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Analysis of FGM beams by means of a unified formulation

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Abstract. This paper proposes several axiomatic refined theories for the linear static analysis of beams made of functionally graded materials. A bi-directional variation upon the cross-section is accounted for. Via a unified formulation, a generic N-order approximation is assumed for the displacement unknown variables over the beam cross-section. The governing differential equations and the boundary conditions are derived in terms of a fundamental nucleo that does not depend upon the approximation order. A Navier type, closed form solution is adopted. Beams undergo bending and torsional loadings. Deep beams are investigated. Comparisons with three-dimensional finite element models are given. The numerical investigation shows that the proposed unified formulation yields the complete three-dimensional displacement and stress fields as long as the appropriate approximation order is considered.

1. Introduction

Functionally Graded Materials (FGMs) are composed of two or more materials whose volume fraction changes gradually along desired spatial directions resulting in a smooth and continuous change in the effective properties. This allows a tailored material design that broadens the structural design space by implementing a multi-functional response with a minimal weight increase. Also, the study of beam structures represents an important topic of research since many primary and secondary structural elements, such as aircraft wings, helicopter rotor blades or robot arms can be idealised as beams. A general account of FGMs (design, fabrication and applications) can be found in Suresh and Mortensen [1], Miyamoto et al. [2] and Watanabe et al. [3]. Regarding the modelling of FGMs-based beams, Chakraborty et al. [4] developed a finite element based on Timoshenko's beam theory in which the shape functions have been derived from the general exact solution of the static governing equations. Exponential and polynomial gradation of mechanical and thermal properties along the through-the-

thickness direction were accounted for. Static, free vibration and wave propagation analyses were carried out. Li [5] proposed a unified approach to the formulation of Timoshenko's and Euler-Bernoulli's beam models. Young's modulus varied along the transverse coordinate according to a power-law function. Kadoli et al. [6] proposed a fined element based on a third-order approximation of the axial displacement and constant transverse displacement for the static analysis of beams made of metal-ceramic FGMs. Components' volume fraction was supposed to vary according to a power-law function. A discrete layer approach was adopted to account for material gradation. As far as elasticity solutions are concerned, Sankar [7] solved the plane elasticity equations exactly. An Euler-Bernoulli type theory was also derived. The shear stress component was obtained via integration of the indefinite equilibrium equations. Simply supported, FGM beams subjected to sinusoidal loadings were investigated. Young's modulus was supposed to vary exponentially along the thickness direction, while Poisson's ratio was constant. Zhu and Sankar [8] adopted a combined Fourier series-Galerkin method for the analysis of simply supported FGM-based beams for which the Young's modulus was a third-order polynomial function of the through-the-thickness coordinate. The present paper proposes a systemic manner of formulating axiomatically refined beam models accounting for material gradation above the cross-section. Via a concise notation for the kinematic field, the governing differential equations and the corresponding boundary conditions are reduced to a "fundamental nucleo" in terms of the displacement components. The fundamental nucleo does not depend upon the approximation order. This is, therefore, assumed as free parameter of the formulation. This formulation is named Carrera's Unified Formulation and it was previously applied to the modelling of isotropic beam structures (see Carrera and Giunta [9] and Carrera et al. [10]). Displacement-based theories that account for non-classical effects, such as transverse shear and in- and out-of-plane warping of the cross-section, can be formulated. It is worth mentioning that no special warping functions need to be assumed. Navier's, closed form solution is adopted to solve the governing differential equation. Slender and deep beams are investigated. Isotropic FGMs are accounted for. Young's modulus is considered to vary exponentially along two perpendicular directions on the cross-section. Poisson's ration is constant, since its effect on the deformation is much less than that of Young's modulus (see Delale and Erdogan [11]). The proposed models are validated towards three-dimensional finite element models that have been appositely developed.

