x^2\,u_{x_4} + xz\,u_{x_5} + z^2\,u_{x_6} \tag{8}$$

The 1D model described by Eq. 8 has 18 generalized displacement variables: three constant, six linear, and nine parabolic terms. The order N of the expansion is arbitrary and is set as an input of the analysis. The choice of N for a given structural problem is usually made through a convergence study.
LE models exploit Lagrange polynomials to build 1D higher-order theories. In this paper, two types of cross-section polynomial sets are adopted: nine-point elements, L9, and six-point elements, L6. The isoparametric formulation is exploited to deal with arbitrary shaped geometries. The L9 interpolation functions are given by [23]:

$$\begin{aligned}F_\tau &= \tfrac{1}{4}(r^2+r\,r_\tau)(s^2+s\,s_\tau) \qquad \tau = 1,3,5,7\\ F_\tau &= \tfrac{1}{2}s_\tau^2(s^2-s\,s_\tau)(1-r^2)+\tfrac{1}{2}r_\tau^2(r^2-r\,r_\tau)(1-s^2) \qquad \tau = 2,4,6,8\\ F_\tau &= (1-r^2)(1-s^2) \qquad \tau = 9\end{aligned} \tag{9}$$

where r and s range from -1 to $+1$ and where r_τ and s_τ are the natural coordinates of the interpolation points above the cross-section. One component of the displacement field given by an L9 element is:

$$u_x = F_1\,u_{x_1} + F_2\,u_{x_2} + F_3\,u_{x_3} + F_4\,u_{x_4} + F_5\,u_{x_5} + F_6\,u_{x_6} + F_7\,u_{x_7} + F_8\,u_{x_8} + F_9\,u_{x_9} \tag{10}$$

where $u_{x_1},...,u_{x_9}$ are the displacement variables of the problem and they represent the translational displacement components of each of the nine points of the L9 element. This means that LE models provide elements that only have displacement variables. For the sake of brevity L6 polynomial models are not described here, they can be found in [18].

FE Formulation and the Fundamental Nucleus

The FE approach is herein adopted to discretize the structure along the y-axis, this process is conducted via a classical finite element methodology via the Principle of Virtual Displacements. In a compact notation the stiffness matrix can be written as:

$$
\begin{aligned}
\boldsymbol{K}^{ij\tau s} = \; & I_l^{ij} \triangleleft \left(\boldsymbol{D}_{np}^T F_\tau \boldsymbol{I}\right)\left[\tilde{\boldsymbol{C}}_{np}\left(\boldsymbol{D}_p F_s \boldsymbol{I}\right) + \tilde{\boldsymbol{C}}_{nn}\left(\boldsymbol{D}_{np} F_s \boldsymbol{I}\right)\right] + \\
& \left(\boldsymbol{D}_p^T F_\tau \boldsymbol{I}\right)\left[\tilde{\boldsymbol{C}}_{pp}\left(\boldsymbol{D}_p F_s \boldsymbol{I}\right) + \tilde{\boldsymbol{C}}_{pn}\left(\boldsymbol{D}_{np} F_s \boldsymbol{I}\right)\right] \triangleright_\Omega + \\
& I_l^{ij_{,y}} \triangleleft \left[\left(\boldsymbol{D}_{np}^T F_\tau \boldsymbol{I}\right)\tilde{\boldsymbol{C}}_{nn} + \left(\boldsymbol{D}_p^T F_\tau \boldsymbol{I}\right)\tilde{\boldsymbol{C}}_{pn}\right] F_s \triangleright_\Omega \boldsymbol{I}_{\Omega y} + \\
& I_l^{i_{,y}j} \boldsymbol{I}_{\Omega y} \triangleleft F_\tau \left[\tilde{\boldsymbol{C}}_{np}\left(\boldsymbol{D}_p F_s \boldsymbol{I}\right) + \tilde{\boldsymbol{C}}_{nn}\left(\boldsymbol{D}_{np} F_s \boldsymbol{I}\right)\right] \triangleright_\Omega + \\
& I_l^{i_{,y}j_{,y}} \boldsymbol{I}_{\Omega y} \triangleleft F_\tau \tilde{\boldsymbol{C}}_{nn} F_s \triangleright_\Omega \boldsymbol{I}_{\Omega y}
\end{aligned}
\tag{11}
$$

