1. Introduction

Vibration damping consists of a challenging task to increase fatigue life and comfort of advanced structures. The variation of the properties of the material composing the structure provides a valuable approach to this task. Only, very limited options are available for such changes in the material properties. The present work proposes the use of NiTi shape memory alloys for this purpose. Presenting an exhaustive mathematical model for these structures is suffering from the nonlinearity behavior which is due to the phase transformation. Therefore, in the most cases, the proposed models are accompanied with a lot of simplifying assumptions. In the study by Jafari and Ghasvand [1] dynamic analysis of SMA beam under a moving load was investigated. In their study, the effect of hysteretic loops was modeled by substitution of an equivalent damping ratio in the equation of motion. The dynamic analysis of SMA beam was studied by Hashemi and Khadem [2]. They considered the effect of the phase transformation. But In their research they assumed that, the beam behaves like a one-degree-of freedom system. Zbicik [3] investigated the response history of SMA beam under impulse loading. He employed the rheological scheme for modeling the behavior of SMA material. He assumed that the material properties of SMA are constant.

Recently, many considerations have been focused on the improvement in the properties of the composite structures by shape memory alloys. In the study by Rogers and Barker [4] SMA wires were used to control the frequency of a graphite/epoxy multilayered beam. Upon the heating of SMA wires, an axial force was generated in the beam because of the shape memory effect. They showed that, the fundamental frequency of the beam was increased significantly by utilizing 15% volume fraction of SMA wires. Baz et al. [5] demonstrated that the SMA wires embedded to composite beams have a capability to control their natural frequencies. The effect of pre-strain, and also the effect of temperature on the SMA wires, was taken to account in their study. The effect of SMA wires on the controlling of the buckling and frequency analysis of composite beam was studied by Baz et al. [6]. They found that the buckling load of a flexible composite beam was increased up to three times the uncontrolled beam. Epps and
Chandra [7] implemented an experimental–analytical investigation on the composite beams embedded with SMA wires. They demonstrated that, as the volume fraction of SMA wires increases the natural frequency of SMA composite beams increases. However, it cannot be found any details on the damping properties of SMA wires. Khalili et al. [8] studied the nonlinear dynamic response of sandwich beam with SMA hybrid composite skins under impulse loading. In their research a new element was proposed using the high order sandwich panel theory. They investigate the influence of the SMA wires on the vibration suppression of sandwich beam considering the phase transformation effects. In the research conducted by Ostachowicz et al. [9], dynamic and buckling analysis of composite plates embedded with SMA wires was investigated. They showed that the SMA wires have a significant effect on the natural frequencies and the thermal buckling of these structures. Lee and Lee [10] studied the buckling and post-buckling analysis multilayered composite plates embedded with SMA wires. They showed that the critical load of the composite plates is increased by activation of the SMA wires. Cho and Rhee [11] presented the nonlinear finite element model for static analysis of shape memory alloy wire reinforced hybrid laminate composite shells. They used the Brison’s constitutive equation based on the iterative method for modeling the behavior of SMA wires.

The use of multilayer composite structures has been continuously growing in the recent years. Multilayered composite structures are utilized in many components of automotive, aerospace, and transportation vehicles. In recent years, many theories devoted to analyze the multilayer composite structures. Kirchhoff [12] (CLT) and Reissner–Mindlin [13,14] (FSDT) plate theories are not suitable to analysis the multilayered composite structures [15]. Because they cannot satisfy the continuity of transverse stresses between two adjacent laminate. In addition, these theories are not able to fulfill the zig-zag manner of the displacement distribution along the thickness direction. These conditions are called C00 requirements in Ref. [16]. In this regard, a lot of theories have been presented for modification the FSDT [17–23]. These theories are known as higher-order shear deformation theories (HSDT). Khalili et al. [24] modified high-order theory for sandwich panels HSAPT, by applying first-order shear deformation theory for face-sheets and used the improved HSAPT to study the free vibration and low velocity response of sandwich panels. A lot of finite-element models are proposed using the HSDT models [17–23,25]. It should be mentioned that, a closed-form solutions can be found only in some few cases, especially for the linear problems with specific boundary conditions [26]. Two-dimensional theories are divided to some categories, based on the unknown variables. If the displacement field is only unknown, the corresponding theories are known as classical models and the governing equations are derived using the Principle of Virtual Displacements (PVD). If the transverse stresses are also assumed as unknowns, the corresponding theories are known as mixed theories [27,28]. Carrera et al. [16,29–33] presented a unified formulation (UF) of multilayered theories, for both the PVD and RMVT formulations. This can be referred as Equivalent-Single-Layer (ESL), if the unknown variables are considered for the whole plate, or (LW), if the unknown variables are considered for each layer, individually.

In this study, the nonlinear dynamic analysis of composite multilayered plate embedded with SMA wires is investigated based on the Carrera’s unified formulation. The instantaneous phase transformation effects are considered for all the points on the plate for the first time. The Brinson’s SMA constitutive equation is used to model the pseudoelastic behavior of shape memory alloys wires. In the present study, the (RMVT) is utilized to derivation of the governing equations. The governing equations of motion and the kinetic relations of phase transformation are coupled with each other. Therefore, a transient finite-element-based method beside an iterative incremental procedure is presented to study the dynamic response of multilayered composite plate embedded with SMA wires. Finally, a new program code is written in MATLAB software in order to dynamic analysis of composite plate embedded with SMA wires.