2. Preliminaries

Cross-sections, Ω, that are obtainable as union of N_{Ω^k} rectangular sub-domains:

$$\Omega = \bigcup_{k=1}^{N_{\Omega^k}} \Omega^k \tag{1}$$

with:

$$\Omega^k = \left\{ (y,z) : y_1^k \leq y \leq y_2^k ; z_1^k \leq z \leq z_2^k \right\} \tag{2}$$

are considered. Terms $\left\{ \left(y_i^k, z_j^k \right) : i,j = 1,2 \right\}$ are the coordinates of the corner points of a k sub-domain. Superscript 'k' represents the cross-section sub-domain index, while, as

subscript, it stands for summation over the range $(1, N_{\Omega^k})$. A Cartesian reference system is adopted: y- and z-axis are two orthogonal directions laying on the cross-section. The x coordinate is coincident to the axis of the beam. It is bounded such that $0 \leq x \leq l$, being l the axial extension of the beam. The displacement field is:

$$\mathbf{u}^T(x,y,z) = \left\{ \begin{array}{ccc} u_x(x,y,z) & u_y(x,y,z) & u_z(x,y,z) \end{array} \right\} \tag{3}$$

in which u_x, u_y, and u_z are the displacement components along x-, y- and z-axes. Superscript 'T' represents the transposition operator. The stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\varepsilon}$, vectors are grouped into vectors $\boldsymbol{\sigma}_n$, $\boldsymbol{\varepsilon}_n$ that act on the cross-section:

$$\begin{aligned} \boldsymbol{\sigma}_n^T &= \left\{ \begin{array}{ccc} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \end{array} \right\}, \\ \boldsymbol{\varepsilon}_n^T &= \left\{ \begin{array}{ccc} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \end{array} \right\} \end{aligned} \tag{4}$$

and $\boldsymbol{\sigma}_p$, $\boldsymbol{\varepsilon}_p$ acting on planes orthogonal to Ω:

$$\begin{aligned} \boldsymbol{\sigma}_p^T &= \left\{ \begin{array}{ccc} \sigma_{yy} & \sigma_{zz} & \sigma_{yz} \end{array} \right\}, \\ \boldsymbol{\varepsilon}_p^T &= \left\{ \begin{array}{ccc} \varepsilon_{yy} & \varepsilon_{zz} & \varepsilon_{yz} \end{array} \right\} \end{aligned} \tag{5}$$

Linear strain-displacement relations are accounted for:

$$\boldsymbol{\varepsilon}_n = \mathbf{D}_{np}\mathbf{u} + \mathbf{D}_{nx}\mathbf{u}, \quad \boldsymbol{\varepsilon}_p = \mathbf{D}_p\mathbf{u} \tag{6}$$

where $\mathbf{D}_{np}$, $\mathbf{D}_{nx}$, and $\mathbf{D}_p$ are the following differential matrix operators:

$$\mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial}{\partial y} & 0 & 0 \\ \dfrac{\partial}{\partial z} & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nx} = \mathbf{I}\frac{\partial}{\partial x}, \quad \mathbf{D}_p = \begin{bmatrix} 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix} \tag{7}$$

and $\mathbf{I}$ is the unit matrix. Under the hypothesis of isotropic linear elastic FGMs, the generalised Hooke law holds. According to Eqs. (4) and (5), it reads:

$$\begin{aligned} \boldsymbol{\sigma}_p &= \mathbf{C}_{pp}(y,z)\,\boldsymbol{\varepsilon}_p + \mathbf{C}_{pn}(y,z)\,\boldsymbol{\varepsilon}_n, \\ \boldsymbol{\sigma}_n &= \mathbf{C}_{np}(y,z)\,\boldsymbol{\varepsilon}_p + \mathbf{C}_{nn}(y,z)\,\boldsymbol{\varepsilon}_n \end{aligned} \tag{8}$$

Young's modulus, E, is supposed to vary versus y and z according to the following exponential law:

$$E(y,z) = E_0 e^{(\alpha_1 y + \beta_1)} e^{(\alpha_2 z + \beta_2)} \tag{9}$$

Poisson's ratio, ν, is considered to be constant. Matrixes $\mathbf{C}_{pp}$, $\mathbf{C}_{nn}$, $\mathbf{C}_{pn}$ and $\mathbf{C}_{np}$ in Eqs. (8) are:

$$\begin{gathered} \mathbf{C}_{pp} = \begin{bmatrix} C_{22} & C_{23} & 0 \\ C_{23} & C_{33} & 0 \\ 0 & 0 & C_{44} \end{bmatrix} e^{(\alpha_1 y + \alpha_2 z)}, \quad \mathbf{C}_{nn} = \begin{bmatrix} C_{11} & 0 & 0 \\ 0 & C_{66} & 0 \\ 0 & 0 & C_{55} \end{bmatrix} e^{(\alpha_1 y + \alpha_2 z)}, \\ \mathbf{C}_{pn} = \mathbf{C}_{np}^T = \begin{bmatrix} C_{12} & 0 & 0 \\ C_{13} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{(\alpha_1 y + \alpha_2 z)} \end{gathered} \tag{10}$$

