A Unified Finite Element Approach for Generalized Coupled Thermoelastic Analysis of 3D Beam-Type Structures, Part 1: Equations and Formulation

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abstract

An innovative 1D finite element (FE) approach is developed to analyze the 3D static, transient, and dynamic problems in the coupled and uncoupled thermoelasticity for the nonhomogeneous anisotropic materials. The Galerkin method is directly applied to the governing equations to obtained a weak formulation of the thermoelasticity problems with arbitrary loads and boundary conditions. To surmount the restrictions of the classical beam theories, a 1D FE procedure is proposed in the context of the Carrera Unified Formulation (CUF). Since coupled *thermoelastic analyses are computationally demanding, the proposed 1D FE approach can be employed as a powerful means to simulate the generalized coupled thermoelastic behavior of structures. This methodology, indeed, reduces the 3D problems to 1D models with 3D-like accuracies and very low computational costs. The Lord-Shulman and the Green-Lindsay models are considered as the generalized theories of thermoelasticity. Furthermore, as simplified cases, the classical coupled, dynamic uncoupled, quasi-static uncoupled and steady-state uncoupled theories of thermoelasticity may be derived from the formulation. Moreover, effects of the structural damping can be taken into account in the present formulation. The accuracy of the formulation has been evaluated through numerical simulations and comparisons, which have been presented in a companion paper (Part 2).*

Keywords: Coupled Thermoelasticity; Finite Element Method; Carrera Unified Formulation; Beam Theories.

Introduction

In the *static uncoupled thermoelasticity*, thermal effects on a body are restricted to strains due to a steady-state temperature distribution. As a more general theory of thermoelasticity, considering the transient heat conduction equation leads to time-dependent temperature distributions which can be used to obtain the transient thermal stresses. Such problems are called *quasi-static uncoupled thermoelasticity* problems. Alternatively, if external thermo-mechanical loads applied to the body vary adequately rapidly with the time so that inertia effects are excited, the inertia terms must be taken into account in the equations of motion. This theory is known as the *dynamic uncoupled thermoelasticity*. It is obvious that in all these theories, the temperature field is independently obtained from the heat conduction equation, while the displacement and stress fields are assumed to be dependent of the temperature.

When a structure is exposed to high-speed thermo-mechanical loads, the theories of uncoupled thermoelasticity may not provide entirely true physical behaviors. To avoid this drawback, interactions of the mechanical state of the elastic body on the temperature filed may be also simulated using the theories of *coupled thermoelasticity*. In these theories, the time derivatives of strain appear in the heat conduction equation so as to lead to the coupling between elasticity and energy equations. Accordingly, to find the solution for temperature and displacement fields and finally stresses, these coupled equations must be solved concurrently.

The thermoelasticity equations with the coupling effect was introduced by Duhamel [1] in 1837, for the first time, and then 120 years later, Biot [2], in 1956, presented the theory of classical thermoelasticity based on the principles of the irreversible thermodynamics. A history of thermoelasticity can be further found in the textbooks [3, 4].

Thus, under thermo-mechanical shock loading, the inertia and coupling effects can play important role in the thermoelastic behavior of a body. However, it has been shown that the coupling term may be more effective on the temperature and stress distributions than the inertia term in such situations (see [3, 4]).

Applications of the coupled thermoelasticity in advanced structural design problems have attracted the attention of many researchers during the second half of the last century. These applications can range from aerospace structures to fast-burst reactors, pulsed lasers, and particle accelerators which can supply sudden heat pulses in extremely short periods of time [5]. For instance, in ultra-fast pulsed lasers which is employed for nondestructive detection [6], measurement of material properties [7] and micro-machining [8], the heat pulse may be imposed in an order of Pico-second or less. The nature of the heat transfer mechanism instantaneously after the imposition of the pulse and the resulting temperature distribution at the surface of the body are some matters of interest in such applications.

Due to the parabolic nature of the heat conduction equation in the classical theory of thermoelasticity, the thermal disturbances are predicted to propagate with infinite speed through the elastic body. This prediction may be sufficiently accurate for most engineering applications. However, it is not physically realistic and acceptable in some practical problems involving high thermal loads at extremely short time intervals or very low temperatures near the absolute zero. Indeed, in such cases, the classical theory is not well able to detect thermal wave disturbances (see [9]).

In order to overcome this drawback, several non-classical models of the coupled thermoelasticity with the finite speed of the thermal wave propagation were introduced. Typically, these models are known as the *generalized theories of thermoelasticity*. Among these theories, Lord-Shulman (LS), Green-Lindsay (GL) and Green-Naghdi (GN) are the most well-known models. The detailed discussions of the generalized thermoelasticity with finite wave speeds have been presented in [10].

In general, analytical solutions of the coupled thermoelasticity problems are mathematically laborious, so that many simplifying assumptions may be required to achieve a closed form solution for such problems. A survey of the literature indicates that the number of articles using analytical methods to solve the problems is limited. Most of the analytical studies are restricted to the basic problems such as the infinite space, half-space and layer, where the boundary conditions are simple (see [11, 12]). For some bounded problems with simple initial and boundary conditions, analytical solutions of the coupled thermoelastic equations have been reported by a few investigators. Among them it may be referred to the exact solutions for beam [13], and rectangular plate [14] problems, as well as a one-dimensional (1D) axisymmetric solution for spherical [15], cylindrical [16] and disk [17–19] problems.

Indeed, the exact solution for more sophisticated geometries and boundary conditions as well as for all theories of coupled thermoelasticity is not available in the literature. Accordingly, the development of alternative solution techniques including semi-analytical and numerical methods has been essential. In order to obtain numerical solution of the coupled thermoelastic problems, the finite difference (FD), the finite element (FE) and the boundary element (BE) methods have been used. Among the procedures, however, the FE method is more widely employed for this class of problems due to the adaptability of this method.

The finite element formulation of the thermoelasticity problems can be derived from the variational approach and the weighted residual techniques. For elastic continuum, the variational approach is based on the application of variational calculus, which deals with the extremization of the total potential and kinetic energies, while in the weighted residual methods; the governing equations are multiplied by a weighting function and then averaged over the domain.

In the beginning, based on the variational principle, Wilson and Nickell [20] developed FE formulations for the heat conduction equation without the mechanical coupling term, and Fujino and Ohsaka [21] presented a FE solution to static uncoupled thermoelasticity problems. Later, Nickell and Sackman [22] further presented

FE formulations through the variational approach to solve the coupled thermoelastic equations in a half-space problem. A complete discussion of the variational approach used to thermoelasticity has been presented in the book by Hetnarski and Ignaczak [23].

On the other hand, the weighted residual methods along with an unconditionally stable implicit-explicit procedure were employed to the dynamic coupled thermoelasticity problem by Liu and Chang [24]. Furthermore, Eslami and Salehzadeh [25] applied the weighted residual method based on Galerkin technique to develop a finite element formulation for coupled thermoelasticity. Then, this formulation was employed to solve a 1D rod problem by Eslami and Vahedi [26].

