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Computations and evaluations of higher-order theories for free vibration analysis of beams

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ABSTRACT

This paper deals with higher-order theories for the analysis of free vibration of beam structures. Refined theories are implemented by the application of the Unified Formulation by the first author which allows one to introduce any-order expansions of the displacement unknowns over the beam sections. The selection of the most appropriate theory is made by using a so-called axiomatic–asymptotic approach which permits one to retain only those terms of the displacement expansion which have been established to be significant with respect to an assigned control parameter. The finite element method is used to provide numerical solutions. Various beam sections as well as boundary conditions are considered. Depending on the vibration modes (bending, torsion, etc.), quite different theories are selected. In general, the number of the effective terms of the resultant theories is much lower than the full expansion case amount. The nature of these terms can differ very much as different beam geometries and boundary conditions are considered. It has been concluded that the method proposed appears to be suitable and convenient to establish the most appropriate beam theory for a given problem; it leads, in fact, to the cheapest computational model for a given accuracy.

1. Introduction

One-dimensional (1D) structural models are widely adopted to design slender bodies such as aircraft wings, rotor blades, bridges and wind turbines. 1D theories are, in fact, computationally cheaper and handier than 2D (plate/shell) and 3D (solid) models. The reliability of a 1D model is typically affected by a number of geometrical and material characteristics which require either refined 1D theories or more cumbersome plate, shell and solid models. Typical examples of detrimental factors for a 1D model are poor slenderness, thin-walls, inhomogeneity, anisotropy and concentrated boundary conditions.

1D structural theories are usually referred to as 'beam' models. In a beam model, the 3D structural problem is reduced to 1D by introducing a certain number of unknowns (f_τ) which are exploited to express a generic variable f (e.g. a displacement, stress or strain component). f is usually defined on a point of the cross-section. The unknowns are related to the variable by means of generic base functions (F_τ) defined above the cross-section domain (x, z), that is,

$$f(x,y,z) = F_\tau(x,z) f_\tau(y), \quad \tau = 1,2,\ldots,M \tag{1}$$

where y is the longitudinal direction of the beam and M indicates the number of unknowns. If the three displacement

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components are considered as variables, a possible explicit form of Eq. (1) could be

$$u_x = u_{x_1} + x u_{x_2} + z u_{x_3}$$

$$u_y = u_{y_1} + x u_{y_2} + z u_{y_3}$$

$$u_z = u_{z_1} + x u_{z_2} + z u_{z_3} \tag{2}$$

where nine unknowns $(u_{x_1}, u_{x_2}, \ldots, u_{z_3})$ are exploited.

Classical beam theories are those by Euler–Bernoulli [1] and Timoshenko [2]. These models provide satisfactory accuracies whenever slender and homogenous structures are analyzed in bending. The aforementioned critical features make both models unable to predict the static and dynamic behavior of a structure. In order to preserve the computational advantages of beam models with respect to shells and solids, a number of refinements have been proposed based on different techniques and including a number of non classical effects such as warping, in-plane distortion and shear effects. Amongst them, the most typical methodologies exploit 1. higher-order models, 2. shear correction factors, 3. extension of the de Saint Venant and the Vlasov theories, 4. asymptotic methods and 5. the generalized beam theory (GBT). Excellent and comprehensive reviews on beam models are those by Kapania and Raciti [3] and Ghugal and Shimpi [4]. Brief descriptions of some of the most important works on beam models are hereafter provided; special attention is given to papers related to the free-vibration analysis which is the topic of the present paper.

Higher-order models exploit refined expansions of the problem variables. The refinement process leads to enhanced analysis capabilities with higher computational costs, see Washizu [5]. Third-order models are commonly used and have been proposed by many authors, such as Marur and Kant [6–8], Shi and Lam [9], Kameswara Rao et al. [10], Murthy et al. [11] and Şimşek and Kocatürk [12]. These models usually do not require shear correction factors. Song and Waas [13] conducted free-vibration and buckling analyses by means of higher-order beam models and they pointed out how the adoption of these theories can lead to models laying beyond the elasticity solution; this means, for instance, that higher-order models could provide lower natural frequencies than those foreseen by the elasticity solution. Other higher-order models have been proposed for specific applications. Ganesan and Zabihollah [14,15] developed beam finite elements to deal with different kinds of tapered structures, whereas Piovan et al. [16,17] presented higher-order models for thin-walled curved beams. Refined 1D theories for box beams were proposed by McCarthy and Chattopadhyay [18,19] and Librescu et al. [20,21]. Recently, Lezgy-Nazargah et al. [22] have presented a higher-order beam model for the global–local analysis of composite structures; the global model, based on polynomial or exponential models, is superimposed to a local layer-wise approach.

Shear correction factors have been extensively proposed and adopted in last decades to enhance the global response of classical beam models. A comprehensive analysis on this topic can be found in Timoshenko and Goodier [23]. Recent papers dealing with shear correction factors are those by Goyal and Kapania [24] and Chan et al. [25].

Extensions of Vlasov's theory for thin-walled beams represent another important methodology to improve the capabilities of 1D models. Excellent examples are those proposed by Ambrosini et al. [26,27] and recently by de Borbón et al. [28]. De Saint Venant's based models and the use of warping functions are other important techniques for thin-walled open cross-section beams as shown by Bishop et al. [29].

The asymptotic method exploits an expansion of a characteristic parameter, such as the thickness of the beam, to obtain refined beam models. A reference paper on this method is by Berdichevsky et al. [30]. An important class of beam models derived from asymptotic methods is that known as VABS, see Cesnik et al. [31] and Yu and Hodges [32]. Other models based on the asymptotic method are those proposed by Firouz-Abadi et al. [33] and Kim and Wang [34]. The key feature of this methodology is that the 1D model is governed by variationally consistent and geometrically exact governing equations which provide asymptotically exact stress and strain recovery by means of a beam model having a low number of degrees of freedom. Regular and thin-walled beams can be accounted for.