where the constant stiffness coefficients C_{ij} are:

$$\begin{aligned} C_{ii} &= \frac{1-\nu}{(1+\nu)(1-2\nu)} E_0 e^{(\beta_1+\beta_2)} \quad i=1,2,3 \\ C_{jj} &= \frac{1}{2(1+\nu)} E_0 e^{(\beta_1+\beta_2)} \quad j=4,5,6 \\ C_{ij} &= \frac{\nu}{(1+\nu)(1-2\nu)} E_0 e^{(\beta_1+\beta_2)} \quad i=1,2 \quad j=i+1,3 \end{aligned} \tag{11}$$

3. Hierarchical Beam Theories

The variation of the displacement field towards y- and z-axis is a-priori postulated in the following manner:

$$\mathbf{u}(x,y,z) = F_\tau(y,z)\,\mathbf{u}_\tau(x) \quad \text{with} \quad \tau = 1,\ 2,\ \ldots,\ N_u \tag{12}$$

N_u stands for the number of unknowns and depends upon the theory approximation order, N, that is a free parameter of the formulation. Repeated indexes mean summation. Problem's governing differential equations and boundary conditions can be derived in a unified manner in terms of a single 'fundamental nucleo'. The complexity related to higher than classical approximation terms is tackled and the theoretical formulation is valid for the generic approximation order and approximating functions $F_\tau(y,z)$. In this paper, F_τ are Mac Laurin's polynomials functions, see table 1. The actual

Table 1. Mac Laurin's polynomials via Pascal's triangle.

N	N_u	F_τ
0	1	$F_1 = 1$
1	3	$F_2 = y \quad F_3 = z$
2	6	$F_4 = y^2 \quad F_5 = yz \quad F_6 = z^2$
3	10	$F_7 = y^3 \quad F_8 = y^2 z \quad F_9 = yz^2 \quad F_{10} = z^3$
...	...	...
N	$\frac{(N+1)(N+2)}{2}$	$F_{\frac{(N^2+N+2)}{2}} = y^N \quad F_{\frac{(N^2+N+4)}{2}} = y^{N-1}z \quad \ldots \quad F_{\frac{N(N+3)}{2}} = yz^{N-1}$ $F_{\frac{(N+1)(N+2)}{2}} = z^N$

governing differential equations and boundary conditions due to a fixed approximation order and polynomials' type are obtained straightforwardly via summation of the nucleo corresponding to each term of the expansion. The generic, N-order displacement field is:

$$\begin{aligned} u_x &= u_{x1} + u_{x2}y + u_{x3}z + \cdots + u_{x\frac{(N^2+N+2)}{2}} y^N + \cdots + u_{x\frac{(N+1)(N+2)}{2}} z^N \\ u_y &= u_{y1} + u_{y2}y + u_{y3}z + \cdots + u_{y\frac{(N^2+N+2)}{2}} y^N + \cdots + u_{y\frac{(N+1)(N+2)}{2}} z^N \\ u_z &= u_{z1} + u_{z2}y + u_{z3}z + \cdots + u_{z\frac{(N^2+N+2)}{2}} y^N + \cdots + u_{z\frac{(N+1)(N+2)}{2}} z^N \end{aligned} \tag{13}$$

Higher order models yield a more detailed description of the shear mechanics (no shear correction coefficient is required), the transverse to the section deformations, the coupling of the spatial directions due to Poisson's effect and the torsional mechanics than classical models do. Classical beam models, such as Timoshenko's and Euler-Bernoulli's can be straightforwardly derived from the first-order approximation model. For more details, refer to Carrera and Giunta [9].