Due to the contentious definition of functional for the first law thermodynamics, in deriving the coupled thermoelastic equations by the variational calculus approach, some drawbacks may be incurred. However, the weighted residual techniques such as Galerkin method which convert directly the governing equations to a weak formulation are quite efficient in the convergence rate compared to the conventional method [4].

Although the present study is not a review paper, but in the following, it attempts to survey a number of articles in which the FE method has been used to solve the transient thermoelasticity problems. It should be also noted that the following papers are listed in chronological order and essentially, more emphasis is on the type of problem solved by researchers.

The coupled thermoelastic problem in a long cylinder exposed to a specified thermo-mechanical boundary conditions was solved by Li *et al.* [27]. They used several different techniques for the spatial and time discretization in the FE method to demonstrate the proper numerical techniques for this problem. Carter and

Booker [28] also solved the 1D classical coupled thermoelastic equations for an infinite cylinder. However, all these studies had been done assuming that the thermal disturbances propagate with infinite speed in the elastic medium.

A FE formulation of GL thermoelasticity model was presented by Prevost and Tao [29]. They applied an implicit-explicit scheme to solve the equations for a semi-infinite slab problem subject to surface thermal load. Chen and Weng [30] proposed a transfinite element method, in which the combination of the FE method and Laplace transform technique is employed, to analyze the generalized coupled thermoelasticity problems based on LS, GL and GN models. That is, the problems can be solved in the Laplace transform domain by the FE method and then the transformed solution are numerically inverted to find the physical time domain response. Using this approach, Chen and Weng presented solutions for the cylinder with infinite length and layer problems in [30].

Farhat *et al.* [31] obtained the FE equations for the classical coupled thermoelasticity by Galerkin method and then proposed an implicit-implicit staggered technique to solve the equations. In this paper, the accuracy of the proposed algorithm has been demonstrated by solving half-space and infinitely long shaft problems. Likewise, the FE method along with an explicit time integration architecture were applied by Tamma and Namburu [32] to solve the GL thermoelasticity problem in an 1D half-space.

For a hollow sphere problem subjected to specified boundary conditions, Eslami and Vahedi [33] presented the FE formulation of the classical coupled thermoelasticity under spherical symmetry condition by using Galerkin method. Eslami *et al.* [34] studied the coupled thermoelastic behavior of an axially symmetric cylindrical shell, as well. In these studies, the FE equations were solved by a time marching technique. In addition, using an axisymmetric FE formulation, the coupled thermoelastic response of a functionally graded cylinder subjected to specified boundary conditions was investigated by Reddy and Chin [35].

Cannarozzi and Ubertini [36] derived a variational form of the coupled quasi-static thermoelasticity, in which the elastic equation is stated as the hybrid stress formulation while the mixed flux-temperature formulation is used for the heat equation. In the FE implementation, they developed three quadrilateral elements and assessed characteristics of the proposed approach through some numerical test cases. Based on the first-order shear deformation theory, Chakraborty *et al.* [37] presented a FE formulation for dynamic uncoupled thermoelasticity in functionally graded beam structures. In this paper, a beam element was developed to obtain a convergence stiffness matrix and eliminate the shear locking effect of the element. In addition, Chakraborty and Gopalakrishnan [38] investigated generalized thermoelastic responses in an anisotropic layered medium based on LS and GL theories. They used the spectral FE method to capture the propagation of thermoelastic waves inside the medium.

In a series of papers, using Galerkin FE method, Eslami *et al.* rendered a 1D classical and generalized thermoelasticity solution for annular isotropic [39] and functionally graded [40] disk problems, functionally graded layer problems [41], functionally graded sphere problems [42] and functionally graded beams [43]. These authors employed the same procedure as proposed by [30] to obtain the solutions. In addition, the magneto-thermoelastic behavior of a semi-infinite plate subjected to a magnetic and a thermal shock was investigated by Tian and Shen [44]. They considered the GL model as the generalized thermoelasticity theory and solved the dynamic FE equations directly in time-domain. Abbas *et al.* presented generalized thermoelastic solutions for axisymmetric cylinder [45] and half-space [46] problems based on the LS and GL models. In these studies, the weak formulations were obtained by Galerkin finite element method and then the Newmark time integration scheme was employed to solve the equations. The thermoelastic response of a 1D layered region subjected to thermal shock load was analyzed by Hosseini Zad [47] based on the different theories of coupled thermoelasticity. Darabseh *et al.* [48] considered the coupled thermoelastic problem in a functionally graded thick hollow cylinder under thermal loading. These authors used the GL theory of thermoelasticity and solved the governing equations by using Galerkin FE method. Galerkin FE method along with a traditional time domain integral method were employed by Guo [49] to solve the LS coupled thermoelastic problems in one- and two-dimensional models. Based on the LS and GL generalized theories, Filopoulos *et al.* [50] derived coupled thermoelastic models for nonlocal thermo-mechanical problems in micro-structures. Moreover, they solved a 1D slender bar problem to demonstrate how their models work.

The generalized coupled thermoelastic problem in an axisymmetric infinite cylinder subjected to specified boundary conditions was analyzed by Zenkour and Abbas [51] based on LS theory. In this paper, the transient solution of the FE equations was evaluated directly from the model at any time. The classical coupled thermoelastic problem in a plate subjected to a hypersonic re-entry flow was analyzed by Li *et al.* [52]. They employed the Newmark method and Crank-Nicolson scheme to discretize the equation of motion and heat conduction equation in the time domain, respectively. Furthermore, in this paper, the Rayleigh damping was taken into account in the equation of motion.

Furthermore, the effect of material microstructure on the classic coupled thermoelastic behavior in a 1D half-space was studied by Papathanasiou *et al.* [53]. These investigators used the gradient elasticity theory to model the microstructure influences and applied the FE and time integration methods to solve the governing equations. The classical coupled thermoelastic response in a functionally graded annular plate imposed to lateral thermal shock load was investigated by Jafarinezhad and Eslami [54]. In this paper, the first order shear deformation plate theory was used to obtain the equations of motion and the temperature distribution across the thickness was be approximated by a second order polynomial. These authors utilized the same procedure as used in [30, 39–43].

Analysis of the works reviewed above concerning finite element solution of the coupled thermoelasticity problems may lead to the following inferences

- Due to use of the coupled thermoelasticity theories in advanced structural design, they are still topics of active research.
- After over half a century of application of FE method in thermoelasticity, this method is still applied as a powerful numerical tool in such problems.
- The major presented studies deal with the coupled thermoelasticity response in the basic problems including an infinite medium, a half-space and a layer as well as in the axisymmetric problems. Moreover, two- or three-dimensional coupled thermoelastic solutions for some simple problems may be found in just a few number of articles.
- Most of the investigators applied Galerkin technique to the governing equations to obtain a weak formulation of the problem, especially for the theories of generalized thermoelasticity.