Schardt [35] introduced the Generalized Beam Theory (GBT). This method is based on a piecewise description of the cross-section displacement field and accounts for in-plane distortions. The cross-section displacement field of a thin-walled beam is assumed as a linear combination of deformation modes defined on a number of cross-section nodes. A recent paper on GBT devoted to the free-vibration analysis is by Bebiano et al. [36].

This work is embedded in the framework of the Carrera Unified Formulation (CUF) for higher-order 1D models [37]. CUF has been developed during the last decade, initially for plate/shell models [38,39], recently for 1D models [40]. The unique contribution given by CUF models is due to their hierarchical capabilities which make the choice of the expansion functions (F_τ) and their order arbitrary. This means that any-order structural models can be implemented with no need of formal changes in the problem equations and matrices. CUF can therefore deal with arbitrary geometries, boundary conditions and material characteristics with no need of *ad hoc* formulations. CUF is particularly advantageous when different cross-section geometries or boundary conditions have to be considered. As the structural configuration changes, in fact, the beam model to be adopted can be chosen by a straightforward convergence analysis on the order of the beam model. Static [41–43] and free-vibration [44,45] analyses showed the enhanced capabilities of CUF 1D models which are able to detect shell- and solid-like solutions for different structural models including thin-walled models under point loads and shell-like natural modes. A further extension of the present formulation [46,47] allowed us to deal with open cross-sections, boundary conditions enforced on lateral edges and layer-wise approaches. In those papers a combination of locally refined models was exploited above the beam cross-section in order to obtain discontinuous spatial derivatives which are fundamental to analyze realistic beams such as helicopter blades. Also, the possibility of locally refining the beam model allowed us to optimize the computational costs of the model for a given structural problem.

The present paper analyzes the effectiveness of higher-order terms in detecting the free-vibration behavior of compact and thin-walled isotropic structures. The methodology exploited is the so-called mixed axiomatic–asymptotic technique which has been recently proposed by Carrera and Petrolo [48]. This method permits us to evaluate the contribution of each term of the expansion on the prediction of the mechanical response of a structure. Moreover, parametric studies can be easily conducted on a number of parameters including slenderness ratios, thicknesses and boundary conditions. Previous works dealt with the static analysis of plates [49,50] and beams [51], while the present paper presents the results obtained for free vibrations. The analysis of the effectiveness of each term allows us to detect reduced higher-order models which are able to provide results within a given accuracy.

2. CUF 1D formulation

The transposed displacement vector is defined as

$$\mathbf{u}(x,y,z) = \{u_x \;\; u_y \;\; u_z\}^T \tag{3}$$

where x, y, and z are orthonormal axes as shown in Fig. 1. Stress, $\boldsymbol{\sigma}$, and strain, $\boldsymbol{\epsilon}$, components are grouped as follows:

$$\boldsymbol{\sigma} = \{\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \sigma_{xy} \;\; \sigma_{xz} \;\; \sigma_{yz}\}^T, \quad \boldsymbol{\epsilon} = \{\epsilon_{xx} \;\; \epsilon_{yy} \;\; \epsilon_{zz} \;\; \epsilon_{xy} \;\; \epsilon_{xz} \;\; \epsilon_{yz}\}^T \tag{4}$$

Linear strain–displacement relations are used,

$$\boldsymbol{\epsilon} = \mathbf{D}\mathbf{u} = (\mathbf{D}_y + \mathbf{D}_\Omega)\mathbf{u} \tag{5}$$

where

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \\ \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \end{bmatrix} = \mathbf{D}_\Omega + \mathbf{D}_y \tag{6}$$

The Hooke law is exploited to obtain stress components,

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon} \tag{7}$$

The components of **C** are the material coefficients whose explicit expressions are not reported here for the sake of brevity, they can be found in [52].

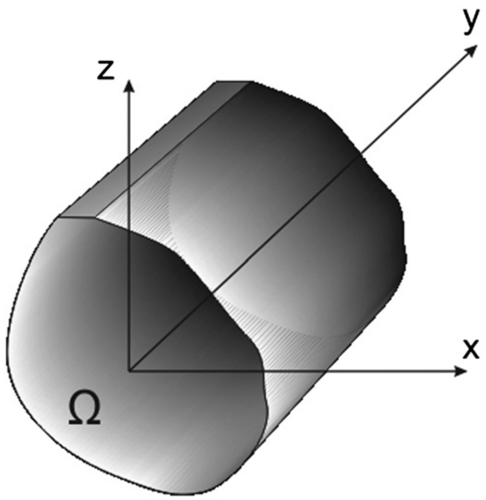


Fig. 1. Reference system of the beam.

2.1. Hierarchical higher-order models

In the CUF framework, the displacement field is the expansion of generic functions F_τ,

$$\mathbf{u} = F_\tau \mathbf{u}_\tau, \quad \tau = 1,2,\ldots,M \tag{8}$$

where F_τ vary above the cross-section. $\mathbf{u}_\tau$ is the displacement vector and M stands for the number of terms of the expansion. According to the Einstein notation, the repeated subscript, τ, indicates summation. The choice of F_τ determines the class of 1D CUF models to adopt. Two classes of 1D models have been developed, the Taylor Expansion class (TE) and the Lagrange Expansion class (LE). Only TE are adopted in this work, detailed descriptions of LE can be found in [47,46].