4. Governing Equations

The governing differential equations and the boundary conditions are obtained in terms of the displacement components through the Principle of Virtual Displacements:

$$\delta L_i = \delta L_e \tag{14}$$

L_i represents the strain energy. L_e stands for the external work. δ stands for a virtual variation. By substitution of the geometrical relations, Eqs. (6), the material constitutive equations, Eqs. (8), and the unified hierarchical approximation of the displacements, Eq. (12), and after integration by parts, the virtual strain energy becomes:

$$\delta L_i = \int_l \delta \mathbf{u}_\tau^T \; \mathbf{K}^{\tau s} \; \mathbf{u}_s \; dx + \left[\delta \mathbf{u}_\tau^T \; \mathbf{\Pi}^{\tau s} \; \mathbf{u}_s\right]_{x=0}^{x=l} \tag{15}$$

The components of the differential matrix $\mathbf{K}^{\tau s}$ are:

$$\begin{aligned}
K_{xx}^{\tau s} &= \left(J_{\tau,ys,y}^{66k} + J_{\tau,zs,z}^{55k} - J_{\tau s}^{11k} \frac{\partial^2}{\partial x^2} \right)_k, \\
K_{yy}^{\tau s} &= \left(J_{\tau,ys,y}^{22k} + J_{\tau,zs,z}^{44k} - J_{\tau s}^{66k} \frac{\partial^2}{\partial x^2} \right)_k, \\
K_{zz}^{\tau s} &= \left(J_{\tau,ys,y}^{44k} + J_{\tau,zs,z}^{33k} - J_{\tau s}^{55k} \frac{\partial^2}{\partial x^2} \right)_k, \\
K_{xy}^{\tau s} &= \left(-J_{\tau s,y}^{12k} + J_{\tau,ys}^{66k} \right)_k \frac{\partial}{\partial x}, \quad K_{yx}^{\tau s} = \left(J_{\tau,ys}^{12k} - J_{\tau s,y}^{66k} \right)_k \frac{\partial}{\partial x}, \\
K_{xz}^{\tau s} &= \left(-J_{\tau s,z}^{13k} + J_{\tau,zs}^{55k} \right)_k \frac{\partial}{\partial x}, \quad K_{zx}^{\tau s} = \left(J_{\tau,zs}^{13k} + J_{\tau s,z}^{55k} \right)_k \frac{\partial}{\partial x}, \\
K_{yz}^{\tau s} &= \left(J_{\tau,ys,z}^{23k} + J_{\tau,zs,y}^{44k} \right)_k, \quad K_{zy}^{\tau s} = \left(J_{\tau,zs,y}^{23k} + J_{\tau,ys,z}^{44k} \right)_k
\end{aligned} \tag{16}$$

Subscripts preceded by comma represent derivation versus the corresponding spatial coordinates. The generic term $J_{\tau,(\phi)s,(\xi)}^{ghk}$ with $\phi, \xi = y, z$ and $g, h = 1, 2, \ldots, 6$ is the cross-section inertial momentum of a k sub-domain accounting for the material gradation:

$$J_{\tau,(\phi)s,(\xi)}^{ghk} = \int_{\Omega^k} C_{gh}^k e^{(\alpha_1 y + \alpha_2 z)} F_{\tau,(\phi)} F_{,(\xi)} \; d\Omega \tag{17}$$

As far as the boundary conditions are concerned, the components of $\mathbf{\Pi}^{\tau s}$ are:

$$\begin{gathered}\Pi_{xx}^{\tau s}=\left(J_{\tau s}^{11k}\right)_k\frac{\partial}{\partial x},\quad \Pi_{xy}^{\tau s}=\left(J_{\tau s,y}^{12k}\right)_k,\quad \Pi_{xz}^{\tau s}=\left(J_{\tau s,z}^{13k}\right)_k,\\ \Pi_{yy}^{\tau s}=\left(J_{\tau s}^{66k}\right)_k\frac{\partial}{\partial x},\quad \Pi_{yx}^{\tau s}=\left(J_{\tau s,y}^{66k}\right)_k,\quad \Pi_{yz}^{\tau s}=0,\\ \Pi_{zz}^{\tau s}=\left(J_{\tau s}^{55k}\right)_k\frac{\partial}{\partial x},\quad \Pi_{zx}^{\tau s}=\left(J_{\tau s,z}^{55k}\right)_k,\quad \Pi_{zy}^{\tau s}=0\end{gathered}\tag{18}$$

The virtual work done by the external loadings is obtained in a similar manner. For sake of brevity it is not reported here. For more details refer to Carrera and Giunta [9] The differential equations and related boundary conditions are solved via a Navier type solution:

$$\left(u_{x\tau},u_{y\tau},u_{z\tau}\right)=F_\tau\left(U_{x\tau}\cos\left(\frac{m\pi}{l}x\right),U_{y\tau}\sin\left(\frac{m\pi}{l}x\right),U_{z\tau}\sin\left(\frac{m\pi}{l}x\right)\right)\tag{19}$$

upon assumption that the external loadings vary towards x in a similar manner. In the case of a pressure loading p_{yy}, for instance:

$$p_{yy}=P_{yy}\sin\left(\frac{m\pi}{l}x\right)\tag{20}$$

where m represents the half-wave number along the beam axis. $\{U_{i\tau}:i=x,y,z\}$ are the maximal amplitudes of the displacement components and P_{yy} the maximal amplitude of the surface loading.

