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Variable Kinematic Shell Elements for the Analysis of Electro-Mechanical Problems

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The present article considers the linear static analysis of both composite plate and shell structures embedding piezoelectric layers by means of a shell finite element with variable through-the-thickness kinematic. The refined models used are grouped in the Unified Formulation by Carrera (CUF) and they permit to accurately describe the distribution of displacements and stresses along the thickness of the multilayered shell. The shell element has nine nodes and the mixed interpolation of tensorial components (MITC) method is employed to contrast the membrane and shear locking phenomenon. The governing equations are derived from the principle of virtual displacement (PVD) and the finite element method (FEM) is employed to solve them. Cross-ply multilayered plates and cylindrical shells embedding piezoelectric layers are analyzed with simply-supported boundary conditions and subjected to sensor and actuator configurations. Various thickness ratios are considered. The results, obtained with different theories contained in the CUF, are compared with both the elasticity solutions given in literature and the analytical solutions obtained using the CUF and the Navier's method. From the analysis, one can conclude that the shell element based on the CUF is very efficient and its use is mandatory with respect to the classical models in the study of multilayered structures embedding piezo-layers.

Keywords: shells, finite element method, piezoelectric materials, mixed interpolated tensorial components, Carrera's Unified Formulation, layer-wise, laminated composites

1. Introduction

Piezoelectric materials have the ability to convert mechanical energy into electrical energy, and vice versa. For the last 50 years, the use of piezoelectric components as electro-mechanical transducers in sensors as well as in actuator applications has been continuously increasing. More recently, piezoelectrics have been considered among the most suitable materials for extending the structural capabilities beyond the purely passive load carrying one. Vibration and noise suppression, controlled active deformation, and health monitoring are among the most important applications of these "intelligent" structural components. Analytical solutions for general smart structural problems are a very difficult task and they exist, only for a very few specialized and idealized cases. Meanwhile, the finite element method (FEM), has become the most widely used technique to model various physical processes, including piezoelectricity. The introduction of piezoelectric material into a passive structure naturally leads to a multilayered component, and it has been recognized that classi-

cal models are not suitable for an accurate design of such structures, see for example the review article of Noor and Burton [1] and the references cited herein. Interlaminar continuity (IC) of the transverse stresses as well as the related discontinuity of the slopes of the displacement distributions in thickness direction at the layers interfaces are the main effects arising in multilayered structures, which cannot be captured by classical formulations based on Love first approximation theories (LFAT), see, e.g., [2]. Many refined theories for plates and shells have been proposed in order to meet the modeling requirements—known as C_z^0 -requirements—posed by these characteristics; further details can be found, e.g., in the monograph of Reddy [3] and in the paper of Carrera [4]. For curved structures, Koiter [5] recognized the importance of transverse stress effects even for homogeneous shells and recommended the inclusion of such effects whenever a consistent higher-order model has to be proposed. The fundamentals for the modeling of piezoelectric materials have been given in many contributions, in particular, in the pioneering works of Mindlin [6], EerNisse [7], Tiersten and Mindlin [8], and in the monograph of Tiersten [9]. The embedding of piezoelectric layers into plates and shells sharpens the requirements of an accurate modeling of the resulting adaptive structure due to the localized electro-mechanical coupling, see, e.g., the review of Saravanos and Heyliger [10]. Therefore, within the framework of two-dimensional approaches, layerwise descriptions have been often proposed either for the electric field only (see, e.g., the works of Kapuria [11] and of Ossadzew-David and

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Touratier [12]) or for both the mechanical and electrical unknowns (e.g., Heyliger et al. [13]). Ballhause et al. [14] showed that a fourth-order assumption for the displacements leads to the correct closed form solution. They conclude that the analysis of local responses requires at least a layer-wise description of the displacements, see also [15]. Benjeddou et al. [16] emphasized that a quadratic electric potential through the plate thickness satisfies the electric charge conservation law exactly. An attempt to mathematically substantiate axiomatic two-dimensional piezoelectric shell formulations by the means of asymptotic expansions can be found in the book of Rogacheva [17]. An exhaustive overview of the many different modeling approaches and solution techniques for laminated piezoelectric plates and shells is far beyond the scope of this article; more details on this topic can be found, e.g., in the already cited review of Saravanos and Heyliger, and in the surveys of Gopinathan et al. [18] and of Benjeddou [19]. For the last, years, interest has been emerging for mixed formulations involving also stresses and dielectric displacements as primary variables, see for example the recent works of Lammering and Mesecke-Rischmann [20] and of Benjeddou and Andrianarison [21]. Some of the latest contributions to the finite elements (FEs) analysis of piezoelectric plates that includes a first shear deformation theory (FSDT) description of displacements and a layerwise (LW) form of the electric potential was developed by Sheikh et al. [22]. The numerical, membrane and bending behavior of FEs that are based on FSDTs were analyzed by Auricchio et al. [23] in the framework of a suitable variational formulation. The third-order theory of higher order theories type was applied by Thornburgh and Chattopadhyay [24] to derive FEs that take into consideration the electro-mechanical coupling. Similar elements have more recently been considered by Shu [25]. The extension of the third-order zig-zag Ambartsumian multilayered theory to finite analysis of electro-mechanical problems has been proposed by Oh and Cho [26]. An extension to piezoelectricity of numerically efficient plate/shell elements based on the mixed interpolation of tensorial components formulation has recently been provided by Kogl and Bucalem [27, 28]. Some of the latest contributions to the FEs analysis of piezoelectric shells that are based on exact geometry solid-shell element with the first-order 7-parameter equivalent single-layer theory was developed by Kulikov and Plotnikova [29], and a piezoelectric solid shell element with a mixed variational formulation and a geometrically nonlinear theory was developed by Klinkel and Wagner [30].

A new shell finite element is presented in this article for the analysis of composite structures with piezo-layers. It is based on the Carrera's Unified Formulation (CUF), which has been developed by Carrera for multi-layered structures [31]. Many works have been devoted to the extension of the CUF. Among others, Carrera [32] extends principle of virtual displacement (PVD) and Reissner mixed variational theorem (RMVT) variational statements to piezo-laminated plates. The modeling of piezo-laminated plates using LW mixed FEs was then proposed in [33]. Subsequently, an extension of the RMVT to piezoelectric laminates with analytical results was published in [34]. Mixed FEs for static and dynamics analysis of piezoelectric plates have been provided in [35], where only transverse stresses were modeled by RMVT. Mixed FEs with direct

evaluation of transverse electric displacement have been provided in [36]. LW theories contained in the CUF have been implemented in the present shell finite element. An analytical solution type has already been successfully employed in combination with the Unified Formulation for piezoelectric plates [14] and piezoelectric shells [15]. The cylindrical geometry is considered and the mixed interpolation of tensorial components (MITC) method [37, 38] is used to contrast the membrane and shear locking. The governing equations for the static analysis of composite structures are derived from the PVD, in order to apply the finite element method (FEM). Some composite cylindrical shells are analyzed and the results, obtained with the different models contained in the CUF, are compared with the analytical solution and the exact solution given in the literature.

2. Unified Formulation

The main feature of CUF [32, 39, 40] is the unified manner in which the displacement variables are handled. According to CUF, the displacement field and the potential field are written by means of approximating functions in the thickness direction as follows:

$$\mathbf{u}^k(\alpha, \beta, z) = F_\tau(z) \mathbf{u}_\tau^k(\alpha, \beta), \quad \delta \mathbf{u}^k(\alpha, \beta, z) = F_s(z) \delta \mathbf{u}_s^k(\alpha, \beta),$$

$$\tau, s = 0, 1, \dots, N, \quad (1)$$

$$\Phi^k(\alpha, \beta, z) = F_\tau(z) \Phi_\tau^k(\alpha, \beta), \quad \delta \Phi^k(\alpha, \beta, z) = F_s(z) \delta \Phi_s^k(\alpha, \beta),$$

$$\tau, s = 0, 1, \dots, N, \quad (2)$$

where (α, β, z) is a curvilinear coordinates reference system, defined in the next section, and the displacement $\mathbf{u} = \{u, v, w\}$ and the potential Φ are referred to such system. δ indicates the virtual variation and k identifies the layer. F_τ and F_s are the so-called thickness functions depending only on z . \mathbf{u}_s and Φ_s are the unknown variables depending on the coordinates α and β . τ and s are sum indexes and N is the order of expansion in the thickness direction assumed for the variables.

In the case of LW models, the displacement and the potential are defined at k -layer level:

$$\mathbf{u}^k = F_t \mathbf{u}_t^k + F_b \mathbf{u}_b^k + F_r \mathbf{u}_r^k = F_\tau \mathbf{u}_\tau^k,$$

$$\tau = t, b, r, \quad r = 2, \dots, N. \quad (3)$$

$$\Phi^k = F_t \Phi_t^k + F_b \Phi_b^k + F_r \Phi_r^k = F_\tau \Phi_\tau^k,$$

$$\tau = t, b, r, \quad r = 2, \dots, N. \quad (4)$$

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2}. \quad (5)$$

The top (t) and bottom (b) values of the displacements and the potential are used as unknown variables and one can impose the following compatibility conditions:

$$u_t^k = u_b^{k+1}, \quad \Phi_t^k = \Phi_b^{k+1}, \quad k = 1, \quad N_l - 1. \quad (6)$$

The LW models, in respect to the equivalent single layer models (ESL), allow the zig-zag form of the variables distribution in layered structures to be modeled.

3. MITC9 Shell Element

In this section, the derivation of a shell finite element for the analysis of multilayered structures is presented. The element is based on LW theories contained in the Unified Formulation. A nine-nodes shell element is considered. After an overview in scientific literature about the methods that permit to withstand the membrane and shear locking, the MITC technique has been adopted for this element.

3.1. Shell Geometry

Shells are bi-dimensional structures in which one dimension (in general the thickness in z direction) is negligible with respect to the other two in-plane dimensions. Geometry and the reference system are indicated in Figure 1. By considering multilayered structures, the square of an infinitesimal linear segment in the layer, the associated infinitesimal area, and volume are given by:

$$\begin{aligned} ds_k^2 &= H_\alpha^k d\alpha_k^2 + H_\beta^k d\beta_k^2 + H_z^k dz_k^2, \\ d\Omega_k &= H_\alpha^k H_\beta^k d\alpha_k d\beta_k, \\ dV &= H_\alpha^k H_\beta^k H_z^k d\alpha_k d\beta_k dz_k, \end{aligned} \quad (7)$$

where the metric coefficients are:

$$H_\alpha^k = A^k(1 + z_k/R_\alpha^k), \quad H_\beta^k = B^k(1 + z_k/R_\beta^k), \quad H_z^k = 1. \quad (8)$$

k denotes the k -layer of the multilayered shell. R_α^k and R_β^k are the principal radii of the midsurface of the layer k . A^k and B^k are the coefficients of the first fundamental form of Ω_k (Γ_k is the Ω_k boundary). In this article, the attention has been restricted to shells with constant radii of curvature (cylindrical, spherical, toroidal geometries) for which $A^k = B^k = 1$, in particular, cylindrical shells have been considered. Details for shells are reported in [17].

Geometrical relations can be expressed in matrix form as:

$$\begin{aligned} \epsilon_p &= [\epsilon_{\alpha\alpha}, \epsilon_{\beta\beta}, \epsilon_{\alpha\beta}] = (\mathbf{D}_p + \mathbf{A}_p)\mathbf{u}, \\ \epsilon_n &= [\epsilon_{\alpha z}, \epsilon_{\beta z}, \epsilon_{zz}] = (\mathbf{D}_{np} + \mathbf{D}_{nz} - \mathbf{A}_n)\mathbf{u}, \end{aligned} \quad (9)$$

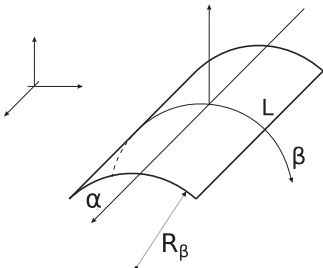


Fig. 1. Geometry of the shell.

where the differential operators are defined as follows:

$$\begin{aligned} \mathbf{D}_p &= \begin{bmatrix} \frac{\partial_\alpha}{H_\alpha} & 0 & 0 \\ 0 & \frac{\partial_\beta}{H_\beta} & 0 \\ \frac{\partial_\beta}{H_\beta} & \frac{\partial_\alpha}{H_\alpha} & 0 \end{bmatrix}, & \mathbf{D}_{np} &= \begin{bmatrix} 0 & 0 & \frac{\partial_\alpha}{H_\alpha} \\ 0 & 0 & \frac{\partial_\beta}{H_\beta} \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathbf{D}_{nz} &= \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \end{aligned} \quad (10)$$

$$\mathbf{A}_p = \begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha R_\alpha} \\ 0 & 0 & \frac{1}{H_\beta R_\beta} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_n = \begin{bmatrix} \frac{1}{H_\alpha R_\alpha} & 0 & 0 \\ 0 & \frac{1}{H_\beta R_\beta} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

The geometrical relations between electric field \mathcal{E} and potential Φ are defined as follows:

$$\begin{aligned} \mathcal{E}_p &= [\mathcal{E}_\alpha, \mathcal{E}_\beta]^T = -\mathbf{D}_{ep} \Phi, \\ \mathcal{E}_n &= [\mathcal{E}_z]^T = -\mathbf{D}_{en} \Phi, \end{aligned} \quad (12)$$

where the differential operators are defined as follows:

$$\mathbf{D}_{ep} = \begin{bmatrix} \frac{\partial_\alpha}{H_\alpha} \\ \frac{\partial_\beta}{H_\beta} \end{bmatrix}, \quad \mathbf{D}_{en} = [\partial_z].$$

3.2. MITC Method

A shell finite element is here presented for the analysis of composite structures. It is based on the CUF, which was developed by Carrera for multilayered structures [31].

According to the finite element method, the displacement and the potential components are interpolated on the nodes of the element by means of the Lagrangian shape functions N_i :

$$\delta \mathbf{u}_s = N_i \delta \mathbf{q}_{u_s i}, \quad \mathbf{u}_\tau = N_j \mathbf{q}_{u_\tau j} \quad \text{with } i, j = 1, \dots, 9, \quad (13)$$

$$\delta \Phi_s = N_i \delta q_{\Phi_s i}, \quad \Phi_\tau = N_j q_{\Phi_\tau j} \quad \text{with } i, j = 1, \dots, 9, \quad (14)$$

where $\mathbf{q}_{u_\tau j} = (q_{u_\tau}, q_{v_\tau}, q_{w_\tau})$ and $\delta \mathbf{q}_{u_s i} = (\delta q_{u_s}, \delta q_{v_s}, \delta q_{w_s})$ are the nodal displacements and their virtual variations of the mechanical variables, and $q_{\Phi_\tau j} = (q_{\Phi_\tau})$ and $\delta q_{\Phi_s i} = (\delta q_{\Phi_s})$ are the nodal potential and their virtual variation for the electric variable. Substituting in the geometrical relations (9) one has:

$$\begin{aligned} \epsilon_p &= F_\tau (\mathbf{D}_p + \mathbf{A}_p) (N_i) \mathbf{q}_{\tau i}, \\ \epsilon_n &= F_\tau (\mathbf{D}_{n\Omega} - \mathbf{A}_n) (N_i) \mathbf{q}_{\tau i} + F_{\tau,3} \mathbf{A}_{nz} (N_i) \mathbf{q}_{\tau i}. \end{aligned} \quad (15)$$

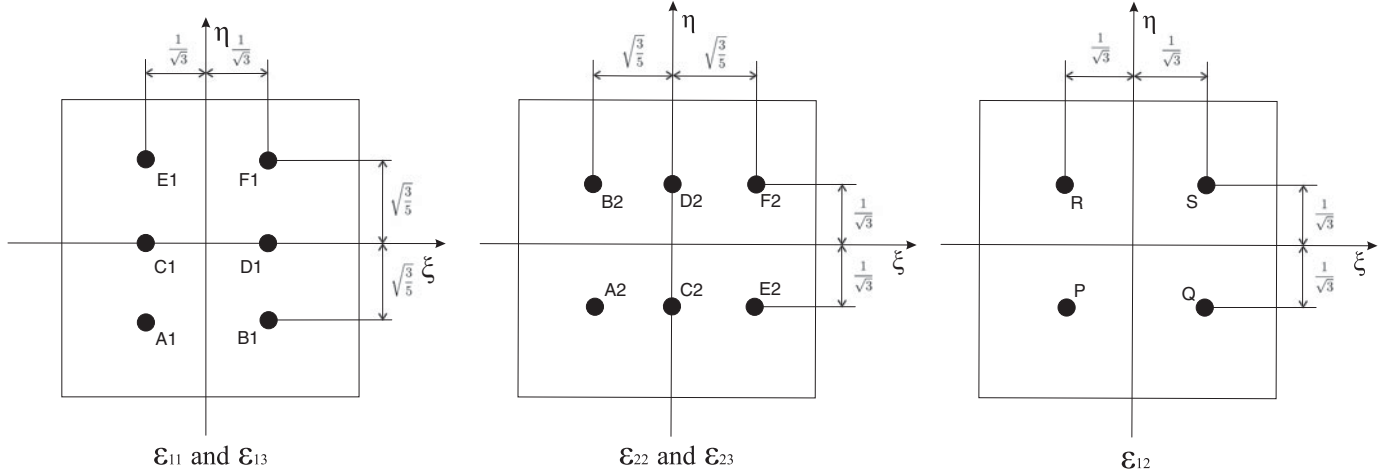


Fig. 2. Tying points for the MITC9 shell finite element.

Considering the local coordinate system (ξ, η) , the MITC shell elements [41, 42] are formulated by using, instead of the strain components directly computed from the displacements, an interpolation of these within each element using a specific interpolation strategy for each component. The corresponding interpolation points, called tying points, are shown in Figure 2 for the MITC9 shell element.

The interpolating functions are Lagrangian and are calculated by imposing that the function assumes the value 1 in the corresponding tying point and 0 in the others. These are arranged in the following arrays:

$$\begin{aligned} N_{m1} &= [N_{A1}, N_{B1}, N_{C1}, N_{D1}, N_{E1}, N_{F1}], \\ N_{m2} &= [N_{A2}, N_{B2}, N_{C2}, N_{D2}, N_{E2}, N_{F2}], \\ N_{m3} &= [N_P, N_Q, N_R, N_S]. \end{aligned} \quad (16)$$

From this point on, the subscripts $m1$, $m2$, and $m3$ indicate quantities calculated in the points $(A1, B1, C1, D1, E1, F1)$, $(A2, B2, C2, D2, E2, F2)$, and (P, Q, R, S) , respectively. Therefore, the strain components are interpolated as follows:

$$\begin{aligned} \boldsymbol{\epsilon}_p &= \begin{bmatrix} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \epsilon_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} N_{m1} & 0 & 0 \\ 0 & N_{m2} & 0 \\ 0 & 0 & N_{m3} \end{bmatrix} \begin{bmatrix} \epsilon_{\alpha\alpha m1} \\ \epsilon_{\beta\beta m2} \\ \epsilon_{\alpha\beta m3} \end{bmatrix}, \\ \boldsymbol{\epsilon}_n &= \begin{bmatrix} \epsilon_{\alpha z} \\ \epsilon_{\beta z} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} N_{m1} & 0 & 0 \\ 0 & N_{m2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{\alpha z m1} \\ \epsilon_{\beta z m2} \\ \epsilon_{zz} \end{bmatrix}, \end{aligned} \quad (17)$$

where the strains $\epsilon_{\alpha\alpha m1}$, $\epsilon_{\beta\beta m2}$, $\epsilon_{\alpha\beta m3}$, $\epsilon_{\alpha z m1}$, and $\epsilon_{\beta z m2}$ are expressed by means of Eqs. (15) and the shape functions N_i are calculated in the tying points.

4. Constitutive Equations

The second step towards the governing equations is the definition of the constitutive equations that permit the stresses

and the electric displacements to be expressed by means of the strains and the electric fields. The generalized Hooke's law is considered, by employing a linear constitutive model for infinitesimal deformations. A linear coupling of the electric fields is employed to complete stresses equations and to describe the electric displacements equations. In a composite material, these equations are obtained in material coordinates $(1, 2, 3)$ for each layer k and then rotated in the general curvilinear reference system (α, β, z) .

