Static analysis of functionally graded sandwich plates according to a hyperbolic theory considering Zig-Zag and warping effects

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In this paper, a variation of Murakami’s Zig-Zag theory is proposed for the analysis of functionally graded plates. The new theory includes a hyperbolic sine term for the in-plane displacements expansion and accounts for through-the-thickness deformation, by considering a quadratic evolution of the transverse displacement with the thickness coordinate. The governing equations and the boundary conditions are obtained by a generalization of Carrera’s Unified Formulation, and further interpolated by collocation with radial basis functions. Numerical examples on the static analysis of functionally graded sandwich plates demonstrate the accuracy of the present approach. The thickness stretching effect on such problems is studied.

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1. Introduction

The strong difference of mechanical properties between faces and core in sandwich structures (or layered composites) introduces a discontinuity of the deformed core-faces planes at the interfaces. This is known as Zig-Zag (ZZ) effect. Such discontinuities make difficult the use of classical theories such as Kirchhoff [1] or Reissner–Mindlin [2,3] type theories (see the books by Zenkert [4], and Vinson [5] to trace accurate responses of sandwich structures). Two possibilities can be used to capture the ZZ effect (see the overviews by Burton and Noor [6], Noor et al. [7], Altenbach [8], Librescu and Hause [9], Vinson [10], and Demasi [11]): the so-called layer-wise models, and a Zig-Zag function (ZZF) in the framework of mixed multilayered plate theories. An historical review on ZZ theories has been provided by Carrera [12].

The first alternative can be computational expensive for laminates with large number of layers as the degrees-of-freedom increase as the number of layers increases. Considering the second alternative, Murakami [13] proposed a ZZF that is able to reproduce the slope discontinuity. Equivalent single layer models with only displacement unknowns can be developed on the basis of ZZF. A review of early developments on the application of ZZF has been provided in the review article by Carrera [14]. The advantages of analyze multilayered anisotropic plate and shells using the ZZF as well as the Finite Element implementation have been discussed by Carrera [15]. Further studies on the use of Murakami’s Zig-Zag function (MZZF) have been documented in [15–17].

The use of alternative methods to the Finite Element Methods for the analysis of plates, such as the meshless methods based on radial basis functions (RBFs) is attractive due to the absence of a mesh and the ease of collocation methods. The use of radial basis function for the analysis of structures and materials has been previously studied by numerous authors [18–34].

Carrera’s Unified Formulation (CUF) was proposed in [14,35,36] for laminated plates and shells and extended to functionally graded (FG) plates in [37–39]. The present formulation is a generalization of the original CUF in the sense that considers different displacement fields for in-plane and out-of-plane displacements. In this paper the application of ZZF to bending analysis of thin and thick FG sandwich plates is studied. A new displacement theory is used, considering a quadratic variation of the transverse displacements (allowing for through-the-thickness deformations), and introducing a hyperbolic sine term in the in-plane displacement expansion. This can be seen as a variation of the original Murakami’s ZZ displacement field. CUF is combined with RBFs for the static analysis: the principle of virtual displacements is used under CUF to obtain the governing equations and boundary equations and these are interpolated by collocation with RBFs.

The paper is organized as follows. The problem we are dealing with is introduced in Section 2. Then, the state-of-the-art review on the use of Zig-Zag functions and the displacement field of the

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present shear deformation theory is presented in Section 3. For the sake of completeness CUF and the radial basis functions collocation technique for the static analysis of FG plates are briefly reviewed in Sections 4 and 5, respectively. Numerical examples on the static analysis of simply supported functionally graded sandwich square plates are presented and discussed in Section 6. These include the computation of the displacements and stresses of sandwich plates with FGM in the core or in the skins, considering several material power-law exponents, side-to-thickness ratios and skin-core-skin ratios as well. Final conclusions are presented in Section 7.

2. Problem formulation

Consider a rectangular plate of plan-form dimensions \(a\) and \(b\) and uniform thickness \(h\). The co-ordinate system is taken such that the \(x-y\) plane \((z = 0)\) coincides with the midplane of the plate \((z \in [-h/2,h/2])\). The plate is subjected to a transverse mechanical load applied at the top of the plate.

Two different types of functionally graded sandwich plates are studied: sandwich plates with FG core and sandwich plates with FG skins.

In the sandwich plate with FG core the bottom skin is fully metal (isotropic) and the top skin is fully ceramic (isotropic as well). The core layer is graded from metal to ceramic so that there are no interfaces between core and skins, as illustrated in Fig. 1. The volume fraction of the ceramic phase in the core is obtained by adapting the typical polynomial material law as:

\[
V_c = \left(0.5 + \frac{z}{h_c}\right)^p
\]

where \(z_c \in [h_1,h_2]\), \(h_c = h_2 - h_1\) is the thickness of the core, and \(p > 0\) is the power-law exponent that defines the gradation of material properties across the thickness direction as shown in Fig. 3 (left).