2. Constitutive equation of the SMA wires

In this study, the constitutive equation of shape memory alloys has been proposed by Brinson [34] is utilized. This constitutive equation presents the relation between the stress (σ), strain (ε), temperature (T) and martensite fraction (ml) as follows:

\[ \sigma - \sigma_0 = E(\xi)(\epsilon) - E(\xi_0)(\epsilon_0) + \Omega(\xi)(\xi_0) - \Omega(\xi_0)(\xi) + \theta(T - T_0) \]

(1)

where \(E(\xi)\) and \(\theta\) are the Young's modulus and the thermoelastic coefficient, respectively. The subscript 0 implies the initial conditions of the corresponding term. \(E(\xi)\) can be expressed as follows [34]:

\[ E(\xi) = E_A + \xi(E_M - E_A) \]

(2)

where \(E_M\) and \(E_A\) are the Young’s modulus of the shape memory alloys in the martensite and austenite phases, respectively.

In addition, \(\Omega(\xi)\) is the transformation tensor and can be written in terms of Young’s modulus as follows:

\[ \Omega(\xi) = -\xi_0 E(\xi) \]

(3)

where \(\xi_0\) is the maximum strain that can be recovered completely.

In the model of Brinson, the martensite volume fraction is separated into two parts as follows:

\[ \xi = \xi_A + \xi_T \]

(4)

where \(\xi_A\) indicates the fraction of the martensite that is induced by stress and \(\xi_T\) indicates the fraction of the martensite that is induced by temperature. Kinetic relations of the phase transformation (see Fig. 1) are expressed as follows [34]:

For conversion to martensite:

\[ \frac{1 - \xi_T}{2} \cos \left( \frac{\pi}{\sigma_{cr}^T - \sigma_T} (\sigma - \sigma_T - C_{M}(T - M)) \right) + \frac{1 + \xi_0}{2} \]

\[ \xi_T = \xi_{T0} - \frac{\xi_T}{1 - \xi_0} (\xi - \xi_0) \]

(5)

For \(T < M\) and \(\sigma_{cr} < \sigma < \sigma_{cr}^*\)

![Fig. 1. Pseudoelastic behavior of shape memory alloys [35].](image-url)
\[ \tilde{\xi}_r = \frac{1 - \tilde{\xi}_0}{2} \cos \left( \frac{\pi}{\sigma^r_{xy} - \sigma^r_{xy}} (\sigma - \sigma^r_{xy}) \right) + \frac{1 + \tilde{\xi}_0}{2} \] (6)

\[ \tilde{\xi}_t = \tilde{\xi}_{r0} - \tilde{\xi}_0 (\tilde{\xi}_t - \tilde{\xi}_0) + \Delta_{\xi_t} \]

For \( M_t < T < M_t \) and \( T < T_0 \)

\[ \Delta_{\xi_t} = \frac{1 - \tilde{\xi}_{r0}}{2} (\cos(\omega(T - M_t)) + 1) \]

Else \( \Delta_{\xi_t} = 0 \)

For conversion to austenite:

For \( T > A_t \) and \( C_t(T - A_t) < \sigma < C_t(T - A_a) \)

\[ \xi = \frac{\tilde{\xi}_0}{2} \cos \left( a_0 (T - A_t - \frac{\sigma}{C_t}) + 1 \right) \]

\[ \tilde{\xi}_r = \frac{\tilde{\xi}_0}{C_0} (\tilde{\xi}_0 - \tilde{\xi}) \]

\[ \tilde{\xi}_t = \frac{\tilde{\xi}_{r0}}{C_0} (\tilde{\xi}_{r0} - \tilde{\xi}) \]

where \( \sigma^r_{xy} \) and \( \sigma^r_{xy} \) are the critical stresses for the start and end of the phase transformation, respectively.

### 3. Unified formulation and finite element analysis

The Unified Formulation (UF) allows to unify a lot of two-dimensional theories for plates, by separation of the unknown variables into functions which depend only on the thickness coordinate \( z \), and the ones coincide with the in-plane coordinates \((x, y)\) [16,33]. In the Carrera’s unified formulation (CUF) the global unknown \( u(x, y, z) \) and its variation \( \delta u(x, y, z) \) can be expressed as follows [16]:

\[
\left( x, y, z \right) F_r(z) \left( x, y, z \right) \quad \delta u(x, y, z) = F_t(z) \delta u(x, y, z)
\]

with \( \tau, s = t, b, r \) and \( r = 2, \ldots, N \) (8)

Bold letters stand for arrays and the summing convention is expressed by repeated indices \( \tau \) and \( s \). Subscripts \( t, b, r \) and \( r \) represent the top, the bottom and the higher order terms of the expansion, respectively. The expansion of \( N \) can be written from first to fourth order. In this study, Reissner’s mixed variational theorem (RMVT) is utilized to enforce the interlaminar continuity of the shear and normal stresses between the two adjacent layers a priori. Most of the available two dimensional theories are divided into two categories; equivalent single layer models (ESL) and layer-wise models (LW). In the ESL models, the \( z \) power expansion is utilized for displacement field as follows:

\[ u = u_0 + z^r u_r, \quad r = 1, 2, \ldots, N \] (9)