Therefore, the in-plane stresses $\boldsymbol{\sigma}_p = [\sigma_{\alpha\alpha}, \sigma_{\beta\beta}, \sigma_{\alpha\beta}]$, the transverse stresses $\boldsymbol{\sigma}_n = [\sigma_{\alpha z}, \sigma_{\beta z}, \sigma_{zz}]$, and the electric displacements $\boldsymbol{D} = [D_\alpha, D_\beta, D_z]$ after the rotation are:

$$\boldsymbol{\sigma}_{pC}^k = \mathbf{C}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{pn}^k \boldsymbol{\epsilon}_{nG}^k - \mathbf{e}_{pp}^{kT} \boldsymbol{\mathcal{E}}_{pG}^k - \mathbf{e}_{np}^{kT} \boldsymbol{\mathcal{E}}_{nG}^k, \quad (18)$$

$$\boldsymbol{\sigma}_{nC}^k = \mathbf{C}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{nn}^k \boldsymbol{\epsilon}_{nG}^k - \mathbf{e}_{pn}^{kT} \boldsymbol{\mathcal{E}}_{pG}^k - \mathbf{e}_{nn}^{kT} \boldsymbol{\mathcal{E}}_{nG}^k, \quad (19)$$

$$\boldsymbol{D}_{pC}^k = \mathbf{e}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{e}_{pn}^k \boldsymbol{\epsilon}_{nG}^k + \boldsymbol{\epsilon}_{pp}^k \boldsymbol{\mathcal{E}}_{pG}^k + \boldsymbol{\epsilon}_{pn}^k \boldsymbol{\mathcal{E}}_{nG}^k, \quad (20)$$

$$\boldsymbol{D}_{nC}^k = \mathbf{e}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{e}_{nn}^k \boldsymbol{\epsilon}_{nG}^k + \boldsymbol{\epsilon}_{np}^k \boldsymbol{\mathcal{E}}_{pG}^k + \boldsymbol{\epsilon}_{nn}^k \boldsymbol{\mathcal{E}}_{nG}^k, \quad (21)$$

where

$$\begin{aligned} \mathbf{C}_{pp}^k &= \begin{bmatrix} C_{11}^k & C_{12}^k & C_{16}^k \\ C_{12}^k & C_{22}^k & C_{26}^k \\ C_{16}^k & C_{26}^k & C_{66}^k \end{bmatrix}, & \mathbf{C}_{pn}^k &= \begin{bmatrix} 0 & 0 & C_{13}^k \\ 0 & 0 & C_{23}^k \\ 0 & 0 & C_{36}^k \end{bmatrix}, \\ \mathbf{C}_{np}^k &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13}^k & C_{23}^k & C_{36}^k \end{bmatrix}, & \mathbf{C}_{nn}^k &= \begin{bmatrix} C_{55}^k & C_{45}^k & 0 \\ C_{45}^k & C_{44}^k & 0 \\ 0 & 0 & C_{33}^k \end{bmatrix}. \end{aligned} \quad (22)$$

$$\begin{aligned} \mathbf{e}_{pp} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \mathbf{e}_{pn} &= \begin{bmatrix} e_{15} & e_{14} & 0 \\ e_{25} & e_{24} & 0 \end{bmatrix}, \\ \mathbf{e}_{np} &= \begin{bmatrix} e_{31} & e_{32} & e_{36} \end{bmatrix}, & \mathbf{e}_{nn} &= \begin{bmatrix} 0 & 0 & e_{33} \end{bmatrix}. \end{aligned} \quad (23)$$

$$\begin{aligned} \boldsymbol{\epsilon}_{pp} &= \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{bmatrix}, & \boldsymbol{\epsilon}_{pn} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \boldsymbol{\epsilon}_{np} &= \begin{bmatrix} 0 & 0 \end{bmatrix}, & \boldsymbol{\epsilon}_{nn} &= \begin{bmatrix} \epsilon_{33} \end{bmatrix}. \end{aligned} \quad (24)$$

The material coefficients C_{ij} depend on the Young's moduli E_1 , E_2 , and E_3 ; the shear moduli G_{12} , G_{13} , and G_{23} ; and Poisson moduli ν_{12} , ν_{13} , ν_{23} , ν_{21} , ν_{31} , and ν_{32} that characterize the layer material. The piezoelectric material is characterized by the piezoelectric coefficients e_{ij} and the permittivity coefficients ϵ_{ij} .

5. Governing Equations

This section presents the derivation of the governing finite element stiffness matrix based on the PVD in the case of multilayered shell structures subjected to electro-mechanical loads.

The PVD for a multilayered piezoelectric structure reads:

$$\int_V \left(\delta \epsilon_{pG}^T \sigma_{pC} + \delta \epsilon_{nG}^T \sigma_{nC} - \delta \mathcal{E}_{pG}^T \mathcal{D}_{pC} - \delta \mathcal{E}_{nG}^T \mathcal{D}_{nC} \right) \times dV = \delta L_e, \quad (25)$$

where Ω_k and A_k are the integration domains in the plane and in the thickness direction. The first member of the equation represents the variation of the internal work, while the second member is the external work. In order to refer the integration domains to the midsurface of each layer in the curvilinear coordinate system, one has to introduce the parameters H_α , H_β as follows:

$$\int_{\Omega_k} \int_{A_k} \left\{ \delta \epsilon_{pG}^k T \sigma_{pC}^k + \delta \epsilon_{nG}^k T \sigma_{nC}^k - \delta \mathcal{E}_{pG}^k T \mathcal{D}_{pC}^k - \delta \mathcal{E}_{nG}^k T \mathcal{D}_{nC}^k \right\} H_\alpha H_\beta d\Omega_k dz = \delta L_e. \quad (26)$$

Substituting the constitutive Eqs. (18)–(21), the geometrical relations written via the MITC method (17) and applying the CUF 1 and the finite element method (FEM) approximation (13), one obtains the following governing equations:

$$\delta \mathbf{u}_s^k : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k + \mathbf{K}_{u\Phi}^{k\tau s} \Phi_\tau^k = \mathbf{P}_{us}^k, \quad (27)$$

$$\delta \Phi_s^k : \mathbf{K}_{\Phi u}^{k\tau s} \mathbf{u}_\tau^k + \mathbf{K}_{\Phi\Phi}^{k\tau s} \Phi_\tau^k = \mathbf{P}_{\Phi s}^k. \quad (28)$$

In compact form:

$$\delta \mathbf{q}_s^k : \mathbf{K}^{k\tau s} \mathbf{q}_\tau^k = \mathbf{P}_s^k, \quad (29)$$

where

$$\mathbf{K}^{k\tau s} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\Phi} \\ \mathbf{K}_{\Phi u} & \mathbf{K}_{\Phi\Phi} \end{bmatrix}^{k\tau s}. \quad (30)$$

The mechanical part $\mathbf{K}_{uu}^{k\tau sij}$ is a 3×3 matrix, the coupling matrices $\mathbf{K}_{u\Phi}^{k\tau sij}$, $\mathbf{K}_{\Phi u}^{k\tau sij}$ have dimension 3×1 and 1×3 respectively, and the electrical part $\mathbf{K}_{\Phi\Phi}^{k\tau sij}$ is a 1×1 matrix. The global matrix $\mathbf{K}^{k\tau sij}$ is called fundamental nucleus, and its explicit expression is given in the appendix. This is the basic

element from which the stiffness matrix of the whole structure is computed. For the expansion of the fundamental nucleus on the indexes τ and s and the assembling procedure at a multilayer level for LW approaches, the reader can refer to [32]. $\mathbf{q}_\tau^k = (q_{u_\tau}^k, q_{v_\tau}^k, q_{w_\tau}^k, q_{\Phi_\tau}^k)$ is the vector of the nodal mechanical displacements and the nodal electric potential. \mathbf{P}_s^k is the fundamental nucleus for the external load.

6. Numerical Results and Discussion

The model introduced, unlike a 3D degenerate approach, does not involve an approximation of the geometry of the shell and it describes accurately the curvature of the shell. However, the locking phenomenon is still present. In this work, such a model is combined with a simple displacement formulation for the analysis of composite structures. The refined theories contained in CUF, coupled with the MITC method, permit to increase the degree of approximation by increasing the order of expansion of displacements in the thickness direction and the number of used elements. In electro-mechanical problems it is necessary to impose the value of the electric potential variable at the top and bottom positions. To obtain this, layerwise models with Legendre polynomials are employed. The efficiency of layer-wise models is tested with the FE scheme, and the numerical results are compared with the ones obtained with the 3D elasticity approach. In this direction, two kinds of reference problems are considered: the composite square plate embedding piezoelectric layers at the top and bottom positions, analytically analyzed by Ballhause et al. [14], and single and multilayered cylinders embedding piezoelectric layers, analytically analyzed. Both of them are evaluated in sensor and actuator configuration. For the sensor case, a bi-sinusoidal transverse normal pressure is applied at the top surface:

$$p_z^+ = \hat{p}_z^+ \sin(m\pi\alpha/a) \sin(n\pi\beta/b), \quad (31)$$

with amplitude $\hat{p}_z^+ = 1$ and wave numbers $m = 1$, $n = 1$ for the plate and $n = 8$ for the shell. The potential at the top and bottom positions is imposed by $\Phi_t = \Phi_b = 0$.

For the actuator case, a bi-sinusoidal electric potential is imposed at the top surface:

$$\Phi^+ = \hat{\Phi}_z^+ \sin(m\pi\alpha/a) \sin(n\pi\beta/b), \quad (32)$$

with amplitude $\hat{\Phi}_z^+ = 1$ and wave numbers $m = 1$, $n = 1$ for the plate and $n = 8$ for the shell. The potential at the bottom position is imposed by $\Phi_b = 0$. No mechanical load is applied.

The two problems are briefly described in the following sections. In this article, some acronyms are used in tables and figures. The first letter indicates the multilayer approach, layerwise (L). The second letter refers to the employed variational statement: in this work only the principle of virtual displacements D is used. Only for the multilayered shell, reference solutions are calculated with the full Reissner mixed variational theory, so the second letters are FM . The number N

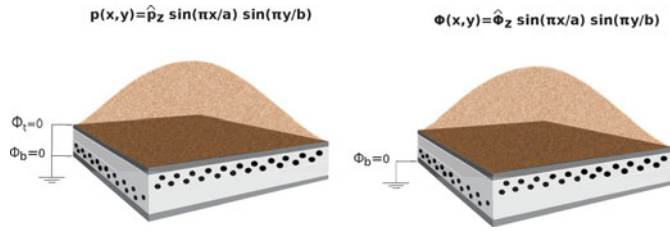


Fig. 3. Piezo plate sensor and actuator cases.

indicates the order of expansion used in the thickness direction (from 1 to 4). For references solutions obtained with an analytical method, a subscript a is used.

6.1. Multilayered Plate

The structure analyzed by Ballhause et al. [14] (see Figure 3) is a composite square plate embedding piezoelectric layers made globally of four layers, a core in Gr/Ep composite material by two orthotropic layers with lamination ($0^\circ/90^\circ$), and the skins in PZT-4 by two piezoelectric layers. In respect to the total thickness, a single piezoelectric skin is thick $h_p = 0.1h_{tot}$, while the single core layer is thick $h_c = 0.4h_{tot}$. The physical properties of the shell are given in Table 1.

Due to the symmetry of both the geometry and the load, a quarter of the plate is analyzed and the following symmetry and boundary conditions (simply-supported) are applied:

$$\begin{aligned}
 u_\tau(0, y) &= 0, \\
 v_\tau(x, 0) &= 0, \\
 w_\tau(x, b/2) &= 0, \\
 w_\tau(a/2, y) &= 0, \\
 u_\tau(x, b/2) &= 0, \\
 v_\tau(a/2, y) &= 0,
 \end{aligned} \tag{33}$$

Table 1. Physical data for multilayered plate and cylindrical shell

Properties	PZT-4	Gr/EP
E_{11} (GPa)	81.3	132.38
E_{22} (GPa)	81.3	10.756
E_{33} (GPa)	64.5	10.756
ν_{12}	0.329	0.24
ν_{13}	0.432	0.24
ν_{23}	0.432	0.49
G_{44} (GPa)	25.6	3.606
G_{55} (GPa)	25.6	5.6537
G_{66} (GPa)	30.6	5.6537
e_{15} (C/m ²)	12.72	0
e_{24} (C/m ²)	12.72	0
e_{31} (C/m ²)	-5.20	0
e_{32} (C/m ²)	-5.20	0
e_{33} (C/m ²)	15.08	0
$\tilde{\epsilon}_{11}/\epsilon_0$	1475	3.5
$\tilde{\epsilon}_{22}/\epsilon_0$	1475	3.0
$\tilde{\epsilon}_{33}/\epsilon_0$	1300	3.0

Table 2. Transverse normal displacement $w * 10^{11}$ evaluated along the thickness in ($z = 0$), electric potential $\Phi * 10^3$ evaluated along the thickness in ($z = 0$), transverse normal electric displacement $\mathcal{D}_z * 10^9$ evaluated along the thickness at top ($z = +h/2$) (plate with 4 layers; sensor case)

	a/h	2	4	10	100
		Reference [14]			
w	$LD4_a$	4.9113	30.029	582.06	4,675,300
	$FSDT_a$	2.8575	18.488	423.29	3,668,700
Φ	$LD4_a$	0.9103	6.1084	44.471	4580.2
	$FSDT_a$	0.78657	2.6580	15.044	1470.3
\mathcal{D}_z	$LD4_a$	0.0256	0.0161	0.0139	0.0136
	$FSDT_a$	0.0615	0.0401	-0.1174	-18.729
w	LD4	4.9112	30.0286	582.1298	4,675,118.5
	LD3	4.9112	30.0285	582.1298	4,675,118.5
	LD2	4.8954	29.9812	581.8951	4,675,095.5
	LD1	4.8087	29.8512	579.2426	4,647,075.5
Φ	LD4	0.9106	6.1107	44.4934	4586.8311
	LD3	0.9103	6.1102	44.4920	4586.8311
	LD2	0.8948	6.0899	44.4760	4586.7891
	LD1	0.8599	6.0320	44.1802	4559.1191
\mathcal{D}_z	LD4	0.0252	0.0140	0.0051	-0.0140
	LD3	0.0255	0.0142	0.0054	-0.0144
	LD2	0.0271	0.0156	0.0064	-0.0205
	LD1	-0.0665	-0.0890	-0.2905	-23.8867

with $\tau = 0, 1, \dots, N$.

The results are presented for different thickness ratios $a/h = 2, 4, 10, 100$. A mesh grid of 12×12 elements is taken to ensure the convergence of the solution.

In general, the results approach to the exact solution by increasing the order of expansion N for the various thickness ratios, see Tables 2 and 4. One can note that the element does not suffer the locking phenomenon even when the plate is very thin ($a/h = 100$), see Tables 3 and 5. Only the LD4 model is able to exactly reproduce the analytical solution in the case of thick shell. The behavior of shear and transverse stresses σ_{xz} , σ_{zz} is nonlinear along the thickness of the plate. The continuity conditions are reached only by increasing the order of expansion N , see Figures 4–10. Only for the actuator case of very thin plates ($a/h = 100$), the LD4 model is not able to fulfill the continuity conditions of transverse stresses σ_{zz} at the interfaces between layers, see Figure 11. To overcome this problem, a mixed variational principle could be used. For the description of the electric potential Φ , it is necessary to use the higher-order model, LD4, to describe the nonlinear behavior in thick plates ($a/h = 2$), see Figures 12 and 13. On the other hand, for thin plates ($a/h = 100$), a layerwise description of lower order is sufficient, see Figures 14 and 15. The electric displacement \mathcal{D}_z has a nonlinear behavior for every thickness ratio in the sensor case; it is necessary for a higher-order model to obtain the continuity interface, see Figures 16 and 17. For the actuator case, due to its linear behavior, the electric displacement can be described by a lower-order, layerwise model, see Figures 18 and 19.

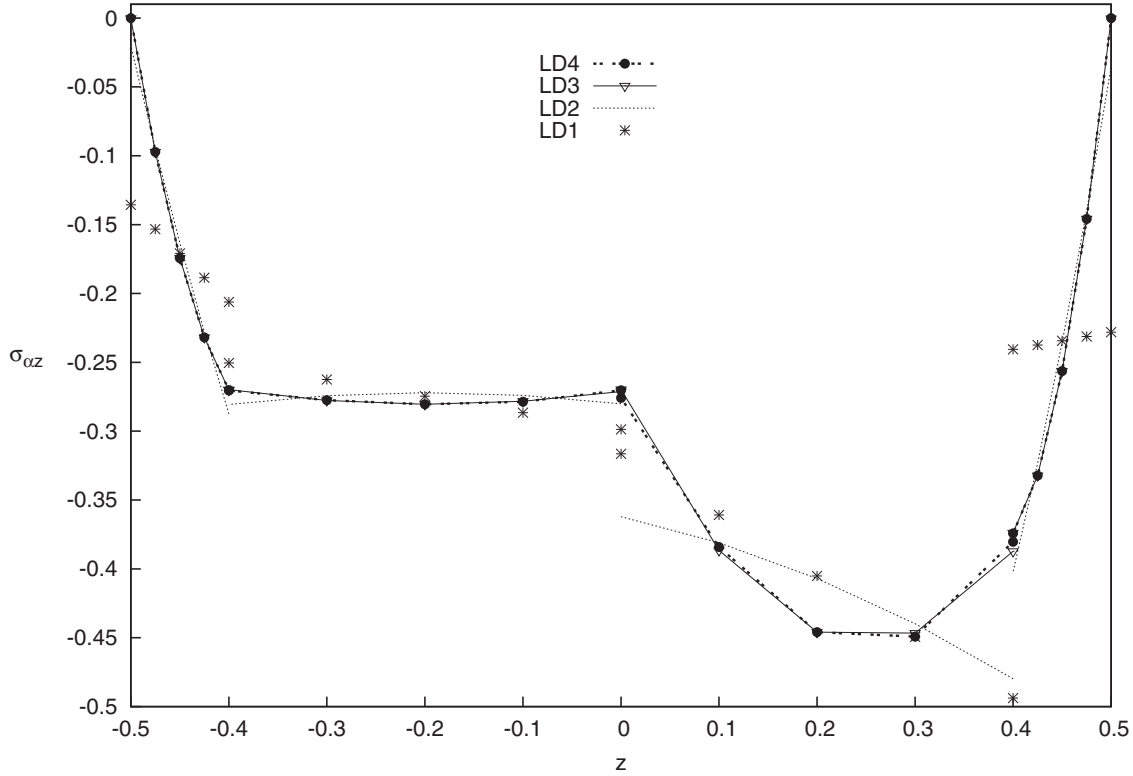


Fig. 4. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Sensor case.

6.2. Multilayered Piezoelectric Cylinder

The structure analyzed by Carrera and Brischetto [43] (see Figure 20) is a mono-layered piezoelectric cylinder, with the material properties given in Table 1. Moreover, a three-layer configuration cylinder with the core in Gr/Ep composite material is analyzed. For multilayered cases, reference solutions

Table 3. Principal stresses σ_{xx} , σ_{zz} evaluated along the thickness at top ($z = +h/2$) and shear stress σ_{xz} evaluated along the thickness in ($z = 0$) (plate with 4 layers; sensor case)

		a/h	2	4	10	100
			Reference [14]			
σ_{xx}	$LD4_a$	3.2207	6.5642	32.771	3142.1	
	$LD4_a$	-0.26995	-0.68720	-1.8540	-18.832	
σ_{xz}	LD4	3.2240	6.5710	32.8145	3146.8381	
	LD3	3.2263	6.5719	32.8153	3146.8376	
	LD2	3.2088	6.5659	32.8137	3146.8433	
	LD1	3.5228	7.0093	34.3028	3271.7583	
σ_{zz}	LD4	-0.2758	-0.6902	-1.8576	-18.8640	
	LD3	-0.2697	-0.6862	-1.8560	-18.8638	
	LD2	-0.3620	-0.8157	-2.1371	-21.6073	
	LD1	-0.3165	-0.7094	-1.8693	-18.9304	
σ_{zz}	LD4	1.0001	1.0001	1.0001	0.9955	
	LD3	1.0052	1.0015	1.0008	0.9957	
	LD2	1.0334	1.0268	1.0270	1.0757	
	LD1	2.1237	2.3562	5.1631	342.3075	

Table 4. Transverse normal displacement $w * 10^{11}$ evaluated along the thickness in ($z = 0$), electric potential Φ evaluated along the thickness in ($z = 0$), transverse normal electric displacement $\mathcal{D}_z * 10^9$ evaluated along the thickness at top ($z = +h/2$) (plate with 4 layers; actuator case)

		a/h	2	4	10	100
			Reference [14]			
w	$LD4_a$	-1.7475	-1.4707	-1.3697	-1.3493	
	$FSDT_a$	-13.923	-14.107	-14.159	-14.169	
Φ	$LD4_a$	0.3330	0.4477	0.4910	0.4999	
	$FSDT_a$	0.3219	0.4461	0.4908	0.4999	
\mathcal{D}_z	$LD4_a$	-9.4085	-2.4184	-0.4168	-0.0370	
	$FSDT_a$	-3.6667	-0.9566	-0.1816	-0.0347	
w	LD4	-1.7476	-1.4708	-1.3697	-1.3494	
	LD3	-1.7476	-1.4707	-1.3697	-1.3494	
	LD2	-1.7291	-1.4663	-1.3691	-1.3493	
	LD1	-2.1030	-1.5963	-1.4297	-1.3971	
Φ	LD4	0.3330	0.4477	0.4910	0.4999	
	LD3	0.3330	0.4477	0.4910	0.4999	
	LD2	0.3331	0.4477	0.4910	0.4999	
	LD1	0.3241	0.4468	0.4910	0.4999	
\mathcal{D}_z	LD4	-9.4104	-2.4185	-0.4167	-0.0370	
	LD3	-9.4047	-2.4182	-0.4167	-0.0370	
	LD2	-9.3822	-2.4167	-0.4166	-0.0370	
	LD1	-5.2969	-1.3815	-0.2504	-0.0353	

Table 5. Principal stresses σ_{xx} , σ_{zz} evaluated along the thickness at top ($z = +h/2$) and shear stress σ_{xz} evaluated along the thickness in ($z = 0$) (plate with 4 layers; actuator case)

a/h		2	4	10	100	
						Reference [14]
σ_{xx}	LD4 _a	3.8162	1.1180	0.1680	-0.0246	
σ_{xz}	LD4 _a	0.0864	0.0239	0.0020	0.0000	
σ_{xx}	LD4	3.8404	1.1249	0.1693	-0.0246	
	LD3	3.8618	1.1262	0.1693	-0.0246	
	LD2	3.9439	1.1312	0.1695	-0.0246	
	LD1	12.4636	3.3532	0.5272	-0.0210	
σ_{xz}	LD4	0.0929	0.0241	0.0020	0.0000	
	LD3	0.0651	0.0227	0.0020	0.0000	
	LD2	0.1824	0.0359	0.0028	0.0000	
	LD1	0.0215	0.0029	0.0004	0.0000	
σ_{zz}	LD4	0.0006	0.0001	0.0000	0.0000	
	LD3	0.0271	0.0019	0.0001	0.0000	
	LD2	0.0547	0.0045	0.0001	0.0000	
	LD1	8.1925	2.1772	0.3540	0.0036	

are evaluated with an analytical layerwise full mixed approach (LWFM). With the LWFM it is possible to impose the interface continuity of both mechanical transverse stresses and normal electric displacement variables.