In sandwich plates with FG skins the core is fully ceramic (isotropic) and skins are composed of a functionally graded material across the thickness direction. The bottom skin varies from a metal-rich surface \((z = -h/2)\) to a ceramic-rich surface while the top skin face varies from a ceramic-rich surface to a metal-rich surface \((z = h/2)\), as illustrated in Fig. 2. There are no interfaces between core and skins. The volume fraction of the ceramic phase in the skins is obtained as:

\[
V_c = \begin{cases} 
\left(\frac{z - h_0}{h_2 - h_0}\right)^p, & z \in [-h/2,h_1] \\
\left(\frac{z - h_0}{h_2 - h_0}\right)^p, & z \in [h_2,h/2] 
\end{cases}
\]

where \(p \geq 0\) is a scalar parameter that allows the user to define gradation of material properties across the thickness direction of the skins. The \(p = 0\) case corresponds to the (isotropic) fully ceramic plate.

The sandwich plate with FG skins may be symmetric or non-symmetric about the mid-plane as we may vary the thickness of each face. Fig. 3 (right) shows a non-symmetric sandwich with volume fraction defined by the power-law (2) for various exponents \(p\), in which top skin thickness is the same as the core thickness and the bottom skin thickness is twice the core thickness. Such thickness relation is denoted as 2-1-1. A bottom-core-top notation is
being used. 1-1-1 means that skins and core have the same thickness. In both sandwich plates the volume fraction for the metal phase is given as $V_m = 1 - V_c$.

### 3. A new hyperbolic sine ZZF theory

#### 3.1. The Zig-Zag function

The Murakami’s Zig-Zag function $Z(z)$ depends on the adimensioned layer coordinate, $z_k$, according to the following formula:

$$Z(z) = (-1)^k z_k$$  \hspace{1cm} (3)

$z_k$ is defined as $z_k = \frac{h_k}{h}$, where $z_k$ is the layer thickness coordinate and $h_k$ is the thickness of the $k$th layer. $Z(z)$ has the following properties:

1. It is a piece-wise linear function of layer coordinates $z_k$.
2. $Z(z)$ has unit amplitude for the whole layers.
3. The slope $Z(z) = \frac{dz}{dz}$ assumes opposite sign between two adjacent layers. Its amplitude is layer thickness independent.

#### 3.2. Overview on Murakami’s Zig-Zag theories

In 1986, a refinement of FSDT by inclusion of ZZ effects and transverse normal strains was introduced in Murakami’s original ZZF [13], defined by the following displacement field:

$$\begin{align*}
  u &= u_0 + z u_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0 + z w_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) w_2
\end{align*}$$  \hspace{1cm} (4)

where $u$ and $v$ are the in-plane displacements and $w$ is the transverse displacement. The involved unknowns are $u_0, u_1, u_2, v_0, v_1, v_2, w_0, w_1$, and $w_2$; $u_0$, $u_1$, and $w_0$ are translations of a point at the midplane; $u_1$, $v_1$, and $w_1$ are rotations as in the typical FSDT; and the additional degrees of freedom $u_2$ and $v_2$ have a meaning of displacement. $z_k, z_{k+1}$ are the bottom and top $z$-coordinates at each layer.

More recently, another possible FSDT theory has been investigated by Carrera [15] and Demasi [16], ignoring the through-the-thickness deformations:

$$\begin{align*}
  u &= u_0 + z u_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0
\end{align*}$$  \hspace{1cm} (5)

with $u_0, u_1, u_2, v_0, v_1, v_2, w_0, z_k$, and $z_{k+1}$ as before.

Ferreira et al. [40] and Rodrigues et al. [41] used a ZZF theory involving the following expansion of displacements

$$\begin{align*}
  u &= u_0 + z u_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0 + z w_1 + z^2 w_2
\end{align*}$$  \hspace{1cm} (6)

This represents a variation of the Murakami’s original theory, allowing for a quadratic evolution of the transverse displacement across the thickness direction. Furthermore, Ferreira et al. [42] used two higher order ZZ theories allowing for a quadratic evolution of the transverse displacement across the thickness direction as well and involving the following displacement fields:

$$\begin{align*}
  u &= u_0 + z u_1 + z^2 u_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + z^2 v_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0 + z w_1 + z^2 w_2
\end{align*}$$  \hspace{1cm} (7)

$$\begin{align*}
  u &= u_0 + z u_1 + \sinh \left( \frac{h_k}{h} \right) u_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + \sinh \left( \frac{h_k}{h} \right) v_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0 + z w_1 + z^2 w_2
\end{align*}$$  \hspace{1cm} (8)

In Eqs. (7) and (8), $w_2$ denote higher-order translations and $u_3$ and $v_3$ denote rotations. $u_0, u_1, u_2, v_0, v_1, v_2, w_0, w_1, u_3$, and $v_3$ are as in (4)–(6).