The index 0 represents the displacement values related to the reference surface of the plate. Linear and higher-order expansions in the \( z \)-direction are described using the \( r \)-polynomials. \( N \) is a free parameter by which the demanded model can be obtained. To write all the models in a unified formulation, the relation (8) is represented as follows:

\[ \mathbf{u} = F_t u_t + F_b u_b + F_r u_r = F_s u_s, \quad \tau = t, b, r, \quad r = 2, \ldots, N - 1 \] (10)

By Comparison of the above relation with Eq. (8), it can be seen that index \( b \) presents the values related to \((u_b = u_0)\), while index \( t \) stands for the highest-order term \((u_t = u_0)\). Therefore, \( F_t \) polynomials can be defined as follows:

\[ F_b = 1, \quad F_t = Z^r, \quad F_r = Z^r, \quad r = 2, \ldots, N - 1 \] (11)

The above expansions can be used for all the multilayer plate that is coincided with the ESL models. For achieving the LW models, the expansion in Eq. (8) must be utilized for every layer, individually.

The Taylor expansion is not appropriate for description of the LW models, since, in the Taylor expansion, the displacements at the interfaces are not as unknowns to enforce the continuity between the two adjacent layers. For this purpose, a combination of Legendre polynomials [36–38] can be utilized as the thickness functions [16]:

\[ u^k = F_s u^k_s + F_b u^k_b + F_r u^k_r = F_t u^k_t, \quad \tau = t, b, r, \quad r = 2, \ldots, N, \quad k = 1, 2, \ldots, N_1 \] (12)

where \( N_1 \) is the number of the layers constituting the plate. The indices \( t \) and \( b \) stand for the values corresponding to the top layer and the bottom layer, respectively. The thickness functions \( F_t(z) \) are expressed for the kth layer. The Legendre polynomials \( F_t(z) \) are:

\[ P_0 = 1, \quad P_1 = z, \quad P_2 = \frac{3z^2 - 1}{2}, \quad P_3 = \frac{5z^3}{2}, \quad \frac{3z^2}{2}, \quad \frac{35z^4}{8} - \frac{15z^2}{4} + \frac{3}{8} \] (13)

where \( z \) and \( h_b \) are the local coordinate and the thickness of kth layer, respectively. Therefore, \( -1 \leq z \leq 1 \), in this regard see Fig. 2. The thickness functions are obtained using the Legendre polynomials as follows [16]:

\[ F_t = \frac{P_0 + P_1}{2}, \quad F_1 = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N \] (14)

Therefore, these functions have the following characteristic:

\[ z_k = \begin{cases} 1 & F_t = 1, \quad F_b = 0, \quad F_r = 0 \\ -1 & F_t = 0, \quad F_b = 1, \quad F_r = 0 \end{cases} \] (15)

Consequently, the continuity of the displacement at the two adjacent layers is fulfilled as follows:

\[ u^k_t = u^{k+1}_b, \quad k = 1, 2, \ldots, N_1 - 1 \] (16)

The transverse stresses are considered for every layer, independently, regardless of the ESL and LW models. For this purpose, the LW explanation that is used for displacements is utilized for the transverse stresses to enforce the interlaminar continuity for both the ESL and LW models:

\[ \sigma^t_{nm} = F_s \sigma^t_{nm} + F_b \sigma^t_{nm} + F_r \sigma^t_{nr}, \quad \tau = t, b, r, \quad r = 2, \ldots, N, \quad k = 1, 2, \ldots, N_1 \] (17)

Therefore, the continuity of the transverse stresses at the two adjacent layers is fulfilled as follows:

\[ \sigma^t_{nm} = \sigma^{t+1}_{nm}, \quad k = 1, 2, \ldots, N_1 - 1 \] (18)

For abbreviating the name of the models, the new acronym is introduced. To achieve this, the first parameter of the acronym is corresponding to the type of the model; therefore, letter L means LW and E means ESL. The second parameter identifies the type of the variational statements which is used in derivation of the governing equations; in this regard, M means (RMVT). The third parameter shows the number \( N \) which identifies the order of model. For example LLM3 means LW theory using RMVT with third order of displacement and stress fields in the layer.

In this study, a quadratic nine-nodes finite element is used to approximate the displacements and the transverse stresses as follows:

\[ u^i = F_9 N_i q^i_t \quad \sigma^t_{nm} = F_9 N_i g^i_t, \quad i = 1, \ldots, 9 \] (19)

where \( N_i \) are the Lagrange quadratic shape functions, also:

\[ q^i_t = [q^i_{nt}, q^i_{nb}, q^i_{nr}]^T \quad g^i_t = [g^i_{nt}, g^i_{nb}, g^i_{nr}]^T, \quad i = 1, \ldots, N_b \] (20)

where \( q^i_t \) and \( g^i_t \) are the nodal unknown of the element for the kth layer.
4. Theoretical formulation and methods

Using the Hamilton’s principle, it can be found that:

\[ \delta L_{\text{int}} - \delta L_{\text{ext}} = 0 \]

(21)

where, \( L_{\text{int}} \) is the internal work, \( L_{\text{ext}} \) is the work done by the inertial force and \( L_{\text{ext}} \) is the work done by the external force. The total internal work is given by the mechanical work and the work due to the phase transformation.

\[ L_{\text{int}} = L_{\text{int,m}} + L_{\text{int,sma}} \]

(22)

where, \( L_{\text{int,m}} \) is the work done by the mechanical stresses and \( L_{\text{int,sma}} \) is the work due to the phase transformation of SMA wires.