TE models are based on Taylor-like polynomial expansions, $x^i z^j$, of the displacement field above the cross-section of the structure (i and j are positive integers). The three displacement components of a generic N-order are then expressed by

$$u_x = \sum_{N_i=0}^{N} \left(\sum_{M=0}^{N_i} x^{N-M} z^M u_{x_{(N(N+1)+M+1)/2}} \right)$$

$$u_y = \sum_{N_i=0}^{N} \left(\sum_{M=0}^{N_i} x^{N-M} z^M u_{y_{(N(N+1)+M+1)/2}} \right)$$

$$u_z = \sum_{N_i=0}^{N} \left(\sum_{M=0}^{N_i} x^{N-M} z^M u_{z_{(N(N+1)+M+1)/2}} \right) \tag{9}$$

The order N of the expansion is arbitrary and is set as an input of the analysis. A convergence study is usually needed to choose N for a given structural problem. For example, the second-order model, $N=2$, has the following kinematic model:

$$u_x = u_{x_1} + x u_{x_2} + z u_{x_3} + x^2 u_{x_4} + xz u_{x_5} + z^2 u_{x_6}$$

$$u_y = u_{y_1} + x u_{y_2} + z u_{y_3} + x^2 u_{y_4} + xz u_{y_5} + z^2 u_{y_6}$$

$$u_z = u_{z_1} + x u_{z_2} + z u_{z_3} + x^2 u_{z_4} + xz u_{z_5} + z^2 u_{z_6} \tag{10}$$

The 1D model described by Eq. (10) has 18 generalized displacement variables: three constant, six linear, and nine parabolic terms.

2.2. FE formulation and the fundamental nucleus

The FE approach is herein adopted to discretize the structure along the y-axis, this process is conducted via a classical finite element methodology via the Principle of Virtual Displacements. The shape functions, N_i, and the nodal displacement vector, $\mathbf{q}_{\tau i}$, are used and the displacement vector becomes

$$\mathbf{u}(x,y,z) = N_i(y) F_\tau(x,z) \mathbf{q}_{\tau i}, \quad i = 1,2,\ldots,K \tag{11}$$

With

$$\mathbf{q}_{\tau i} = \{ q_{u_{x_{\tau i}}} \;\; q_{u_{y_{\tau i}}} \;\; q_{u_{z_{\tau i}}} \}^T \tag{12}$$

K is the number of the nodes on the element. For the sake of brevity, the explicit forms of the shape functions N_i are not reported here, they can be found in [53]. Elements with four nodes (B4) are used in this paper, that is, a cubic approximation along the y-axis is adopted.

The stiffness matrix is obtained via the Principle of Virtual Displacements,

$$\delta L_{\text{int}} = \delta L_{\text{ine}} \tag{13}$$

where L_{int} stands for the strain energy and L_{ine} is the work of the inertial loadings. δ stands for the virtual variation. The virtual variation of the strain energy is given by

$$\delta L_{\text{int}} = \int_V (\delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma})\, dV$$

$$= \int_V \delta \mathbf{q}_{\tau i}^T [\mathbf{D}^T (N_i(y) F_\tau(x,z) \mathbf{I})] \mathbf{C} [\mathbf{D}(N_j(y) F_s(x,z) \mathbf{I})] \mathbf{q}_{sj}\, dV \tag{14}$$

By introducing Eq. (5) into Eq. (14), it is possible to rewrite the virtual variation of L_{int} as

$$\delta L_{\text{int}} = \delta \mathbf{q}_{\tau i}^T \left\{ \int_V [(\mathbf{D}_\Omega + \mathbf{D}_y)^T (F_\tau(x,z) N_i(y) \mathbf{I})] \mathbf{C} [(\mathbf{D}_\Omega + \mathbf{D}_y)(N_j(y) F_s(x,z) \mathbf{I})]\, dV \right\} \mathbf{q}_{sj}$$

$$= \delta \mathbf{q}_{\tau i}^T \left\{ \int_l \left(N_i(y) \left(\int_\Omega [\mathbf{D}_\Omega^T (F_\tau(x,z) \mathbf{I})] \mathbf{C} [\mathbf{D}_\Omega (F_s(x,z) \mathbf{I})]\, d\Omega \right) N_j(y) \right) dy \right.$$

$$+\int_l \left(N_i(y)\left(\int_\Omega [\mathbf{D}_\Omega^T(F_\tau(x,z)\mathbf{I})]\mathbf{C}F_s(x,z)\,d\Omega\right)\mathbf{D}_y(N_j(y)\mathbf{I})\right)\,dy$$
$$+\int_l \left(\mathbf{D}_y^T(N_i(y)\mathbf{I})\left(\int_\Omega F_\tau(x,z)\mathbf{C}[\mathbf{D}_\Omega(F_s(x,z)\mathbf{I})]\,d\Omega\right)N_j(y)\right)\,dy$$
$$+\int_l \left(\mathbf{D}_y^T(N_i(y)\mathbf{I})\left(\int_\Omega F_\tau(x,z)\mathbf{C}F_s(x,z)\,d\Omega\right)\mathbf{D}_y(N_j(y)\mathbf{I})\right)\,dy\Bigg\}\mathbf{q}_{sj} \tag{15}$$

where Ω is the cross-section domain. The variation of the internal work is then written by means of the CUF fundamental nucleus,

$$\delta L_{\text{int}} = \delta\mathbf{q}_{\tau i}^T\mathbf{K}^{ij\tau s}\mathbf{q}_{sj} \tag{16}$$

where $\mathbf{K}^{ij\tau s}$ is the stiffness matrix in the form of the fundamental nucleus. The explicit forms of the nine components of $\mathbf{K}^{ij\tau s}$ are not reported here, they can be found in [47].

No assumptions on the approximation order have been done to obtain the fundamental nucleus. It is therefore possible to obtain refined 1D models without changing the formal expression of the nucleus components. This is the key-point of CUF which permits, with only nine FORTRAN statements, to implement any-order 1D theories.

The work of the inertial loadings is similarly obtained to compute the mass matrix as shown in [44].

The global matrices can be indicated as **K** for the stiffness matrix and **M** for the mass matrix. Harmonic solutions are introduced which lead to an eigenvalue problem,

$$(-\omega_k^2\mathbf{M}+\mathbf{K})a_k = 0 \tag{17}$$

where ω_k if the k-th natural oscillatory frequency associated to the k-th eigenvector, a_k.