#### 3.3. The hyperbolic sine ZZF shear deformation theory

All previous cited work using ZZ functions deals with laminated plates or shells. In the present work a new hyperbolic sine ZZF theory is introduced for the analysis of functionally graded sandwich plates. The choice of the new displacement field is based on previous work by the authors and the role of the Zig-Zag effect on sandwich structures. The authors have successfully used a hyperbolic sine quasi-3D shear deformation theory accounting for thickness stretching without the Zig-Zag effect in the study of functionally graded plates [43]. The present theory adds the terms to consider the Zig-Zag effect. The present theory is based on the following displacement field:

$$\begin{align*}
  u &= u_0 + z u_1 + \sinh \left( \frac{h_k}{h} \right) u_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) u_2 \\
  v &= v_0 + z v_1 + \sinh \left( \frac{h_k}{h} \right) v_3 + (-1)^k \frac{h_k}{h} \left( z - \frac{1}{2} (z_k + z_{k+1}) \right) v_2 \\
  w &= w_0 + z w_1 + z^2 w_2
\end{align*}$$  \hspace{1cm} (9)
The involved unknowns have the same meaning as in equations (7) and (8). The expansion of the degrees of freedom $u_0, u_1, u_3, v_0, v_1, v_3, w_0, w_1, w_2$ are functions of the thickness coordinate only. These are layer-independent, unlike those of $u_2$ and $v_2$, as illustrated in Figs. 4 and 5. Fig. 4 shows the meaning of the unknows in the in-plane displacements expansion in present theory: $u_0, u_1, v_1$ (translations), $u_3$ and $v_3$ (rotations). In Fig. 5 one can visualize that this $ZZF$ correspondence to a rotation per layer.

4. The Unified Formulation for the static analysis of FG sandwich plates

In this section it is shown how to obtain the fundamental nuclei under CUF, which allows the derivation of the governing equations and boundary conditions for FG plates.

4.1. Functionally graded materials

A conventional FG plate considers a continuous variation of material properties over the thickness direction by mixing two different materials [44]. The material properties of the FG plate are assumed to change continuously throughout the thickness of the plate, according to the volume fraction of the constituent materials. Although one can use CUF for one-layer, isotropic plate, we consider a multi-layered plate. In fact, the sandwiches in study present three physical layers, $k_p = 1, 2, 3$, each containing a different displacement field. Nevertheless, we are dealing with functionally graded materials and becomes mandatory to model the continuous variation of properties across the thickness direction. A considerable number of layers is needed to ensure correct computation of material properties at each thickness position, and for that reason we consider $N_p = 91$ virtual (mathematical) layers of constant thickness. In the following, $k_p$ refers to physical layers and $k = 1, \ldots, 91$ refers to virtual layers.

The CUF procedure applied to FG materials starts by evaluating the volume fraction of the two constituents for each layer. Then, a homogenization technique is employed to find the values of the modulus of elasticity, $E^p$, and Poisson’s ratio, $\nu^p$, of each layer.

To describe the volume fractions an exponential function can be used as in [45], or the sigmoid function as proposed in [46]. In the present work a power-law function is used as most researchers do [47–50]. In the typical FG plate the power-law function defines the volume fraction of the ceramic phase as:

$$V_c = (0.5 + \frac{z}{h})^p$$  (10)

where $z \in [-h/2, h/2]$, $h$ is the thickness of the plate, and $p$ is a scalar parameter that allows the user to define gradation of material properties across the thickness direction. In both sandwich plates, the volume fraction of the ceramic phase of the FG layers are obtained by adapting the typical power-law. Furthermore, we need to compute the volume fraction for each layer. In the sandwich plate with FG core case, (1) becomes:

$$V_c^k = \begin{cases} 0, & \text{in the bottom skin} \\ (0.5 + \frac{z}{h})^p, & \text{in the core} \\ 1, & \text{in the top skin} \end{cases}$$  (11)

$$V_c^k = \begin{cases} (\frac{z}{h} - \frac{h}{2})^p, & \text{in the bottom skin} \\ 1, & \text{in the core} \\ (\frac{z}{h} - \frac{h}{2})^p, & \text{in the top skin} \end{cases}$$  (12)

where $z$ is the thickness coordinate of a point of each (virtual) core layer, and $h_c$ and $p$ are as in (1).

Considering (2), for the sandwich plate with FG skins case one has:

$$E^k(z) = E_{m}V_m + E_{c}V_c; \quad \nu^k(z) = \nu_{m}V_m + \nu_{c}V_c$$  (13)

Other homogenization procedures could be used, for example the Mori–Tanaka one [51,52].

4.2. Modeling of the displacement components

According to the Unified Formulation by Carrera, the three displacement components $u_k = (u, v)$ and $u_k = (w)$ and their relative variations are modeled as:

$$(u_k, u_k, u_k) = F_E \left( u_{zt}, u_{zt}, u_{zt} \right)$$

$$(\partial u_k, \partial u_k, \partial u_k) = F_s \left( \partial u_{zt}, \partial u_{zt}, \partial u_{zt} \right)$$  (14)

Resorting to the displacement field in Eq. (9), we choose vectors $F_1 = \left[ 1 \ z \ \sinh \left( \frac{z}{h} \right) \right]$ for in-plane displacements and $F_2 = \left[ 1 \ z \ z^2 \right]$ for displacement $w$. In this case, thickness-stretching is considered. For the thickness effect study, in the case that thickness-stretching is not allowed, the vector for transverse displacement is replaced with $F_1 = 1$, meaning that we are considering the expansion $w = w_0$ in the displacement field.