The geometrical parameters of the considered cylinder are $a = 40$, $b = 2\pi R_\beta$ and, $R_\beta = 10$.

Due to the symmetry of both the geometry and the load, an octave of the cylinder is studied (1/2 in the axial direction and

1/4 in the hoop direction). The following symmetry conditions are applied:

$$\begin{aligned} v_\tau(\alpha, 0) &= 0, \\ u_\tau(0, \beta) &= 0, \\ v_\tau(\alpha, R_\beta \pi/2) &= 0, \end{aligned} \quad (34)$$

and the following boundary conditions are prescribed:

$$v_\tau(a/2, \beta) = w_\tau(a/2, \beta) = 0, \quad (35)$$

with $\tau = 0, 1, \dots, N$.

The results are presented for these cases for different thickness ratios $R/h = 2, 4, 10, 100$. A mesh grid of 12×12 elements is taken to ensure the convergence of the solution.

The results lead to similar conclusions made for the plate: the solution converges to the exact solution by increasing the order of expansion N ; the LW models are able to give good results, see Tables 6 and 7. If one considers the stresses, the behavior is the same: higher-order layerwise models are necessary to match the reference solution in the thick shells, see Tables 8 and 9. In this case, the use of the LD4 model becomes mandatory. The LD4 model only is able to fulfill the continuity conditions of shear stresses $\sigma_{\alpha z}$ at the interfaces between layers, while the lower-order model gives a completely wrong result, even if the shell is very thin ($R/h = 100$), see Figures 21–24. The LD4 model is not able to fulfill the

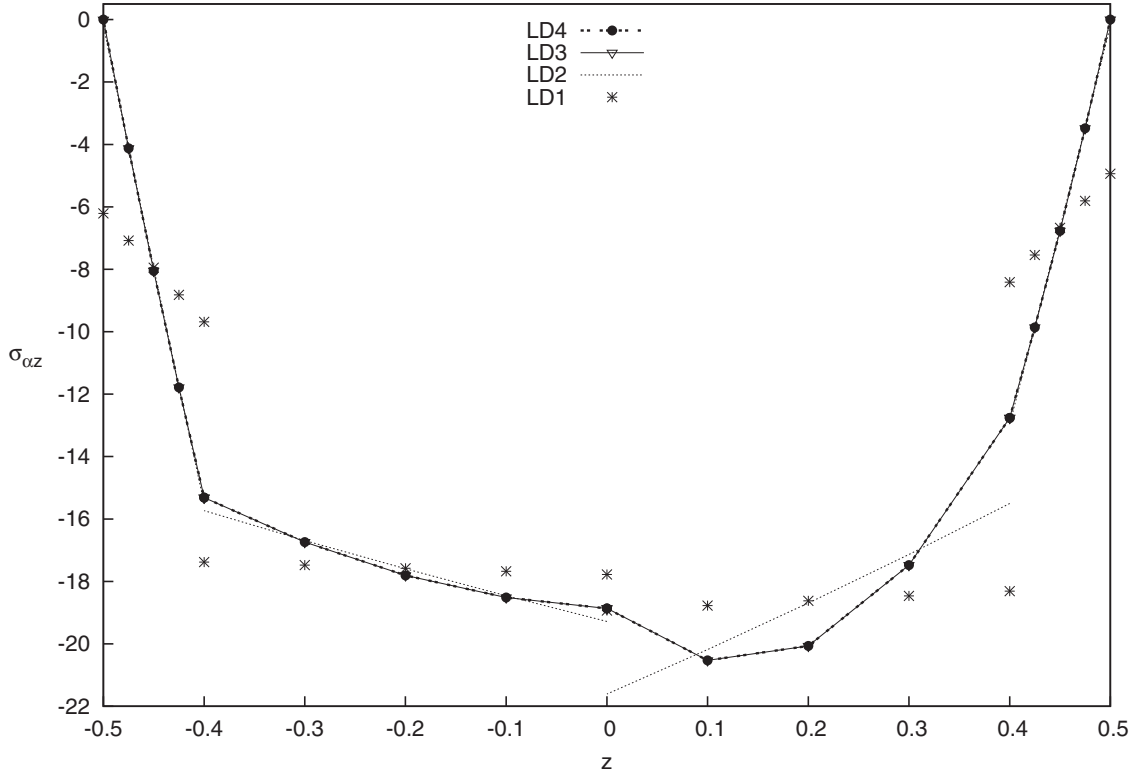


Fig. 5. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio (a/h) = 100. Plate with four layers. Sensor case.

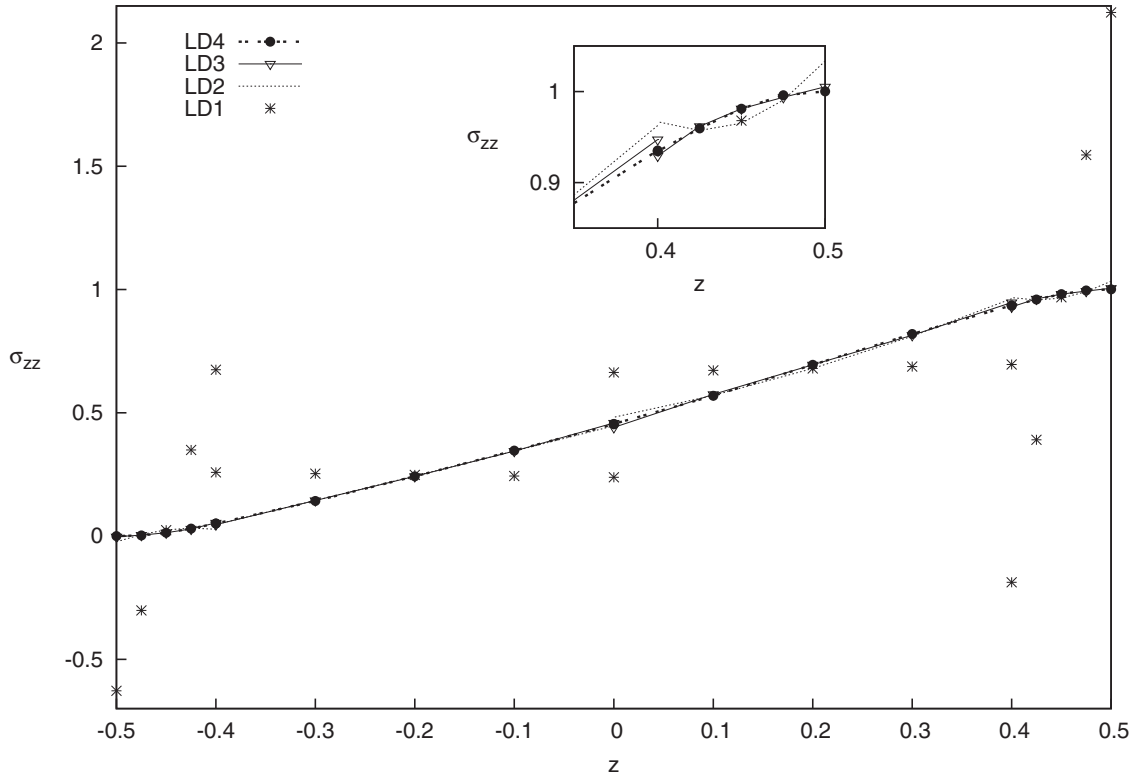


Fig. 6. Transverse stress σ_{zz} along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Sensor case.

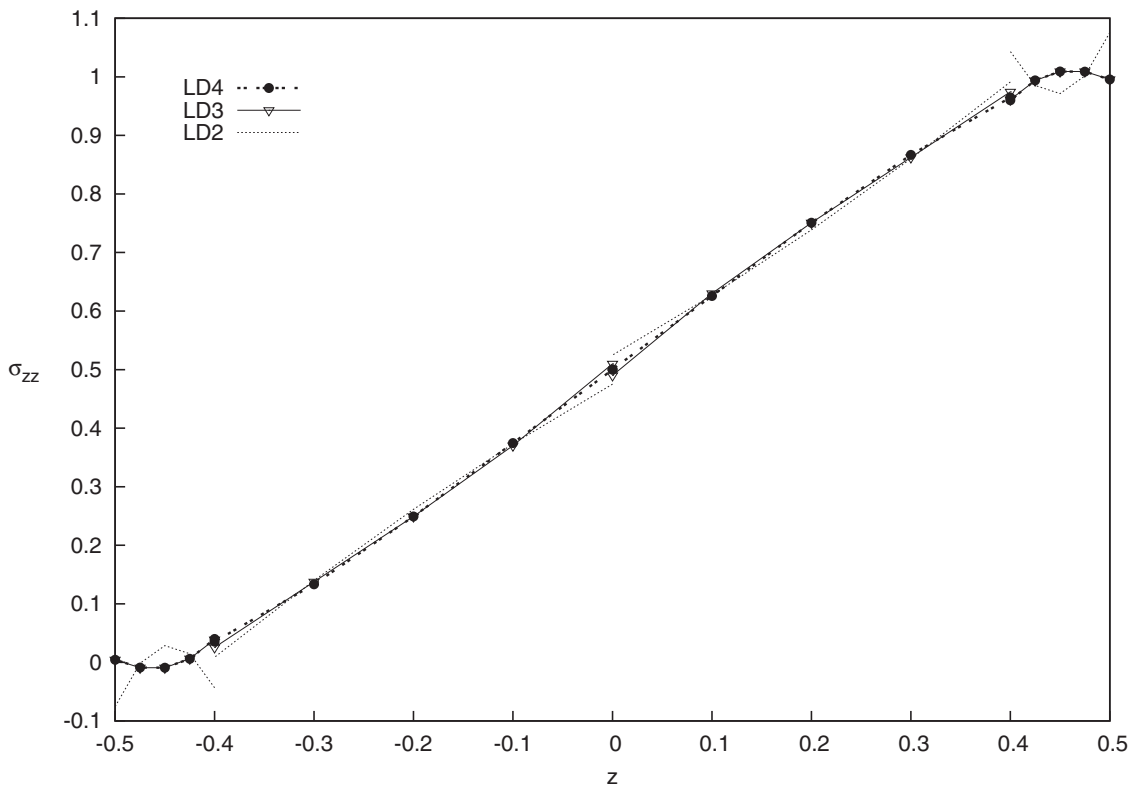


Fig. 7. Transverse stress σ_{zz} along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Sensor case.

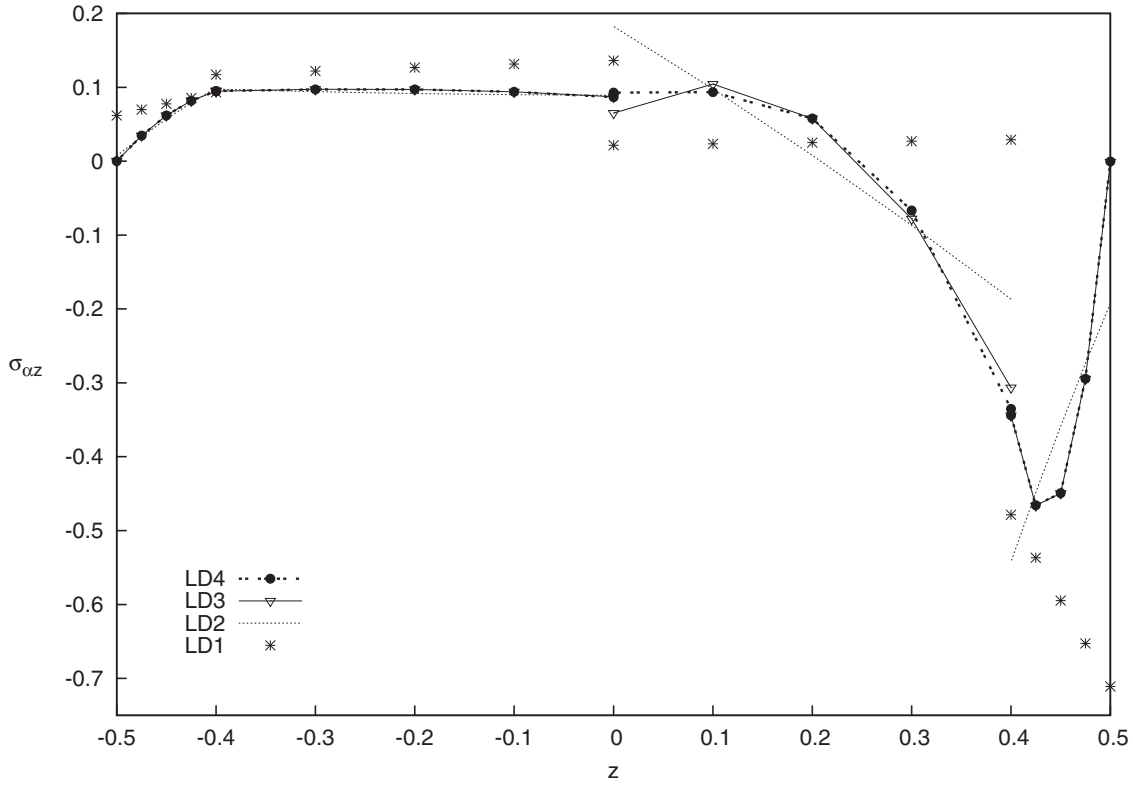


Fig. 8. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Actuator case.

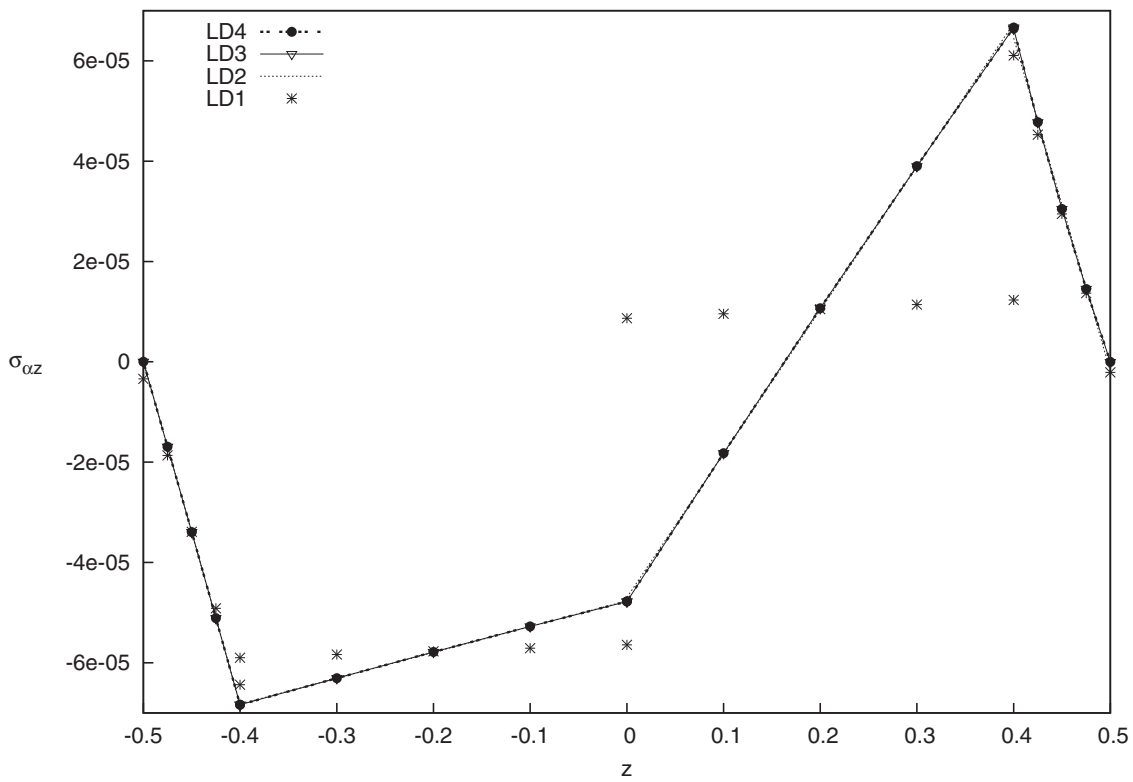


Fig. 9. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Actuator case.

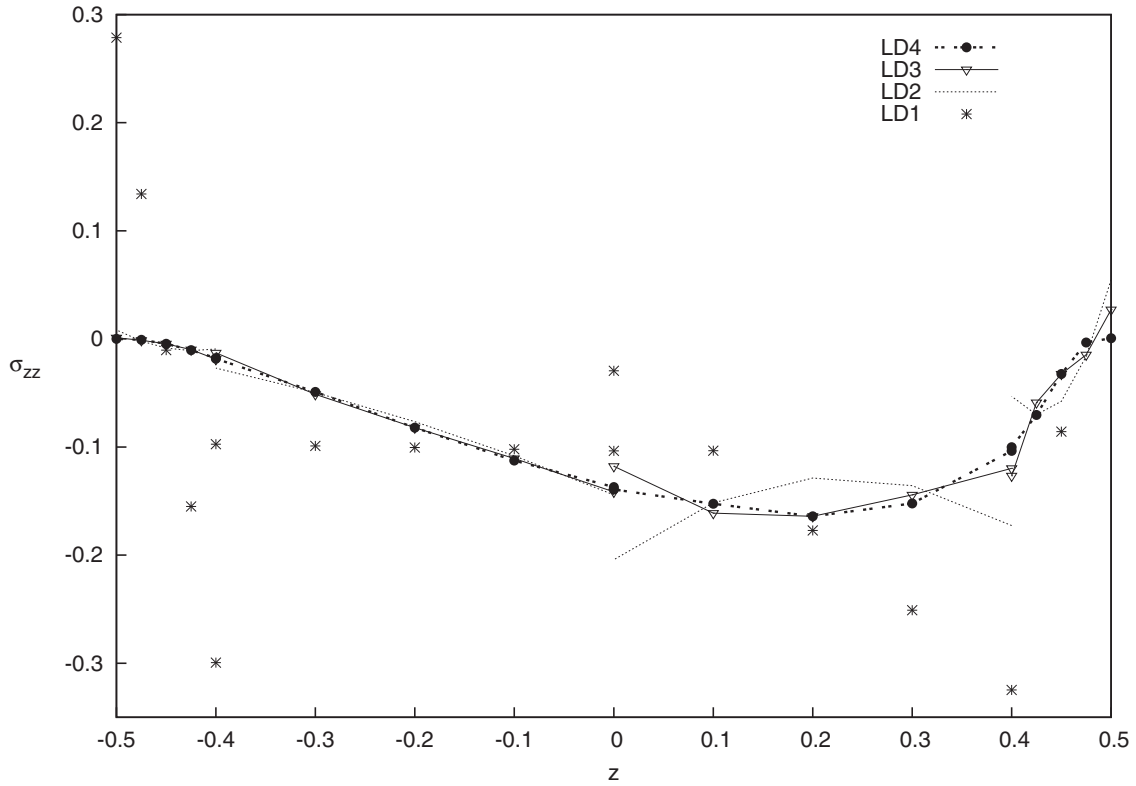


Fig. 10. Transverse stress σ_{zz} along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Actuator case.