3. Mixed axiomatic/asympthotic method to evaluate the effectiveness of model variables

The technique exploited for the analysis of the effectiveness of each generalized variable of a refined 1D model is briefly described in this section. More details can be found in previous works on plate [48] and beam models [51,37]. The method developed is referred to as '*mixed axiomatic/asympthotic*' because it allows one to obtain asymptotic-like results by means of axiomatically built models. In other words, CUF is used to generate higher-order models (axiomatically) and the effect of each term of the expansion is evaluated for different sets of parameters, such as thickness, loadings and geometrical boundary conditions.

The procedure adopted can be summarized by the following steps:

1. The problem data are fixed ⇒ *Loadings*, *BCs*, *Materials*
2. CUF is used to generate the governing equations for the considered theories ⇒ $u = F_\tau u_\tau$, $\mathbf{K}^{ij\tau s}$
3. A theory is fixed and used to establish the accuracy ⇒

$N=0$	$N=1$		$N=2$		
u_{x_1}	$u_{x_2}x$	$u_{x_3}z$	$u_{x_4}x^2$	$u_{x_5}xz$	$u_{x_6}z^2$
u_{y_1}	$u_{y_2}x$	$u_{y_3}z$	$u_{y_4}x^2$	$u_{y_5}xz$	$u_{y_6}z^2$
u_{z_1}	$u_{z_2}x$	$u_{z_3}z$	$u_{z_4}x^2$	$u_{z_5}xz$	$u_{z_6}z^2$

4. Each term is deactivated in turn ⇒

▲	▲	▲	▲	▲	▲
▲	▲	△	▲	▲	▲
▲	▲	▲	▲	▲	▲

5. The effect of the absence of a given term on the natural frequencies is evaluated. If the absence of a term does not corrupt a given vibration mode, the term will be considered inactive for that mode. ⇒

Active term	Inactive term
Yes, ▲	No, △

6. All the active terms are retained to build a reduced higher-order model ⇒

△	▲	△	▲	△	▲
▲	▲	△	△	▲	△
△	▲	▲	△	▲	△

⇒

$$u_x = xu_{x_2} + x^2u_{x_4} + z^2u_{x_6}$$
$$u_y = u_{y_1} + xu_{y_2} + xzu_{y_5}$$
$$u_z = xu_{z_2} + zu_{z_3} + xzu_{z_5}$$

The graphic notation introduced makes use of black symbols to indicate active terms and white symbols for deactivated terms. The elimination of a term, as well as the evaluation of its effectiveness in the analysis, can be obtained either by rearranging the rows and columns of the stiffness matrix or by exploiting a penalty technique. The accuracy of a reduced model is evaluated by computing the error E_f which is defined as

$$E_f = \left\|\frac{f-f_{\text{ref}}}{f_{\text{ref}}}\right\| \times 100 \tag{18}$$

where f_{ref} denotes the frequency computed trough the reference model. This approach is herein applied by making different parameters vary. The influence of the following structural characteristics is evaluated:

1. Transversal section shape.
2. Boundary conditions.

4. Results and discussion

Numerical examples are carried out in this section. A number of structural layouts are considered and the influence of each displacement variable on the solution was evaluated. A preliminary free vibration analysis is first conducted on deep and thin-walled structures in order to evaluate the proper benchmark solution to be used to obtain reduced models. Results obtained via the mixed axiomatic/asympthotic approach are then presented.

4.1. Preliminary assessments

A rectangular cantilevered beam was first considered. The geometry of the cross-section is shown in Fig. 2a and the related dimensions are given in Table 1. The length of the beam, L, is equal to 10 m. An isotropic material was adopted, the Young modulus, E, is equal to 69 GPa, the Poisson ratio, ν, is equal to 0.33 and the density, ρ, is equal to 2700 Kg/m^3. Table 2 presents the first four natural frequencies of the structure from different models. In particular, a SOLID model in MSC Nastran was used to compare the results from classical models (EBBT and TBT) and refined models up to the fourth-order. The effect of the Poisson locking correction on EBBT, TBT and $N=1$ is shown in Table 3 where the first two natural frequencies are reported. Fig. 3 shows a mesh convergence study which was conducted on the first frequency by means of an $N=2$ model. The analytical solution was obtained by means of the Euler–Bernoulli model.

A semicircular thin-walled cantilevered beam was then analyzed in order to provide comparisons with results from the open literature. The radius, r, is equal to 0.0245 m, the thickness, t, is equal to 0.004 m, and the length, L, is equal to 0.82 m. The material was assumed isotropic with the Young modulus equal to 68.9 GPa, the Poisson ratio equal to 0.3 and the density equal to 2700 Kg/m^3. The first two uncoupled frequencies are reported in Table 4 where results from different models are compared with those by Jun et al. [54]. A more detailed comparison between 1D CUF models and beam models from open literature can be found in Carrera et al. [37].

These preliminary results suggest the following:

1. A general good agreement was found between the present 1D formulation and the results from the SOLID model and the open literature.
2. EBBT, TBT and $N=1$ provide frequencies which are lower than those from higher-order models. This is due to the