4.3. Strains

Strains are separated into in-plane and normal components, denoted respectively by the subscripts $p$ and $n$. The mechanical strains in the $k$th layer can be related to the displacement field $u^k = \left\{ u_z^k, u_z^k, u_z^k \right\}$ via the geometrical relations ($G$):

$$\varepsilon_{zz} = \left( \varepsilon_{zz}, \varepsilon_{yy}, \gamma_{xy} \right) = D_p^k u^k,$$  (15)

$$\varepsilon_{zz} = \left( \gamma_{zz}, \gamma_{xy}, \varepsilon_{xx} \right) = D_{np}^k u^k,$$

wherein the differential operator arrays are defined as follows:
If \( \varepsilon_{zz} = 0 \) is considered, thickness-stretching is not allowed. In this case, \( e_{kk}^t \) and the differential operator array \( D_{kk}^t \) remain as before, but the other strains are reduced to

\[
e_{kk}^t = [\gamma_{xz}, \gamma_{yz}]^{1/2} = (D_{kk}^t + D_{kn}^t) \mathbf{u}^t,
\]

wherein the differential operator arrays are defined as:

\[
D_{kk}^t = \begin{bmatrix} 0 & 0 & \partial_x \varepsilon_{kk}^t \\ 0 & 0 & \partial_y \varepsilon_{kk}^t \\ \partial_x \partial_y \varepsilon_{kk}^t \end{bmatrix}, \quad D_{kn}^t = \begin{bmatrix} \partial_x \varepsilon_{kn}^t \\ \partial_y \varepsilon_{kn}^t \\ 0 \end{bmatrix}, \quad D_{nn}^t = \begin{bmatrix} 0 & 0 & \partial_x \varepsilon_{nn}^t \\ 0 & 0 & \partial_y \varepsilon_{nn}^t \\ \partial_x \partial_y \varepsilon_{nn}^t \end{bmatrix}.
\]

#### 4.4. Elastic stress–strain relations

To define the constitutive equations (G), stresses are separated into in-plane and normal components as well. The 3D constitutive equations are given as:

\[
\sigma_{pc}^t = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}] = C_{pp}^t \varepsilon_{pc}^t + C_{kn}^t e_{kn}^t
\]

\[
\sigma_{nc}^t = [\sigma_{xz}, \sigma_{yz}, -\sigma_{zz}] = C_{np}^t \varepsilon_{pc}^t + C_{nm}^t e_{nc}^t
\]

with

\[
C_{pp}^t = \begin{bmatrix} C_{11}^t & C_{12}^t & 0 \\ C_{12}^t & C_{22}^t & 0 \\ 0 & 0 & C_{66}^t \end{bmatrix}, \quad C_{pm}^t = \begin{bmatrix} 0 & 0 & C_{33}^t \\ 0 & 0 & C_{23}^t \\ 0 & 0 & 0 \end{bmatrix},
\]

\[
C_{np}^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{33}^t & C_{44}^t & C_{55}^t \end{bmatrix}, \quad C_{nm}^t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{33}^t & C_{44}^t & C_{55}^t \end{bmatrix},
\]

and the \( C_{kk}^t \) are the three-dimensional elastic constants

\[
C_{11}^t = C_{33}^t = C_{66} = \frac{E^t}{1-\nu^t(1-\nu^t)},
\]

\[
C_{12}^t = C_{23}^t = \frac{E^t}{2(1+\nu^t)},
\]

\[
C_{44}^t = C_{55}^t = \frac{E^t}{2(1-\nu^t)},
\]

where the modulus of elasticity and Poisson’s ratio were defined in (13), and \( G \) is the shear modulus \( G = \frac{E^t}{2(1+\nu^t)} \).

For the \( \varepsilon_{zz} = 0 \) case, the plane-stress case is used:

\[
\sigma_{pc}^t = [\sigma_{xx}, \sigma_{yy}, 0] = C_{pp}^t \varepsilon_{pc}^t
\]

\[
\sigma_{nc}^t = [\sigma_{xz}, \sigma_{yz}, 0] = C_{np}^t \varepsilon_{pc}^t
\]

with

\[
C_{pp}^t = \begin{bmatrix} C_{55}^t & 0 \\ 0 & C_{44}^t \end{bmatrix}, \quad C_{np}^t = \begin{bmatrix} 0 & 0 & C_{33}^t \\ 0 & 0 & C_{23}^t \\ 0 & 0 & 0 \end{bmatrix}
\]

and the \( C_{kk}^t \) are the plane-stress reduced elastic constants:

\[
C_{11}^t = C_{22}^t = \frac{E^t}{1-\nu^t}, \quad C_{12}^t = \frac{E^t}{1+\nu^t}, \quad C_{44}^t = C_{55}^t = \frac{E^t}{2(1-\nu^t)}.
\]