In the present study, the (RMVT) is utilized in derivation of the governing equations [27,28]. For the study of SMA hybrid multilayered plate, the RMVT is explained as follows:

\[ \delta L_{\text{int,m}} = \sum_{k=1}^{N} \int_{D_k} \left\{ \delta \varepsilon^{\text{eq}}_{\text{sc}}(z) \varepsilon^{\text{eq}}_{\text{sc}}(z) + \delta \varepsilon^{\text{eq}}_{\text{nc}}(z) \varepsilon^{\text{eq}}_{\text{nc}}(z) \right\} dz \]

(23)

The statement \( \left( \delta \varepsilon^{\text{eq}}_{\text{sc}}(z) \varepsilon^{\text{eq}}_{\text{sc}}(z) - \varepsilon^{\text{eq}}_{\text{sc}}(z) \right) \) is Lagrange Multiplier which fulfils the compatibility of the transverse strains \( e_{nz} \), \( \Omega \), and \( A_{nn} \) imply the in-plane and the thickness domains of the lamina, respectively. Index \( C \) means that the corresponding terms are obtained using the geometrical relations, while \( C \) denotes the corresponding terms that are obtained by the constitutive equations. Also, index \( M \) implies that the stress components are considered a priori. In the framework of CUF, the strains are as follows:

\[ \varepsilon^{\text{eq}}_{\text{sc}}(z) = \left[ \varepsilon_{xx}(z), \varepsilon_{yy}(z), \varepsilon_{xy}(z) \right]^T = D_p \varepsilon^o(z) \]

(24a)

\[ \varepsilon^{\text{eq}}_{\text{nc}}(z) = \left[ \gamma_{xz}(z), \gamma_{xy}(z), \varepsilon_{zz}(z) \right]^T = (D_{np} + D_{nn}) \varepsilon^o(z) \]

(24b)

where the indices \( p \) and \( n \) imply the in-plane and normal components, respectively. The operators \( D_p, D_{np}, \) and \( D_{nn} \) are the differential matrices which are defined as follows:

\[ D_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \quad D_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{bmatrix}, \quad D_{nn} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_x & 0 \\ \partial_x & \partial_y & \partial_z \end{bmatrix} \]

(25)

Also, for the stress components, it can be written:

\[ \sigma^a_p(z) = C_{pp}^{(e)}(z) \varepsilon^o_p(z) + C_{pm}^{(e)}(z) \varepsilon^o_m(z) \]

(26a)

\[ \sigma^a_m(z) = C_{mp}^{(e)}(z) \varepsilon^o_p(z) + C_{mm}^{(e)}(z) \varepsilon^o_m(z) \]

(26b)

where \( \sigma^a_p, \sigma^a_n \) and the matrices \( C_p^{(e)}, C_m^{(e)}, C_{pp}, C_{pm}, C_{mp}, \) and \( C_{mm} \) are:

\[ \sigma^a_p(z) = \begin{bmatrix} \sigma^a_{xx}(z), \sigma^a_{yy}(z), \sigma^a_{xy}(z) \end{bmatrix} \]

(27a)

\[ C_p(z) = \begin{bmatrix} C_{pp}(z) & C_{pm}(z) & C_{mp}(z) \\ C_{pm}(z) & C_{mm}(z) & C_{mp}(z) \\ C_{mp}(z) & C_{mp}(z) & C_{mm}(z) \end{bmatrix} \]

(27b)

where \( C_{ij} \) denotes the stiffness of the \( i \)th layer of the plate. The Young's modulus and other properties of a SMA hybrid composite layer are obtained as follows [39]:

\[ E_i(z) = E_k + E_L(z) \]

(28a)

\[ E_L(z) = E_k(1 - \sqrt{k_c} - 1) \]

(28b)

\[ G_{kl}(z) = G_{kl}(1 - k_c(z) + k_c(z)) + k_c(z) \]

(28c)

\[ \nu_{ij} = \frac{k_c(z) + k_c(z)}{2} \]

(28d)

where indices 's' and 'c' imply the shape memory and the composite medium, respectively. Since the displacements and transverse stresses are unknown in RMVT; therefore the constitutive equations must be expressed as follows:

\[ \sigma^a_p(z) = C_{pp}^{(e)}(z) e^o_p(z) + C_{pm}^{(e)}(z) e^o_m(z) \]

(29a)

\[ \varepsilon^a_p(z) = C_{pp}^{(e)}(z) e^o_p(z) + C_{pm}^{(e)}(z) e^o_m(z) \]

(29b)

The coefficients \( C_{pp}, C_{pm}, \) and \( C_{mm} \) in the above relations are defined as follows:

\[ \bar{C}_{pp}^{(e)}(z) = C_{pp}^{(e)}(z) - C_{pm}^{(e)}(z) C_{mm}^{(e)}(z) \]

(30a)

\[ \bar{C}_{pm}^{(e)}(z) = -C_{pm}^{(e)}(z) \]

(30b)

After substitution of Eqs. (19), (20), (24) and (29) in Eq. (23), one gets the following:
\[ \delta L_{\text{int}}^k = \int_{\Omega} \left\{ \delta q_{ij}^k(\xi) \left[ D_{ij}^p(N, I) \mathbf{Z}_{ij}^{\text{int}}(\xi) - D_{ij}^p(N, I) \mathbf{q}_{ij}^k(\xi) \right] \right\} d\Omega \]

where the following layer's stiffness and compliance have been introduced:

\[
\left( \mathbf{Z}_{pp}^{\text{int}}(\xi), \mathbf{Z}_{pn}^{\text{int}}(\xi), \mathbf{Z}_{np}^{\text{int}}(\xi) \right) = \left( C_{pp}^n(\xi), C_{pn}^n(\xi), C_{np}^n(\xi) \right) E_{\text{ts}}
\]

In the above relation, the symbols \( \ldots \in \Omega \) implies the integrals on domain \( \Omega \) and also:

\[
E_{\text{ts}} = \int_{\Omega} (F_{1} F_{2}) d\xi
\]

Therefore, \( \delta L_{\text{int}}^k \) can be expressed as:

\[ \delta L_{\text{int}}^k = \int_{\Omega} \left\{ \delta q_{ij}^k(\xi) \left[ K_{ij}^{\text{int}}(\xi) \mathbf{q}_{ij}^k(\xi) + K_{ij}^{\text{int}}(\xi) \mathbf{g}_{ij}^k(\xi) \right] \right\} d\Omega \]

where

\[ K_{ij}^{\text{int}}(\xi) = \int_{\Omega} \mathbf{D}_{ij}^p(N, I) \mathbf{Z}_{ij}^{\text{int}}(\xi) d\Omega \]

Above relations are \([3 \times 3]\) 'fundamental nuclei' such that the whole stiffness structure can be obtained by expanding the indices and then assembling the mentioned nuclei. The components of this nucleus can be expressed as follows:

\[ K_{ij}^{\text{int}}(\xi) = \int_{\Omega} \left[ \mathbf{D}_{ij}^p(N, I) \mathbf{Z}_{ij}^{\text{int}}(\xi) \right] d\Omega \]

It should be mentioned that due to the location dependency of the coefficients \( \mathbf{Z}_{ij}^{\text{int}}(\xi) \), they must be remain in the integral domain. The mentioned integrals and also the integral in the thickness direction are computed by the Gaussian quadrature method. To overcome the shear locking problem, the selective reduced integration technique is employed in this study [40].

\[ \delta L_{\text{int}}^k \] is obtained as follows:

\[ \delta L_{\text{int}}^k = \int_{A_k} \int_{\Omega} \left\{ \left( \delta \sigma_{\text{xax}}(\xi) \sigma_{\text{xax}}(\xi) + \mathbf{x}_{\text{mag}}(\xi) \mathbf{\sigma}_{\text{mag}}(\xi) \right) d\xi d\Omega \right\}
\]

where \( \sigma_{\text{xax}}(\xi) \) and \( \mathbf{\sigma}_{\text{xax}}(\xi) \) are the stresses due to the phase transformation in the x and y directions, respectively. These stresses can be expressed as follows:

\[ \sigma_{\text{xax}}(\xi) = k_{\text{xax}} x^k(\xi) \]

\[ \mathbf{\sigma}_{\text{xax}}(\xi) = k_{\text{xax}} x^k(\xi) \]

where \( k_{\text{xax}}, k_{\text{yax}} \) imply the volume fraction of the SMA wires in the x and y directions, respectively. The new parameters \( \mathbf{x}^k(i = x, y) \), which are due to the phase transformation, are explained as follows:

\[ \mathbf{x}^k(\xi) = \mathbf{E}^k(\xi) \mathbf{A}^k(\xi) \]

In the above relations \( \mathbf{E}^k(i = x, y) \) and \( \mathbf{A}^k(i = x, y) \) are the martensite volume fraction and the Young's modulus of the SMA wires in the x and y directions, respectively. In the framework of CUF, \( \delta \mathbf{e}_{\text{mag}}(\xi) \) and \( \delta \mathbf{\sigma}_{\text{mag}}(\xi) \) can be obtained by the following relations:

\[ \delta \mathbf{e}_{\text{mag}}(\xi) = N_{\text{xax}} \mathbf{E}^k \delta \mathbf{e}_{\text{xax}}(\xi) \]

\[ \delta \mathbf{\sigma}_{\text{mag}}(\xi) = N_{\text{xax}} \mathbf{E}^k \delta \mathbf{\sigma}_{\text{xax}}(\xi) \]
After substitution of Eqs. (38), (39) and (40) in Eq. (37), one gets the following:

\[
\delta L_{\text{M SMA}}^k = \delta \mathbf{q}_{\text{M SMA}}^T (\xi) \mathbf{P}_{\text{M SMA}}^k \delta \mathbf{u}^k (\xi)
\]

where

\[
\mathbf{P}_{\text{M SMA}}^k = \left[ \begin{array}{c} \mathbf{p}_{\text{q}}^k (\xi) \\
\mathbf{p}_{\text{r}}^k (\xi) \\
0 \\
0 
\end{array} \right]
\]  
(42a)

\[
\mathbf{p}_{\text{q}}^k (\xi) = \left[ \begin{array}{c} \mathbf{p}_{\text{q}}^k (\xi) \\
\mathbf{p}_{\text{q}}^k (\xi) \\
\mathbf{p}_{\text{q}}^k (\xi) \\
\mathbf{p}_{\text{q}}^k (\xi) 
\end{array} \right]
\]  
(42b)

where

\[
\mathbf{p}_{\text{q}}^k (\xi) = \int_{\Omega_k} N_i \mathbf{q}_{\text{M SMA}}^k (\xi) \, d\Omega
\]

(42c)

and

\[
\mathbf{p}_{\text{r}}^k (\xi) = \int_{\Omega_k} \mathbf{q}_{\text{M SMA}}^k (\xi) \, d\Omega
\]

(42d)

\[
\delta \mathbf{q}_{\text{M SMA}}^T (\xi) : \mathbf{K}_{\text{M SMA}}^k (\xi) \mathbf{q}_{\text{M SMA}}^k (\xi) + \mathbf{M}_{\text{M SMA}}^k \delta \mathbf{q}_{\text{M SMA}}^k (\xi) = \mathbf{P}_{\text{M SMA}}^k \delta \mathbf{u}^k (\xi)
\]

(56a)

\[
\delta \mathbf{q}_{\text{M SMA}}^T (\xi) : \mathbf{K}_{\text{M SMA}}^k (\xi) \mathbf{q}_{\text{M SMA}}^k (\xi) + \mathbf{M}_{\text{M SMA}}^k \delta \mathbf{q}_{\text{M SMA}}^k (\xi) = 0
\]

(56b)

According to the above equations, it can be observed that stiffness/compliance matrices and also some force vector are variable with the martensite volume fraction and so, these stiffness/compliance matrices and force vector are instantaneous and dependent on the position of every point on the plate. At the same time, the martensite volume fraction is dependent on the stress and consequently the displacement values, therefore, these stiffness/compliance matrices and force vector are unknown. In other words, the governing equations of motion and the kinetic relations of the phase transformation are coupled with each other which makes the governing equations physically nonlinear. Figs. 3–5 are presented the assembling scheme for the fundamental nucleus. It must be mentioned that the transverse stress unknowns are eliminated using the static condensation method [33]. Therefore, Eqs. (56) can be expressed as follows:

\[
\mathbf{K}(\xi) \mathbf{q}(\xi) + \mathbf{M} \mathbf{q}(\xi) = \mathbf{P}(\xi)
\]

(57)

where

\[
\mathbf{K}(\xi) = [\mathbf{K}_{\text{M SMA}}(\xi) - \mathbf{K}_{\text{M SMA}}(\xi)(\mathbf{K}_{\text{M SMA}}(\xi))^{-1}\mathbf{K}_{\text{M SMA}}(\xi)]
\]

(58)

4.1. Time discretization

The stiffness matrix and the force vector in Eq. (57) are physically nonlinear, and therefore, an incremental solution technique beside an iterative method is utilized for solving the coupled equations.

The Newmark scheme is employed for time discretization of Eq. (57) by the following form [41]:

\[
\mathbf{q}_{m+1} = \mathbf{q}_m + \Delta t^2 \mathbf{q}_{m+1} + 1/2\Delta t^2 \mathbf{q}_{m+1}
\]

(59a)

\[
\mathbf{q}_{m+1} = \mathbf{q}_m + \Delta t \mathbf{q}_{m+1}
\]

(59b)

where \( \mathbf{q} \) and \( \mathbf{q}_m \) are the displacement and velocity vectors, respectively. Also, one has:

![Fig. 3. Assembling scheme related to \( \mathbf{K}_{\text{M SMA}} \) and \( \mathbf{M} \) in ESL models.](image-url)
\[ \begin{align*}
\mathbf{q}_{m+1} &= a_3 (\mathbf{q}_{m,1} - \mathbf{q}_m) - a_4 \mathbf{q}_m - a_5 \mathbf{q}_m \\
\mathbf{q}_{m+1} &= \mathbf{q}_m + a_1 \mathbf{q}_m + a_2 \mathbf{q}_m \tag{64a}
\end{align*} \]

with \( a_1 = \lambda \Delta t \) and \( a_2 = (1 - \lambda) \Delta t \).

### 4.2. The proposed iterative incremental approach

In order to solve the nonlinear coupled equations, the following steps are proposed:

1. At the first step, the initial values of \( \mathbf{q}_0 \), \( \mathbf{q}_0 \), and \( \{ \xi \}_0 \) are assumed (they are usually assumed as zero).
2. Determine the initial acceleration \( \mathbf{q}_0 \) by Eq. (63).
3. Define the new time \( t_{m+1} = t_m + \Delta t_{m+1} \) and put \( \{ \xi \}_m = \{ \xi \}_m \).
4. Update the material properties and \( \mathcal{E}_0 (i=x,y) \) using the \( \{ \xi \}_m \).
5. Calculate the \( \mathbf{q}_{m+1} \) by Eq. (60) based on the \( \{ \xi \}_m \).
6. Compute \( \{ \sigma \}_m \) using \( \mathbf{q}_{m+1} \) and \( \{ \xi \}_m \).
7. Calculate \( \{ \xi \}_m \) using the kinetic relations of the phase transformation based on the \( \{ \sigma \}_m \). (It is explained in Fig. 6).
8. This iterative scheme continues until a specified convergence criterion is obtained. For this aim the following criterion can be used:

\[ \text{max} \left( \frac{|\xi - \xi_{m+1}|}{\xi_{m+1}} \right) < \delta \tag{65} \]

where the index \( p \) indicates the predictor, \( k \) indicates the layer \( k \) and \( \xi_{y,m+1}(j=x,y) \) implies the martensite volume fraction of the \( j \)-th gauss point in the \( x \) and \( y \) directions, respectively, for the element. Also, \( \delta \) is a small numeral. When the convergence criterion is fulfilled, \( \mathbf{q}_{m+1} \) and \( \mathbf{q}_{m,1} \) are determined by Eq. (64). Then, the next time increment is started and the foregoing steps are repeated from step 3 to step 8. If the convergence criterion is not fulfilled, a new approximation of \( \{ \xi \}_m \) is computed using the relaxation method as:

9. \( \{ \xi \}_m = \{ \xi \}_m \) and steps 4 to 8 are repeated again, till the convergence criterion is fulfilled.

In order to identify the condition of the phase transformation on every gauss point of the plate, the algorithm scheme in Fig. 6 is utilized at any time increment.

### 5. Numerical results and discussion

A new code is written in MATLAB software for derivation the results based on the above formulations. This section consists of two examples. At the first example for verifications the present finite model, a particular problem is studied and compared with the published results. In the second example a nonlinear dynamic analysis of SMA multilayered plate is implemented. Some parametric studies such as length-to-thickness ratio, plate aspect ratio and also the effect of different boundary conditions, upon the loss factors are investigated.

**Example 1.** A static analysis of a simply supported composite plate with \([0^\circ/90^\circ]\), lay-up without SMA wires, is studied in order to validate the present two dimensional finite element model. In this paper it is assumed that all of the layers of the plate have equal thickness. The material properties of the lamina are as follows:

\[ \begin{align*}
E_1 &= 25, \quad G_{12} = \frac{G_{13}}{E_2} = 0.5, \quad G_{23} = 0.2, \quad v_{12} = v_{13} = v_{23} = 0.25
\end{align*} \]
The square composite plate is subjected to bi-sinusoidal load over the top surface of the plate in $z$ direction as follow:

$$P_z = p_z \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right)$$

In this example the deflection is normalized as follows:

$$U(x, y, z) = \frac{u_1 100h^3E_2}{p, a^4}$$

<table>
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**Example 2.** This example deals with the dynamic analysis of the composite plate embedded with SMA wires. The composite plate has a $[0^\circ/90^\circ/90^\circ/0^\circ]$ lay-up. According to the Fig. 2 the layers are numbered from the bottom layer to the top layer. The material properties of the layers are as follows [44]:

$$E_1 = 50 \text{ GPa}, \quad E_2 = E_3 = 10 \text{ GPa}, \quad G_{12} = G_{13} = G_{23} = 5 \text{ GPa},$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25, \quad \rho = 1600 \text{ kg/m}^3$$

The geometrical dimensions of the plate are as follows:

$$a = b = 1 \text{ m}, \quad h = 0.05 \text{ m}$$

The SMA wires are embedded in the layers 1 and 8 in the x direction and the layers 2 and 7 in the y direction. The volume fraction of SMA wires is chosen 40% for each lamina. The material properties of the SMA wires are shown in Table 2. The boundary conditions are simply supported and a step impulsive pressure is applied on the top surface of the plate in the $z$ direction. The amplitude of uniform pressure is $P = 3 \text{ MPa}$. Fig. 7 shows the stress–strain history at the midpoint $(a/2, b/2)$ of the top lamina considering the pseudoelastic effect of the SMA wires. As can be seen, the stress–strain diagram exhibits the hysteresis loops. Fig. 8 shows the variation of the martensite volume fraction (MVF) of the SMA wires with time for the mentioned point. In Fig. 9, the time response of the deflection at the center point of the plate $u_z(a/2, b/2, 0)$ is presented. From this figure, it can be observed that, the amplitude of vibration decreases regularly, such that at the time $t = 0.06 \text{ s}$, the amplitude is reduced to 68% of its value at the first peak ($t = 0.0034 \text{ s}$). This phenomenon is because of the hysteresis loops which dissipate the energy gradually. In addition, from Fig. 7, it can be seen that as the plate vibrates, the region of dissipated energy diminishes gradually and therefore the rate of reduction in amplitudes decreases. The vibration goes until the stress–strain curve arrives the linear-elastic state and after this situation, the plate vibrates with constant amplitude...
and more reduction in vibration amplitude cannot be seen (it can be seen in Fig. 12). In other words, due to the dissipation of energy by the SMA wires, the amplitude of the vibration goes to decrease, until it is arriving to the linear-elastic states of the wires and after this situation, it vibrates with constant amplitude.

In this part the effect of $a/h$ on the dynamic response of the square simply supported SMA hybrid composite plate is investigated. At the first step, for selecting a suitable model, a maximum deflection at the first peak at the center of the plate, $u_{z}(a/2, b/2, 0)$, for different $a/h$ ratios is presented in the Table 3 based on the different models. Since, the problem in this study is nonlinear; therefore, the run time of the program is very high. Hence, for saving the computational cost, the selected models in Table 3 are used for the next analysis. For example, the dynamic response of the plate with $a/h = 20$ and $a/b = 1$ and SSSS boundary conditions.

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<th>$a/h$</th>
<th>IP (MPa)</th>
<th>EM1</th>
<th>EM2</th>
<th>EM3</th>
<th>LM1</th>
<th>LM2</th>
<th>LM3</th>
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<td>0.0904</td>
<td>EM1</td>
</tr>
</tbody>
</table>

and more reduction in vibration amplitude cannot be seen (it can be seen in Fig. 12). In other words, due to the dissipation of energy by the SMA wires, the amplitude of the vibration goes to decrease, until it is arriving to the linear-elastic states of the wires and after this situation, it vibrates with constant amplitude.

In this part the effect of $a/h$ on the dynamic response of the square simply supported SMA hybrid composite plate is investigated. At the first step, for selecting a suitable model, a maximum deflection at the first peak at the center of the plate, $u_{z}(a/2, b/2, 0)$, for different $a/h$ ratios is presented in the Table 3 based on the different models. Since, the problem in this study is nonlinear; therefore, the run time of the program is very high. Hence, for saving the computational cost, the selected models in Table 3 are used for the next analysis. For example, the dynamic response of the plate with $a/h = 20$ and $a/b = 1$.
a/h = 20 is shown in Fig. 10 for different models. As can be seen, a good agreement is observed between the different models, so it can be concluded that the accuracy of the model EM1 with less computational cost is suitable for analyzing the plate with a/h = 20. It should be mentioned that the intensity of the pressure (IP) for different a/h ratios is such that the maximum deflection at the first peak for different a/h ratios is in the same range. Loss factor of the SMA multilayered plate is computed using the measurement of the vibration amplitude within the vibration and the following relation:

$$\zeta = \frac{1}{2\pi} \ln \left( \frac{x_1 - x_{\text{mean}}}{x_{n-1} - x_{\text{mean}}} \right)$$  \hspace{1cm} (66)

Fig. 11 shows the variation of the loss factor with a/h ratio. It can be seen that as the a/h ratio increases, the loss factor decreases. In other words, as the thickness increases, the capacity of SMA wires for dissipating the energy increases. The reason for this phenomenon is that, as the thickness increases, the value of the stress in the SMA wires increases, and therefore reaches its critical magnitude for transformation.

It is known that the boundary conditions can have an important effect on the analysis of the composite multilayered plate. For this aim, the boundary conditions SSSS, CCC and CSCS are investigated, where S and C stand for the simply supported and clamped boundary conditions, respectively. The results from the CCC and CSCS boundary conditions are shown in Figs. 12 and 13, respectively. The intensity of pressure is 9, 6.5 and 3 MPa for CCC, CSCS and SSSS, respectively. It can be seen that, the capability of CCC boundary conditions in damping is higher than the other boundary conditions, such that the loss factor is 2.67%, 2.48% and 1.58% for CCC, CSCS and SSSS boundary conditions, respectively. Therefore, as the stiffness of the edges increases, the capacity of SMA wires for dissipating the energy increases.

Plate aspect ratio (a/b) has an important effect on the analysis of the SMA hybrid composite plate. In this regard, the composite plate has a [0°/90°]/90°/0°] lay-up and the SMA wires are embedded in the layers 1 and 8 in the x direction, which the volume fraction of SMA wires is chosen 40% for each lamina. Fig. 14 shows the variation of the loss factor with different a/b ratios for a/h = 20 and SSSS boundary conditions. The IP and the suitable model are presented in Table 4. It can be observed from Fig. 14 that, as a/b ratio increases, the loss factor decreases. In other words, when the length of the plate is shorter than the width of the plate, the SMA wires show more capacity for dissipating the energy.

### Table 4

<table>
<thead>
<tr>
<th>a/b (MPa)</th>
<th>IP</th>
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<tr>
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</table>

**6. Conclusion**

In this research, a nonlinear dynamic analysis of the multilayer composite plate embedded with SMA wires is implemented in the frame work of Carrera’s Unified Formulation (CUF), considering the instantaneous phase transformation effects, for every point on the plate. The CUF has the capability to unify many theories in a unified form which can be differed by the order of expansion and definition of the variables in the thickness direction. This can be (ESL), if the unknown variables are considered for the whole plate, or (LW), if the unknown variables are considered for each layer, individually. The Brinson’s SMA constitutive equation is used to model the pseudoelastic effect of the SMA wires. The governing equations are derived from the Reissner Mixed Variational Theorem (RMVT) in order to enforce the interlaminar continuity of transverse shear and normal stresses between two adjacent layers.
The governing equations of motion and the kinetic relations of phase transformation are coupled with each other. Therefore, an iterative incremental finite-element-based scheme is proposed for solving the coupled equations. The Newmark method is employed to time discretization of the governing equations. The parametric effects of length-to-thickness ratio, plate aspect ratio and also the effect of different boundary conditions, upon the loss factors are investigated. Results show that, as the length-to-thickness ratio and the plate aspect ratio increases, the loss factor decreases. Also, it can be concluded that, as the stiffness of edges increases, the capacity of SMA wires to dissipate the energy increases.

References

[34] Lagoudas DC. Shape memory alloys, modeling and engineering applications. USA: Springer; 2008.