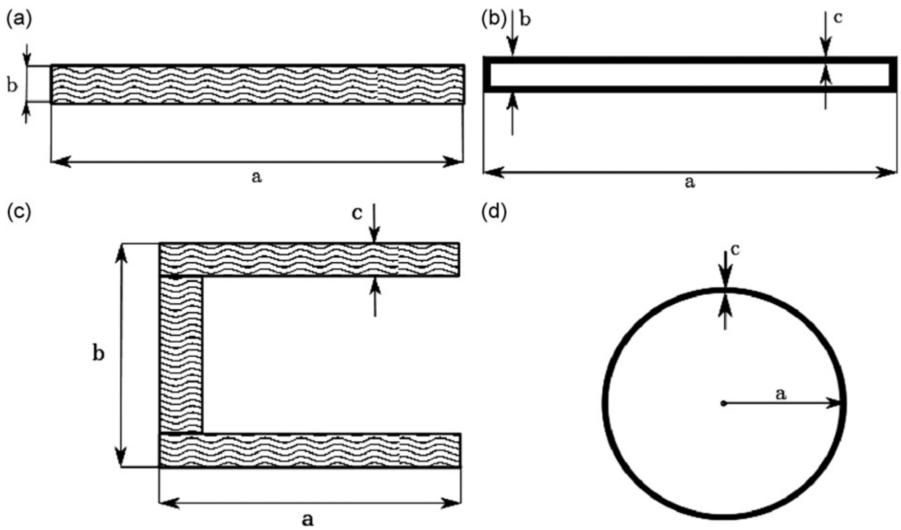


Fig. 2. Beam cross-sections considered: (a) Rectangular, (b) Rectangular thin-walled, (c) C-shaped, (d) Annular.

Table 1
Dimensions of the cross-sections considered.

Cross-section	a (m)	b (m)	c (m)
Rectangular	1.0	0.1	n/a
Rectangular thin-walled	1.0	0.1	0.005
C-shaped	1.0	1.0	0.005
Annular	1.0	n/a	0.02

Table 2
First four natural frequencies of a cantilevered rectangular beam via different structural models.

Model	1st Frequency (H)	2nd Frequency (Hz)	3rd Frequency (Hz)	4th Frequency (Hz)
3D FEM	0.8325	5.2141	8.0181	14.5998
EBBT	0.8166	5.1170	8.1504	14.3251
TBT	0.8165	5.1151	8.1090	14.3156
$N=1$	0.8165	5.1151	8.1090	14.3129
$N=2$	0.8318	5.2117	8.1529	14.6444
$N=3$	0.8263	5.1748	8.1447	14.4927
$N=4$	0.8255	5.1702	8.1443	14.4793

Table 3
First two natural frequencies of a cantilevered rectangular beam via different structural models; EBBT, TBT and $N=1$ without Poisson Locking correction.

Model	1st Frequency (Hz)	2nd Frequency (Hz)
EBBT	0.9940	6.2286
TBT	0.9939	6.2253
$N=1$	0.9939	6.2252
$N=2$	0.8318	5.2117
$N=3$	0.8263	5.1748
$N=4$	0.8255	5.1702

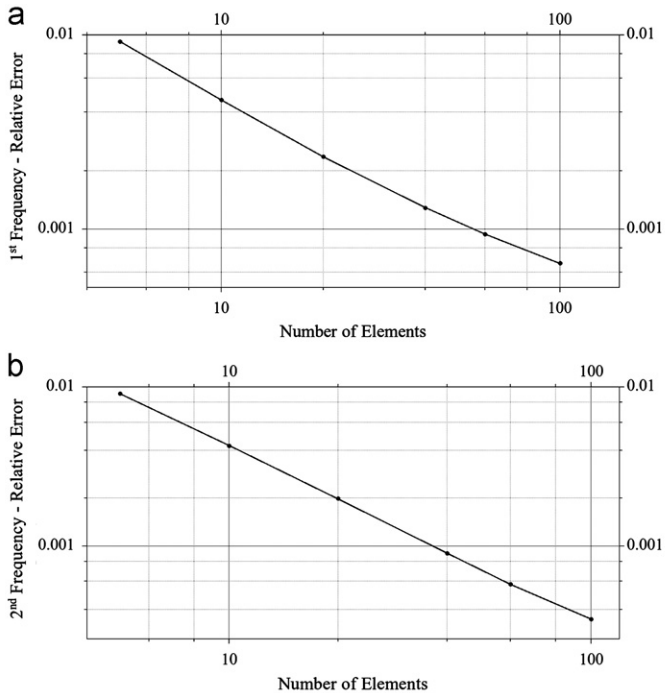


Fig. 3. Mesh convergence analysis for the first two frequencies of the rectangular beam.

Table 4
First two uncoupled natural frequencies of a semicircular cantilevered beam via different structural models, Hz.

	Jun el al. [54]	EBBT	$N=2$	$N=4$
1st	31.8	31.86	31.98	31.00
2nd	199.31	199.57	199.07	193.28

correction of the Poisson locking that is activated only for those models. A more detailed analysis of the Poisson locking can be found in [37].

3. The $N=4$ model offers significant improvements in the computation of the frequencies especially if a thin-walled structure is considered. The fourth-order model will be then adopted as reference for the mixed axiomatic/asympthotic method.
4. A 20 B4 mesh offers a good convergence behavior and will be adopted hereafter.

4.2. Reduced models for different cross-section geometries

Refined theories for different cross-section geometries are developed in this section. Four geometries were considered, rectangular, rectangular thin-walled, C-shaped and annular. The geometrical characteristics are given in Fig. 2 and Table 1. Cantilevered beams were considered and an isotropic material was adopted, the Young modulus, E, is equal to 69 GPa, the Poisson ratio, ν, is equal to 0.33 and the density, ρ, is equal to 2700 Kg/m^3. The length of the beam is equal to 10 m. Three natural modes were considered to evaluate the influence of each generalized variables on the solution,

1. The frequency related to the first bending mode along the 'z' direction, hereafter referred to as *Flexural z*.
2. The frequency related to the first bending mode along the 'x' direction, hereafter referred to as *Flexural x*.
3. The frequency related to the first torsional mode along, hereafter referred to as *Torsional*.

Table 5 shows the influence of a number of expansion terms on the modes considered for the rectangular cross-section beam. The first row is related to the full $N=4$ model and, since it is adopted as reference, this model is not affected by errors. The second row shows the influence of the constant term along the x-direction, u_{x1}. The bending mode along z and the torsional mode are not affected by the absence of this term ($E_f=0.0$), whereas the bending mode along x cannot be detected by such a model. The third row presents the influence of the constant term along y, u_{y1}. The absence of this term does not corrupt the three modes considered. The fourth row is related to the bilinear term along z, u_{z5}. This generalized variable affects the bending mode along x since $E_f=1.12$ percent. Such a procedure allows us to evaluate the influence of each term on each mode. Unless otherwise specified, a term is considered inactive in computing a mode if its absence yields $E_f < 0.1\%$.

Table 5
Influence of different terms on the natural frequencies of the rectangular cross-section beam.

Reduced model															E_f %		
															Flexural z	*Flexural x*	*Torsional*
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	0.00	0.00	0.00
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▿	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	0.00	X	0.00
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▿	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	0.00	0.00	0.00
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			
▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾	0.00	1.12	0.00
▾	▾	▾	▾	▿	▾	▾	▾	▾	▾	▾	▾	▾	▾	▾			

Table 6 shows the reduced models for the rectangular beam, these models were obtained by retaining the active terms for a given mode. The second column presents the ratio between the number of effective terms, M_e, and the total number of terms M_{tot}. In a finite element formulation, M_e represents the number of degrees of freedom per node of the structural model, that is, its computational cost. The last column presents the error on the solution given by the reduced model. This error is not necessarily null because of the combined effect of the terms which were not retained. The explicit expression of the beam model needed to detect the first bending mode along z is

$$u_x = xzu_{x_5} + x^3zu_{x_{12}}$$

$$u_y = zu_{y_3} + xu_{y_2} + xzu_{y_5}$$

$$u_z = u_{z_1} + x^2u_{z_4} + z^2u_{z_6} + x^4u_{z_{11}} \tag{19}$$

Such a model requires only seven terms out of 45. It has to be underlined that the reduced model related to the bending along x requires less terms than the one in Eq. (19) because of the higher flexibility of the rectangular cross-section along z.

Table 6
Reduced models for the rectangular cross-section beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
▽	▽	▽	▽	▼	▽	▽	▽	▽	▽	▽	▼	▽	▽	▽		
▽	▽	▼	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	7/45	0.01
▼	▽	▽	▼	▽	▼	▽	▽	▽	▽	▼	▽	▽	▽	▽		
1^{st} Flexural x																
▼	▽	▽	▼	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽		
▽	▼	▽	▽	▽	▽	▼	▽	▽	▽	▽	▽	▽	▽	▽	5/45	0.02
▽	▽	▽	▽	▼	▽	▽	▽	▽	▽	▽	▽	▽	▽	▽		
1^{st} Torsional																
▽	▽	▼	▽	▽	▽	▽	▼	▽	▼	▽	▽	▽	▽	▽		
▽	▽	▽	▽	▼	▽	▽	▽	▽	▽	▽	▼	▽	▼	▽	9/45	0.00
▽	▼	▽	▽	▽	▽	▼	▽	▼	▽	▽	▽	▽	▽	▽		

Table 7
Reduced models for the rectangular thin-walled cross-section beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
○	○	○	○	●	○	○	○	○	○	○	●	○	●	○		
○	○	●	○	○	○	○	●	○	●	○	○	○	○	○	10/45	0.01
●	○	○	●	○	●	○	○	○	○	●	○	○	○	○		
1^{st} Flexural x																
●	○	○	●	○	○	○	○	○	○	○	○	○	○	●		
○	●	○	○	○	○	●	○	○	○	○	○	○	○	○	6/45	0.05
○	○	○	○	●	○	○	○	○	○	○	○	○	○	○		
1^{st} Torsional																
○	○	●	○	○	○	○	●	○	●	○	○	○	○	○		
○	○	○	○	●	○	○	○	○	○	○	●	○	●	○	8/45	0.00
○	●	○	○	○	○	○	○	●	○	○	○	○	○	○		

The torsional mode is the most cumbersome for this cross-section configuration,

$$u_x = zu_{x_3} + x^2 z u_{x_8} + z^3 u_{x_{10}}$$

$$u_y = xzu_{y_5} + x^3 z u_{y_{12}} + xz^3 u_{y_{14}}$$

$$u_z = xu_{z_2} + x^3 u_{z_7} + xz^2 u_{z_9} \tag{20}$$

Table 8
Reduced models for the C-shaped cross-section beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
△	△	▲	△	▲	△	△	▲	△	▲	△	▲	△	▲	△		
△	△	▲	△	▲	△	△	▲	△	▲	△	▲	△	▲	△	20/45	0.03
▲	▲	△	▲	△	▲	▲	△	▲	△	▲	△	▲	△	△		
1^{st} Flexural x																
▲	▲	△	▲	△	▲	△	△	△	△	△	△	△	△	▲		
▲	▲	△	▲	△	▲	▲	△	△	△	▲	△	▲	△	▲	17/45	0.02
△	△	▲	△	▲	△	△	△	△	▲	△	△	△	▲	△		
1^{st} Torsional																
△	△	▲	△	▲	△	△	▲	△	▲	△	▲	△	▲	△		
△	△	▲	△	▲	△	△	▲	△	▲	△	▲	△	▲	△	19/45	0.02
▲	▲	△	▲	△	▲	▲	△	▲	△	▲	△	△	△	△		

Table 9
Reduced models for the annular beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
□	□	□	□	■	□	□	□	□	□	□	■	□	■	□		
□	□	■	□	□	□	□	■	□	■	□	□	□	□	□	7/45	8.11
■	□	□	□	□	□	□	□	□	□	□	□	□	□	□		
1^{st} Flexural z^*																
□	□	□	□	■	□	□	□	□	□	□	■	□	■	□		
□	□	■	□	□	□	□	■	□	■	□	□	□	□	□	9/45	0.04
■	□	□	■	□	■	□	□	□	□	□	□	□	□	□		
1^{st} Flexural x^*																
■	□	□	■	□	■	□	□	□	□	□	□	□	□	□		
□	■	□	□	□	□	■	□	■	□	□	□	□	□	□	9/45	0.05
□	□	□	□	■	□	□	□	□	□	□	■	□	■	□		
Torsional																
□	□	■	□	□	□	□	□	□	□	□	□	□	□	□		
□	□	□	□	□	□	□	□	□	□	□	□	□	□	□	3/45	0.00
■	■	□	□	□	□	□	□	□	□	□	□	□	□	□		