#### 4.5. Principle of virtual displacements

In the framework of the Unified Formulation, the Principle of Virtual Displacements (PVD) for the pure-mechanical case is written as:

\[
\delta \mathbf{u}_j^t : \mathbf{K}_uw^{t} \mathbf{u}^t = \delta \mathbf{l}_w^t
\]

where the fundamental nucleus \( \mathbf{K}_uw^{t} \) is obtained as:
The radial basis function (\(\phi\)) approximation of a function (\(u\)) is given by
\[
\tilde{u}(x) = \sum_{i=1}^{N} \tilde{z}_i \phi(\|x - y_i\|_2), x \in \mathbb{R}^n
\]  
(39)

where \(y_i, i = 1, \ldots, N\) is a finite set of distinct points (centers) in \(\mathbb{R}^n\).

The most common RBFs are

Cubic: \(\phi(r) = r^3\)

Thin plate splines: \(\phi(r) = r^2 \log(r)\)

Wendland functions: \(\phi(r) = (1 - r^2)^p(r)\)

Gaussian: \(\phi(r) = e^{-\alpha r^2}\)

Multiquadrics: \(\phi(r) = \sqrt{c^2 + r^2}\)

Inverse Multiquadrics: \(\phi(r) = (c^2 + r^2)^{-1/2}\)

where the Euclidian distance \(r\) is real and non-negative and \(c\) is a positive shape parameter. In the present work, we consider the compact-support Wendland function defined as
\[
\phi(r) = (1 - c r)^{2(2c r)} + 25(c r^2) + 8c r + 1
\]  
(40)

The shape parameter (\(c\)) is obtained by an optimization procedure, as detailed in Ferreira and Fasshauer [54].

Considering \(N\) distinct interpolations, and knowing \(u(x_j), j = 1, 2, \ldots, N\), we find \(u_j\) by the solution of a \(N \times N\) linear system
\[
A\tilde{u} = \tilde{f}
\]  
(41)

where \(A = \left[ \phi(\|x - y_i\|_2) \right]_{i,j=1}^{N \times N}, \quad \tilde{x} = [x_1, x_2, \ldots, x_N]^T\) and \(\tilde{u} = [u(x_1), u(x_2), \ldots, u(x_N)]^T\).

Consider a linear elliptic partial differential operator \(L\) acting in a bounded region \(\Omega\) in \(\mathbb{R}^n\) and another operator \(L_b\) acting on a boundary \(\partial \Omega\). In the static problems we seek the computation of displacements (\(\tilde{u}\)) from the global system of equations
\[
L\tilde{u} = \tilde{f} \text{ in } \Omega
\]  
(42)
\[
L_b \tilde{u} = \tilde{g} \text{ on } \partial \Omega
\]  
(43)

The right-hand side of (42) and (43) represent the external forces applied on the plate and the boundary conditions applied along the perimeter of the plate, respectively. The PDE problem defined in (42) and (43) will be replaced by a finite problem, defined by an algebraic system of equations, after the radial basis expansions.

The solution of a static problem by radial basis functions considers \(N_b\) nodes in the domain and \(N_b\) nodes on the boundary, with a total number of nodes \(N = N_N + N_b\). In the present work, a \(9^2\) Chebyshev grid is employed (see Fig. 6) and a square plate is com-
puted with side length $a = 2$. For a given number of nodes per side $(N + 1)$ they are generated by MATLAB code as:

$$x = \cos(\pi \cdot j / N); y = x;$$

One advantage of such mesh is the concentration of points near the boundary.

We denote the sampling points by $x_i \in \Omega$, $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $i = N + 1, \ldots, N$. At the points in the domain we solve the following system of equations

$$\sum_{j=1}^{N} a_{ij} \phi(||x - y||_2) = f(x_j), \quad j = 1, 2, \ldots, N_j \quad (44)$$

or

$$\mathcal{L}^t \mathbf{a} = \mathbf{F} \quad (45)$$

where

$$\mathcal{L}^t = [\mathcal{L}^t \phi(||x - y||_2)]_{N \times N} \quad (46)$$

At the points on the boundary, we impose boundary conditions as

$$\sum_{j=1}^{N} a_{nj} \phi(||x - y||_2) = g(x_j), \quad j = N + 1, \ldots, N \quad (47)$$

or

$$\mathbf{B} \mathbf{a} = \mathbf{G} \quad (48)$$

where

$$\mathbf{B} = [\mathcal{L}_{\partial \phi}][||x_{N+1} - y||_2]]_{N \times N}$$

Therefore, we can write a finite-dimensional static problem as

$$\begin{bmatrix} \mathcal{L}^t \\ \mathbf{B} \end{bmatrix} \mathbf{a} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \quad (49)$$

By inverting the system (49), we obtain the vector $\mathbf{a}$. We then obtain the solution $\mathbf{u}$ using the interpolation Eq. (39).