where:

$$
\boldsymbol{I}_{\Omega y} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad\qquad \triangleleft \ldots \triangleright_\Omega = \int_\Omega \ldots \, d\Omega \tag{12}
$$

$$
\left(I_l^{ij}, I_l^{ij_{,y}}, I_l^{i_{,y}j}, I_l^{i_{,y}j_{,y}}\right) = \int_l \left(N_i N_j, N_i N_{j_{,y}}, N_{i_{,y}} N_j, N_{i_{,y}} N_{j_{,y}}\right) dy \tag{13}
$$

$\mathbf{K}^{ij\tau s}$ is the stiffness matrix in the form of the fundamental nucleus, its components can be found in [20]. As far as the formal expression of the fundamental nucleus is concerned, it has to be underlined that:

- It does not depend on the expansion order.
- It does not depend on the choice of the F_τ expansion polynomials.

These are the key-point of CUF which permits, with only nine FORTRAN statements, to implement any-order multiple class theories.

Finite Element Application of TE and LE Models to Composites

As far as nonhomogeneous composite structures are concerned, two modeling approaches are adopted in this paper:

1. An Equivalent Single Component approach (ESC) where a homogenization of the properties of each component is conducted by summing the contributions of each component in the stiffness matrix. It is important to highlight that, if layered structures are considered, the present ESC will provide the Equivalent Single Layer approach (ESL).
2. A Component-Wise approach (CW) where the homogenization is just conducted at the interface level. If layered structures are considered, CW will provide the Layer-Wise approach (LW).

Both assembly procedures of the stiffness matrix in the framework of CUF are graphically shown in Fig. 2 in case of layered structures, both procedures are still valid for ESC and CW, respectively. TE and LE models can be used in an ESC manner. LE models obtain an CW description straightforwardly by considering different sets of L-elements per each component, i.e. one for the fiber and another one for the matrix. The homogenization is then conducted in correspondence to the shared interface cross-section nodes. CW is here obtained only by means of LE. It should be underlined that TE could also allow us to obtain CW. In this case, further equations imposing interface conditions should be added to solve the structural problem.

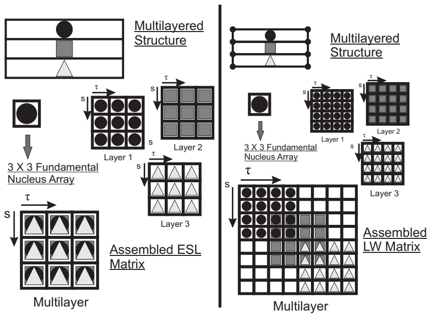

Figure 2: ESL and LW assembly scheme

Results and Discussion

A structural model having the typical components of a fiber-matrix system is analyzed in this section. A cantilevered beam is considered with a square cross-section. The length of the beam, L, is equal to 1 m and the edge of the cross-section, b, is equal to 0.1 m. The cross-section geometry and its discretization via Lagrange elements is shown in Fig. 3, L9 and L6 elements are adopted. The circular portion is made of an orthotropic material having E_L = 202.038 GPa, E_T = 12.134 GPa, ν_{LT} = 0.2128, ν_{TT} = 0.2704, G_{LT} = 8.358 GPa, G_{TT} = 4.776 GPa, where 'L' denotes a direction parallel to the fibers and 'T' perpendicular to the fibers. The surrounding portion of the cross-section is considered as being made of an isotropic material with E = 3.252 GPa and ν = 0.355. A point load is applied at the center point of the cross-section at the free tip of the structure, the load acts along the z-direction. The analysis is conducted by means of different CUF 1D models, both TE and LE, and results are compared with those from a SOLID model in Ansys.
Table 1 shows the vertical displacement of the loaded point and the maximum axial stress evaluated at the clamped cross-section at the top point of the circular component. Shear stress distributions at the clamped cross-section are given in Figs. 4, 5.