5. Numerical Results

A square cross-section is considered, see Fig. 1. Dimension a along y and z axes equals 0.1 m. Deep beams ($l/a=5$) are investigated. A unit maximal amplitude (1 MPa) for the surface loading is assumed. The material exhibits a gradation along y and z directions both: $E_0=1000$ MPa, α_i such that $E\left(a,0\right)/E_0=E\left(0,a\right)/E_0=3$ and $\beta_i=0$ with $i=1$ and 2, see Fig. 2. Poisson's ratio equals 0.3. The proposed models are compared with a three-dimensional FEM solution (addressed as FEM 3D) developed via MSC.Nastran commercial code. The eight-node brick element "HEXA8" is used (see MSC.Nastran User's Guide [12]). Elements' sides measure $2\cdot10^{-3}$ m. Each element is considered as homogeneous by referring to the material properties at its centre point. Fig. 3 presents the cross-section deformation at $x/l=0.5$. Fourth-order model matches the reference FEM solution. The shape of the deformation is due to the Young modulus variation law: points at $z/a=1$ are stiffer than those at $z/a=0$. Beam undergoes bending and torsion. A first-order model yields an accurate description of the bending stress component σ_{xx} as shown in Figs. 4 and 5, where results are presented in the form of colour maps. Results are computed at beam mid-span. Normal stress component σ_{yy} computed via FEM 3D solution and fifth-order model is presented in Figs. 6 and 7. A good approximation of the normal stress component σ_{zz} is obtained via a sixth-order model as shown in Figs. 8 and 9. Shear stress component σ_{xz} via FEM 3D model and fifth-order theory is presented in Figs. 10 and 11. Results are evaluated for $x/l=0$. In Figs. 12 and 13, the colour maps of the shear stress component σ_{xy} are presented. A fifth-order model

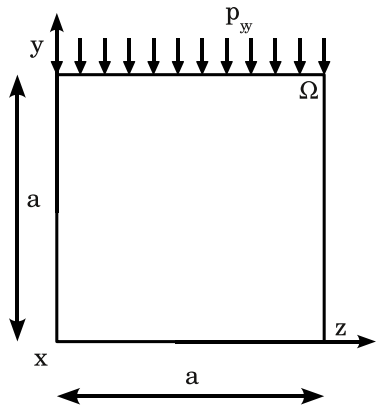

Figure 1. Square cross-section geometry and loading.

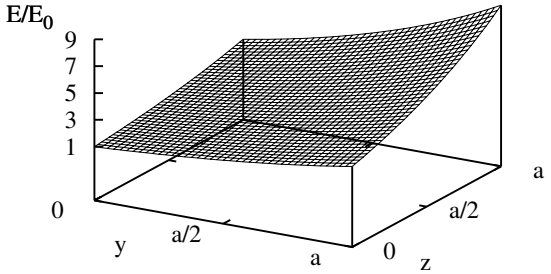

Figure 2. Young modulus variation along the cross-section.

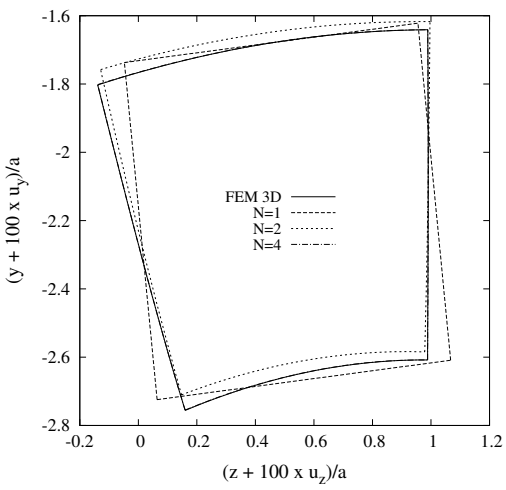

Figure 3. In-plane warping at $x/l = 0.5$.

has been used. As shown via the comparison with the reference FEM 3D solution, the proposed models yield an accurate description of the three-dimensional stress state. As far as the computational time is concerned, the proposed analytical models require less than a second, regardless the approximation order. The FEM solution based on the proposed models, not reported here, is obtained in few seconds for a very fine mesh. For the reference FEM 3D solution, the computational time is about five minutes.