$*$ models obtained by retaining terms giving $E_f < 0.01$ %

The beam model that is necessary to detect all these three modes is

$$u_x = u_{x_1} + z u_{x_3} + x^2 u_{x_4} + xz u_{x_5} + x^2 z u_{x_8} + z^3 u_{x_{10}} + x^3 z u_{x_{12}}$$

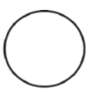
$$u_y = x u_{y_2} + z u_{y_3} + xz u_{y_5} + x^3 u_{y_7} + x^3 z u_{y_{12}} + xz^3 u_{y_{14}}$$

$$u_z = u_{z_1} + x u_{z_2} + x^2 u_{z_4} + xz u_{z_5} + z^2 u_{z_6} + x^3 u_{z_7} + xz^2 u_{z_9} + x^4 u_{z_{11}} \tag{21}$$

Tables 7–9 present the reduced models for the remaining three geometries. The annular model (Table 9) required a smaller E_f since it was observed that the reduced model obtained with $E_f < 0.1$ percent was affected by an error equal to 8.11 percent on the bending mode. It has to be underlined how the symmetry of the annular cross-section makes the reduced beam models for bending symmetric. Table 10 shows the reduced models obtained by summing each of the three reduced models for a given cross-section, that is, the reduced models needed to detect all the modes considered. For instance, the beam model needed to detect first bending and torsional modes of an annular cross-section beam is

$$u_x = u_{x_1} + z u_{x_3} + x^2 u_{x_4} + xz u_{x_5} + z^2 u_{x_6} + x^3 z u_{x_{12}} + xz^3 u_{x_{14}}$$

Table 10
Combined reduced models for beams with different cross-sections.

<table>
<tr><th rowspan="2">Cross-Section</th><th rowspan="2">Reduced Model</th><th colspan="3">E_f %</th></tr>
<tr><th>Flexural z</th><th>Flexural x</th><th>Torsional</th></tr>
<tr><td></td><td>$M_e = 21$
▼ ▽ ▼ ▼ ▼ ▽ ▽ ▼ ▽ ▼ ▽ ▼ ▽ ▽ ▽
▽ ▼ ▼ ▽ ▼ ▽ ▼ ▽ ▽ ▽ ▽ ▼ ▽ ▼ ▽
▼ ▼ ▽ ▼ ▼ ▼ ▼ ▽ ▼ ▽ ▼ ▽ ▽ ▽ ▽</td><td>0.01</td><td>0.03</td><td>0.00</td></tr>
<tr><td></td><td>$M_e = 24$
● ○ ● ● ● ○ ○ ● ○ ● ○ ● ○ ● ●
○ ● ● ○ ● ○ ● ● ○ ● ○ ● ○ ● ○
● ● ○ ● ● ● ○ ○ ● ○ ● ○ ○ ○ ○</td><td>0.01</td><td>0.05</td><td>0.00</td></tr>
<tr><td></td><td>$M_e = 37$
▲ ▲ ▲ ▲ ▲ ▲ △ ▲ △ ▲ △ ▲ △ ▲ ▲
▲ ▲ ▲ ▲ ▲ ▲ ▲ ▲ △ ▲ ▲ ▲ ▲ ▲ ▲
▲ ▲ ▲ ▲ ▲ ▲ ▲ △ ▲ ▲ ▲ △ ▲ ▲ △</td><td>0.02</td><td>0.02</td><td>0.01</td></tr>
<tr><td></td><td>$M_e = 20$
■ □ ■ ■ ■ ■ □ □ □ □ □ ■ □ ■ □
□ ■ ■ □ □ □ ■ ■ ■ ■ □ □ □ □ □
■ ■ □ ■ ■ ■ □ □ □ □ □ ■ □ ■ □</td><td>0.04</td><td>0.05</td><td>0.00</td></tr>
</table>

$$u_y = xu_{y_2} + zu_{y_3} + x^3u_{y_7} + x^2zu_{y_8} + xz^2u_{y_9} + z^3u_{y_{10}}$$

$$u_z = u_{z_1} + xu_{z_2} + x^2u_{z_4} + xzu_{z_5} + z^2u_{z_6} + x^3zu_{z_{12}} + xz^3u_{z_{14}} \tag{22}$$

These results suggest the following:

1. It is confirmed that different sets of displacement variables are needed to detect different modes.
2. Thin walls and the asymmetry of the cross-section play a fundamental role in determining the number of terms needed to detect a given mode. In particular the asymmetry seems to be of primary importance.
3. As general guidelines, it can be stated that if asymmetric cross-section are considered full models should be adopted, whereas symmetric geometries should be analyzed by means of reduced models since large reductions of the computational costs are possible.