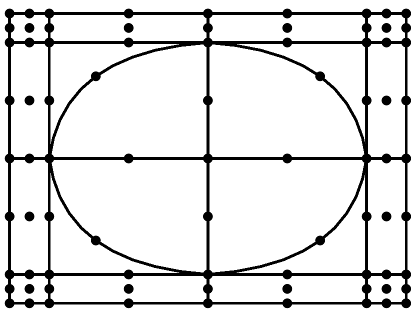

Figure 3: Cross-section geometry and Lagrange element distributions

These results make evident the validity of the present formulation to detect accurate displacement and stress fields of composite structures composed by different components some of them being 'soft-in-shear'. Moreover, the computational costs of CUF 1D models are extremely lower than those needed for SOLID model computations. LE provide more accurate results than TE as far as stress components are considered.

Conclusions

This paper presented the application of CUF 1D models to fiber-reinforced composite structures. Two classes of CUF 1D models were employed, the TE which are based on Taylor-like expansions of the

Model	$u_z \times 10^1$ $[mm]$	$\sigma_{yy} \times 10^4$ $[N/mm^2]$	DOFs
		CUF 1D - TE	
EBBT	-7.811	1.894	1089
TBT	-7.835	1.894	1815
N=1	-7.907	1.894	1089
N=2	-7.853	1.875	2178
N=3	-7.869	2.583	3630
N=4	-7.913	2.413	5445
		CUF 1D - LE	
	-7.933	2.072	7533
		SOLID	
	-7.812	2.020	142779

Table 1: Displacement and axial stress values obtained from different CUF 1D models and compared with those from a SOLID model

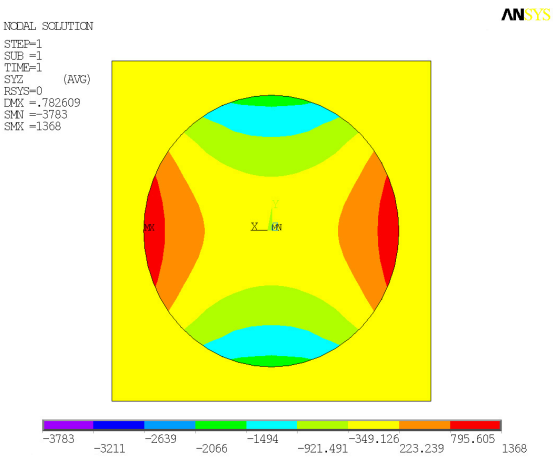

Figure 4: Shear stress (σ_{yz}) distribution from a SOLID model

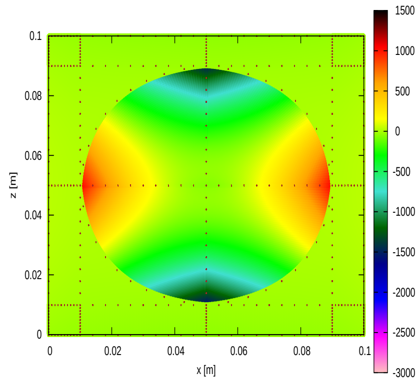

Figure 5: Shear stress (σ_{yz}) distribution from a CUF 1D model, LE

cross-section kinematics, and the LE which exploits Lagrange polynomials to discretize the cross-section geometry. A cantilever beam under bending was considered to simulate a typical fiber-matrix system. Results were compared with those from a SOLID model. The following conclusions hold:

1. The present 1D formulation is able to detect 3D-like displacement and stress distributions of composites.

2. CUF 1D computational costs are extremely lower than those necessary for SOLID models.

3. LE models are particularly indicated to model the fiber-matrix level of composites since they provide a so called Compoents-Wise description of the structure.

Future works should deal with the analysis of more complex and realistic composites having a larger number of fibers, moreover, failure and multiscale analyses should be implemented.

Acknowledgement

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