6. Conclusions

A unified formulation for modelling of beam structures made of isotropic functionally graded materials has been proposed. Higher order models that account for shear deformations and in- and out-of-plane warping can be formulated straightforwardly.

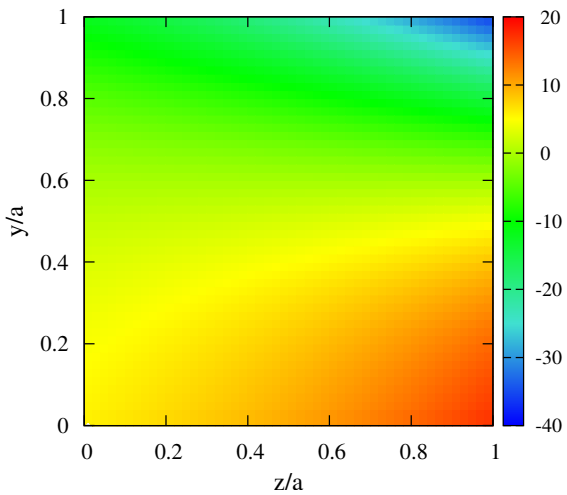

Figure 4. σ_{xx} (MPa) at mid-span via FEM 3D.

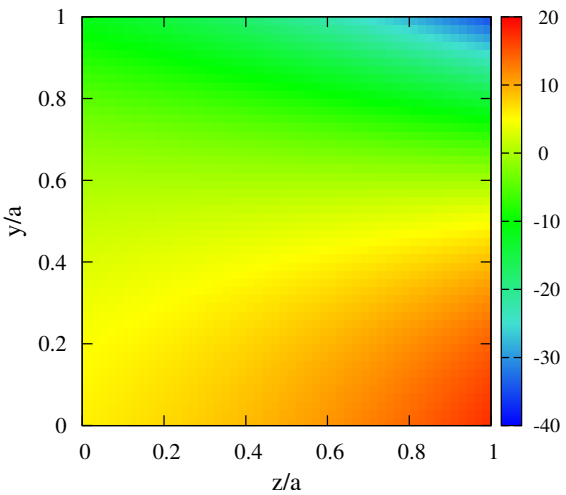

Figure 5. σ_{xx} (MPa) at mid-span via first-order model.

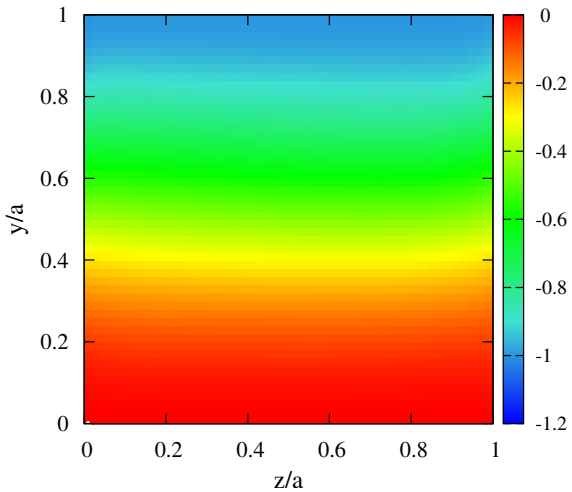

Figure 6. σ_{yy} (MPa) at mid-span via FEM 3D.

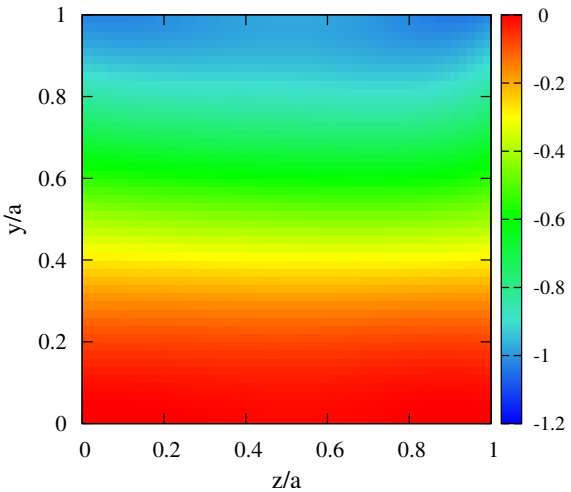

Figure 7. σ_{yy} (MPa) at mid-span via fifth-order model.