The radial basis collocation method follows a simple implementation procedure. Taking Eq. (49), we compute

$$\mathbf{a} = \left[ \frac{\mathcal{L}^t}{\mathbf{B}} \right]^{-1} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \quad (50)$$

This $\mathbf{a}$ vector is then used to obtain solution $\mathbf{u}$, by using (39). If derivatives of $\mathbf{u}$ are needed, such derivatives are computed as

$$\frac{\partial \mathbf{u}}{\partial x} = \sum_{j=1}^{N} \phi_j \quad (51)$$

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} = \sum_{j=1}^{N} \phi_j \quad (52)$$

In the present collocation approach, we need to impose essential and natural boundary conditions. Consider, for example, the condition $w = 0$, on a simply supported or clamped edge. We enforce the conditions by interpolating as

$$w = 0 \rightarrow \sum_{j=1}^{N} \phi_j = 0 \quad (53)$$

Other boundary conditions are interpolated in a similar way.

6. Numerical examples

In this section the shear deformation plate theory is combined with radial basis functions collocation for the static analysis of functionally graded sandwich plates. Displacements and stresses of simply supported (SSSS) square ($a = b = 2$) sandwich plates with FGM in the core or in the skins, both symmetric and unsymmetric, are analyzed. Various side-to-thickness ratios, power-law exponents, and skin-core-skin thickness ratios are considered. The plate is subjected to a bi-sinusoidal transverse mechanical load, $p = p_0 \cos \left( \frac{2\pi}{l} \right) \cos \left( \frac{2\pi}{h} \right)$ (see Fig. 6), applied at the top of the plate.

As stated before, all numerical examples are performed employing a Chebyshev grid and the Wendland function as defined in (40) with an optimized shape parameter. The plate is a sandwich, physically divided into 3 layers, but we consider 91 virtual layers. The power-law function is used to describe the volume fraction of the metal and ceramic phases (see (1) and (2)) and the material homogenization technique adopted is the law of mixtures (13), the same used in the references.

The following material properties are used:

- Zirconia Young’s modulus: $E_z = 151$ GPa
- Aluminum Young’s modulus: $E_m = 70$ GPa
- Alumina Young’s modulus: $E_c = 380$ GPa

with Poisson’s ratio constant $\nu = 0.3$. Only Young’s modulus needs a homogenization technique.

An initial study was performed for each type of sandwich to show the convergence of the present approach and select the number of Chebyshev points to use in the computation of the static problems problems.

6.1. Sandwich with FG core

The static analysis of sandwich plates with FG core is now performed. In the following examples the materials are aluminum (55) and alumina (56). The thickness of each skin layer is $h_i = 0.1h$ and the core layer thickness is $h = 0.8h$, i.e., we are dealing with a 1-8-1 sandwich.

The non-dimensional parameters used are:

$$\tilde{w} = \frac{10E_h h^3}{a^2 p_0}, \quad \text{evaluated at the center of the plate}$$

$$\tilde{\sigma}_{xx} = \frac{h}{a p_0} \sigma_{xx}, \quad \text{evaluated at the center of the plate}$$

$$\tilde{\sigma}_{xy} = \frac{h}{a p_0} \sigma_{xy}, \quad \text{evaluated at the corner of the plate}$$

$$\tilde{\sigma}_{zz} = \frac{h}{a p_0} \sigma_{zz}, \quad \text{evaluated at the midpoint of the side}$$

$$\tilde{\sigma}_{zz}, \quad \text{evaluated at center of the plate}$$

Two convergence studies were performed, varying the exponent power-law $p$ and the side-to-thickness ratio $a/h$. Table 1 refers to $p = 1$ and $a/h = 4$ and Table 2 refers to $p = 10$ and $a/h = 100$. A $15^2$ grid was chosen for the following static problems.

Table 3 and Figs. 7 and 8 refer to the out-of-plane displacement. In Table 3 we tabulate the values of the deflection obtained with