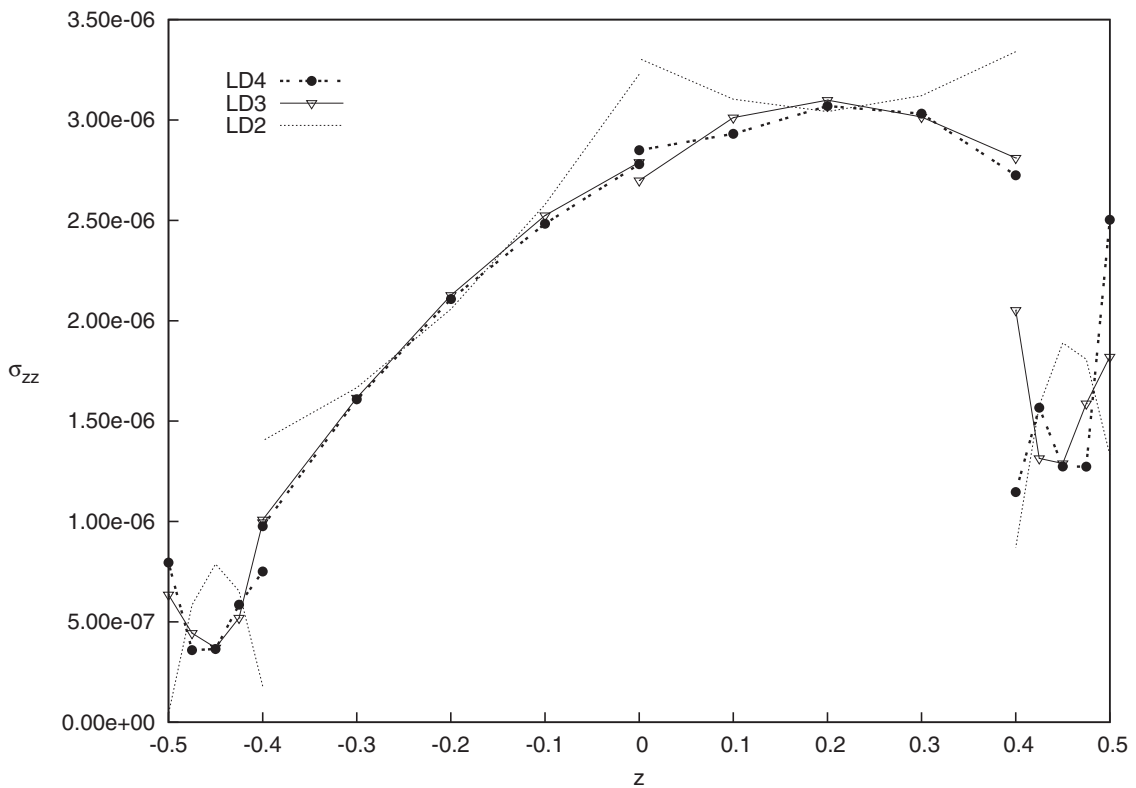


Fig. 11. Transverse stress σ_{zz} along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Actuator case.

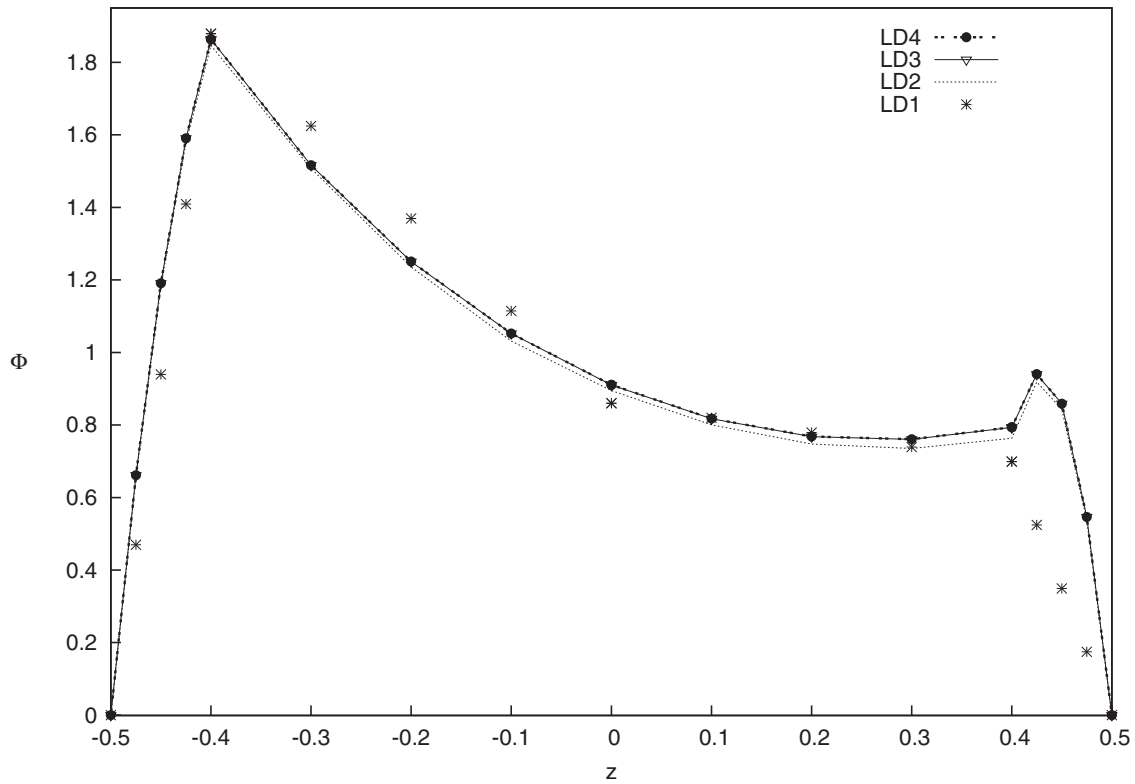


Fig. 12. Electric potential Φ along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Sensor case.

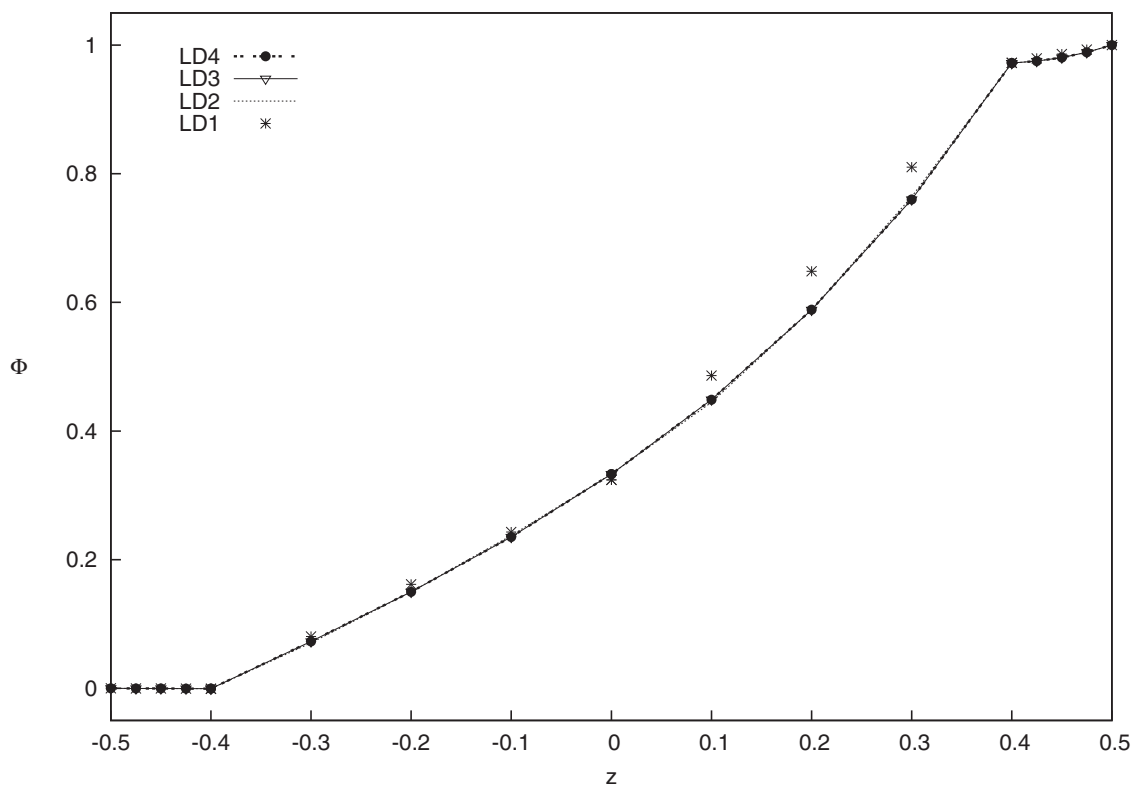


Fig. 13. Electric potential Φ along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Actuator case.

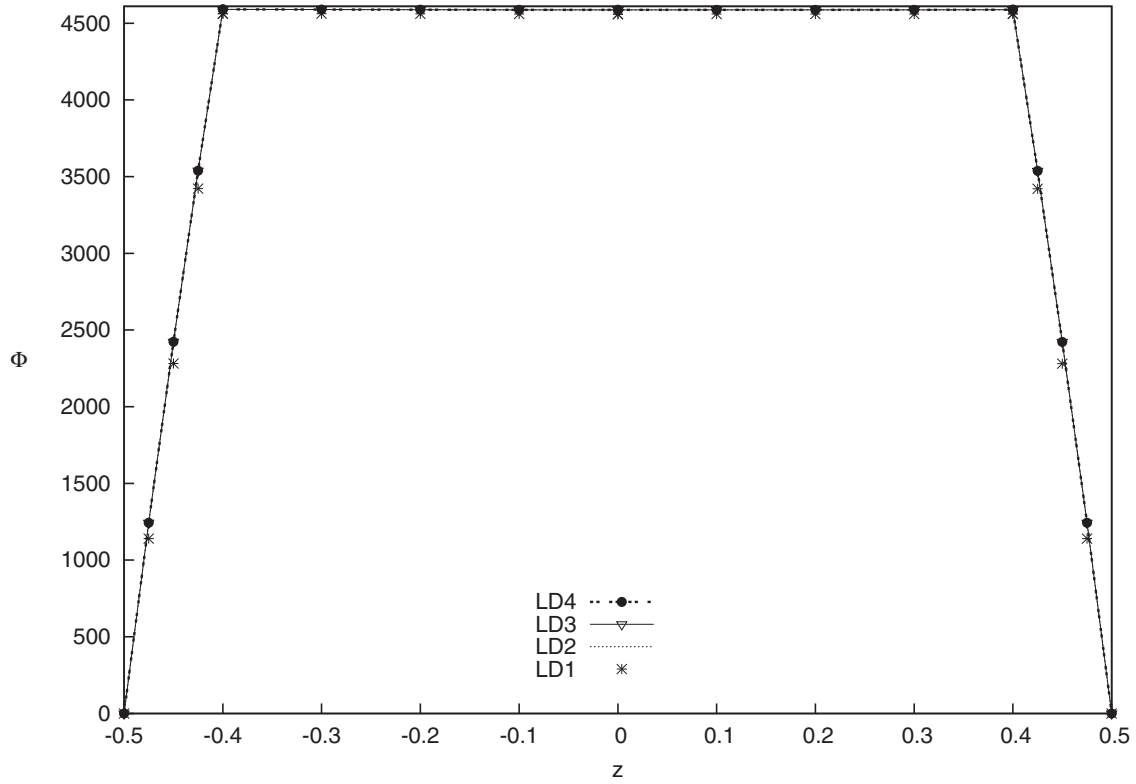


Fig. 14. Electric potential Φ along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Sensor case.

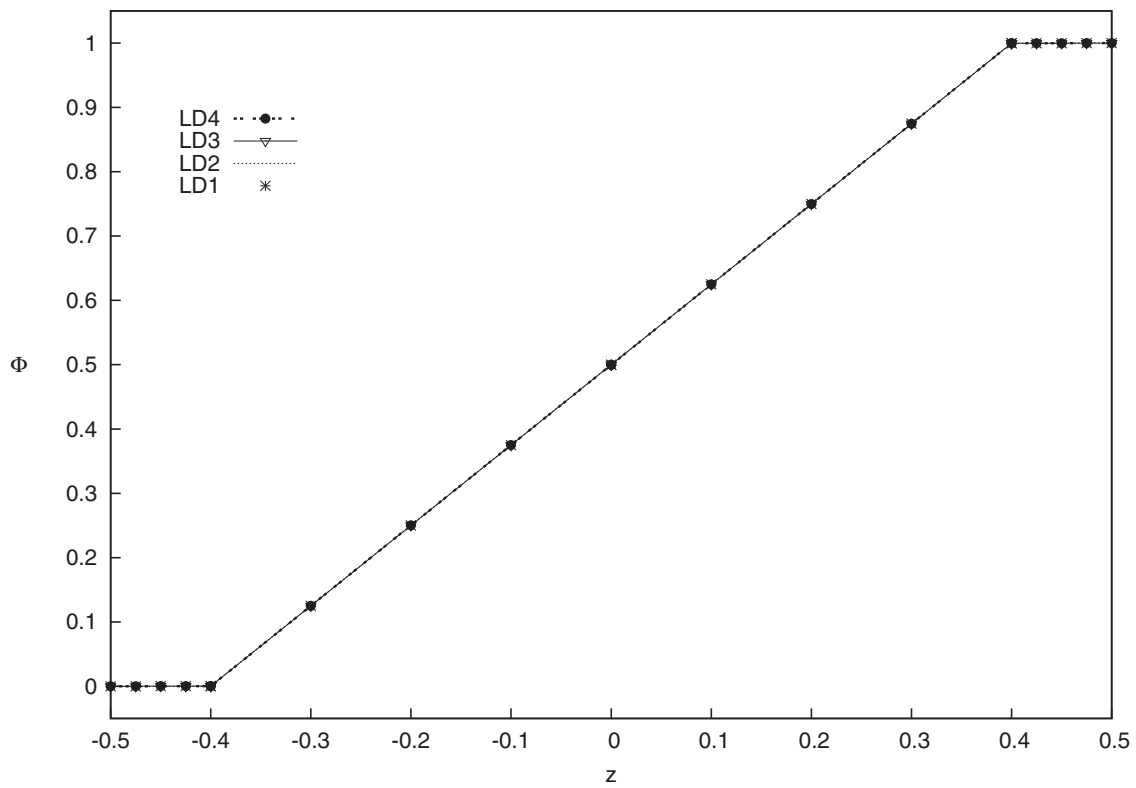


Fig. 15. Electric potential Φ along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Actuator case.

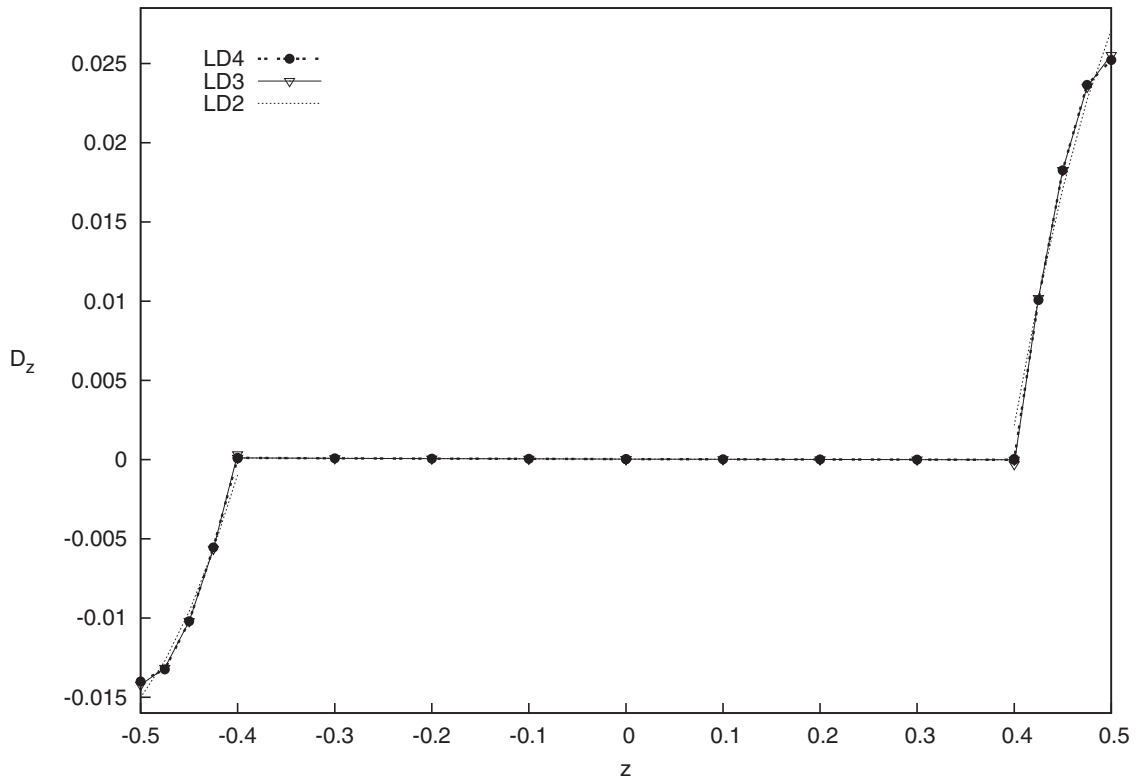


Fig. 16. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Sensor case.

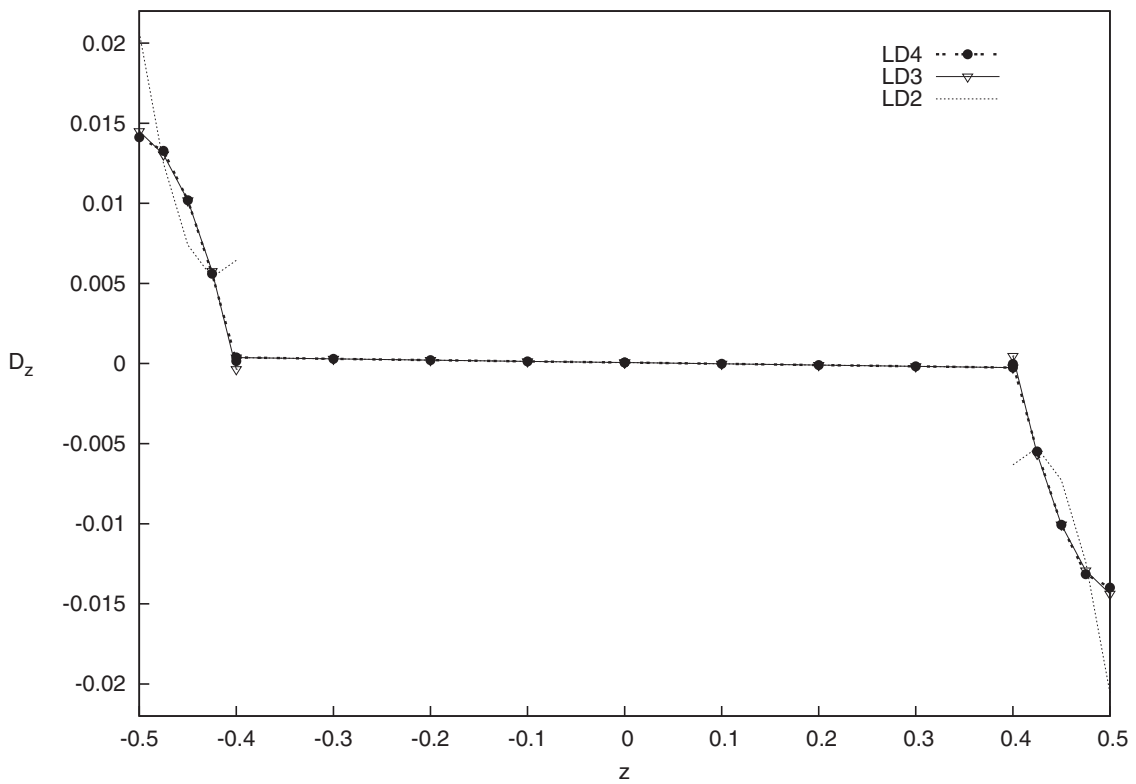


Fig. 17. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Sensor case.

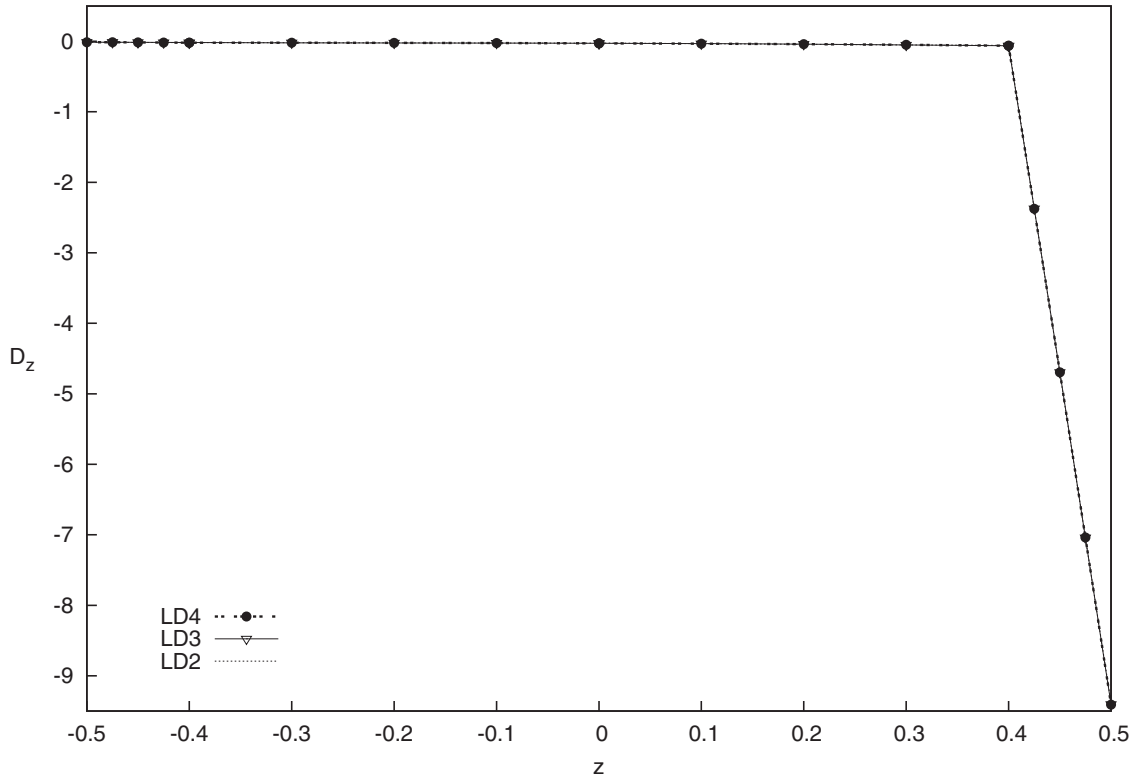


Fig. 18. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(a/h) = 2$. Plate with four layers. Actuator case.