Closed form, Navier type solution has been adopted. Deep beams under a bending-torsion loading have been investigated. Young modulus have been assumed to vary versus the cross-section coordinates according to an exponential law function. It has been shown that the proposed formulation yields results as accurate as desired through

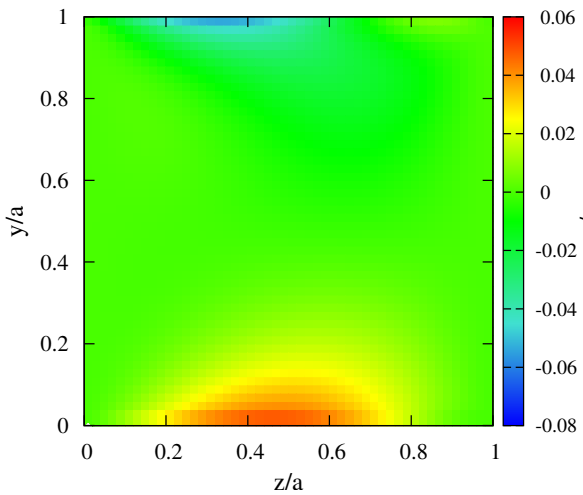

Figure 8. σ_{zz} (MPa) at mid-span via FEM 3D.

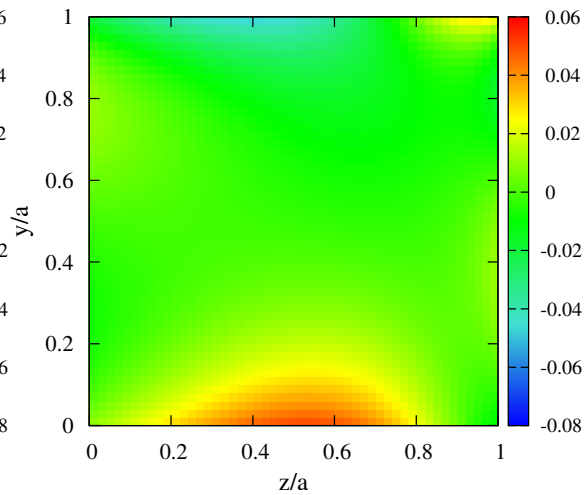

Figure 9. σ_{zz} (MPa) at mid-span via sixth-order model.

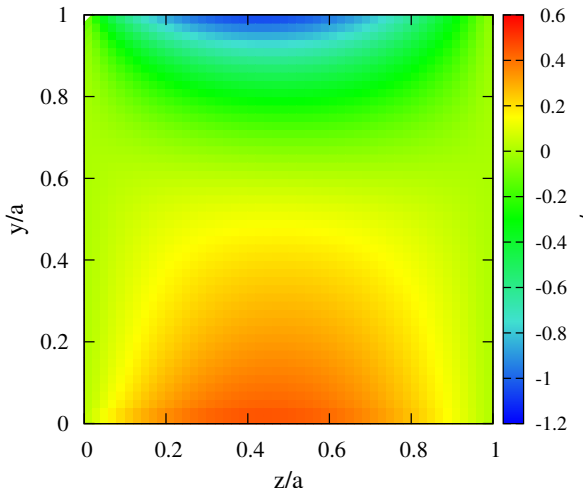

Figure 10. σ_{xz} (MPa) at $x/l = 0$ via FEM 3D.