In more practical problems, the geometry of structures as well as boundary and loading conditions are complex. Furthermore, in recent years, advanced new materials are used in modern structures under thermo-mechanical shock loads, so that, the 1D and 2D FE models are not able to provide all the desired information. Accordingly, the 3D FE modeling techniques may be required for a detailed coupled thermoelastic analysis in such cases.

Notwithstanding substantial advances in the field of computing power, a 3D FE model still imposes large computational costs, especially during a time-consuming transient solution of the coupled thermoelastic problems. This can explain why there is a growing interest in the development of refined FE models to 3D solution of such problems with lower computational efforts.

A general approach which can be employed to develop refined finite element models has been suggested in the book by Carrera *et al.* [55]. They introduced the Carrera Unified Formulations (CUF) in which the FE methods are formulated on the basis of a class of theories of structures. In fact, Carrera *et al.* [56] first developed a unified formulation for the 2D FE method to overcome the limitations of classical theories of plates and shells. A 1D FE method in framework of the CUF was later extended by Carrera *et al.* [57] based on the beam model to go beyond the classical beam theories. Indeed, the CUF has been able to enhance the capabilities of the 1D and 2D conventional finite element methods, so that using these refined methods leads to 3D-like solutions but with lower computational costs. Furthermore, analysis of multi-field problems such as mechanical, thermal, electric and magnetic fields, as well as of layered structures is of other outstanding features of the CUF models. Thus, using 2D FE-CUF approach, the static uncoupled thermoelasticity problems were solved in multilayered plates [58] and laminated composites shells and plates [59]. In addition, the static uncoupled thermoelastic behavior in laminated beams was successfully investigated by Giunta *et al.* [60] where a class of 1D FE-CUF models was employed to model the 3D structure. However, in all these papers the heat conduction equation is assumed in a steady-state condition and independent of the equation of motion in which the inertia effects are also ignored. Moreover, recently, the authors has successfully extend a class of the 1D FE-CUF models for static uncoupled thermoelastic analysis of some complex structures with non-uniform cross-sections such as variable thickness disks and rotors in [61, 62]. Accordingly, the FE-CUF models may be able to analyze more complicated thermoelasticity problems containing the coupling and inertia effects.

The main objective of the present paper is to propose a refined FE approach in the context of the 1D CUF for 3D solution of the generalized coupled thermoelasticity problems. Since among all of the generalized theories, the LS and GL theories are of most interest to researchers, only these two theories are considered in the present study. The paper has been organized as follows. In Section 2, the basic equations of the generalized theories of thermoelasticity in a general form are presented. Section 3 quotes application of the Galerkin technique to obtained a weak formulation of the thermoelasticity problems. Section 4 briefly presents 1D FE-CUF approach for the coupled thermoelastic problems. In Sections 5, the FE equations are obtained in CUF Form as the so-called fundamental nuclei and then the formulation is extracted for the different theories of thermoelasticity, and finally Section 6 gives a summary of conclusion.

Furthermore, a companion paper [63] provides some numerical evaluations related to the FE formulation proposed herein.

Governing Basic Equations

The equation of motion for a 3D elastic body in the physical coordinate system (x, y, z) is stated as follows [4, 52]

$$\sigma_{ij,j} + X_i = \rho \ddot{u}_i + \zeta \dot{u}_i \tag{1}$$

where σ_{ij} and u_i are stress and displacement components, respectively. X_i denotes body forces per unit volume, ρ is mass density, and ζ stands for the damping coefficient of the material. Likewise, the superscript dot $(\cdot)$ and the subscript comma (,) indicate the derivatives with respect to the time (t) and the space variables (x, y, z), respectively.
In addition, the strain-displacement relations within the linear context of small deformation theory are expressed as [4]

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

and according to GL generalized theory, Hooke's law for a nonhomogeneous anisotropic thermoelastic material can be written as [4]

$$\sigma_{ij} = C_{ijpq}\epsilon_{pq} - \beta_{ij}(T + t_1 \dot{T}) \tag{3}$$

where C_{ijpq} is a fourth-order tensor containing all the elastic coefficients of a general nonhomogeneous anisotropic material. $\beta_{ij} = C_{ijpq}\alpha_{pq}$ is the second order tensor of thermoelastic moduli where in α_{ij} is the coefficient of thermal expansion tensor. In Eq. (3), T denotes the temperature change relative to the reference temperature

T_0, so that this temperature difference creates thermal strain. Also, t_1 is one of the two relaxation times defined by Green and Lindsay.

On the other hand, the energy balance equation can be expressed as [4]

$$q_{i,i} = R - T_0\dot{\mathrm{S}} \tag{4}$$

where q_i is heat flux vector and R stands for internal heat source per unit volume per unit time. S denotes entropy per unit volume and is given by the following relationship [4]

$$\mathrm{S} = \frac{\rho c}{T_0}(T + t_2\dot{T}) + \beta_{ij}\epsilon_{ij} - \frac{1}{T_0}\tilde{c}_i T_{,i} \tag{5}$$

Here, c is specific heat, while t_2 and $\tilde{c}_i$ are another relaxation time and a vector of material new constants, respectively, proposed by Green and Lindsay. Moreover, based on LS and GL theories of thermoelasticity the heat conduction equation for an anisotropic material can be stated as [4]

$$q_i + t_0\dot{q}_j = -\kappa_{ij}T_{,j} - \tilde{c}_i\dot{T} \tag{6}$$

where κ_{ij} is the thermal conductivity tensor and t_0 is relaxation time associated with LS theory.

Equations (1) to (3) may be combined to give the equation of motion in term of the displacement components as

$$(C_{ijkl}u_{k,l})_{,j} - (\beta_{ij}T)_{,j} - (t_1\beta_{ij}\dot{T})_{,j} + X_i = \rho\ddot{u}_i + \zeta\dot{u}_i \tag{7}$$

Likewise, by using Eqs. (4)-(6) and Eq. (2), the energy equation can be expressed in terms of the temperature and displacement fields as

$$\begin{aligned}&\rho c(t_0+t_2)\ddot{T}+\rho c\dot{T}-2\tilde{c}_i\dot{T}_{,i}-(\kappa_{ij}T_{,j})_{,i}\\&+t_0T_0\beta_{ij}\ddot{u}_{i,j}+T_0\beta_{ij}\dot{u}_{i,j}=R+t_0\dot{R}\end{aligned}\tag{8}$$

Equations (7) and (8) constitute the governing system of equations for the generalized coupled thermoelasticity problems based on the LS (for $t_1 = t_2 = \tilde{c}_i = 0$) and GL (for $t_0 = 0$) theories in an anisotropic and nonhomogeneous medium. In these equations, the derivatives of the relaxation times and $\tilde{c}_i$ with respect to position variables are ignored. Thus, the four coupled equations, including three equations of motion and one heat conduction equation, under specified initial and boundary conditions must be simultaneously solved for the three unknown displacement components (u_i) and the one unknown temperature change (T).