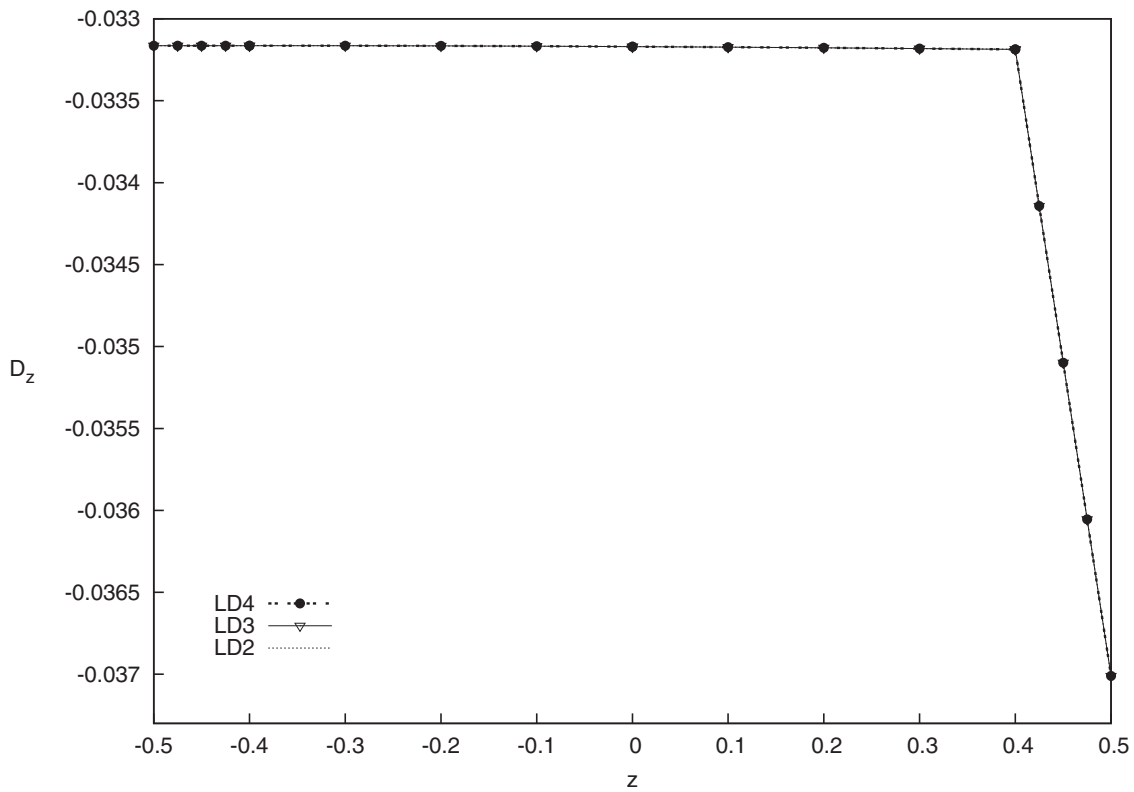


Fig. 19. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(a/h) = 100$. Plate with four layers. Actuator case.

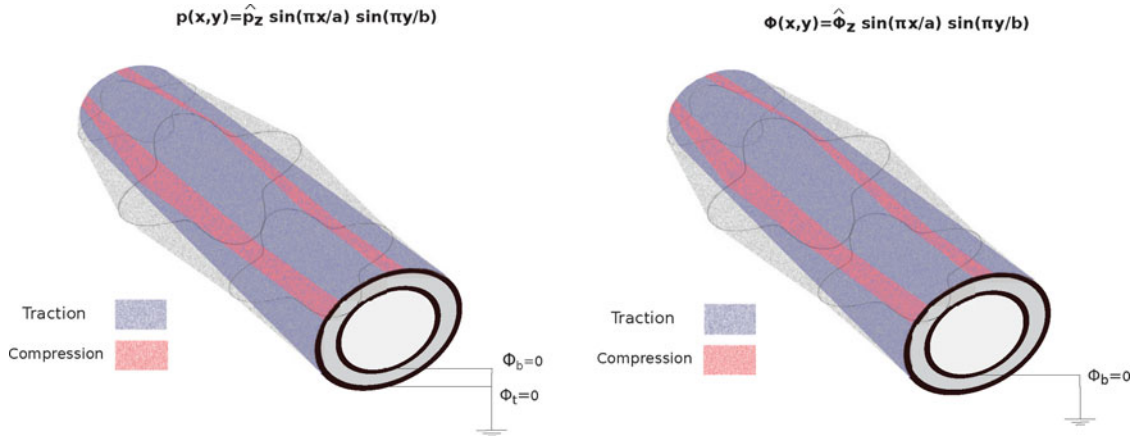


Fig. 20. Cylindrical piezo shell, sensor, and actuator cases.

continuity conditions of transverse stresses σ_{zz} at the interfaces between layers, see Figures 25–28. To overcome this problem, a mixed variational principle could be used. For the description of the electric potential Φ , it is necessary to use the higher-order model, LD4, to describe the nonlinear behavior of thick shell ($R/h = 2$), see Figures 29 and 30; on the other hand, for thin shell ($R/h = 100$) a layerwise description of lower order is sufficient, see Figures 31 and 32. The electric displacement \mathcal{D}_z has a nonlinear behavior for every thickness

ratio in the sensor case, and higher-order models are not sufficient to describe it correctly along the thickness for thick shell ($R/h = 2$), see Figure 33. For the thin cylinder ($R/h = 100$), the continuity interface cannot be reached, see Figure 34. For the actuator case, in spite of its linear behavior, it is necessary for a higher-order model to obtain the continuity interface of the electric displacement, see Figures 35 and 36. To overcome this problem, a mixed variational principle could be used.

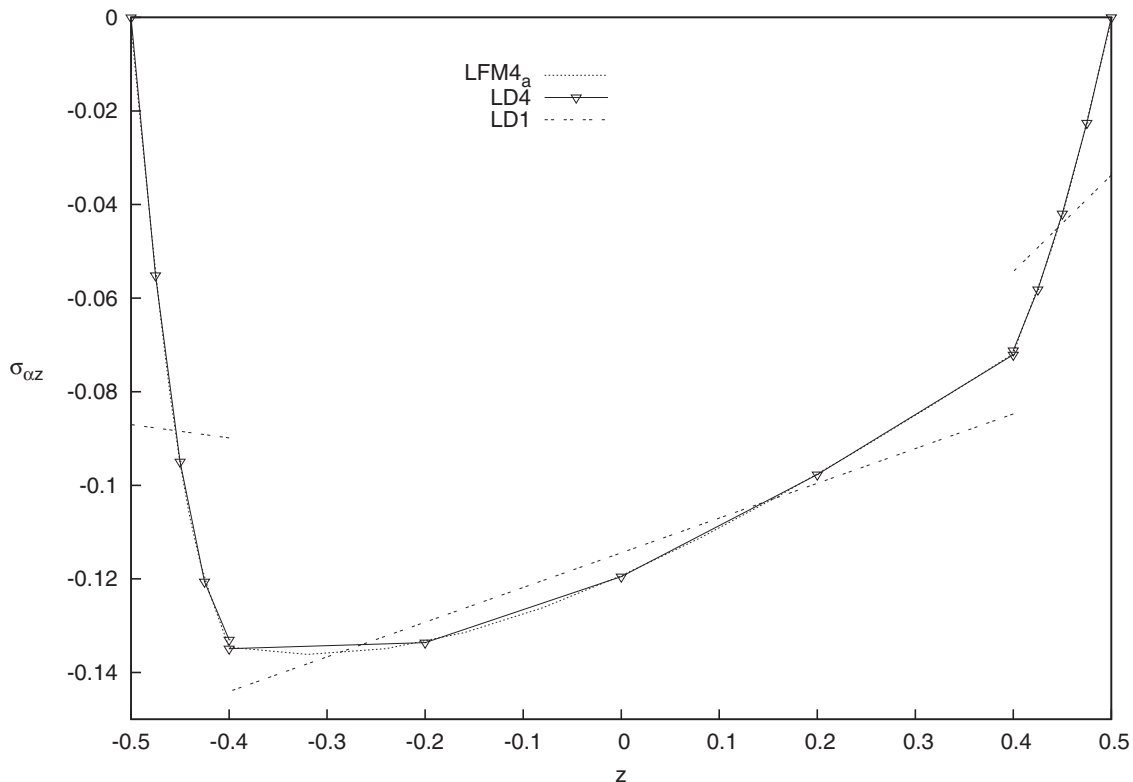


Fig. 21. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio (R/h) = 2. Shell with three layers. Sensor case.

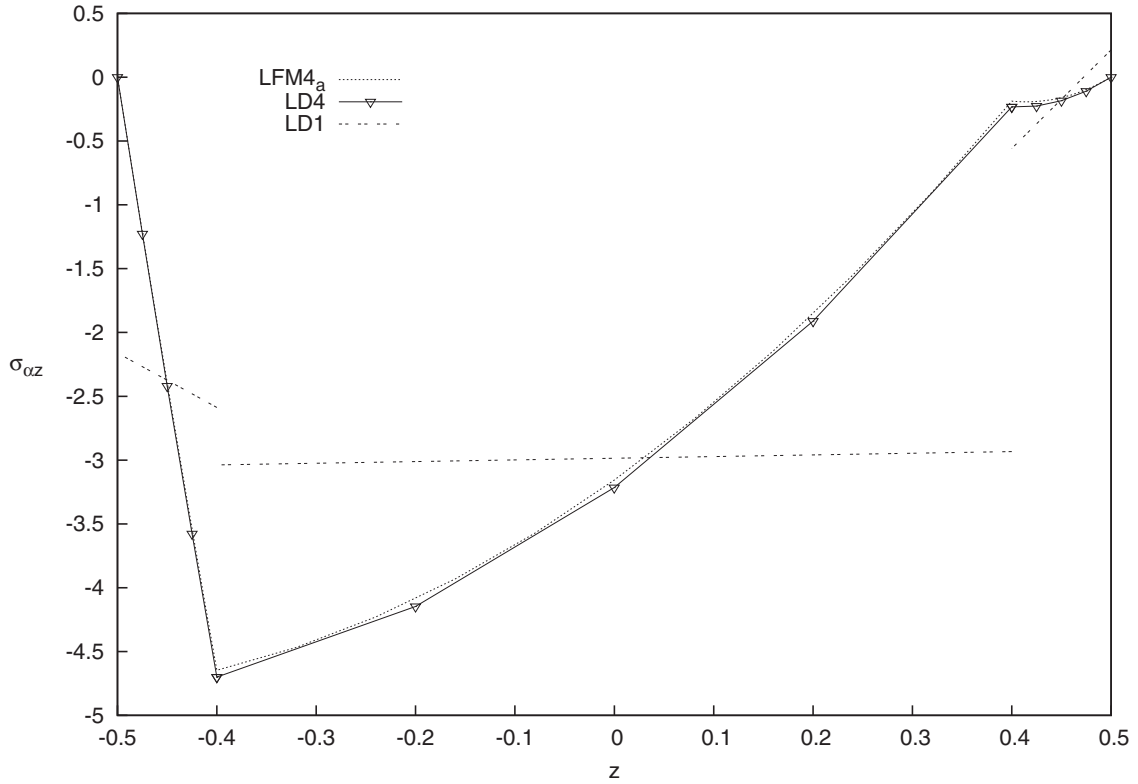


Fig. 22. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Sensor case.

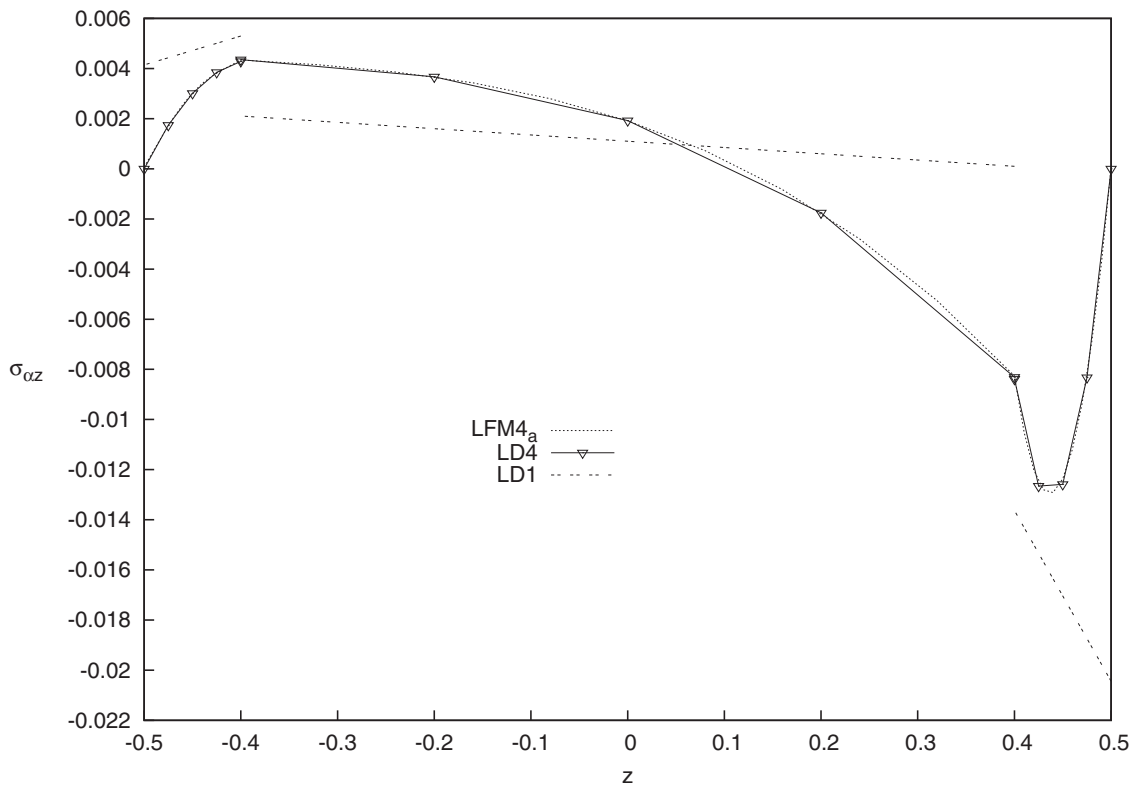


Fig. 23. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Actuator case.

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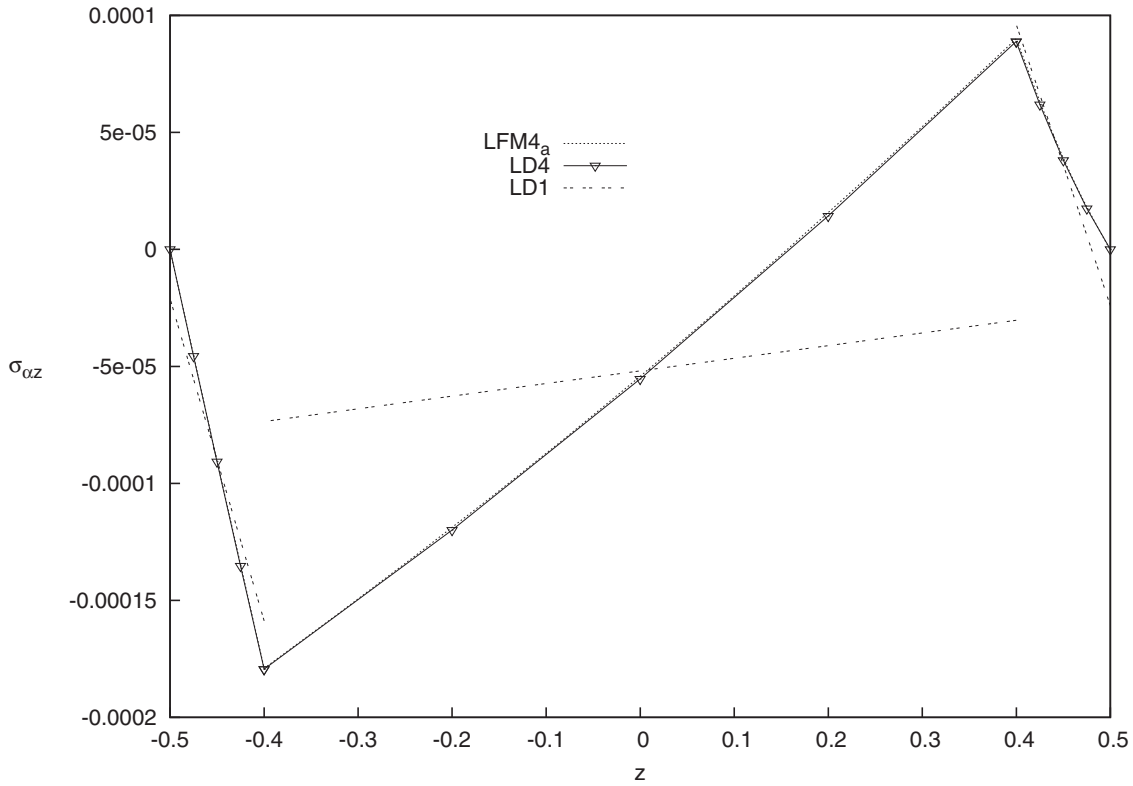


Fig. 24. Shear stress $\sigma_{\alpha z}$ along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Actuator case.

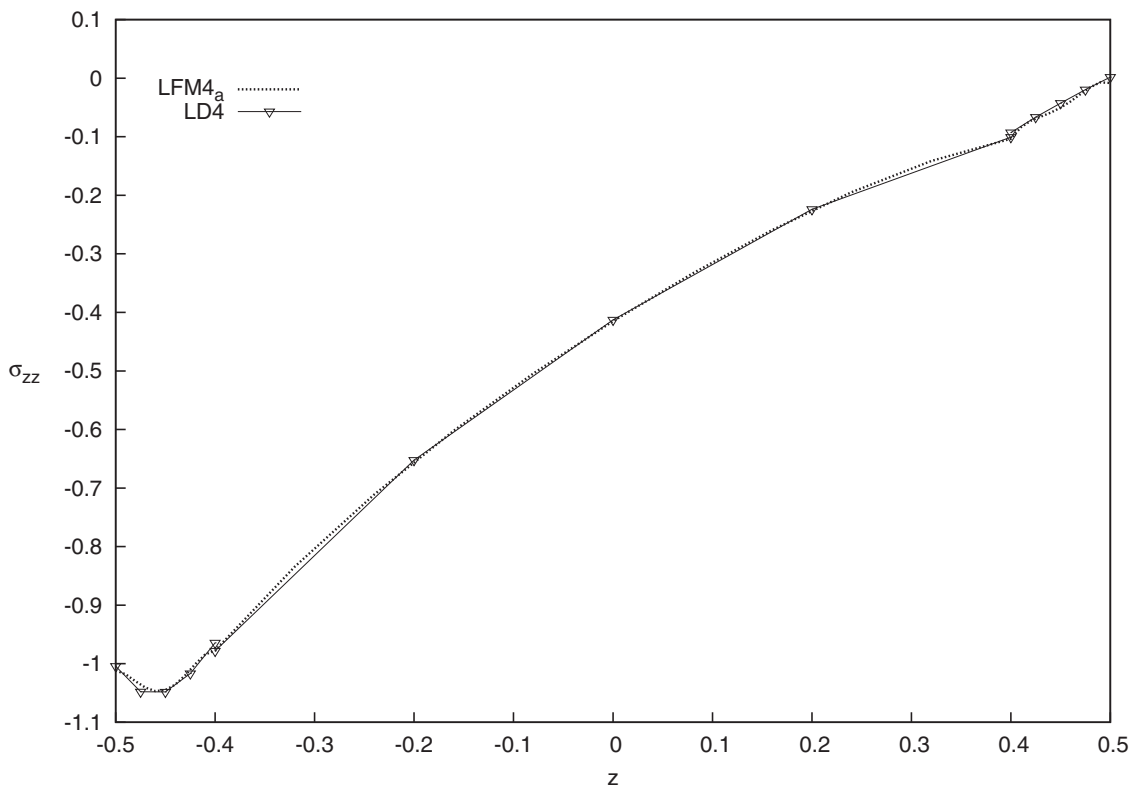


Fig. 25. Transverse stress σ_{zz} along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Sensor case.