<table>
<thead>
<tr>
<th>Grid</th>
<th>$9^2$</th>
<th>$11^2$</th>
<th>$13^2$</th>
<th>$15^2$</th>
<th>$17^2$</th>
<th>$19^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{w}(0)$</td>
<td>0.7411</td>
<td>0.7417</td>
<td>0.7417</td>
<td>0.7417</td>
<td>0.7417</td>
<td>0.7417</td>
</tr>
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<td>0.6224</td>
<td>0.6236</td>
<td>0.6235</td>
<td>0.6236</td>
<td>0.6236</td>
<td>0.6236</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{xy}(0)$</td>
<td>0.3263</td>
<td>0.3164</td>
<td>0.3164</td>
<td>0.3164</td>
<td>0.3164</td>
<td>0.3164</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{yy}(0)$</td>
<td>0.2329</td>
<td>0.2333</td>
<td>0.2332</td>
<td>0.2332</td>
<td>0.2332</td>
<td>0.2332</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{zz}(0)$</td>
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<td>0.2748</td>
<td>0.2747</td>
<td>0.2747</td>
<td>0.2747</td>
<td>0.2747</td>
</tr>
<tr>
<td>$\tilde{\sigma}_{zz}(0)$</td>
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<td>0.2193</td>
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</tr>
<tr>
<td>$\tilde{\sigma}_{yy}(0)$</td>
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<td>0.3164</td>
<td>0.3165</td>
<td>0.3164</td>
<td>0.3164</td>
<td>0.3164</td>
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<tr>
<td>$\tilde{\sigma}_{zz}(0)$</td>
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<td>0.2333</td>
<td>0.2332</td>
<td>0.2332</td>
<td>0.2332</td>
<td>0.2332</td>
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</tbody>
</table>
present approach for various power-law exponents $p$ and side-to-thickness ratios $a/h$, and compare with available references. In Fig. 7, the thickness-stretching effect on the deformed of the simply supported sandwich square plate with FG core, with $p = 1$ and $a/h = 10$, is visualized. Figure is the plot of the top $(z = h/2)$ of the plate. Fig. 8 presents the out-of-plane displacement through the thickness direction, for a sandwich with FG core with side-to-thickness ratio $a/h = 4$, varying the exponent power-law value $p$

Table 3

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>$\epsilon_{xx}$</th>
<th>$p = 1$</th>
<th>$p = 4$</th>
<th>$p = 5$</th>
<th>$p = 10$</th>
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</thead>
<tbody>
<tr>
<td>Ref. LD4 [38]</td>
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<td>1.1327</td>
<td>1.2232</td>
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</tr>
<tr>
<td>Ref. LM4 [38]</td>
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<td>1.1329</td>
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</tr>
<tr>
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<td></td>
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<tr>
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<tr>
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<tr>
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<td>1.1753</td>
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<tr>
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<td></td>
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<td></td>
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<tr>
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<td>Ref. LM4 [38]</td>
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</tr>
<tr>
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<td>0.8045</td>
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</table>

Fig. 7. Deformed of the SSSS sandwich square plate with FG core ($p = 1$, $a/h = 10$), subjected to sinusoidal load at the top, according to the hyperbolic sine ZZ theory, considering and disregarding thickness-stretching.
with present approach for various exponents of the power-law \( p \) and side-to-thickness ratios \( a/h \). In figures we present stresses through the thickness direction of a SSSS sandwich square plate with FG core, \( a/h = 100 \) according to the hyperbolic sine ZZ theory, for several values of \( p \). In all tables, results obtained with present hyperbolic sine ZZ theory and RBF collocation are in good agreement with references.

### 6.2. Sandwich with FG skins

We now focus on sandwich plates with isotropic core and FG skins. All examples consider a sandwich plate made of aluminum (55) and zirconia (54) and with side-to-thickness ratio \( a/h = 10 \). Ta-
core, subjected to sinusoidal load at the top, according to the hyperbolic sine ZZ theory, for several values of \( p \). The non-dimensional displacements and stresses are given as

\[
\begin{align*}
\bar{w} &= \frac{10hE_0}{a^2p_z} w, \\
\bar{u} &= \frac{10hE_0}{a^2p_z} u, \\
\bar{\sigma}_{zz} &= \frac{h}{a^2p_z} \sigma_{zz}, \\
\end{align*}
\]

Table 10

<table>
<thead>
<tr>
<th>p</th>
<th>( \omega(0) )</th>
<th>11(^2)</th>
<th>13(^2)</th>
<th>15(^2)</th>
<th>17(^2)</th>
<th>19(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3489</td>
<td>0.3490</td>
<td>0.3490</td>
<td>0.3490</td>
<td>0.3490</td>
<td>0.3490</td>
</tr>
<tr>
<td>( \sigma_{xx} )</td>
<td>1.5917</td>
<td>1.5880</td>
<td>1.5893</td>
<td>1.5889</td>
<td>1.5891</td>
<td>1.5891</td>
</tr>
<tr>
<td>( \sigma_{zz} )</td>
<td>0.2673</td>
<td>0.2667</td>
<td>0.2669</td>
<td>0.2668</td>
<td>0.2668</td>
<td>0.2668</td>
</tr>
</tbody>
</table>

Table 11

<table>
<thead>
<tr>
<th>p</th>
<th>( \omega(0) )</th>
<th>2-1-2</th>
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<th>1-1-1</th>
<th>2-2-1</th>
<th>1-2-1</th>
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<td>0</td>
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<td>0.3490</td>
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<td>0.3490</td>
<td>0.3490</td>
</tr>
<tr>
<td>( \sigma_{xx} )</td>
<td>1.5917</td>
<td>1.5880</td>
<td>1.5893</td>
<td>1.5889</td>
<td>1.5891</td>
<td>1.5891</td>
</tr>
<tr>
<td>( \sigma_{zz} )</td>
<td>0.2673</td>
<td>0.2667</td>
<td>0.2669</td>
<td>0.2668</td>
<td>0.2668</td>
<td>0.2668</td>
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</tbody>
</table>

Fig. 12. \( \sigma_{zz} \) through the thickness direction of a SSSS sandwich square plate with FG core, \( a/h = 100 \), subjected to sinusoidal load at the top, according to the hyperbolic sine ZZ theory, for several values of \( p \).