FE Formulation through Galerkin Technique

To obtain a FE formulation of the governing equations (7) and (8), Galerkin technique may be utilized. In implementation of the conventional FE method, the 3D domain with the volume V can be discretized into a finite number of regular 3D solid elements. Thus, the components of displacement and temperature change in each base element can be approximated by identical shape functions as follows

$$\begin{aligned}&u_i^{(e)}(x,y,z,t)=\phi_m(x,y,z)U_i^m(t)\\&T^{(e)}(x,y,z,t)=\phi_m(x,y,z)\Theta^m(t)\end{aligned}\tag{9}$$

where $U_i^m(t)$ and $\Theta^m(t)$ are the displacement vector and the temperature change at each nodal point of the element. $\phi_m(x, y, z)$ denotes shape functions in the base element. It is noted that in these approximations, time and space variables are separated into distinct functions. Furthermore, the repeated subscript $m\,(m = 1, \cdots, r)$ is a dummy index and indicates summation while r stands for the number of nodal points in the element [4, 64].

According to Galerkin method, multiplying both sides of Eq. (1) by the shape functions ϕ_m and then integrating over volume of the element, yields

$$\int_{V^{(e)}} \left(\sigma_{ij,j} + X_i - \rho\ddot{u}_i - \zeta\dot{u}_i\right)\phi_m dV = 0 \tag{10}$$

On the first term of Eq. (10) the divergence theorem can be applied as

$$\int_{V^{(e)}} \left(\sigma_{ij,j}\right)\phi_m dV = \int_{S^{(e)}} \sigma_{ij} n_j \phi_m dS - \int_{V^{(e)}} \phi_{m,j}\sigma_{ij} dV \tag{11}$$

where n_j is the unit vector normal to the boundary surface of the element $S^{(e)}$. Substituting relation (11) into Eq. (10) gives

$$\begin{aligned} &\int_{S^{(e)}} \sigma_{ij} n_j \phi_m dS - \int_{V^{(e)}} \phi_{m,j}\sigma_{ij} dV + \int_{V^{(e)}} \left(X_i \phi_m\right) dV \\ &- \int_{V^{(e)}} \left(\rho\ddot{u}_i \phi_m\right) dV - \int_{V^{(e)}} \left(\zeta\dot{u}_i \phi_m\right) dV = 0 \end{aligned} \tag{12}$$

Furthermore, by using the relationship between the traction vector (t_i^n) acting on an arbitrary surface and the stress tensor, the first integral in Eq. (12) may be expressed as

$$\int_{S^{(e)}} \sigma_{ij} n_j \phi_m dS = \int_{S^{(e)}} \mathrm{t}_i^n \phi_m dS \tag{13}$$

Therefore, Eq. (12) can be rewritten as follows

$$\begin{aligned} &\int_{V^{(e)}} (\rho \ddot{u}_i \phi_m)\, dV + \int_{V^{(e)}} (\zeta \dot{u}_i \phi_m)\, dV + \int_{V^{(e)}} (\phi_{m,j} \sigma_{ij})\, dV \\ &= \int_{V^{(e)}} X_i \phi_m dV \; + \int_{S^{(e)}} \mathrm{t}_i^n \phi_m dS \end{aligned} \tag{14}$$

Similarly, applying Galerkin method to the energy equation (8) gives

$$\begin{aligned} &\int_{V^{(e)}} \Big(\rho c (t_0 + t_2) \ddot{T} + \rho c \dot{T} - 2 \tilde{c}_i \dot{T}_{,i} - \left(\kappa_{ij} T_{,j}\right)_{,i} \\ &+ t_0 T_0 \beta_{ij} \ddot{u}_{i,j} + T_0 \beta_{ij} \dot{u}_{i,j} - R - t_0 \dot{R} \Big) \phi_m dV = 0 \end{aligned} \tag{15}$$

where the weak form of the term $\left(\kappa_{ij} T_{,j}\right)_{,i}$ can be written according to the divergence theorem as

$$\int_{V^{(e)}} \left(\kappa_{ij} T_{,j}\right)_{,i} \phi_m dV = \int_{S^{(e)}} \left(\kappa_{ij} T_{,j} n_i \phi_m\right) dS - \int_{V^{(e)}} \left(\kappa_{ij} T_{,j} \phi_{m,i}\right) dV \tag{16}$$

and likewise substituting this form into Eq. (15) and rearranging the terms result in the following

$$\begin{aligned}
&\int\limits_{V^{(e)}} (t_0 T_0 \beta_{ij} \ddot{u}_{i,j} \phi_m) dV + \int\limits_{V^{(e)}} (t_0 \rho c \ddot{T} \phi_m) dV + \int\limits_{V^{(e)}} (t_2 \rho c \ddot{T} \phi_m) dV \\
&+ \int\limits_{V^{(e)}} (T_0 \beta_{ij} \dot{u}_{i,j} \phi_m) dV + \int\limits_{V^{(e)}} (\rho c \dot{T} \phi_m) dV - \int\limits_{V^{(e)}} (2 \tilde{c}_i \dot{T}_{,i} \phi_m) dV \\
&+ \int\limits_{V^{(e)}} (\kappa_{ij} T_{,j} \phi_{m,i}) dV = \int\limits_{S^{(e)}} (q_i n_i \phi_m) dS + \int\limits_{V^{(e)}} (R \phi_m) dV + \int\limits_{V^{(e)}} (t_0 \dot{R} \phi_m) dV
\end{aligned} \tag{17}$$

The system of equations (17) and (14) as well as the stress-strain relations (3) may be expressed in vector form as

$$\begin{aligned}
&\int\limits_{V^{(e)}} (\rho \ddot{\mathbf{u}} \phi_m) dV + \int\limits_{V^{(e)}} (\zeta \dot{\mathbf{u}} \phi_m) dV + \int\limits_{V^{(e)}} (\mathbf{D}^{\mathrm{T}} \phi_m \boldsymbol{\sigma}) dV \\
&= \int\limits_{V^{(e)}} (\mathbf{X} \phi_m) dV + \int\limits_{S^{(e)}} (\mathbf{t} \phi_m) dS
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\int\limits_{V^{(e)}} (t_0 T_0 \boldsymbol{\beta}^{\mathrm{T}} \mathbf{D} \ddot{\mathbf{u}} \phi_m) dV + \int\limits_{V^{(e)}} (t_0 \rho c \ddot{T} \phi_m) dV + \int\limits_{V^{(e)}} (t_2 \rho c \ddot{T} \phi_m) dV \\
&+ \int\limits_{V^{(e)}} (T_0 \boldsymbol{\beta}^{\mathrm{T}} \mathbf{D} \dot{\mathbf{u}} \phi_m) dV + \int\limits_{V^{(e)}} (\rho c \dot{T} \phi_m) dV - \int\limits_{V^{(e)}} (2 \tilde{\mathbf{c}}^{\mathrm{T}} \boldsymbol{\nabla} \dot{T} \phi_m) dV \\
&+ \int\limits_{V^{(e)}} (\boldsymbol{\nabla}^{\mathrm{T}} T \boldsymbol{\kappa} \boldsymbol{\nabla} \phi_m) dV = \int\limits_{S^{(e)}} (\mathbf{q}^{\mathrm{T}} \mathbf{n} \phi_m) dS + \int\limits_{V^{(e)}} (R \phi_m) dV + \int\limits_{V^{(e)}} (t_0 \dot{R} \phi_m) dV
\end{aligned} \tag{19}$$

$$\boldsymbol{\sigma} = \mathbf{C} \boldsymbol{\epsilon} - \boldsymbol{\beta} (T + t_1 \dot{T}) \tag{20}$$