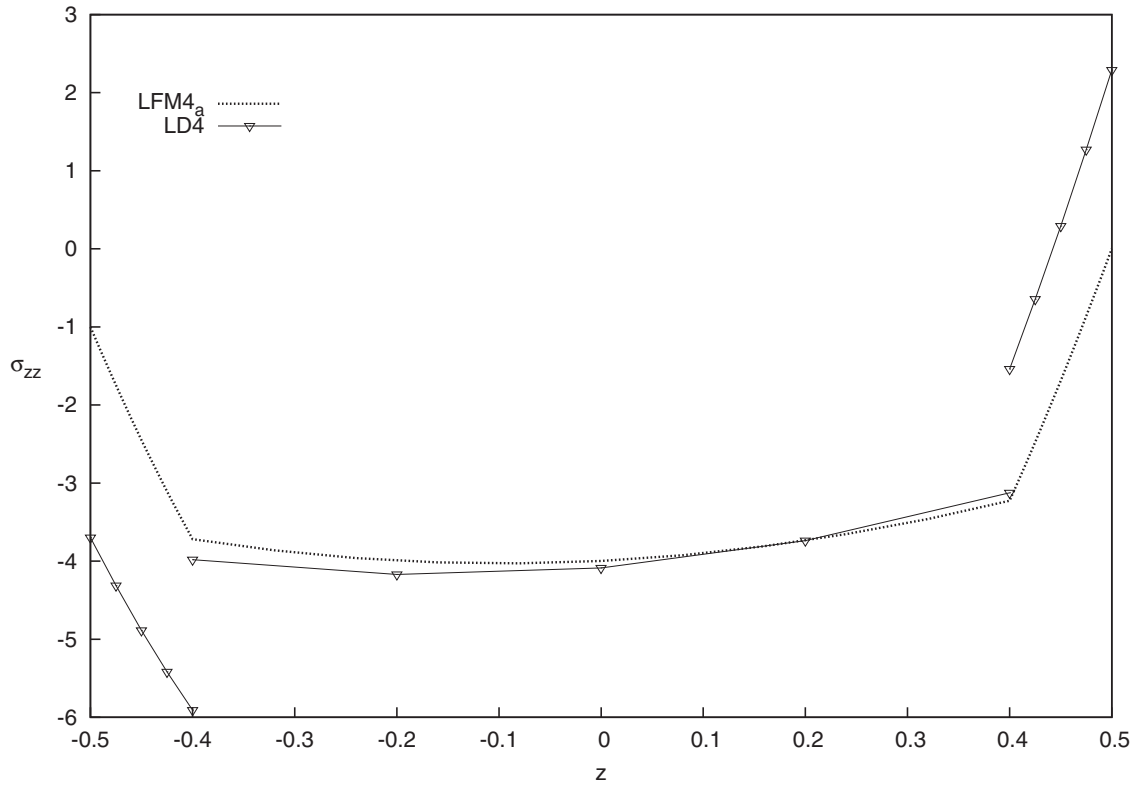


Fig. 26. Transverse stress σ_{zz} along the thickness, with thickness ratio (R/h) = 100. Shell with three layers. Sensor case.

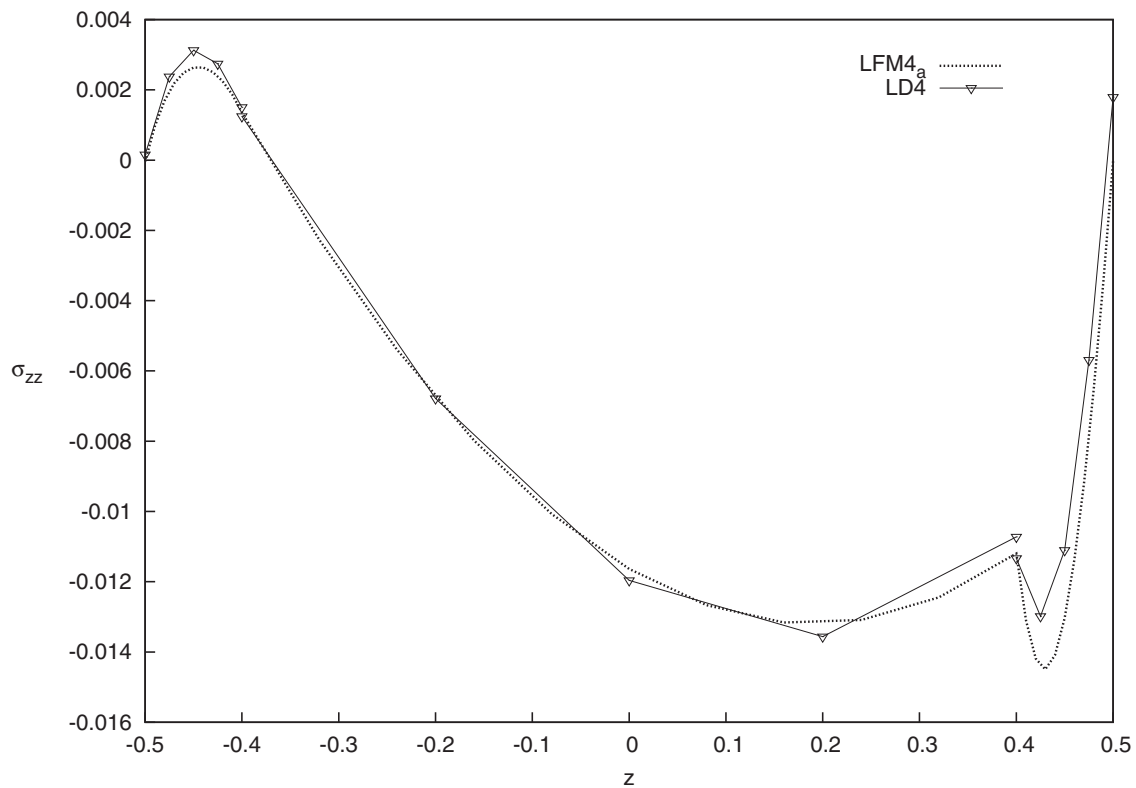


Fig. 27. Transverse stress σ_{zz} along the thickness, with thickness ratio (R/h) = 2. Shell with three layers. Actuator case.

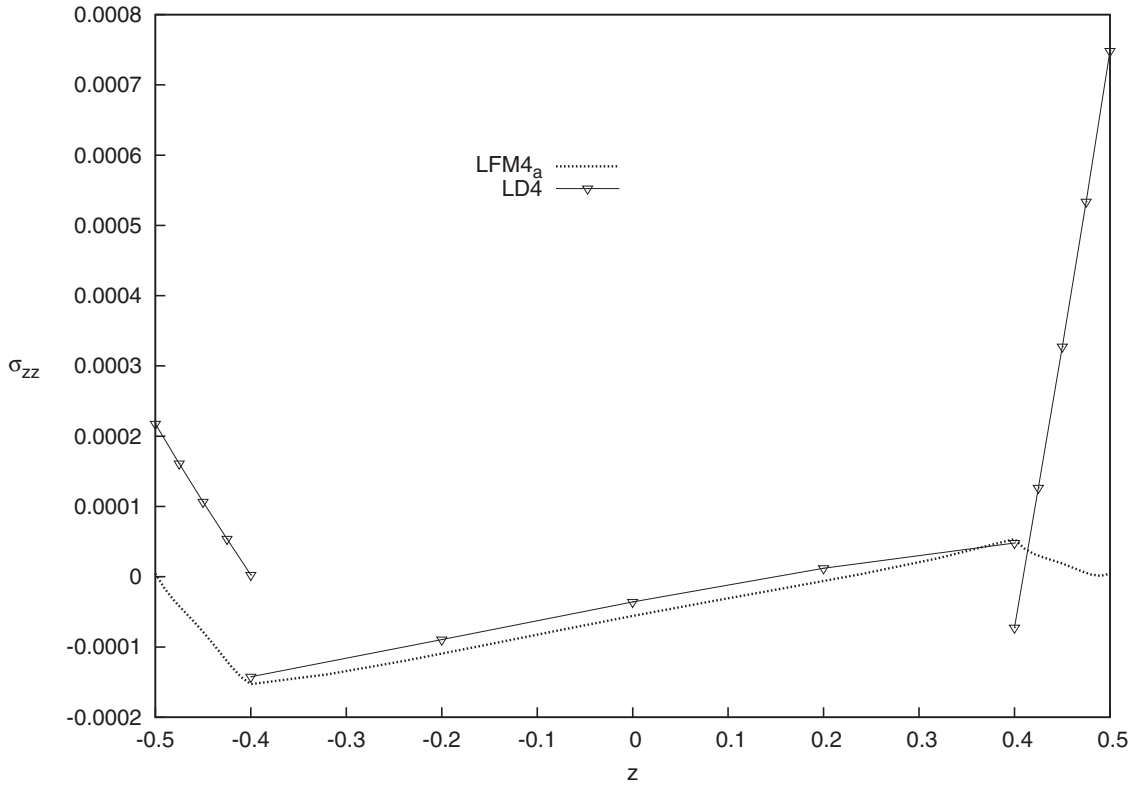


Fig. 28. Transverse stress σ_{zz} along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Actuator case.

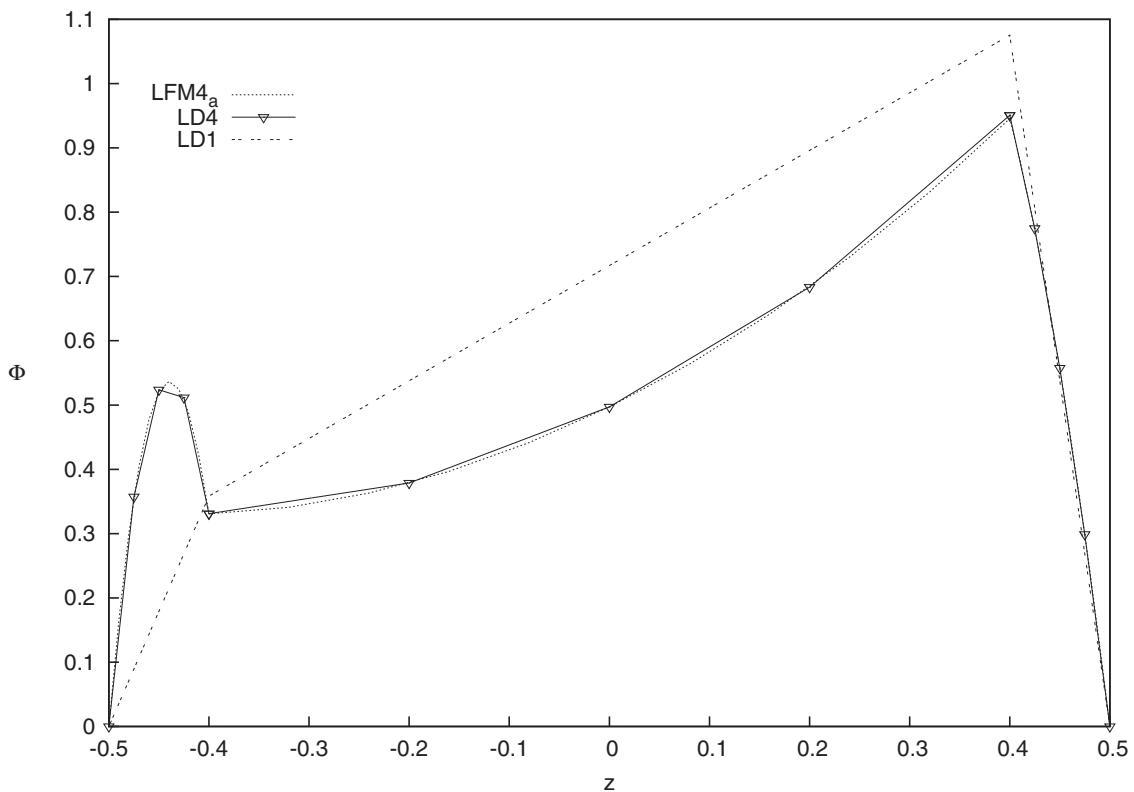


Fig. 29. Electric potential Φ along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Sensor case.

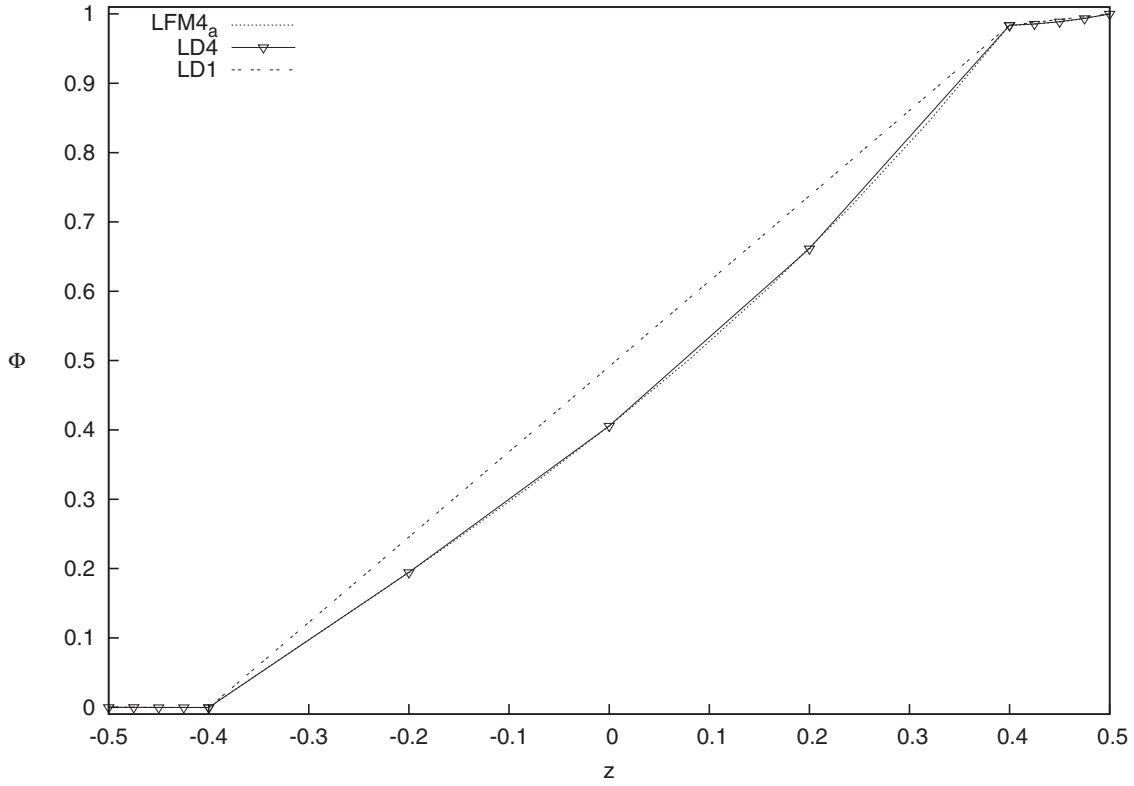


Fig. 30. Electric potential Φ along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Actuator case.

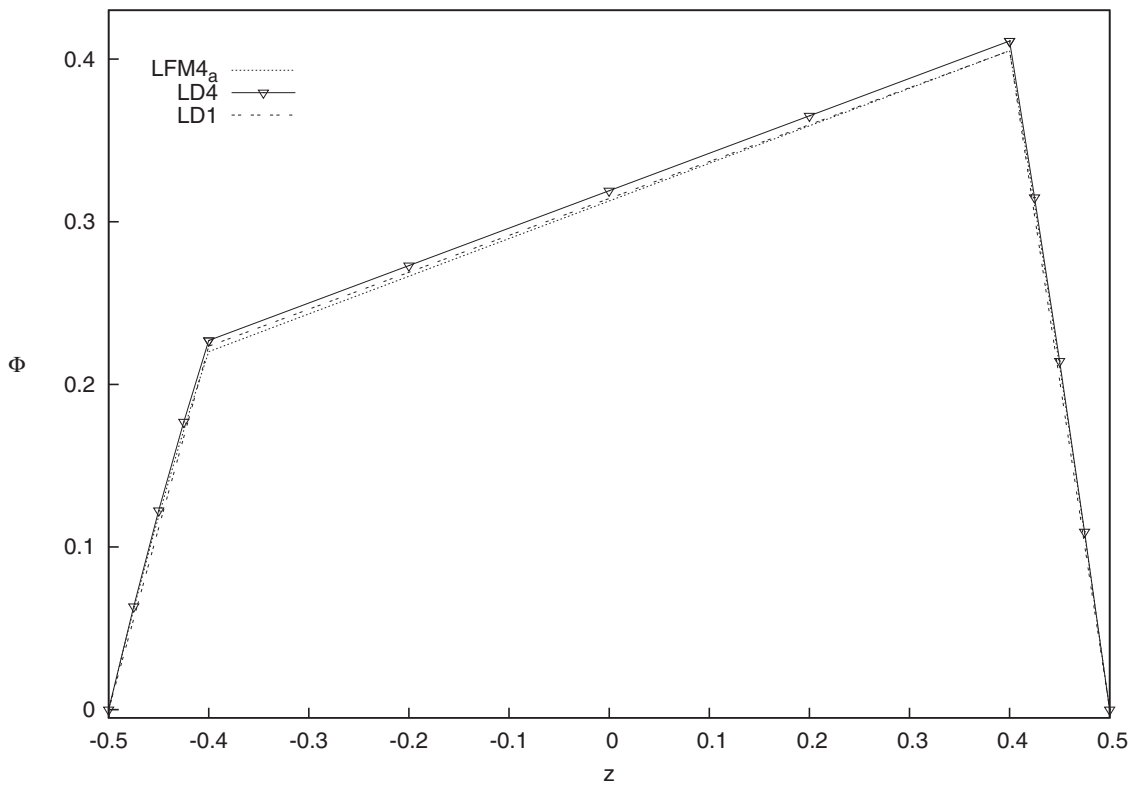


Fig. 31. Electric potential Φ along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Sensor case.

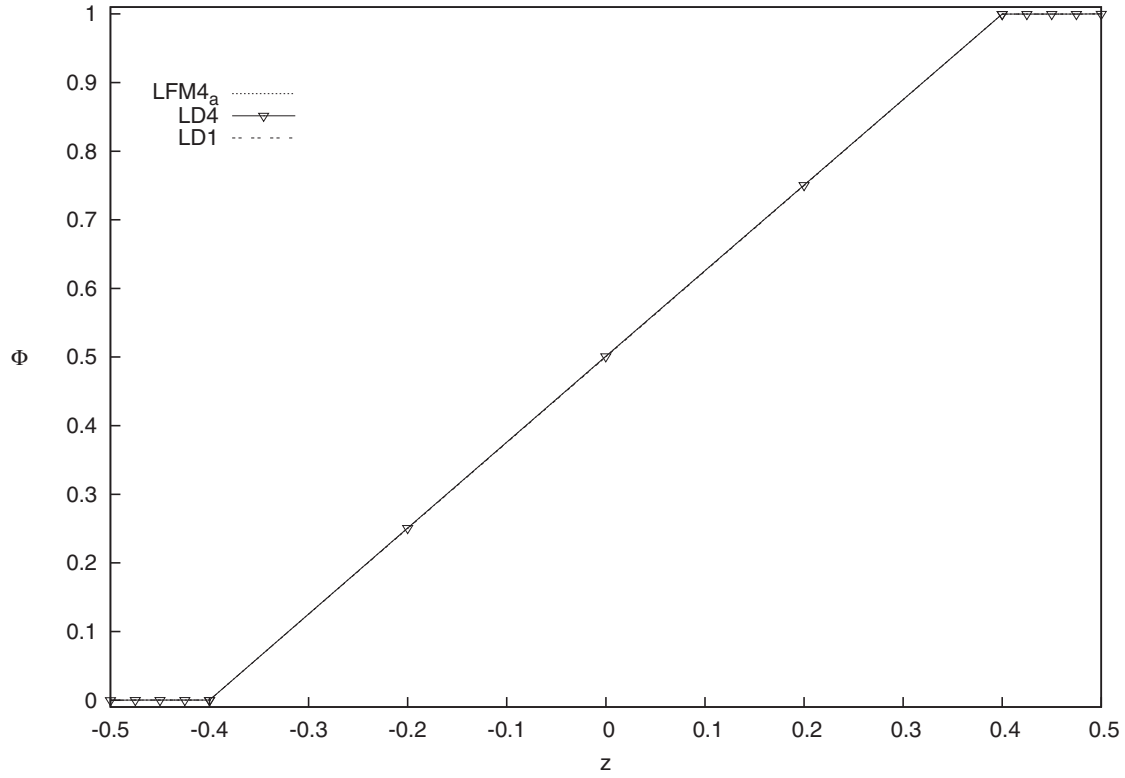


Fig. 32. Electric potential Φ along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Actuator case.