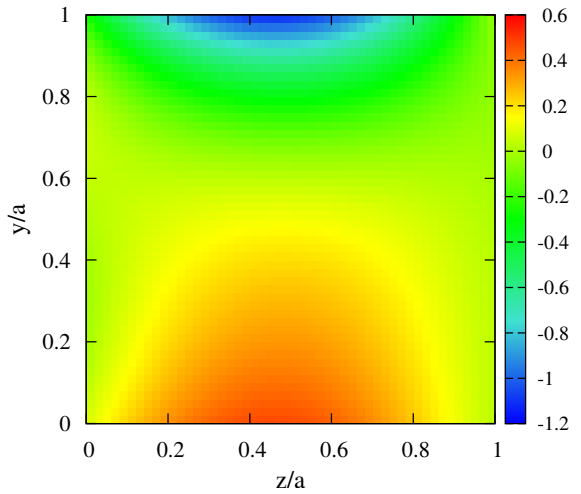

Figure 11. σ_{xz} (MPa) at $x/l = 0$ via fifth-order model.

an appropriate choice of the approximation order. Higher order models match the reference three-dimensional FEM solution. The efficiency of the models is very high. The computational time is of the order of 10^{-1} seconds, while three-dimensional FEM models require several minutes. The assumption of a polynomial law function for the

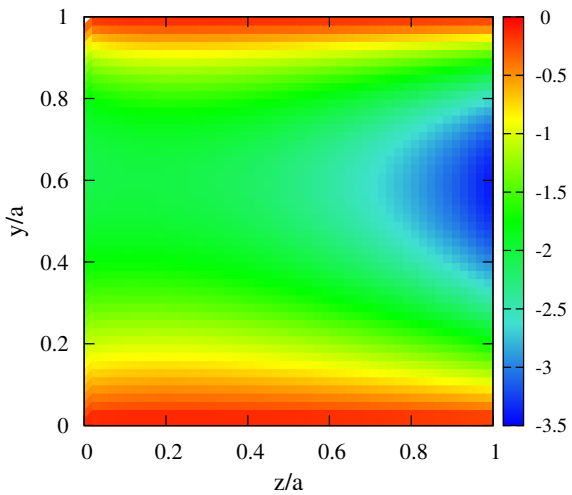

Figure 12. σ_{xy} (MPa) at $x/l = 0$ via FEM 3D.

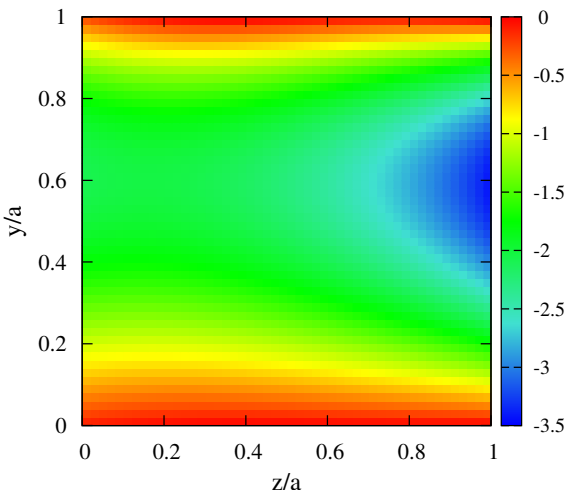

Figure 13. σ_{xy} (MPa) at $x/l = 0$ via fifth-order model.

material properties will be matter of further investigation. In such a manner, a generic gradation law can be accounted for via its polynomial approximation.

Acknowledgments

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References

[1] Suresh S and Mortensen A 1998 *Fundamentals of Functional Graded Materials* (IOM Communications Limited, London, UK)
[2] Miyamoto Y, Kaysser W A, Rabin B H, Kawasaki A and Ford R G 1999 *Functionally Graded Materials: Design, Processing and Applications* (Kluwer Academic, Boston, MA)
[3] Watanabe R, Nishida T and Hirai T 2003 *Metals and Materials International* **9** 513–519
[4] Chakraborty A, Gopalakrishnan S and Reddy J N 2003 *International Journal of Mechanical Sciences* **45** 519–539
[5] Li X F 2008 *Journal of Sound and Vibration* **318** 1210–1229
[6] Kadoli R, Akhtara K and Ganesanb N 2008 *Applied Mathematical Modelling* **32** 2509–2525
[7] Sankar B V 2001 *Composites Science and Technology* **61** 689–696
[8] Zhu H and Sankar B 2004 *Journal of Applied Mechanics* **71** 421–424
[9] Carrera E and Giunta G 2010 *International Journal of Applied Mechanics* **2**
[10] Carrera E, Giunta G, Nali P and Petrolo M 2010 *Computers and Structures* **88** 283–293
[11] Delale F and Erdogan F 1983 *Journal of Applied Mechanics* **50** 609–614
[12] MSC.Nastran 2002 *MSC.Nastran 2001, Getting Started with MSC.Nastran, User's Guide*