Equations (14) and (17), or (18) and (19), represent the general weak formulation containing all possible boundary conditions for the generalized coupled thermoelasticity problems.

1D FE-CUF Approach for the Coupled Thermoelastic Problems

The 3D FE model presented in the previous Section, however, leads to more accurate results than the traditional 1D or 2D models, but the main drawback of this method is the significant increase of degrees of freedom (DOF) and, consequently, computational efforts. The computational competence is definitely reduced in a 3D model with enormous DOF especially in an iterative solution scheme of the dynamic coupled thermoelasticity problems. To lower the computational costs of such problems without loss of accuracy, refined 1D FE models in the framework of the CUF with 3D capabilities can be developed.

Consider an arbitrary structure subjected to thermo-mechanical shock loads which is located in the rectangular Cartesian coordinate system (x, y, z). As shown in Fig. 1, if the structure can be assumed as a beam along the y-direction, each cross section, whose centroid G, of the beam is defined in the xz-plane and perpendicular to the y-axis.

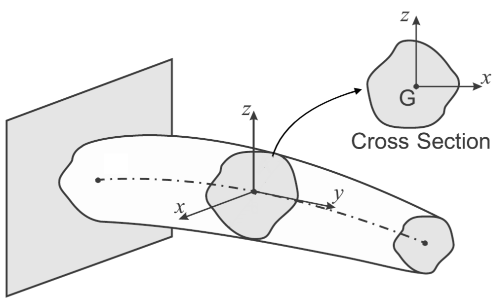

Figure 1: A beam structure with an arbitrary cross section

According to the traditional 1D FE procedure, the structure can be discretized into a finite number of 1D beam elements along the y-axis. In this case, as illustrated in Fig. 2-a, the approximate displacement and temperature fields in each element can be obtained by the beam shape functions $N_m(y)$ as

$$\begin{aligned} \mathbf{u} &= N_m \mathbf{u}^m \\ T &= N_m T^m \end{aligned} \tag{21}$$

in which $\mathbf{u}^m$ and T^m are the nodal displacement vector and temperature change, respectively. In addition, the dummy index $m\,(m = 1, \cdots, M)$ denotes the summation and M is the number of nodes in the beam element.

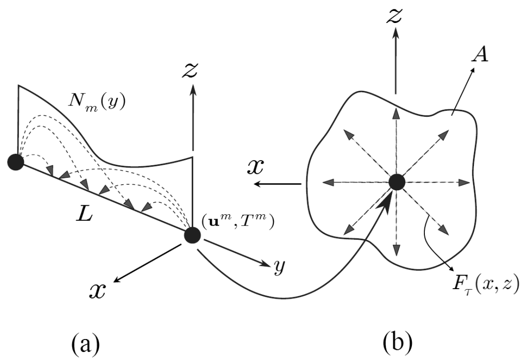

Figure 2: A beam base element

On the other hand, based on the unified formulation for beams presented by Carrea [57], to overcome the limitations of the classical beam theories such as the Euler-Bernoulli and the Timoshenko models, the distributions of displacements and the temperature over the cross section related to each node of the beam element can be described by an expansion of generic functions F_τ as

$$\begin{aligned} \mathbf{u}^m &= F_\tau \, \mathbf{U}^{m\tau} \\ T^m &= F_\tau \Theta^{m\tau} \end{aligned} \tag{22}$$

where F_τ are the functions of the cross section coordinates x and z (see Fig 2-b), $\mathbf{U}^{m\tau}(t) = \{U_x^{m\tau} \quad U_y^{m\tau} \quad U_z^{m\tau}\}^{\mathrm{T}}$ is the generalized displacement vector, and $\Theta^{m\tau}(t)$ denotes the generalized temperature change. Here, $\tau\,(\tau = 1, 2, \cdots, N_{\mathrm{CUF}})$ indicates summation, as well, while N_{CUF} is the number of terms of the expansion. The hierarchical capabilities of the presented unified formulation (22) play a essential role in dealing with variable kinematic models in a compact unified manner. The order of the model is taken into account as a free parameter of the analysis (i.e., as input) in this formulation. In other words, the refined models can be obtained with no need for ad hoc formulations. In Fig. 3, a 3D 8-nodes element is schematically compared with a 1D 2-nodes element refined by the CUF.

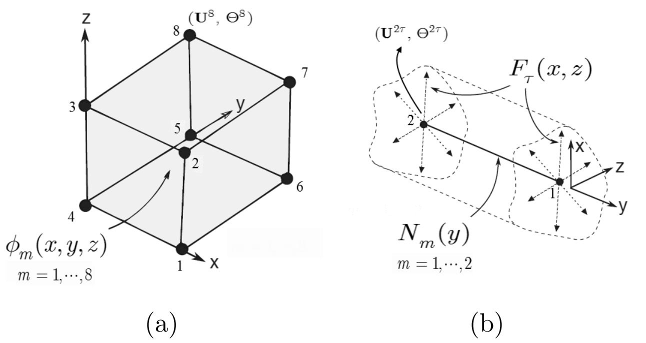

Figure 3: A schematic comparison between a 3D element and a refined 1D element, (a) a 3D 8-nodes element, (b) a refined 1D 2-nodes element.

Thus, comparing the relations (21) and (22) with Eq. (9) results in the following relationship

$$\phi_m(x, y, z) = N_m(y) F_\tau(x, z) \tag{23}$$

In this approach, three types of the beam element, two-, three- and four-nodes, may be used to give a linear, a quadratic and a cubic interpolation function of the displacement and temperature fields along the y-axis, respectively. Likewise, the selection of $F_\tau(x, z)$ and N_{CUF} is arbitrary. That is, various kinds of basic functions including polynomials, harmonics and exponentials of any-order can be assumed to predict the displacement components and temperature change on the beam cross section. For instance, different classes of polynomials such as Taylor, Legendre and Lagrange polynomials are extensively employed as approximation functions in the numerical modeling of structures. More details about the variable kinematic models and the interpolating functions can be found in [57].

FE Equations in CUF Form

The relations (9) and (23) can be substituted into Eqs. (18) and (19) to give

$$\begin{aligned}
&\int_{V^{(e)}} (\rho N_l F_s\, N_m F_\tau)\ddot{\mathbf{U}}^{ls} dV + \int_{V^{(e)}} (\zeta N_l F_s\, N_m F_\tau)\,\dot{\mathbf{U}}^{ls} dV \\
&- \int_{V^{(e)}} (t_1 \mathbf{D}^{\mathrm{T}} N_m F_\tau \boldsymbol{\beta} N_l F_s)\dot{\Theta}^{ls} dV + \int_{V^{(e)}} (\mathbf{D}^{\mathrm{T}} N_m F_\tau \mathbf{C}\mathbf{D} N_l F_s\,)\mathbf{U}^{ls} dV \\
&- \int_{V^{(e)}} (\mathbf{D}^{\mathrm{T}} N_m F_\tau \boldsymbol{\beta} N_l F_s)\Theta^{ls} dV = \int_{V^{(e)}} (\mathbf{X} N_m F_\tau) dV \;+ \int_{S^{(e)}} (\mathbf{t}^n N_m F_\tau) dS
\end{aligned} \tag{24}$$

$$
\begin{aligned}
&\int\limits_{V^{(e)}} (t_0 T_0 \boldsymbol{\beta}^{\mathrm{T}} \mathbf{D} N_l F_s \, N_m F_\tau) \ddot{\mathbf{U}}^{ls} dV \\
&+ \int\limits_{V^{(e)}} (t_0 \rho c N_l F_s N_m F_\tau + t_2 \rho c N_l F_s N_m F_\tau) \, \ddot{\Theta}^{ls} dV \\
&+ \int\limits_{V^{(e)}} (T_0 \boldsymbol{\beta}^{\mathrm{T}} \mathbf{D} N_l F_s \, N_m F_\tau) \dot{\mathbf{U}}^{ls} dV \\
&+ \int\limits_{V^{(e)}} (\rho c N_l F_s N_m F_\tau - 2 \mathbf{c}^{\mathrm{T}} \boldsymbol{\nabla} N_l F_s N_m F_\tau) \dot{\Theta}^{ls} dV \\
&+ \int\limits_{V^{(e)}} (\boldsymbol{\nabla}^{\mathrm{T}} N_l F_s \boldsymbol{\kappa} \boldsymbol{\nabla} N_m F_\tau) \Theta^{ls} dV \\
&= \int\limits_{S^{(e)}} (\mathbf{qn} N_m F_\tau) dS + \int\limits_{V^{(e)}} (R N_m F_\tau) dV + \int\limits_{V^{(e)}} (t_0 \dot{R} N_m F_\tau) dV
\end{aligned}
\tag{25}
$$

here, the indexes s and l are similar to τ and m, respectively, and indicate summation based on Einstein's notation. Equations (24) and (25) render the 1D unified finite element formulation which can be employed to 3D analysis of the generalized coupled thermoelastic problems.

The presented approach enables all the FE matrix and vectors to derived as a condensed notation which is named the so-called *fundamental nucleus* (FN). Indeed, theses fundamental nuclei do not depend on either the order of the expansion or the base functions used. Accordingly, the Eqs. (24) and (25) can be rewritten in matrix form as

$$
\begin{aligned}
&\begin{bmatrix} \mathbf{M}_{UU}^{lm\tau s} & 0 \\ \mathbf{M}_{\Theta U}^{lm\tau s} & \mathbf{M}_{\Theta\Theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{U}}^{ls} \\ \ddot{\Theta}^{ls} \end{Bmatrix} + \begin{bmatrix} \mathbf{G}_{UU}^{lm\tau s} & \mathbf{G}_{U\Theta}^{lm\tau s} \\ \mathbf{G}_{\Theta U}^{lm\tau s} & \mathbf{G}_{\Theta\Theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{U}}^{ls} \\ \dot{\Theta}^{ls} \end{Bmatrix} \\
&+ \begin{bmatrix} \mathbf{K}_{UU}^{lm\tau s} & \mathbf{K}_{U\Theta}^{lm\tau s} \\ 0 & \mathbf{K}_{\Theta\Theta}^{lm\tau s} \end{bmatrix} \begin{Bmatrix} \mathbf{U}^{ls} \\ \Theta^{ls} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{ls} \\ Q^{ls} \end{Bmatrix}
\end{aligned}
\tag{26}
$$

where the FN of the mass, damping and stiffness matrices as well as of the load vector can be expressed as follows

$$
\begin{aligned}
[\mathbf{M}_{UU}^{lm\tau s}]_{3\times3} &= \int_{L^{(e)}}\int_{A^{(e)}} (\rho N_m N_l\,\mathbf{I} F_\tau F_s)\,dAdL \\
[\mathbf{M}_{\Theta U}^{lm\tau s}]_{1\times3} &= \int_{L^{(e)}}\int_{A^{(e)}} (t_0 T_0 N_m N_l[\boldsymbol{\beta}_p^{\mathrm{T}}(\mathbf{D}_p F_s) + \boldsymbol{\beta}_n^{\mathrm{T}}(\mathbf{D}_{np} F_s)] F_\tau)\,dAdL \\
&\quad + \int_{L^{(e)}}\int_{A^{(e)}} (t_0 T_0[\boldsymbol{\beta}_n^{\mathrm{T}} N_m(\mathbf{D}_{ny} N_l) F_s F_\tau])\,dAdL \\
[\mathbf{M}_{\Theta\Theta}^{lm\tau s}]_{1\times1} &= \int_{L^{(e)}}\int_{A^{(e)}} (\rho c t_0 N_m N_l F_\tau F_s)\,dAdL \\
&\quad + \int_{L^{(e)}}\int_{A^{(e)}} (\rho c t_2 N_m N_l F_\tau F_s)\,dAdL
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
[\mathbf{G}_{UU}^{lm\tau s}]_{3\times3} &= \int_{L^{(e)}}\int_{A^{(e)}} (\zeta N_m N_l\,\mathbf{I} F_\tau F_s)\,dAdL \\
[\mathbf{G}_{U\Theta}^{lm\tau s}]_{3\times3} &= -\int_{L^{(e)}}\int_{A^{(e)}} (t_1 N_m N_l[(\mathbf{D}_p^{\mathrm{T}} F_\tau \mathbf{I})\boldsymbol{\beta}_p + (\mathbf{D}_{np}^{\mathrm{T}} F_\tau \mathbf{I})\boldsymbol{\beta}_n] F_s)\,dAdL \\
&\quad - \int_{L^{(e)}}\int_{A^{(e)}} (t_1(\mathbf{D}_{ny}^{\mathrm{T}} N_m) N_l[F_\tau \boldsymbol{\beta}_n F_s])\,dAdL \\
[\mathbf{G}_{\Theta U}^{lm\tau s}]_{1\times3} &= \int_{L^{(e)}}\int_{A^{(e)}} (T_0 N_m N_l[\boldsymbol{\beta}_p^{\mathrm{T}}(\mathbf{D}_p F_s) + \boldsymbol{\beta}_n^{\mathrm{T}}(\mathbf{D}_{np} F_s)] F_\tau)\,dAdL \\
&\quad + \int_{L^{(e)}}\int_{A^{(e)}} (T_0[\boldsymbol{\beta}_n^{\mathrm{T}} N_m(\mathbf{D}_{ny} N_l) F_s F_\tau)\,dAdL \\
[\mathbf{G}_{\Theta\Theta}^{lm\tau s}]_{1\times1} &= \int_{L^{(e)}}\int_{A^{(e)}} (\rho c N_m N_l F_\tau F_s)\,dAdL \\
&\quad - \int_{L^{(e)}}\int_{A^{(e)}} (2\tilde{\mathbf{c}}^{\mathrm{T}} N_m[\boldsymbol{\nabla}_n N_l] F_\tau F_s)\,dAdL \\
&\quad - \int_{L^{(e)}}\int_{A^{(e)}} (2\tilde{\mathbf{c}}^{\mathrm{T}} N_m N_l F_\tau[\boldsymbol{\nabla}_p F_s])\,dAdL
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
[\mathbf{K}_{UU}^{lm\tau s}]_{3\times3} =& \int_{L^{(e)}}\int_{A^{(e)}} \Big(N_m N_l \left[(\mathbf{D}_{np}^{\mathrm{T}} F_\tau \mathbf{I})[\mathbf{C}_{nn}(\mathbf{D}_{np} F_s \mathbf{I}) + \mathbf{C}_{np}(\mathbf{D}_p F_s \mathbf{I})]\right. \\
&\quad + \left.(\mathbf{D}_p^{\mathrm{T}} F_\tau \mathbf{I})[\mathbf{C}_{pp}(\mathbf{D}_p F_s \mathbf{I}) + \mathbf{C}_{pn}(\mathbf{D}_{np} F_s \mathbf{I})]\right]\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big(N_m(\mathbf{D}_{ny} N_l)[(\mathbf{D}_{np}^{\mathrm{T}} F_\tau \mathbf{I})\mathbf{C}_{nn} + (\mathbf{D}_p^{\mathrm{T}} F_\tau \mathbf{I})\mathbf{C}_{pn}]F_s\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big((\mathbf{D}_{ny}^{\mathrm{T}} N_m) N_l F_\tau [\mathbf{C}_{np}(\mathbf{D}_p F_s \mathbf{I}) + \mathbf{C}_{nn}(\mathbf{D}_{np} F_s \mathbf{I})]\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big((\mathbf{D}_{ny}^{\mathrm{T}} N_m)(\mathbf{D}_{ny} N_l) F_\tau \mathbf{C}_{nn} F_s\Big)\, dAdL \\
[\mathbf{K}_{U\Theta}^{lm\tau s}]_{3\times1} =& -\int_{L^{(e)}}\int_{A^{(e)}} \Big(N_m N_l[(\mathbf{D}_p^{\mathrm{T}} F_\tau)\boldsymbol{\beta}_p + (\mathbf{D}_{np}^{\mathrm{T}} F_\tau)\boldsymbol{\beta}_n]F_s\Big)\, dAdL \\
&- \int_{L^{(e)}}\int_{A^{(e)}} \Big((\mathbf{D}_{ny}^{\mathrm{T}} N_m) N_l F_\tau \boldsymbol{\beta}_n F_s\Big)\, dAdL \\
[\mathbf{K}_{\Theta\Theta}^{lm\tau s}]_{1\times1} =& \int_{L^{(e)}}\int_{A^{(e)}} \Big(N_m N_l (\boldsymbol{\nabla}_p^{\mathrm{T}} F_s)\boldsymbol{\kappa}(\boldsymbol{\nabla}_p F_\tau)\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big((\boldsymbol{\nabla}_n^{\mathrm{T}} N_l)(N_m)\boldsymbol{\kappa}(\boldsymbol{\nabla}_p F_\tau)F_s\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big(N_l(\boldsymbol{\nabla}_n N_m)\boldsymbol{\kappa}(\boldsymbol{\nabla}_p^{\mathrm{T}} F_s)F_\tau\Big)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} \Big((\boldsymbol{\nabla}_n^{\mathrm{T}} N_l)(\boldsymbol{\nabla}_n N_m)F_\tau \boldsymbol{\kappa} F_s\Big)\, dAdL
\end{aligned}
\tag{29}
$$

$$
\begin{aligned}
\{\mathbf{F}^{m\tau}\}_{3\times1} =& \int_{L^{(e)}}\int_{A^{(e)}} (\mathbf{X} N_m F_\tau)\, dAdL + \int_{S^{(e)}} (\mathbf{t} N_m F_\tau) dS \\
\{Q^{m\tau}\}_{1\times1} =& \int_{S^{(e)}} (\mathbf{q}^{\mathrm{T}}\mathbf{n} N_m F_\tau) dS + \int_{L^{(e)}}\int_{A^{(e)}} (R N_m F_\tau)\, dAdL \\
&+ \int_{L^{(e)}}\int_{A^{(e)}} (t_0 \dot{R} N_m F_\tau)\, dAdL
\end{aligned}
\tag{30}
$$

In the expressions (27)-(30), $\mathbf{I}$ represents the identity matrix. The subscript p denotes the in-plane components over a cross section of the structure, while n indicates the normal components to the cross section. Accordingly, the matrices $\mathbf{D}_p$, $\mathbf{D}_{np}$ and $\mathbf{D}_{ny}$ and the vectors $\boldsymbol{\nabla}_p$ and $\boldsymbol{\nabla}_n$ can be defined as

$$\mathbf{D}_p = \begin{bmatrix} 0 & 0 & \partial_z \\ \partial_x & 0 & 0 \\ \partial_z & 0 & \partial_x \end{bmatrix}, \quad \mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{ny} = \begin{bmatrix} 0 & \partial_y & 0 \\ 0 & 0 & \partial_y \\ \partial_y & 0 & 0 \end{bmatrix} \tag{31}$$

and

$$\boldsymbol{\nabla}_p = \{ \ \partial_x \quad 0 \quad \partial_z \ \}^{\mathrm{T}} \ , \ \boldsymbol{\nabla}_n = \{ \ 0 \quad \partial_y \quad 0 \ \}^{\mathrm{T}} \tag{32}$$

Similarly, the grouped elastic coefficients matrix and stress-temperature moduli vector are given as

$$\mathbf{C}_{pp} = \begin{bmatrix} C_{11} & C_{12} & C_{14} \\ C_{21} & C_{22} & C_{24} \\ C_{41} & C_{42} & C_{44} \end{bmatrix} \quad , \quad \mathbf{C}_{nn} = \begin{bmatrix} C_{33} & C_{35} & C_{36} \\ C_{53} & C_{55} & C_{56} \\ C_{63} & C_{65} & C_{66} \end{bmatrix} \quad , \quad \mathbf{C}_{pn} = \begin{bmatrix} C_{13} & C_{15} & C_{16} \\ C_{23} & C_{25} & C_{26} \\ C_{43} & C_{45} & C_{46} \end{bmatrix} \tag{33}$$

$$\boldsymbol{\beta}_p = \{\beta_{zz} \quad \beta_{xx} \quad \beta_{xz}\}^{\mathrm{T}} \qquad \boldsymbol{\beta}_n = \{\beta_{yy} \quad \beta_{yz} \quad \beta_{xy}\}^{\mathrm{T}} \tag{34}$$

where $\mathbf{C}_{np} = \mathbf{C}_{pn}^{\mathrm{T}}$. In order to summarize, the expanded expressions for components of the matrix $\mathbf{C}$ for anisotropic materials are not given here, but they can be found in Ref. [65].

Furthermore, in most practical engineering problems, the structural damping matrix $\mathbf{G}_{UU}^{lm\tau s}$ may be computed by the Rayleigh damping model as [66]

$$\mathbf{G}_{UU}^{lm\tau s} = \zeta_1 \mathbf{M}_{UU}^{lm\tau s} + \zeta_2 \mathbf{K}_{UU}^{lm\tau s} \tag{35}$$

in which $\zeta_1 \mathbf{M}_{UU}$ and $\zeta_2 \mathbf{K}_{UU}$ are the structural mass and stiffness proportional damping terms, respectively, and the parameters ζ_1 and ζ_2 are typically obtained by experiments for materials.

Therefore the FNs (27)-(30) must be expanded with respect to the superscripts m, l, τ and s in order to obtain the FE matrices and vectors of the whole structure. In fact, indexes m and l are exploited to assemble the matrices in the FE procedure while τ and s are used to provide the order of the model. The assembly procedure of the FNs are concerned in the companion paper (Part 2). Thus, the matrix form of the governing equation for the whole structure can be expressed as

$$\mathbf{M}\ddot{\boldsymbol{\Delta}} + \mathbf{G}\dot{\boldsymbol{\Delta}} + \mathbf{K}\boldsymbol{\Delta} = \mathbf{P} \tag{36}$$

where $\mathbf{M}$, $\mathbf{G}$ and $\mathbf{K}$ are the global mass, damping and stiffness matrices. Likewise, $\mathbf{P}$ is the global vector of the applied mechanical and thermal loads and $\boldsymbol{\Delta}$ stands for the global vector of unknowns. Equation (36) of the whole structure can be solved for the nodal displacements and temperature change in the time domain by the standard numerical techniques used to such problems in the literature.

It is noted that in the presented formulation, the thermal and mechanical boundary conditions as well as the body forces and the heat sources are considered as the most general forms. The mechanical boundary conditions may be applied through specified traction vectors or displacements on the boundaries. likewise, the different types of thermal boundary conditions including a known temperature change on a part of the boundary surface, a known heat flux on the boundary and the convection and radiation conditions may be assumed in problems. It is further obvious that the concentrated thermal and mechanical loads can be taken into account as the particular cases of the surface loads. Moreover, the initial thermal and mechanical conditions may be assumed in general form as arbitrary known functions of the space coordinates.

In the unified FE formulation (26), indeed, addition to taking into account the mechanical damping effect by the matrix $\mathbf{G}_{UU}^{lm\tau s}$, six theories of GL, LS, classical, dynamic uncoupled, quasi-static uncoupled and static uncoupled thermoelasticity are included. Accordingly, the generalized theory of thermoelasticity based on the Green-Lindsay and Lord-Shulman can be involved for $t_0 = 0$ and $t_1 = t_2 = \tilde{\mathbf{c}} = 0$, respectively. The four other theories can be provided as particular cases as represented in Table 1. Equation (26) can be simplified to the formulation of the classical coupled thermoelasticity problems by taking $t_1 = t_2 = \tilde{\mathbf{c}} = 0$ and $t_0 = 0$. The classical coupled theory reduces to the *dynamic uncoupled thermaoelasticity* by eliminating the coupling matrix ($\mathbf{G}_{\Theta U}^{lm\tau s}$) from the formulation. The dynamic uncoupled formulation can be employed for the problems in which the rate of imposed thermo-mechanical loads is not rapid enough to generate thermal stress waves. If the inertia forces can be further neglected as $\mathbf{M}_{UU}^{lm\tau s} = 0$, the governing formulation for the *quasi-static uncoupled thermaoelasticity* problems is obtained. Moreover, in a steady-state condition ($\mathbf{G}_{\Theta\Theta}^{lm\tau s} = 0$), the formulation can be more simplified to *static uncoupled thermoelasticity* where thermal stresses are imposed by the deformations due to the steady-state temperature field.

Table 1: Different theories of thermoelasticity through the 1D FE-CUF

	Conditions		Theory
Dynamic Coupled	$t_0 = 0$		Generalized, GL
	$t_1 = t_2 = \tilde{\mathbf{c}} = 0$		Generalized, LS
	$t_0 = 0$ $t_1 = t_2 = \tilde{\mathbf{c}} = 0$		Classical
Uncoupled	$t_0 = 0$ $t_1 = t_2 = \tilde{\mathbf{c}} = 0$	$\mathbf{G}_{\Theta U}^{lm\tau s} = 0$	dynamic
		$\mathbf{M}_{UU}^{lm\tau s} = 0$ $\mathbf{G}_{\Theta U}^{lm\tau s} = 0$	quasi-static
		$\mathbf{M}_{UU}^{lm\tau s} = 0$ $\mathbf{G}_{\Theta U}^{lm\tau s} = 0$ $\mathbf{G}_{\Theta\Theta}^{lm\tau s} = 0$	static

Conclusion

In the framework of the the Carrera Unified Formulation, the 1D FE procedure is developed to the 3D solution of the static, transient, and dynamic problems in the coupled and uncoupled thermoelasticity for the nonhomogeneous anisotropic materials. As particular cases, the generalized theories based on the Lord-Shulman and the Green-Lindsay models, as well as the classical coupled, dynamic uncoupled, quasi-static uncoupled and steady-state uncoupled theories of thermoelasticity can be extracted from the presented formulation. The mechanical damping effect can be further taken into account in the problems. In addition, the thermal and mechanical boundary conditions, the body force and the heat source are considered in the most general forms where no limiting assumption is applied. This generality allows to analyze varieties of more practical thermoelastic problems. Since this approach reduces the 3D problems to the 1D models with 3D-like accuracies and very low computational costs, it may seem to be a competent tool in an iterative solution process of the dynamic coupled thermoelasticity problems.

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