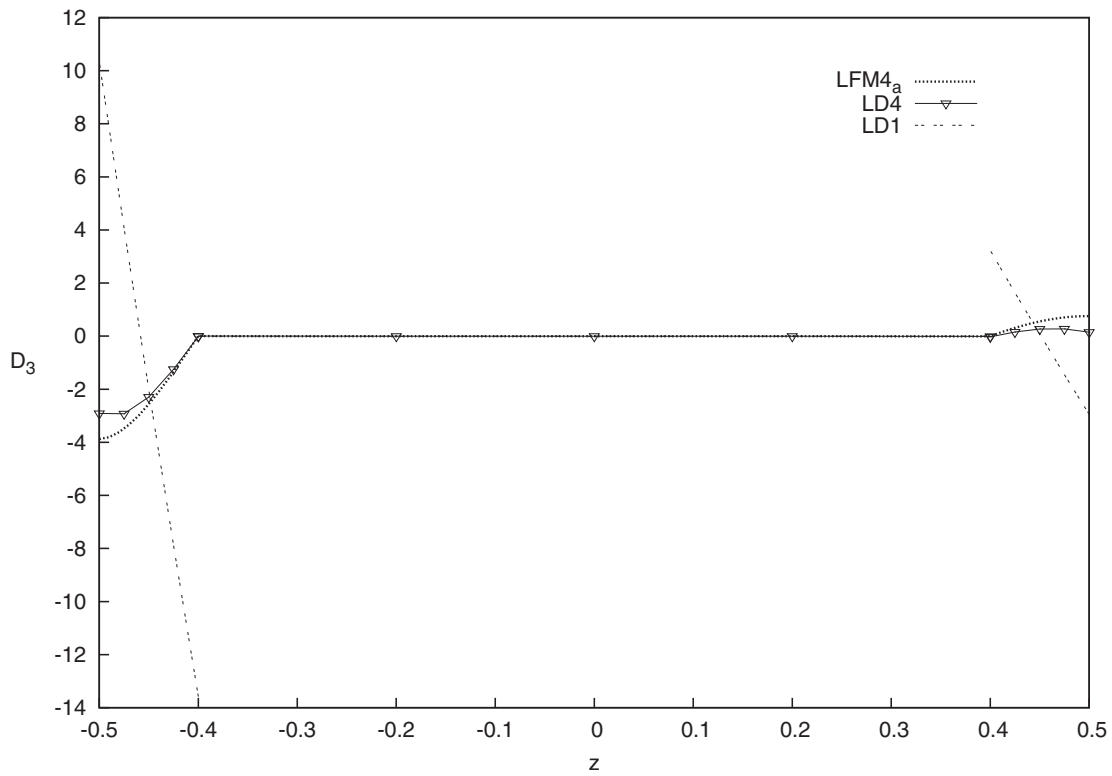


Fig. 33. Electric displacement D_z along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Sensor case.

SOR case.

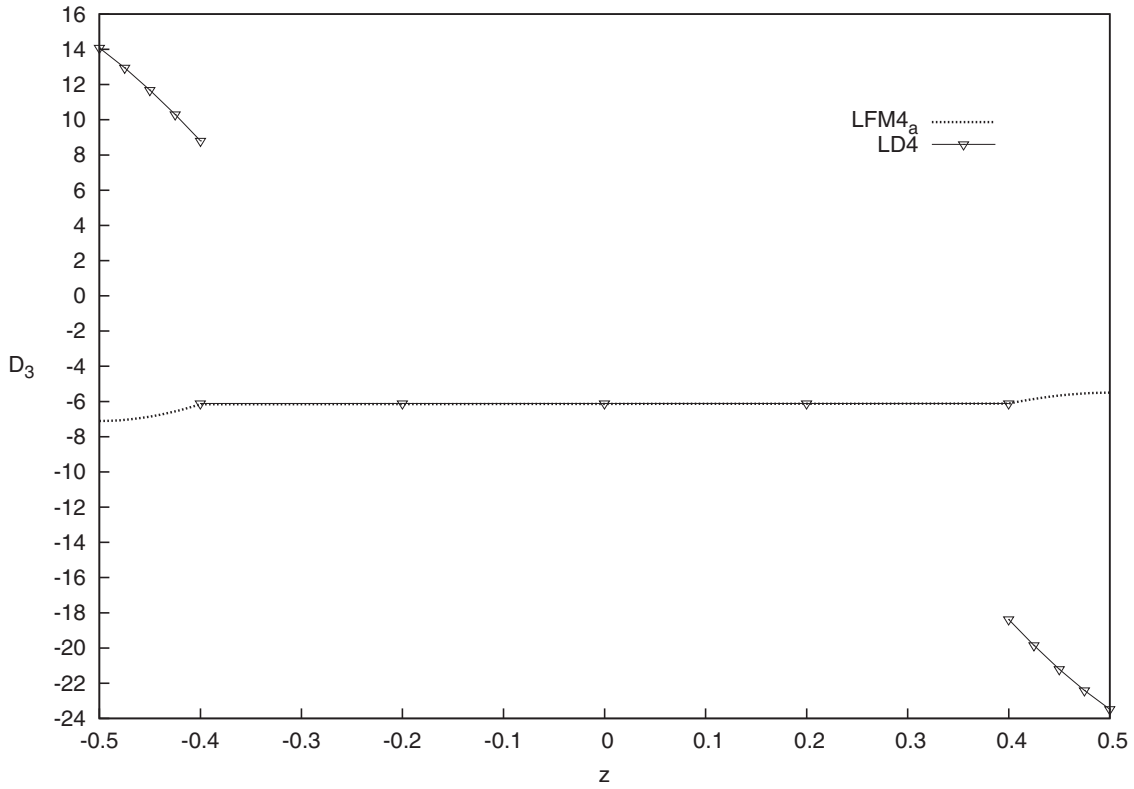


Fig. 34. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Sensor case.

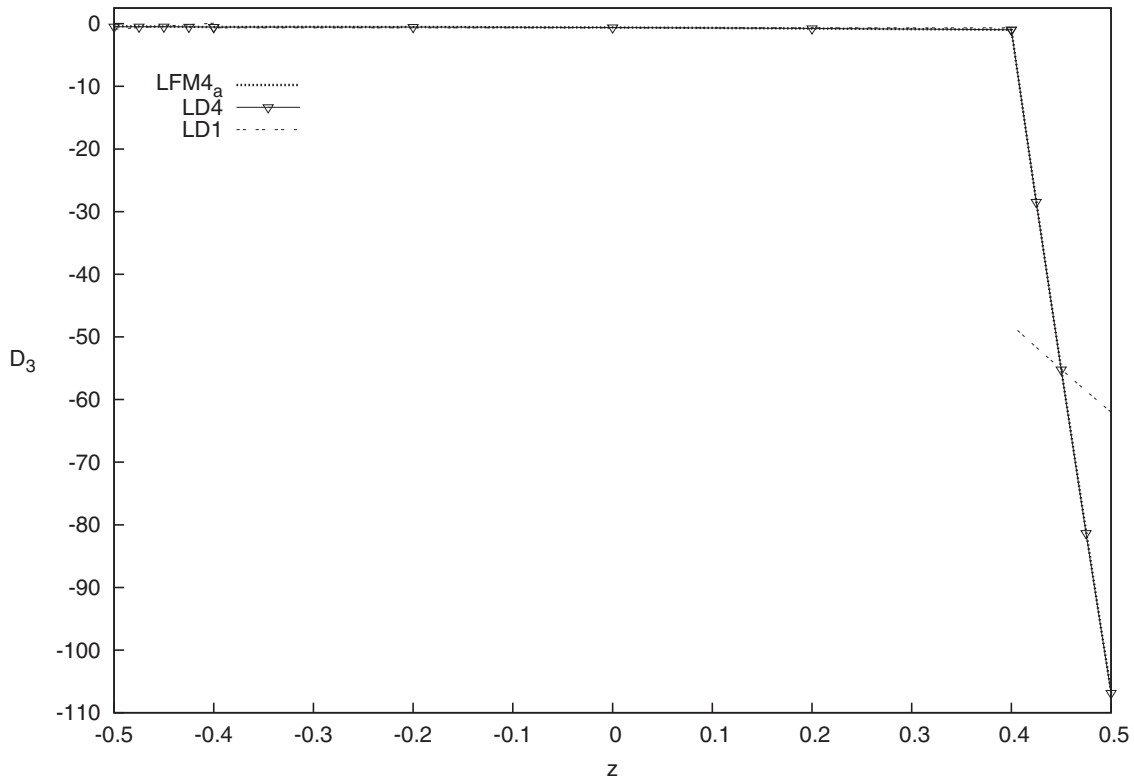


Fig. 35. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(R/h) = 2$. Shell with three layers. Actuator case.

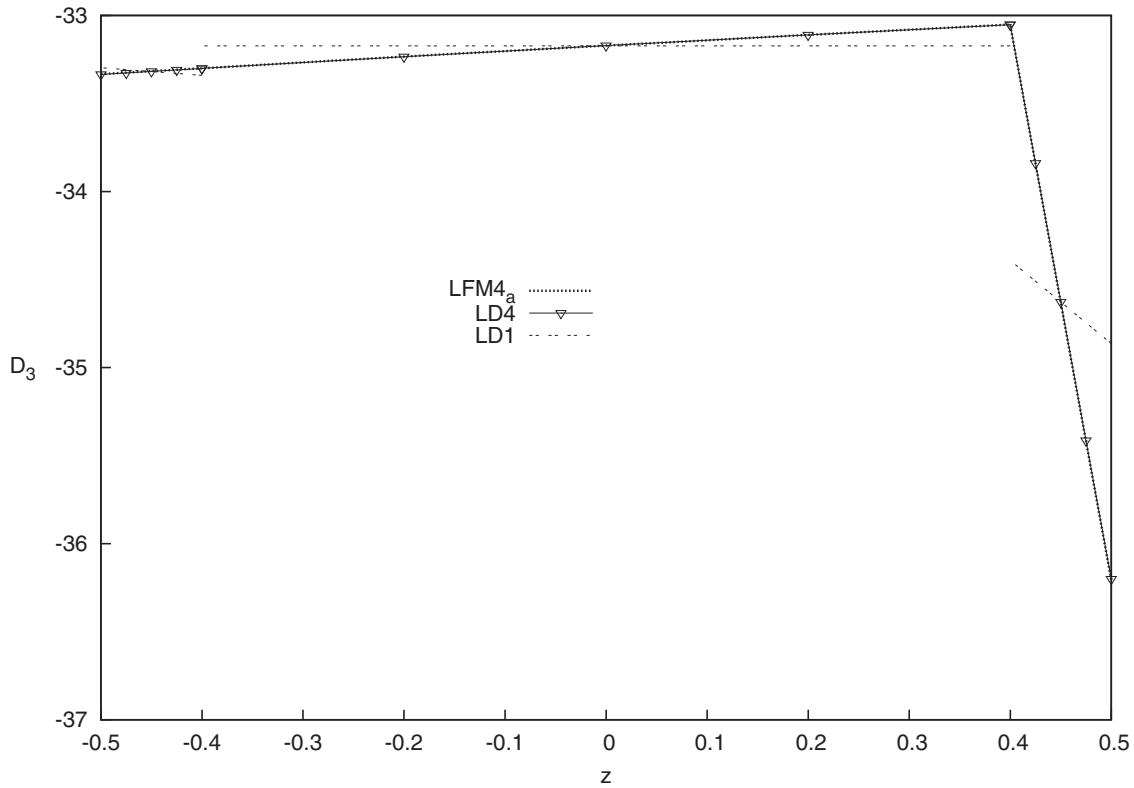


Fig. 36. Electric displacement \mathcal{D}_z along the thickness, with thickness ratio $(R/h) = 100$. Shell with three layers. Actuator case.

Table 6. Transverse normal displacement $w * 10^9$, electric potential Φ , shear stresses $\sigma_{\alpha z}$, $\sigma_{\beta z}$ evaluated along the thickness in $(z = 0)$; transverse normal electric displacement $\mathcal{D}_z * 10^9$ evaluated along the thickness at *top* $(z = +h/2)$; in-plane stresses $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$, $\sigma_{\alpha\beta}$ and transverse normal stress σ_{zz} evaluated along the thickness at $(z = \pm h/2)$ (shell mono-layered piezoelectric cylinder; *Mesh* (10×10) ; sensor case)

R/h		2	4	10	100
			Reference [43]		
$w (10^9)$	<i>LFM4_a</i>	0.0566	0.3332	4.5483	3016.6
σ_{zz}	<i>LFM4_a</i>	-1.1392	-1.0671	-1.0269	-1.0018
Φ	<i>LFM4_a</i>	0.0153	0.0355	0.0942	0.6513
$\mathcal{D}_z (10^9)$	<i>LFM4_a</i>	0.0095	0.0028	-0.1646	-111.76
$w (10^9)$	LD4	0.0566	0.3331	4.5473	3017.2876
$\sigma_{\alpha\alpha}$	LD4	0.4332	2.3837	17.5910	2055.6587
		-1.4003	3.3497	-17.0300	-316.7597
$\sigma_{\beta\beta}$	LD4	0.9068	4.9006	35.0550	2567.2803
		-2.0590	-6.5066	-38.7790	-2542.8062
$\sigma_{\alpha\beta}$	LD4	0.0008	0.0039	0.0291	1.3707
		-0.0008	-0.0040	-0.0329	-4.0802
$\sigma_{\alpha z}$	LD4	-0.1084	-0.2644	-0.7242	-5.1421
$\sigma_{\beta z}$	LD4	0.0046	0.0194	0.0750	0.2104
σ_{zz}	LD4	-0.0798	-0.0469	0.0317	3.2476
		-1.1658	-1.0914	-1.0992	-5.2059
Φ	LD4	0.0153	0.0356	0.0949	0.6696
$\mathcal{D}_z (10^9)$	LD4	0.0054	-0.0179	-0.2668	-111.1786

Table 7. Transverse normal displacement $w * 10^{11}$, electric potential Φ , shear stresses $\sigma_{\alpha z}$, $\sigma_{\beta z}$ evaluated along the thickness in ($z = 0$); transverse normal electric displacement $\mathcal{D}_z * 10^{11}$ evaluated along the thickness at *top* ($z = +h/2$); in-plane stresses $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$, $\sigma_{\alpha\beta}$ and transverse normal stress σ_{zz} evaluated along the thickness at ($z = \pm h/2$) (shell mono-layered piezoelectric cylinder; *Mesh* (10×10); actuator case)

R/h		2	4	10	100
			Reference [43]		
$w (10^{11})$	$LFM4_a$	-9.6220	-11.285	6.4540	11,277
σ_{zz}	$LFM4_a$	0.0431	0.0114	0.0004	-0.0001
Φ	$LFM4_a$	0.3431	0.4611	0.5037	0.5254
$\mathcal{D}_z (10^{11})$	$LFM4_a$	-584.80	-783.99	-1615.6	-16266
$w (10^{11})$	LD4	-9.6219	-11.2852	6.4451	11,275.3779
$\sigma_{\alpha\alpha}$	LD4	-3.2642	-4.4933	-8.9704	-18.4390
		-0.9616	-3.5544	-10.3507	-106.9489
$\sigma_{\beta\beta}$	LD4	0.3518	0.3624	1.3358	94.4157
		0.4177	0.0395	-1.4200	-94.1954
$\sigma_{\alpha\beta}$	LD4	0.0036	0.0042	0.0082	0.1075
		0.0005	0.0019	0.0054	-0.0114
$\sigma_{\alpha z}$	LD4	-0.0207	-0.0207	-0.0324	-0.1905
$\sigma_{\beta z}$	LD4	0.0023	0.0027	0.0014	0.0072
σ_{zz}	LD4	0.0731	0.0311	0.0090	0.1865
		0.1009	0.0371	0.0047	0.0454
Φ	LD4	0.3432	0.4611	0.5037	0.5259
$\mathcal{D}_z (10^{11})$	LD4	-586.0881	-786.2541	-1621.7268	-16,310.8350

7. Conclusion

This article has presented the static analysis of plate and cylindrical composite structures by means of a shell FE based on the CUF [39, 40]. The results have been provided in terms of both displacement, electric potential, transverse stresses, and electric displacement, for various thickness ratios from very thick to very thin shells and the performances of the layerwise theories contained in the CUF have been tested. The conclusions that can be drawn follow:

1. The shell element is locking free when the shell is very thin.
2. The results converge to the exact solution by increasing the order of expansion of the variables in the thickness.
3. When the shell is very thick, the LW models are able to produce good results.
4. The use of LW models is mandatory for both thick and thin shells, if one needs to accurately describe the distribution of transverse shear and normal stresses and transverse electric displacement in the thickness and to satisfy the interlaminar continuity conditions.

Table 8. Transverse normal displacement $w * 10^{11}$, electric potential Φ , shear stress $\sigma_{\alpha z}$, and transverse normal stress σ_{zz} evaluated along the thickness in ($z = 0$). Transverse normal electric displacement $\mathcal{D}_z * 10^{11}$ evaluated along the thickness at *top* ($z = +h/2$). Shell cylinder with 3 layers. sensor case

R/h		2	4	10	100
$w (10^{11})$	$LFM4_a$	30.225	111.91	969.70	403,190
$\sigma_{\alpha z}$	$LFM4_a$	-0.1193	-0.2575	-0.6365	-3.1560
σ_{zz}	$LFM4_a$	-0.415	-0.661	-1.150	-3.997
Φ	$LFM4_a$	0.00497	0.0195	0.0602	0.3127
$\mathcal{D}_z (10^{11})$	$LFM4_a$	0.752	1.104	1.325	-5.495
$w (10^{11})$	LD4	30.223	111.906	969.672	403,287
	LD1	31.60	111.731	947.779	397,293
$\sigma_{\alpha z}$	LD4	-0.1195	-0.2579	-0.6385	-3.2162
	LD1	-0.1144	-0.2383	-0.5845	-2.9859
σ_{zz}	LD4	-0.4129	-0.6597	-1.1529	-4.0875
	LD1	-0.4033	-0.6466	-1.1245	-3.9824
Φ	LD4	0.0050	0.0195	0.0604	0.3190
	LD1	0.0072	0.0212	0.0604	0.3143
$\mathcal{D}_z (10^{11})$	LD4	0.1516	-0.8323	-3.0984	-23.473
	LD1	-2.950	-8.7942	-36.282	-1648.83

Table 9. Transverse normal displacement $w * 10^{11}$, electric potential Φ , shear stress $\sigma_{\alpha z} * 10^4$, and transverse normal stress $\sigma_{zz} * 10^4$ evaluated along the thickness in ($z = 0$); transverse normal electric displacement $\mathcal{D}_z * 10^{11}$ evaluated along the thickness at top ($z = +h/2$) (shell cylinder with 3 layers; actuator case)

R/h		2	4	10	100
$w (10^{11})$	$LFM4_a$	-1.306	-1.400	-1.4667	5.5418
$\sigma_{\alpha z} (10^4)$	$LFM4_a$	19.176	8.4776	1.5865	-0.5423
$\sigma_{zz} (10^4)$	$LFM4_a$	-116.36	-17.086	-0.5748	-0.5571
Φ	$LFM4_a$	0.4058	0.4826	0.5029	0.5009
$\mathcal{D}_z (10^{11})$	$LFM4_a$	-106.61	-66.035	-32.684	-36.209
$w (10^{11})$	LD4	-1.3066	-1.4007	-1.4662	5.5418
	LD1	-1.3646	-1.4450	-1.4849	5.4331
$\sigma_{\alpha z} (10^4)$	LD4	19.250	8.5036	1.5412	-0.5538
	LD1	11.005	5.5673	0.9585	-0.5193
$\sigma_{zz} (10^4)$	LD4	-119.606	-18.605	3.309	-0.357
	LD1	-82.662	-8.323	5.147	-0.490
Φ	LD4	0.4058	0.4827	0.5029	0.5009
	LD1	0.4917	0.4975	0.4996	0.500
$\mathcal{D}_z (10^{11})$	LD4	-106.851	-66.045	-32.642	-36.201
	LD1	-62.002	-38.432	-19.998	-34.863

5. The principle of virtual displacement with layer-wise approach is not sufficient to obtain the continuity interface condition for the transverse stresses for thin plate in sensor case and thin shell for sensor and actuator cases, and the transverse electric displacement for thin shell in sensor case.

Future works could be devoted to consider mixed variational principles in the analysis of shell structures with piezolayers by means of finite elements based on Unified Formulation.

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Appendix: Explicit Form of Stiffness Fundamental Nucleus

In order to write the fundamental nucleus $\mathbf{K}^{k\tau sij}$ in compact form, the following integrals in the domain Ω_k are defined:

$$\begin{aligned}
(W_{m1n1}^k; W_{m1n2}^k; W_{m2n1}^k; W_{m2n2}^k) &= \int_{\Omega_k} (N_{m1}N_{n1}; N_{m1}N_{n2}; N_{m2}N_{n1}; N_{m2}N_{n2}) d\alpha d\beta, \\
(W_{m1n3}^k; W_{m3n1}^k; W_{m3n3}^k; W_{m2n3}^k; W_{m3n2}^k) &= \int_{\Omega_k} (N_{m1}N_{n3}; N_{m3}N_{n1}; N_{m3}N_{n3}; N_{m2}N_{n3}; N_{m3}N_{n2}) d\alpha d\beta, \\
(W_{m1j}^k; W_{m2j}^k; W_{m3j}^k) &= \int_{\Omega_k} (N_{m1}N_j; N_{m2}N_j; N_{m3}N_j) d\alpha d\beta, \\
(W_{in1}^k; W_{in2}^k; W_{in3}^k; W_{ij}^k) &= \int_{\Omega_k} (N_iN_{n1}; N_iN_{n2}; N_iN_{n3}; N_iN_j) d\alpha d\beta, \\
(W_{m1j,\alpha}^k; W_{m1j,\beta}^k; W_{m2j,\alpha}^k; W_{m2j,\beta}^k) &= \int_{\Omega_k} \left(N_{m1} \frac{\partial N_j}{\partial \alpha}; N_{m1} \frac{\partial N_j}{\partial \beta}; N_{m2} \frac{\partial N_j}{\partial \alpha}; N_{m2} \frac{\partial N_j}{\partial \beta} \right) d\alpha d\beta, \\
(W_{i,\alpha n1}^k; W_{i,\beta n1}^k; W_{i,\alpha n2}^k; W_{i,\beta n2}^k) &= \int_{\Omega_k} \left(\frac{\partial N_i}{\partial \alpha} N_{n1}; \frac{\partial N_i}{\partial \beta} N_{n1}; \frac{\partial N_i}{\partial \alpha} N_{n2}; \frac{\partial N_i}{\partial \beta} N_{n2} \right) d\alpha d\beta.
\end{aligned}$$

Moreover, the integrals on the domain A_k , in the thickness direction, are written as:

$$\begin{aligned}
\left(J^{\alpha\tau s}, J_{\alpha}^{\beta\tau s}, J_{\beta}^{\alpha\tau s}, J_{\alpha}^{\beta\tau s}, J_{\beta}^{\alpha\tau s}, J_{\alpha\beta}^{\tau s} \right) &= \int_{A_k} F_{\tau} F_s \left(1, H_{\alpha}^k, H_{\beta}^k, \frac{H_{\alpha}^k}{H_{\beta}^k}, \frac{H_{\beta}^k}{H_{\alpha}^k}, H_{\alpha}^k H_{\beta}^k \right) dz, \\
\left(J^{\alpha\tau z s}, J_{\alpha}^{\beta\tau z s}, J_{\beta}^{\alpha\tau z s}, J_{\alpha}^{\beta\tau z s}, J_{\beta}^{\alpha\tau z s}, J_{\alpha\beta}^{\tau z s} \right) &= \int_{A_k} \frac{\partial F_{\tau}}{\partial z} F_s \left(1, H_{\alpha}^k, H_{\beta}^k, \frac{H_{\alpha}^k}{H_{\beta}^k}, \frac{H_{\beta}^k}{H_{\alpha}^k}, H_{\alpha}^k H_{\beta}^k \right) dz, \\
\left(J^{\alpha\tau s z}, J_{\alpha}^{\beta\tau s z}, J_{\beta}^{\alpha\tau s z}, J_{\alpha}^{\beta\tau s z}, J_{\beta}^{\alpha\tau s z}, J_{\alpha\beta}^{\tau s z} \right) &= \int_{A_k} F_{\tau} \frac{\partial F_s}{\partial z} \left(1, H_{\alpha}^k, H_{\beta}^k, \frac{H_{\alpha}^k}{H_{\beta}^k}, \frac{H_{\beta}^k}{H_{\alpha}^k}, H_{\alpha}^k H_{\beta}^k \right) dz, \\
\left(J^{\alpha\tau z s z}, J_{\alpha}^{\beta\tau z s z}, J_{\beta}^{\alpha\tau z s z}, J_{\alpha}^{\beta\tau z s z}, J_{\beta}^{\alpha\tau z s z}, J_{\alpha\beta}^{\tau z s z} \right) &= \int_{A_k} \frac{\partial F_{\tau}}{\partial z} \frac{\partial F_s}{\partial z} \left(1, H_{\alpha}^k, H_{\beta}^k, \frac{H_{\alpha}^k}{H_{\beta}^k}, \frac{H_{\beta}^k}{H_{\alpha}^k}, H_{\alpha}^k H_{\beta}^k \right) dz.
\end{aligned}$$

The stiffness fundamental nucleus $\mathbf{K}^{\tau sij}$ is:

$$\mathbf{K}^{k\tau sij} = \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{K}_{14} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{K}_{23} & \mathbf{K}_{24} \\ \mathbf{K}_{31} & \mathbf{K}_{32} & \mathbf{K}_{33} & \mathbf{K}_{34} \\ \mathbf{K}_{41} & \mathbf{K}_{42} & \mathbf{K}_{43} & \mathbf{K}_{44} \end{bmatrix}^{k\tau sij}.$$

The elements of the nucleus are:

$$K_{uu_{11}}^{k\tau s} = C_{55}^k N_i^{(m1)} N_j^{(n1)} W_{m1n1}^k J_{\alpha\beta}^{\beta\tau s z} - \frac{C_{55}^k}{R_{\alpha}^k} N_i^{(m1)} N_j^{(n1)} W_{m1n1}^k J_{\beta}^{\alpha\tau s} - \frac{C_{55}^k}{R_{\alpha}^k} N_i^{(m1)} N_j^{(n1)} W_{m1n1}^k J_{\beta}^{\alpha\tau s}$$

$$\begin{aligned}
 K_{uu_{12}}^{k\tau s} &= C_{45}^k N_i^{(m1)} N_j^{(n2)} W_{m1n2}^k J_{\alpha\beta}^{k\tau s_z} - \frac{C_{45}^k}{R_\beta^k} N_i^{(m1)} N_j^{(n2)} W_{m1n2}^k J_{\alpha}^{k\tau s} - \frac{C_{45}^k}{R_\alpha^k} N_i^{(m1)} N_j^{(n2)} W_{m1n2}^k J_{\beta}^{k\tau s_z} \\
 &+ C_{26}^k N_{i,\beta}^{(m3)} N_{j,\beta}^{(n2)} W_{m3n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s} + C_{12}^k N_{i,\alpha}^{(m1)} N_{j,\beta}^{(n2)} W_{m1n2}^k J^{k\tau s} + C_{66}^k N_{i,\beta}^{(m3)} N_{j,\alpha}^{(n3)} W_{m3n3}^k J^{k\tau s} \\
 &+ C_{16}^k N_{i,\alpha}^{(m1)} N_{j,\alpha}^{(n3)} W_{m1n3}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \frac{C_{45}^k}{R_\alpha^k R_\beta^k} N_i^{(m1)} N_j^{(n2)} W_{m1n2}^k J^{k\tau s}, \\
 K_{uu_{13}}^{k\tau s} &= C_{45}^k N_i^{(m1)} N_{j,\beta}^{(n2)} W_{m1n2}^k J_{\alpha}^{k\tau s_z} + C_{55}^k N_i^{(m1)} N_{j,\alpha}^{(n1)} W_{m1n1}^k J_{\beta}^{k\tau s_z} + C_{36}^k N_{i,\beta}^{(m3)} W_{m3j}^k J_{\alpha}^{k\tau s_z} + C_{13}^k N_{i,\alpha}^{(m1)} W_{m1j}^k J_{\beta}^{k\tau s_z} \\
 &- \frac{C_{45}^k}{R_\alpha^k} N_i^{(m1)} N_{j,\beta}^{(n2)} W_{m1n2}^k J^{k\tau s} - \frac{C_{55}^k}{R_\alpha^k} N_i^{(m1)} N_{j,\alpha}^{(n1)} W_{m1n1}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \frac{C_{26}^k}{R_\beta^k} N_{i,\beta}^{(m3)} N_j^{(n2)} W_{m3n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s} \\
 &+ \frac{C_{16}^k}{R_\alpha^k} N_{i,\beta}^{(m3)} N_j^{(n1)} W_{m3n1}^k J^{k\tau s} + \frac{C_{12}^k}{R_\beta^k} N_{i,\alpha}^{(m1)} N_j^{(n2)} W_{m1n2}^k J^{k\tau s} + \frac{C_{11}^k}{R_\alpha^k} N_{i,\alpha}^{(m1)} N_j^{(n1)} W_{m1n1}^k J_{\frac{\beta}{\alpha}}^{k\tau s}, \\
 K_{uu_{21}}^{k\tau s} &= C_{45}^k N_i^{(m2)} N_j^{(n1)} W_{m2n1}^k J_{\alpha\beta}^{k\tau s_z} - \frac{C_{45}^k}{R_\alpha^k} N_i^{(m2)} N_j^{(n1)} W_{m2n1}^k J_{\beta}^{k\tau s_z} - \frac{C_{45}^k}{R_\beta^k} N_i^{(m2)} N_j^{(n1)} W_{m2n1}^k J_{\alpha}^{k\tau s_z} \\
 &+ C_{26}^k N_{i,\beta}^{(m2)} N_{j,\beta}^{(n3)} W_{m2n3}^k J_{\frac{\alpha}{\beta}}^{k\tau s} + C_{66}^k N_{i,\alpha}^{(m3)} N_{j,\beta}^{(n3)} W_{m3n3}^k J^{k\tau s} + C_{12}^k N_{i,\beta}^{(m2)} N_{j,\alpha}^{(n1)} W_{m2n1}^k J^{k\tau s} \\
 &+ C_{16}^k N_{i,\alpha}^{(m3)} N_{j,\alpha}^{(n1)} W_{m3n1}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \frac{C_{45}^k}{R_\alpha^k R_\beta^k} N_i^{(m2)} N_j^{(n1)} W_{m2n1}^k J^{k\tau s}, \\
 K_{uu_{22}}^{k\tau s} &= C_{44}^k N_i^{(m2)} N_j^{(n2)} W_{m2n2}^k J_{\alpha\beta}^{k\tau s_z} - \frac{C_{44}^k}{R_\beta^k} N_i^{(m2)} N_j^{(n2)} W_{m2n2}^k J_{\alpha}^{k\tau s_z} - \frac{C_{44}^k}{R_\beta^k} N_i^{(m2)} N_j^{(n2)} W_{m2n2}^k J_{\alpha}^{k\tau s_z} \\
 &+ C_{22}^k N_{i,\beta}^{(m2)} N_{j,\beta}^{(n2)} W_{m2n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s} + C_{26}^k N_{i,\alpha}^{(m3)} N_{j,\beta}^{(n2)} W_{m3n2}^k J^{k\tau s} + C_{26}^k N_{i,\beta}^{(m2)} N_{j,\alpha}^{(n3)} W_{m2n3}^k J^{k\tau s} \\
 &+ C_{66}^k N_{i,\alpha}^{(m3)} N_{j,\alpha}^{(n3)} W_{m3n3}^k J_{\frac{\beta}{\alpha}}^{k\tau s} + \frac{C_{44}^k}{(R_\beta^k)^2} N_i^{(m2)} N_j^{(n2)} W_{m2n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s}, \\
 K_{uu_{23}}^{k\tau s} &= C_{44}^k N_i^{(m2)} N_{j,\beta}^{(n2)} W_{m2n2}^k J_{\alpha}^{k\tau s_z} + C_{45}^k N_i^{(m2)} N_{j,\alpha}^{(n1)} W_{m2n1}^k J_{\beta}^{k\tau s_z} + C_{23}^k N_{i,\beta}^{(m2)} W_{m2j}^k J_{\alpha}^{k\tau s_z} \\
 &+ C_{36}^k N_{i,\alpha}^{(m3)} W_{m3j}^k J_{\beta}^{k\tau s_z} - \frac{C_{44}^k}{R_\beta^k} N_i^{(m2)} N_{j,\beta}^{(n2)} W_{m2n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s} - \frac{C_{45}^k}{R_\beta^k} N_i^{(m2)} N_{j,\alpha}^{(n1)} W_{m2n1}^k J^{k\tau s} \\
 &+ \frac{C_{22}^k}{R_\beta^k} N_{i,\beta}^{(m2)} N_j^{(n2)} W_{m2n2}^k J_{\frac{\alpha}{\beta}}^{k\tau s} + \frac{C_{12}^k}{R_\alpha^k} N_{i,\beta}^{(m2)} N_j^{(n1)} W_{m2n1}^k J^{k\tau s} + \frac{C_{26}^k}{R_\beta^k} N_{i,\alpha}^{(m3)} N_j^{(n2)} W_{m3n2}^k J^{k\tau s} \\
 &+ \frac{C_{16}^k}{R_\alpha^k} N_{i,\alpha}^{(m3)} N_j^{(n1)} W_{m3n1}^k J_{\frac{\beta}{\alpha}}^{k\tau s}, \\
 K_{uu_{31}}^{k\tau s} &= C_{36}^k N_{j,\alpha}^{(n3)} W_{m3n1}^k J_{\frac{\beta}{\alpha}}^{k\tau s}
 \end{aligned}$$

$$\begin{aligned}
K_{uu32}^{k\tau s} &= C_{23}^k N_{j,\beta}^{(n2)} W_{i n2}^k J_{\alpha}^{k\tau s} + C_{36}^k N_{j,\alpha}^{(n3)} W_{i n3}^k J_{\beta}^{k\tau s} + C_{44}^k N_{i,\beta}^{(m2)} N_j^{(n2)} W_{m2 n2}^k J_{\alpha}^{k\tau s} + C_{45}^k N_{i,\alpha}^{(m1)} N_j^{(n2)} W_{m1 n2}^k J_{\beta}^{k\tau s} \\
&+ \frac{C_{22}^k}{R_{\beta}^k} N_i^{(m2)} N_{j,\beta}^{(n2)} W_{m2 n2}^k J_{\beta}^{k\tau s} + \frac{C_{12}^k}{R_{\alpha}^k} N_i^{(m1)} N_{j,\beta}^{(n2)} W_{m1 n2}^k J_{\alpha}^{k\tau s} + \frac{C_{26}^k}{R_{\beta}^k} N_i^{(m2)} N_{j,\alpha}^{(n3)} W_{m2 n3}^k J_{\alpha}^{k\tau s} \\
&+ \frac{C_{16}^k}{R_{\alpha}^k} N_i^{(m1)} N_{j,\alpha}^{(n3)} W_{m1 n3}^k J_{\alpha}^{k\tau s} - \frac{C_{44}^k}{R_{\beta}^k} N_{i,\beta}^{(m2)} N_j^{(n2)} W_{m2 n2}^k J_{\beta}^{k\tau s} - \frac{C_{45}^k}{R_{\beta}^k} N_{i,\alpha}^{(m1)} N_j^{(n2)} W_{m1 n2}^k J_{\beta}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{uu33}^{k\tau s} &= C_{33}^k W_{i j}^k J_{\alpha\beta}^{k\tau s} + \frac{C_{23}^k}{R_{\beta}^k} N_j^{(n2)} W_{i n2}^k J_{\alpha}^{k\tau s} + \frac{C_{13}^k}{R_{\alpha}^k} N_j^{(n1)} W_{i n1}^k J_{\beta}^{k\tau s} + \frac{C_{23}^k}{R_{\beta}^k} N_i^{(m2)} W_{m2 j}^k J_{\alpha}^{k\tau s} + \frac{C_{13}^k}{R_{\alpha}^k} N_i^{(m1)} W_{m1 j}^k J_{\beta}^{k\tau s} \\
&+ C_{44}^k N_{i,\beta}^{(m2)} N_{j,\beta}^{(n2)} W_{m2 n2}^k J_{\beta}^{k\tau s} + C_{45}^k N_{i,\alpha}^{(m1)} N_{j,\beta}^{(n2)} W_{m1 n2}^k J_{\alpha}^{k\tau s} + C_{45}^k N_{i,\beta}^{(m2)} N_{j,\alpha}^{(n1)} W_{m2 n1}^k J_{\alpha}^{k\tau s} \\
&+ C_{55}^k N_{i,\alpha}^{(m1)} N_{j,\alpha}^{(n1)} W_{m1 n1}^k J_{\alpha}^{k\tau s} + \frac{C_{12}^k}{R_{\alpha}^k R_{\beta}^k} N_i^{(m1)} N_j^{(n2)} W_{m1 n2}^k J_{\alpha}^{k\tau s} + \frac{C_{12}^k}{R_{\alpha}^k R_{\beta}^k} N_i^{(m2)} N_j^{(n1)} W_{m2 n1}^k J_{\alpha}^{k\tau s} \\
&+ \frac{C_{22}^k}{(R_{\beta}^k)^2} N_i^{(m2)} N_j^{(n2)} W_{m2 n2}^k J_{\beta}^{k\tau s} + \frac{C_{11}^k}{(R_{\alpha}^k)^2} N_i^{(m1)} N_j^{(n1)} W_{m1 n1}^k J_{\alpha}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{u\Phi_{14}}^{k\tau s} &= e_{25}^k N_i^{(m1)} W_{m1 j,\beta}^k J_{\alpha}^{k\tau s} + e_{15}^k N_i^{(m1)} W_{m1 j,\alpha}^k J_{\beta}^{k\tau s} + e_{36}^k N_{i,\beta}^{(m3)} W_{m3 j}^k J_{\alpha}^{k\tau s} + e_{31}^k N_{i,\alpha}^{(m1)} W_{m1 j}^k J_{\beta}^{k\tau s} \\
&- \frac{e_{25}^k}{R_{\alpha}^k} N_i^{(m1)} W_{m1 j,\beta}^k J_{\alpha}^{k\tau s} - \frac{e_{15}^k}{R_{\alpha}^k} N_i^{(m1)} W_{m1 j,\alpha}^k J_{\beta}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{u\Phi_{24}}^{k\tau s} &= e_{24}^k N_i^{(m2)} W_{m2 j,\beta}^k J_{\alpha}^{k\tau s} + e_{14}^k N_i^{(m2)} W_{m2 j,\alpha}^k J_{\beta}^{k\tau s} + e_{32}^k N_{i,\beta}^{(m2)} W_{m2 j}^k J_{\alpha}^{k\tau s} + e_{36}^k N_{i,\alpha}^{(m3)} W_{m3 j}^k J_{\beta}^{k\tau s} \\
&- \frac{e_{24}^k}{R_{\beta}^k} N_i^{(m2)} W_{m2 j,\beta}^k J_{\beta}^{k\tau s} - \frac{e_{14}^k}{R_{\beta}^k} N_i^{(m2)} W_{m2 j,\alpha}^k J_{\alpha}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{u\Phi_{34}}^{k\tau s} &= e_{33}^k W_{i j}^k J_{\alpha\beta}^{k\tau s} + \frac{e_{32}^k}{R_{\beta}^k} N_i^{(m2)} W_{m2 j}^k J_{\alpha}^{k\tau s} + \frac{e_{31}^k}{R_{\alpha}^k} N_i^{(m1)} W_{m1 j}^k J_{\beta}^{k\tau s} + e_{24}^k N_{i,\beta}^{(m2)} W_{m2 j,\beta}^k J_{\beta}^{k\tau s} \\
&+ e_{25}^k N_{i,\alpha}^{(m1)} W_{m1 j,\beta}^k J_{\alpha}^{k\tau s} + e_{14}^k N_{i,\beta}^{(m2)} W_{m2 j,\alpha}^k J_{\beta}^{k\tau s} + e_{15}^k N_{i,\alpha}^{(m1)} W_{m1 j,\alpha}^k J_{\alpha}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{\Phi u_{41}}^{k\tau s} &= e_{36}^k N_{j,\beta}^{(n3)} W_{i n3}^k J_{\alpha}^{k\tau s} + e_{31}^k N_{j,\alpha}^{(n1)} W_{i n1}^k J_{\beta}^{k\tau s} + e_{25}^k N_j^{(n1)} W_{i,\beta n1}^k J_{\alpha}^{k\tau s} + e_{15}^k N_j^{(n1)} W_{i,\alpha n1}^k J_{\beta}^{k\tau s} \\
&- \frac{e_{25}^k}{R_{\alpha}^k} N_j^{(n1)} W_{i,\beta n1}^k J_{\alpha}^{k\tau s} - \frac{e_{15}^k}{R_{\alpha}^k} N_j^{(n1)} W_{i,\alpha n1}^k J_{\beta}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{\Phi u_{42}}^{k\tau s} &= e_{32}^k N_{j,\beta}^{(n2)} W_{i n2}^k J_{\alpha}^{k\tau s} + e_{36}^k N_{j,\alpha}^{(n3)} W_{i n3}^k J_{\beta}^{k\tau s} + e_{24}^k N_j^{(n2)} W_{i,\beta n2}^k J_{\alpha}^{k\tau s} + e_{14}^k N_j^{(n2)} W_{i,\alpha n2}^k J_{\beta}^{k\tau s} \\
&- \frac{e_{24}^k}{R_{\beta}^k} N_j^{(n2)} W_{i,\beta n2}^k J_{\beta}^{k\tau s} - \frac{e_{14}^k}{R_{\beta}^k} N_j^{(n2)} W_{i,\alpha n2}^k J_{\alpha}^{k\tau s},
\end{aligned}$$

$$\begin{aligned}
K_{\Phi u_{43}}^{k\tau s} &= e_{33}^k W_{i j}^k J_{\alpha\beta}^{k\tau s} + \frac{e_{32}^k}{R_{\beta}^k} N_j^{(n2)} W_{i n2}^k J_{\alpha}^{k\tau s} + \frac{e_{31}^k}{R_{\alpha}^k} N_j^{(n1)} W_{i n1}^k J_{\beta}^{k\tau s} + e_{24}^k N_{j,\beta}^{(n2)} W_{i,\beta n2}^k J_{\beta}^{k\tau s} \\
&+ e_{14}^k N_{j,\beta}^{(n2)} W_{i,\alpha n2}^k J_{\alpha}^{k\tau s} + e_{25}^k N_{j,\alpha}^{(n1)} W_{i,\beta n1}^k J_{\beta}^{k\tau s} + e_{15}^k N_{j,\alpha}^{(n1)} W_{i,\alpha n1}^k J_{\alpha}^{k\tau s},
\end{aligned}$$

$$K_{\Phi\Phi_{44}}^{k\tau s} = -\varepsilon_{33}^k W_{i j}^k J_{\alpha\beta}^{k\tau s} - \varepsilon_{22}^k W_{i,\beta j,\beta}^k J_{\beta}^{k\tau s} - \varepsilon_{12}^k W_{i,\alpha j,\beta}^k J_{\alpha}^{k\tau s} - \varepsilon_{12}^k W_{i,\beta j,\alpha}^k J_{\beta}^{k\tau s} - \varepsilon_{11}^k W_{i,\alpha j,\alpha}^k J_{\alpha}^{k\tau s}.$$