Table 11
Reduced models for a clamped-clamped rectangular beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
○	○	○	○	○	●	○	○	○	○	○	○	○	○	○	6/45	0.00
○	○	●	○	○	○	○	○	○	●	○	○	○	○	○		
●	○	○	●	○	●	○	○	○	○	○	○	○	○	○		
1^{st} Flexural x																
●	○	○	●	○	●	○	○	○	○	○	○	○	○	○	5/45	0.00
○	●	○	○	○	○	○	○	○	○	○	○	○	○	○		
○	○	○	○	●	○	○	○	○	○	○	○	○	○	○		
1^{st} Torsional																
●	○	●	○	○	○	○	○	○	○	○	○	○	○	○	8/45	0.02
○	●	●	○	○	○	○	○	○	○	○	●	○	●	○		
●	●	○	○	○	○	○	○	○	○	○	○	○	○	○		

Table 12
Reduced models for a hinged-hinged rectangular beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
△	△	△	△	△	▲	△	△	△	△	△	△	△	△	△	5/45	0.00
△	△	▲	△	△	△	△	△	△	△	△	△	△	△	△		
▲	△	△	▲	△	▲	△	△	△	△	△	△	△	△	△		
1^{st} Flexural x																
▲	△	△	▲	△	△	△	△	△	△	△	△	△	△	△	4/45	0.02
△	▲	△	△	△	△	△	△	△	△	△	△	△	△	△		
△	△	△	△	▲	△	△	△	△	△	△	△	△	△	△		
1^{st} Torsional																
△	△	▲	△	△	△	△	△	△	△	△	△	△	△	△	6/45	0.01
△	△	△	△	▲	△	△	△	△	△	△	▲	△	▲	△		
△	▲	△	△	△	△	△	△	▲	△	△	△	△	△	△		

4. As far as the present methodology is concerned, attention has to be paid to the proper choice of E_f since the combined effect of terms having small effects on the solution could not be negligible.

Table 13
Reduced models for a simply-supported rectangular beam.

Reduced Model															M_e/M_{tot}	E_f %
1^{st} Flexural z																
■	■	□	■	□	■	□	□	□	□	□	□	□	□	□		
■	■	□	■	□	■	■	□	■	□	□	□	■	□	□	13/45	0.02
□	□	■	□	■	□	□	□	□	□	□	□	□	□	□		
1^{st} Flexural x																
□	□	■	□	■	□	□	□	□	□	□	□	□	□	□		
□	□	■	□	□	□	□	□	□	■	□	□	□	□	□	8/45	0.00
■	■	□	■	□	■	□	□	□	□	□	□	□	□	□		
1^{st} Torsional																
□	□	■	□	□	□	□	□	□	□	□	□	□	□	□		
□	□	■	□	■	□	□	■	□	■	□	■	□	■	□	9/45	0.00
■	■	□	□	□	□	□	□	□	□	□	□	□	□	□		

Table 14
Combined reduced models for different BCs of the rectangular beam.

BCs	Reduced Model															E_f %		
																Flexural z	Flexural x	Torsional
	$M_e = 14$																	
	●	○	●	●	○	●	○	○	○	○	○	○	○	○	○			
	○	●	●	○	○	○	○	○	○	●	○	●	○	●	○	0.00	0.00	0.02
	●	●	○	●	●	●	○	○	○	○	○	○	○	○	○			
	$M_e = 15$																	
	▲	△	▲	▲	△	▲	△	△	△	△	△	△	△	△	△			
	△	▲	▲	△	▲	△	△	△	△	△	△	▲	△	▲	△	0.00	0.02	0.00
	▲	▲	△	▲	▲	▲	△	△	▲	△	△	△	△	△	△			
	$M_e = 25$																	
	■	■	■	■	■	■	□	□	□	□	□	□	□	□	□			
	■	■	■	■	■	■	■	■	■	■	□	■	■	■	□	0.02	0.00	0.00
	■	■	■	■	■	■	□	□	□	□	□	□	□	□	□			

4.3. Reduced models for different boundary conditions

This section is devoted to the analysis of different boundary conditions. The beam considered is rectangular and it has the same geometric and material characteristics of the one analyzed in the previous section of this paper. Three different boundary conditions were considered,

1. Clamped at both ends.
2. Hinged at both ends. This means that in both ends the displacement components along the cross-section coordinates (i.e. x and z) and the constant axial component (u_{y1}) term are imposed equal to zero.
3. Simply supported. This means that in one end all the displacement components along the cross-section coordinates and the constant axial component term are imposed equal to zero. In the other end, instead, the constant axial component term (u_{y1}) is free.

Tables 11-13 show the reduced models for the three modes considered for all the boundary conditions under investigation. Table 14 presents the combined models required to take into account all the boundary conditions. The following comments can be stated:

1. Boundary conditions play a significant role in the construction of the reduced models. This role is equivalent to that observed for the cross-section geometry.
2. Whereas clamped–clamped and hinged–hinged conditions require similar models, the simply supported condition needs the most cumbersome beam model.
3. In general, significant reductions in the number of generalized variables were obtained for all the boundary conditions considered.

5. Conclusions

This paper has presented the analysis of the effectiveness of higher-order displacement variables of beam models to compute natural modes of vibration. The method proposed is the so-called mixed axiomatic/asymptotic method that allows one to evaluate the role of each term on the solution. The systematic application of this method on each variable led to the development of reduced refined models composed only by the effective terms. The higher-order 1D structural models have been obtained by means of the Carrera Unified Formulation (CUF) which is a hierarchical modeling tool which can deal with arbitrary rich structural models.

Analyses have been conducted on isotropic structures and the effect of different parameters have been investigated. In particular, deep and thin-walled beams have been considered under different boundary conditions. For each problem, reduced models offering the same accuracy of a full fourth-order model have been obtained. The results obtained suggest the following:

1. Refined models offer important improvements in the accuracy of a beam model.
2. In most of the cases considered, important reductions in the number of the displacement variables were obtained. This means that the adoption of the proposed mixed axiomatic/asymptotic approach is a useful tool which preserves the accuracy benefits of higher-order models with significantly smaller computational costs.
3. General guidelines to construct refined models can be retrieved by using the present methodology. The cross-section asymmetry, for instance, represents a primary parameter to be taken into account.

Future investigations should be focused on the analysis of composite structures and the use of optimization techniques to obtain reduced structural models such as the genetic algorithm used for the static analysis of plates [55]. Furthermore, the effect of the choice of the error parameter (E_f) could be investigated in detail since it plays a significant role in the construction of the most appropriate reduced model for a given dynamic beam problem.

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