Fig. 13. \( \sigma_{yy} \) through the thickness direction of a SSSS sandwich square plate with FG core, \( a/h = 100 \), subjected to sinusoidal load at the top, according to the hyperbolic sine ZZ theory, for several values of \( p \).

Fig. 14. \( \sigma_{yz} \) through the thickness direction of a SSSS sandwich square plate with FG core, \( a/h = 100 \), subjected to sinusoidal load at the top, according to the hyperbolic sine ZZ theory, for several values of \( p \).
Two convergence studies were performed, varying the exponent power-law \( p \) and the symmetry of the sandwich. Table 9 refers to the symmetric 2-1-2 plate with \( p = 1 \) and Table 10 refers to the non-symmetric 2-2-1 plate with \( p = 5 \). A 15\(^2\) grid was chosen for the following static problems.

Results referring to the displacements of a sandwich plate with FG skins are presented in Table 11 and Figs. 15–17. In Table 11, the transverse displacement are tabulated and compared with available references, for several values of \( p \) and skin-core-skin thickness ratios. In Fig. 15, the influence of the thickness-stretching on the deformed of the symmetric 1-2-1 simply supported sandwich square plate with FG skins, with \( p = 10 \), subjected to sinusoidal load at the top, is visualized. Fig. 15 is the plot of the bottom \((z = -h/2)\) of the plate. In Figs. 16 and 17 the influence of the

<table>
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<tr>
<th>Source</th>
<th>2-1-2</th>
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<th>1-1-1</th>
<th>2-2-1</th>
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</thead>
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<td>( p = 0 )</td>
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<td>2.05452</td>
<td>2.05452</td>
<td>2.05452</td>
<td>2.05452</td>
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power-law exponent $p$ in the displacements $u_x$ and $w$, respectively, can be visualized. The figures refer to the simply supported 2-1-2 sandwich square plate with FG skins, subjected to sinusoidal load at the top, and presents the displacements through the thickness, according to the hyperbolic sine ZZ theory, for various values of $p$.

The deflection of a simply supported sandwich plate with FG skins increases as the power-law of the material increases. This is seen in Table 11 for all studied plates and in Fig. 16 for a particular one. As the core thickness to the plate thickness ratio increases, the transverse displacement decreases. The results depend on the $C_{15}$ approach.

Table 12 and Fig. 18 present results referring to $\sigma_{xz}$. The values obtained with present hyperbolic sine ZZ theory and RBF collocation are tabulated in Table 12 and compared with available references, for various $p$ and skin-core-skin thickness ratios. Fig. 18 shows the stress through the thickness for the simply supported 2-1-2 sandwich square plate with FG skins, subjected to sinusoidal load at the top, for various values of $p$ (see Figs. 19–23).

In all tables, a good agreement between the present solution and references considered is obtained. (See Table 13).

7. Conclusions

In this paper we presented a study using the radial basis function collocation method to analyze static deformations of thin and thick functionally graded sandwich plates using a variation of Murakami’s Zig-Zag function, considering a hyperbolic sine term for the in-plane displacement expansion and allowing for through-the-thickness deformations. This has not been done before and serves to fill the gap of knowledge in this area.
The obtained results, more significantly in thicker plates, and these demonstrate the accuracy of the present approach. Numerical examples were performed on simply supported sandwich plates, made of functionally graded materials in the core and skin-core-skin thickness ratios. Obtained results were presented in figures and tables and compared with references to thickness and skin-core-skin thickness ratios. Using the Unified Formulation, the plate formulation was easily discretized by radial basis functions collocation. The hardworking of deriving the equations of motion and boundary conditions is eliminated with the present approach. The combination of Carrera’s Unified Formulation and collocation with RBFs proved to be a simple yet powerful alternative to other finite element or meshless methods in the static deformation of thin and thick functionally graded sandwich plates.

Numerical experiments were performed on simply supported sandwich plates, made of functionally graded materials in the core or in the skins, for various material power-law exponents and side-to-thickness and skin-core-thickness ratios. Obtained results were presented in figures and tables and compared with references and these demonstrate the accuracy of present approach. Allow or not extensibility in the thickness direction has influence on the obtained results, more significatively in thicker plates. The $\sigma_{zz}$ should be considered in the formulation, even for thinner functionally graded sandwich plates.

### Acknowledgements

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### References


### Table 13

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<th>$\sigma_{zz}$(0) of a sandwich plate with FG skins, for several exponents $p$ and skin-core-skin ratios.</th>
<th>2-1-2</th>
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References:


