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# Toward Micromechanics of Coupled Fields Materials Containing Functionally Graded Inhomogeneities: Multi-Coating Approach

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**This article presents a micromechanics model for predicting the effective properties of coupled fields composite materials containing ellipsoidal multi-coated inhomogeneities. The model is based on Eshelby's theory of inclusion embedded in an equivalent continuum matrix. The main objective of this work is to build some numerical tools that will be useful for estimating the effective properties of composite materials containing functionally graded inhomogeneities. Two applications of the present micromechanics scheme to two-phase composite materials with electro-elastic and magneto-electro-elastic couplings are presented. The results are in a good agreement with experimental data obtained from the literature.**

**Keywords** micromechanics, composite materials, effective properties, coupled fields, multi-coated inhomogeneities

## 1. INTRODUCTION

Functional composite materials which present a coupling among different fields are widely used in many applications. For example, piezoelectric composites are employed in ultrasonic imaging devices, sensors, and actuators [1]. Such composites inherit the characteristics of functional materials, such as the piezoelectric and piezomagnetic properties, which can be tailored to meet specific applications. The coupling among different fields consists of the capacity to convert the energy stored in the material from one type to another. This effect appears clearly from the constitutive equations.

Researchers have developed many techniques to evaluate the effective properties of composites, initially only for the mechanical properties and then extending the theories to coupled fields properties. An example of such techniques can be found in Bravo-Castillero and Rodriguez-Ramos [2], where the asymptotic homogenization based on a two-scale expansion is developed for thermo-magneto-electro-elastic materials. This mathematically rigorous approach, most of the time, needs a numerical solution in order to obtain the desired homogenized properties. A totally different homogenization approach is presented in Tan and Tong [3], where the hypothesis of uniformity of the fields inside each constituent of the composite material allows the determination of rather simply explicit analytical expression for the equivalent properties. In Aboudi [4], a general homogenization micromechanical method is employed for multiphase thermo-magneto-electro-elastic materials and results are compared with the generalized method of cells and with the Mori-Tanaka predictions [5]. Another useful approach for predicting the effective properties of heterogeneous materials is the statistical continuum approach that takes into account the distribution, shape, and orientation of the composite constituents through correlation functions [6–11].

The micromechanical model presented in this article belongs to the family of the Eshelby tensor-based homogenization techniques. In the literature, there are many techniques of this family that take into account the coupled fields properties. For example, in Li [12] both the Mori-Tanaka and the Self Consistent approaches are considered. The original contribution of the present micromechanics model is the capability to consider inhomogeneities with any number of coatings. The multi-coating homogenization approach has been applied to elastic composite materials [13, 14] and viscoelastic materials [15]. Recently, the multi-coating micromechanics scheme has been applied to predict the effective thermo-electro-elastic properties of piezoelectric composite materials [16].

The article is organized as follows. In section 2., the micromechanics model is presented. A general notation is adopted and the two steps of the homogenization process are described in detail. Section 3. consists in two applications of the proposed homogenization technique to the electro-elastic and

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magneto-electro-elastic effective properties. The numerical effective properties are compared with experimental results from Chan and Unsworth [17].

## 2. MULTI-COATING APPROACH FOR COUPLED FIELDS PROBLEMS

The objective of this section is to define a general framework to compute the effective properties of composite materials containing ellipsoidal multi-coated inhomogeneities with coupled behaviors, such as magneto-electro-elastic (MEE) coupling. A general notation for describing the constitutive behaviors is presented, followed by the description of the two main steps of all micromechanics schemes: (i) localization, which determines the relationship between the microscopic (local) fields and the macroscopic (global) loading, and (ii) homogenization, which employs averaging techniques to approximate the macroscopic behavior.

### 2.1. Constitutive Behavior

The general notation presented in Benveniste and Milton [18] (see also Milton [19], Chapter 6) to represent general linear $n$-coupled field phenomena like magneto-electro-elastic is hereafter used. So, we define $n$ divergence free vector fields, $\mathbf{J}_i(\mathbf{r})$ $(1 \le i \le n)$, and $n$ curl free vector fields, $\mathbf{E}_i(\mathbf{r})$ $(1 \le i \le n)$, that are linked through the following constitutive relation:

$$J_{i\alpha}(\mathbf{r}) = L_{i\alpha j\beta}(\mathbf{r}) E_{j\beta}(\mathbf{r}), \tag{1}$$

where $\alpha$ and $\beta$ are fields indexes assuming the values 1 to $n$, whereas $i$ and $j$ are space indexes assuming the values 1, 2, and 3.

To illustrate this notation, let us consider the constitutive equations of magneto-electro-elastic coupling problem,

$$\begin{cases} \sigma_{il} = C_{ikjl} u_{l,j} + e_{jik}\phi_{,j} + q_{jik}\chi_{,j}, \\ D_i = e_{ilj} u_{l,j} - \kappa_{ij}\phi_{,j} - \lambda_{ij}\chi_{,j}, \\ B_i = q_{ilj} u_{l,j} - \lambda_{ij}\phi_{,j} - \xi_{ij}\chi_{,j}, \end{cases} \tag{2}$$

where $\sigma_{il}$ are the components of the stress tensor; $D_i$ and $B_i$ are, respectively, the components of the electric displacement and magnetic flux vectors; $e_{jik}$ and $q_{jik}$ are, respectively, the components of the piezo-electric and piezo-magnetic properties third order tensors. The elastic properties are denoted by the components, $C_{ijkl}$, of the fourth order stiffness tensor. The second order tensors represented by their components, $\kappa_{ij}$, $\xi_{ij}$, and $\lambda_{ij}$, are the dielectric properties, magnetic permeabilities, and magneto-electric coefficients, respectively. $u_i$, $\phi$, and $\chi$ are the displacement field, electric potential, and magnetic potential, respectively. In the preceding equations and what follows, the comma denotes partial differentiation.

Equation (2) can be rewritten as a coupled field's phenomenon with 5 coupled fields defined as follows:

$$\mathbf{J}_i = \begin{pmatrix} \sigma_{1i} \\ \sigma_{2i} \\ \sigma_{3i} \end{pmatrix} \quad \text{for} \quad i = 1, 2, 3,$$
$$\mathbf{J}_4 = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}, \quad \mathbf{J}_5 = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{3}$$

and

$$\boldsymbol{E}_i = \begin{pmatrix} u_{i,1} \\ u_{i,2} \\ u_{i,3} \end{pmatrix} \quad \text{for} \quad i = 1, 2, 3,$$
$$\boldsymbol{E}_4 = \begin{pmatrix} \phi_{,1} \\ \phi_{,2} \\ \phi_{,3} \end{pmatrix}, \quad \boldsymbol{E}_5 = \begin{pmatrix} \chi_{,1} \\ \chi_{,2} \\ \chi_{,3} \end{pmatrix}. \tag{4}$$

Recall that, by their above definition, the **J**-fields are divergence free and the **E**-fields are curl free. With the following generalized vector field

$$\boldsymbol{U} = \{u_1, u_2, u_3, \phi, \chi\}^\top, \tag{5}$$

and, Eq. (2) can be written in the notation of Eq. (1) as

$$J_{i\alpha} = L_{i\alpha j\beta} E_{j\beta}, \quad E_{j\beta} = U_{\beta,j} \quad \text{with} \quad \alpha, \beta = 1, 2, 3, 4, 5 \quad i, j = 1, 2, 3, \tag{6}$$

where

$$L_{i\alpha j\beta} = \begin{cases} C_{i\alpha j\beta}, & i, \alpha, j, \beta = 1, 2, 3, \\ e_{ji\alpha}, & i, j, \alpha = 1, 2, 3, \quad \beta = 4, \\ q_{ji\alpha}, & i, j, \alpha = 1, 2, 3, \quad \beta = 5, \\ e_{i\beta j}, & i, j, \beta = 1, 2, 3, \quad \alpha = 4, \\ q_{i\beta j}, & i, j, \beta = 1, 2, 3, \quad \alpha = 5, \\ -\kappa_{ij}, & i, j = 1, 2, 3, \quad \alpha = \beta = 4 \\ -\xi_{ij}, & i, j = 1, 2, 3, \quad \alpha = \beta = 5 \\ -\lambda_{ij}, & i, j = 1, 2, 3, \quad \alpha = 4, \beta = 5 \\ -\lambda_{ij}, & i, j = 1, 2, 3, \quad \alpha = 5, \beta = 4. \end{cases} \tag{7}$$

One can represent general $n$-coupled fields problems by [18]

$$\begin{cases} J_{i\alpha} = L_{i\alpha j\beta} E_{j\beta}, \\ E_{j\beta} = U_{\beta,j}, \ \text{with } \alpha, \beta = 1, 2, \cdots, n, \ \text{and } i, j = 1, 2, 3, \\ J_{i\alpha,i} = 0, \end{cases} \tag{8}$$

where $U_\beta$ are the "potentials" and $J_{i\alpha}$ are the "fluxes." In the rest of this article, Roman indices range from 1 to 3 and Greek indices range from 1 to $n$, unless otherwise specified. The symmetry properties for $L_{i\alpha j\beta}$ can be deduced from those of its components, such as $C_{ijkl}$, $e_{ijk}$, $q_{ijk}$, $\kappa_{ij}$, $\xi_{ij}$, and $\lambda_{ij}$ in the above case of magneto-electro-elastic coupling.

Let's consider now a multi-coated inhomogeneity with the core and $(N-1)$ coatings (see Figure 1), whose properties are described in the general setting of Eq. (8) by $\mathbf{L}^1, \mathbf{L}^2, \ldots, \mathbf{L}^N$, respectively. This composite inhomogeneity (denoted here by I) is embedded in an external reference medium with properties denoted by $\mathbf{L}^0$. The reference medium is subjected on its boundary to a potential field, $U_\alpha^\infty$. The main objective of this work is to determine the effective properties, $\mathbf{L}^{eff}$, of this composite material. To this end, the two steps of any micromechanical model to evaluate the effective properties are presented in the following.

## 2.2. Integral Equation and Localization Step

Zeller and Dederichs [20] have proposed to model the composite material shown in Figure 1 as an homogeneous material whose properties vary spatially, that is

$$\mathbf{L}(\mathbf{r}) = \mathbf{L}^0 + \delta\mathbf{L}(\mathbf{r}), \tag{9}$$

where $\mathbf{r} \in V$, $V$ is the volume of the homogeneous medium, $\delta\mathbf{L}(\mathbf{r})$ is the first order spatial variations of the properties. Using the symmetry properties of $L_{i\alpha j\beta}$ in the divergence equation of the "fluxes"

$$J_{i\alpha,i} = 0, \tag{10}$$

one gets the following fundamental equation

$$L^0_{i\alpha j\beta} U_{\beta,ij} = \left(-\delta L_{i\alpha j\beta} E_{j\beta}\right)_{,i}, \tag{11}$$

that can be solved by Green's formalism for the unknowns "potentials," $U_\beta(\mathbf{r})$. The gradient, $E_{i\alpha}(\mathbf{r})$, of $U_\alpha(\mathbf{r})$ is then given as

$$E_{i\alpha}(\mathbf{r}) = E^\infty_{i\alpha} - \int_V \Gamma^0_{i\alpha j\beta}(\mathbf{r}-\mathbf{r}')\delta L_{j\beta k\gamma}(\mathbf{r}')\mathbf{E}_{k\gamma}(\mathbf{r}')d\mathbf{r}', \tag{12}$$

where $\mathbf{\Gamma}^0(\mathbf{r}-\mathbf{r'})$ is the modified Green's pseudo-tensor for this $n-$coupled fields problem. In the case of MEE coupling, the explicit form of $E_{i\alpha}(\mathbf{r})$ is:

$$E_{i\alpha}(\mathbf{r}) = \begin{cases} \varepsilon_{i\alpha} = \frac{1}{2}(u_{i,\alpha} + u_{\alpha,i}), & i, \alpha = 1, 2, 3, \\ \phi_{,i}, & i = 1, 2, 3, \ \ \alpha = 4, \\ \chi_{,i}, & i = 1, 2, 3, \ \ \alpha = 5. \end{cases} \tag{13}$$

Note that, in Eq. (12), the superscript "0" denotes that the Green's pseudo-tensor is computed using the properties, $\mathbf{L}^0$, of the reference medium. The first order spatial variations of the properties, $\delta\mathbf{L}(\mathbf{r})$, is defined as [15]:

$$\delta\mathbf{L}(\mathbf{r}) = \sum_{k=1}^{N} \Delta\mathbf{L}^{(k/0)}\theta_k(\mathbf{r}), \quad \text{with} \quad \Delta\mathbf{L}^{(k/0)} = \left(\mathbf{L}^k - \mathbf{L}^0\right), \tag{14}$$

where the characteristic function, $\theta_k(\mathbf{r})$, of the phase, $k$, occupying the volume, $V_k$, is given by

$$\theta_k(\mathbf{r}) = \begin{cases} 1 & \forall\mathbf{r} \in V_k \\ 0 & \forall\mathbf{r} \notin V_k \end{cases}, \quad \text{with } k \in \{0, 1, 2, \ldots, N\}. \tag{15}$$

For what follows, $V_I$ denotes the volume of the composite inhomogeneity, I, consists of the inhomogeneity and $(N-1)$ coatings and $\varphi_k$ represents the volume fraction of the $k$th phase, such that

$$V_I = \sum_{k=1}^{N} V_k \quad \text{and} \quad \varphi_k = \frac{V_k}{V_I}, \quad k \in \{1, 2, \ldots, N\}. \tag{16}$$

The average value, $\overline{\mathbf{E}}^I$, of $\mathbf{E}(\mathbf{r})$ in the composite inhomogeneity, I, is defined by

$$\overline{\mathbf{E}}^I = \frac{1}{V_I}\int_{V_I} \mathbf{E}(\mathbf{r})d\mathbf{r} = \mathbf{E}^\infty - \mathbf{T}^I(\mathbf{L}^0) : \tau^I, \tag{17}$$

where

$$\begin{cases} \mathbf{T}^I(\mathbf{L}^0) = \dfrac{1}{V_I}\displaystyle\int_{V_I}\int_{V_I} \mathbf{\Gamma}^0(\mathbf{r}-\mathbf{r}')d\mathbf{r}d\mathbf{r}', \\ \boldsymbol{\tau}^I = \sum_{k=1}^{N} \varphi_k \Delta\mathbf{L}^{(k/0)} : \overline{\mathbf{E}}^k, \\ \overline{\mathbf{E}}^k = \dfrac{1}{V_k}\displaystyle\int_{V_k} \mathbf{E}(\mathbf{r})d\mathbf{r}. \end{cases} \tag{18}$$

Let's consider now the localization pseudo-tensors, $\mathbf{A}^I$ and $\mathbf{a}^k$, defined by

$$\overline{\mathbf{E}}^k = a^k : \overline{\mathbf{E}}^I \quad \text{and} \quad \overline{\mathbf{E}}^I = A^I : \mathbf{E}^\infty. \tag{19}$$

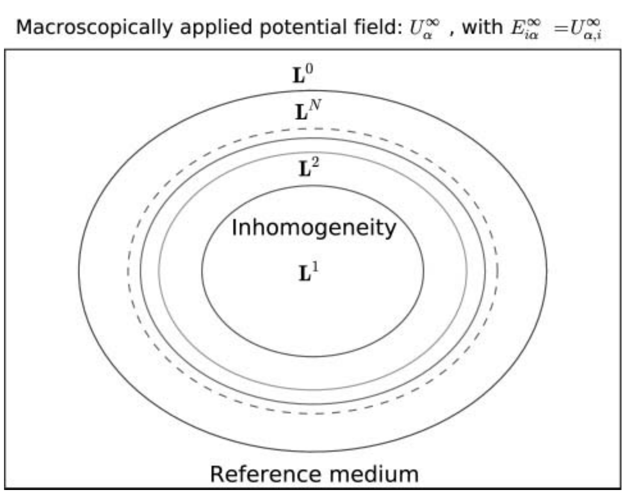


FIG. 1. Topology of the multi-coated inhomogeneity problem. $U_\alpha^\infty$ represents the macroscopically applied potential field and $E^\infty_{i\alpha} = U^\infty_{\alpha,i}$ derives from the potential field.

Making use of Eq. (19) in Eq. (17), one gets

$$\mathbf{A}^I = \left[\mathbf{I}^4 + \mathbf{T}^I(\mathbf{L}^0) : \left(\sum_{k=1}^{N} \varphi_k \Delta\mathbf{L}^{(k/0)} : \mathbf{a}^k\right)\right]^{-1}, \qquad (20)$$

where $\mathbf{I}^4$ is the shorthand notation for the fourth order and the second order identity tensors. To complete the localization step, the local localization pseudo-tensors, $\mathbf{a}^k$, must be known. To this end, let's define the global localization pseudo-tensor, $\mathbf{A}^k$, in the $k$th phase as

$$\overline{E}^k = \mathbf{A}^k : \mathbf{E}^\infty. \qquad (21)$$

It follows from Eq. (19) that

$$\mathbf{A}^k = \mathbf{a}^k : \mathbf{A}^\mathrm{I}. \qquad (22)$$

From the following averaging relation

$$\overline{\mathbf{E}}^\mathrm{I} = \left\langle \overline{\mathbf{E}}^k \right\rangle = \sum_{k=1}^{N} \varphi_k \overline{\mathbf{E}}^k, \qquad (23)$$

one gets the system of equations to solve for $\mathbf{a}^k$ as follows:

$$\left\langle \mathbf{a}^k \right\rangle = \sum_{k=1}^{N} \varphi_k \mathbf{a}^k = \mathbf{I}^4. \qquad (24)$$

Here, the notation, $\langle \boldsymbol{\zeta} \rangle$, denotes the average value of the quantity, $\boldsymbol{\zeta}$, over the whole volume of the composite inhomogeneity, I. Equation (24) constitutes the solution of the posed problem given as function of the unknown $N$ local localization pseudo-tensors, $\mathbf{a}^k$, which can be determined if one takes into account the boundary conditions through the different interfaces in the composite inhomogeneity. Interfacial operators [21] are a very convenient mathematical tool that efficiently calculates the jump of "fluxes" or "potentials" across a material interface (an interface separating two dissimilar materials). These operators are derived by writing the equations for the continuity of "fluxes" and "potentials" across the material interface (hypothesis of perfect interface). The derivation begins with the general case of two solid phases, $k$ and $(k+1)$, with the properties, $\mathbf{L}^k$ and $\mathbf{L}^{k+1}$, separated by a surface with unit normal, $\mathbf{N}$, directed from phase $k$ to phase $(k+1)$.

Using the properties of these two phases, the jump of $\mathbf{E}(\mathbf{r})$ across the material interface is given as follows [21]:

$$E_{i\alpha}^{k+1}(\mathbf{r}) - E_{i\alpha}^{k}(\mathbf{r}) = P_{i\alpha j\beta}^{k+1}\left(L_{j\beta k\gamma}^{k} - L_{j\beta k\gamma}^{k+1}\right)E_{k\gamma}^{k}(\mathbf{r}). \qquad (25)$$

The generalized interfacial operator, $P_{i\alpha j\beta}^{k+1}$, depends only on the constituent material's properties and the unit normal vector of the interface. For the following, let's precise some notations

$$\Omega_j = \bigcup_{k=1}^{j} V_k \quad \text{and} \quad \Delta\mathbf{L}^{(p/q)} = \mathbf{L}^p - \mathbf{L}^q. \qquad (26)$$

Substituting $\mathbf{E}^k(\mathbf{r})$ by the average value, $\overline{\mathbf{E}}^{\Omega_k}$, of $\mathbf{E}(\mathbf{r})$ over the volume, $\Omega_k$, and taking the average, $\overline{\mathbf{E}}^{k+1}$, of $\mathbf{E}(\mathbf{r})$ over the volume, $V_{k+1}$, Eq. (25) yields

$$\overline{\mathbf{E}}^{k+1} = \overline{\mathbf{E}}^{\Omega_k} + \mathbf{T}^{k+1}\left(\mathbf{L}^{k+1}\right) : \Delta\mathbf{L}^{(\Omega_k/k+1)} : \overline{\mathbf{E}}^{\Omega_k}, \qquad (27)$$

where $\mathbf{L}^{\Omega_k}$ is the property of the composite formed by the volume, $\Omega_k$. The expressions of $\mathbf{T}^{k+1}$ and $\overline{\mathbf{E}}^{\Omega_k}$ are given as

$$\mathbf{T}^{k+1}\left(\mathbf{L}^{k+1}\right) = \frac{1}{V_{k+1}} \int_{V_{k+1}} \mathbf{P}^{k+1} \mathrm{d}\mathbf{r}, \qquad (28)$$

$$\overline{\mathbf{E}}^{\Omega_k} = \sum_{i=1}^{k} \frac{V_i}{\Omega_k} \overline{\mathbf{E}}^i = \frac{\sum_{i=1}^{k} \varphi_i \overline{\mathbf{E}}^i}{\sum_{i=1}^{k} \varphi_i}. \qquad (29)$$

It is shown in Cherkaoui et al. [22] that

$$\mathbf{T}^{k+1}\left(\mathbf{L}^{k+1}\right) = \mathbf{T}^{\Omega_k}(\mathbf{L}^{k+1}) - \frac{\sum_{i=1}^{k} \varphi_i}{\varphi_{k+1}}[\mathbf{T}^{\Omega_{k+1}}(\mathbf{L}^{k+1}) - \mathbf{T}^{\Omega_k}(\mathbf{L}^{k+1})], \qquad (30)$$

where

$$\mathbf{T}^{\Omega_p}\left(\mathbf{L}^q\right) = \frac{1}{\Omega_p} \int_{\Omega_p} \int_{\Omega_p} \boldsymbol{\Gamma}\left(\mathbf{L}^q\right)(\mathbf{r} - \mathbf{r}')\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{r}'. \qquad (31)$$

Since $\mathbf{T}^{\Omega_p}(\mathbf{L}^q)$ are not size-dependent but shape dependent, it is obvious that in the specific case of homothetic inhomogeneities, one has

$$\mathbf{T}^{k+1}\left(\mathbf{L}^{k+1}\right) = \mathbf{T}^{\Omega_k}\left(\mathbf{L}^{k+1}\right) = \mathbf{T}^{\Omega_{k+1}}\left(\mathbf{L}^{k+1}\right). \qquad (32)$$

By some algebraic manipulations, one can show that $\Delta\mathbf{L}^{(\Omega_k/k+1)} : \overline{\mathbf{E}}^{\Omega_k}$ in Eq. (27) is given by

$$\Delta\mathbf{L}^{(\Omega_k/k+1)} : \overline{\mathbf{E}}^{\Omega_k} = \frac{\sum_{j=1}^{k} \varphi_j \Delta\mathbf{L}^{(j/k+1)} : \overline{\mathbf{E}}^j}{\sum_{j=1}^{k} \varphi_j}. \qquad (33)$$

Next, let's introduce the pseudo-tensors, $\boldsymbol{\Pi}^k$, defined by

$$\mathbf{a}^k = \boldsymbol{\Pi}^k : \mathbf{a}^1. \qquad (34)$$

Making use of Eqs. (27), (29), and (33), the pseudo-tensors, $\mathbf{\Pi}^k$, are given by the following recurrent relation

$$\begin{cases} \mathbf{\Pi}^1 = \mathbf{I}^4, \\ \mathbf{\Pi}^k = \dfrac{\sum_{j=1}^{k-1} \left(\varphi_j \boldsymbol{\vartheta}^{(j/k)} : \mathbf{\Pi}^j\right)}{\sum_{j=1}^{k-1} \varphi_j}, \end{cases} \tag{35}$$

where the pseudo-tensor, $\boldsymbol{\vartheta}^{(j/k)}$, is defined as

$$\boldsymbol{\vartheta}^{(j/k)} = \mathbf{I}^4 + \mathbf{T}^k\left(\mathbf{L}^k\right) : \Delta\mathbf{L}^{(j/k)}. \tag{36}$$

Finally, Eqs. (24) and (34) give

$$\mathbf{a}^1 = \left(\sum_{k=1}^{N} \varphi_k \mathbf{\Pi}^k\right)^{-1}. \tag{37}$$

The localization step of this micromechanics model is definitively complete once the pseudo-tensor, $\mathbf{a}^1$, is known.

To sum up, the main objective of section 2.2. is to compute the localization pseudo-tensors, $\mathbf{A}^k$, defined by the formula (21). The steps to get $\mathbf{A}^k$ are summarized as follows:

- From the properties, $\mathbf{L}^k$, of each phase, compute $\mathbf{T}^k(\mathbf{L}^k)$ using Eq. (30);
- Compute $\boldsymbol{\vartheta}^{(j/k)}$ from Eq. (36);
- Compute $\mathbf{\Pi}^k$ from Eqs. (35);
- Compute $\mathbf{a}^1$ from Eq. (37) and get $\mathbf{a}^k$ from Eq. (34);
- Compute $\mathbf{A}^{\mathrm{I}}$ from Eq. (20) and finally get $\mathbf{A}^k$ from Eq. (22).

Note also that $\mathbf{T}^{\Omega_p}(\mathbf{L}^q)$ in Eq. (30) can be obtained from Eshelby's pseudo-tensor, $\mathbf{S}^{\Omega_p}(\mathbf{L}^q)$, as

$$\mathbf{T}^{\Omega_p}\left(\mathbf{L}^q\right) = \mathbf{S}^{\Omega_p}\left(\mathbf{L}^q\right) : (\mathbf{L}^q)^{-1}. \tag{38}$$

Expressions for the components of $\mathbf{S}^{\Omega_p}(\mathbf{L}^q)$ in the case of magneto-electro-elastic coupling and ellipsoidal family shape of inhomogeneities can be found in [23].

### 2.3. Homogenization Step and Effective Properties

The effective properties pseudo-tensor, $\mathbf{L}^{\text{eff}}$, is related to the macroscopic fields, $\mathbf{J}^\infty$ and $\mathbf{E}^\infty$, as

$$\mathbf{J}^\infty = \mathbf{L}^{\text{eff}} : \mathbf{E}^\infty. \tag{39}$$

Making use of the relations

$$\begin{cases} \mathbf{J}^\infty = \sum_{k=0}^{N} \varphi_k \overline{\mathbf{J}}^k, \\ \mathbf{E}^\infty = \sum_{k=0}^{N} \varphi_k \overline{\mathbf{E}}^k, \\ \overline{\mathbf{J}}^k = \mathbf{L}^k : \overline{\mathbf{E}}^k, \\ \overline{\mathbf{E}}^k = \mathbf{A}^k : \overline{\mathbf{E}}^\infty, \end{cases} \tag{40}$$

to get Eq. (39), one obtains

$$\mathbf{L}^{\text{eff}} = \sum_{k=0}^{N} \varphi_k \mathbf{L}^k : \mathbf{A}^k. \tag{41}$$

If the reference medium is assumed to be the effective medium and the $N$th coating is a shell of the host material, then the present micromechanics scheme can be referred to as the generalized self-consistent scheme. In this case, Eq. (20) becomes

$$\mathbf{A}^{\mathrm{I}} = \left[\mathbf{I}^4 + \mathbf{T}^{\mathrm{I}}(\mathbf{L}^{\text{eff}}) : \left(\sum_{k=1}^{n} \varphi_k \Delta\mathbf{L}^{(k/\text{eff})} : \mathbf{a}^k\right)\right]^{-1}. \tag{42}$$

Thus, Eq. (41) is a set of nonlinear equations (since the localization pseudo-tensors, $\mathbf{A}^k$, are functions of $\mathbf{L}^{\text{eff}}$) to solve for the effective properties, $\mathbf{L}^{\text{eff}}$.

## 3. NUMERICAL RESULTS FOR ELECTRO-ELASTIC AND MAGNETO-ELECTRO-ELASTIC COUPLINGS

Two different sets of material properties are considered. First, the model is applied to two-phase composite material with electro-elastic coupling. The electro-elastic properties of the constituent materials are shown in Table 1.

Figure 2 shows the effective strain piezo-electric coefficient, $d_{33}$, of Epoxy matrix containing piezo-electric ellipsoidal

TABLE 1
Electro-elastic material properties ($C_{ij}$ in GPa, $e_{ij}$ in C m$^{-2}$, $\kappa_0 = 8.854 \times 10^{-12}$ F/m.)

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{33}$ | $C_{44}$ | $e_{31}$ | $e_{33}$ | $e_{15}$ | $\kappa_{11}/\kappa_0$ | $\kappa_{33}/\kappa_0$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Epoxy | 8 | 4.4 | 4.4 | 8 | 1.8 | 0 | 0 | 0 | 4.2 | 4.2 |
| $BaTiO_3$ | 150 | 66 | 66 | 146 | 4.4 | −4.3 | 17.5 | 11.4 | 1115 | 1260 |
| PZT-4 | 139 | 77.8 | 74.3 | 115 | 25.6 | −5.2 | 15.1 | 12.7 | 730 | 645 |
| PZT-5 | 121 | 75.4 | 75.2 | 111 | 21.1 | −5.4 | 15.8 | 12.3 | 916 | 830 |
| PZT-7A | 148 | 76.2 | 74.2 | 131 | 25.4 | −2.1 | 12.3 | 9.2 | 460 | 235 |

TABLE 2
Magneto-electro-elastic material properties ($C_{ij}$ in GPa, $e_{ij}$ in C m$^{-2}$, $\kappa_{ij}$ in F/m, $q_{ij}$ in N/Am, $\xi_{ij}$ in N s$^2$/C$^2$)

| | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{33}$ | $C_{44}$ |
|---|---|---|---|---|---|
| $BaTiO_3$ | 166 | 77 | 78 | 162 | 43 |
| $CoFe_2O_4$ | 286 | 173 | 170 | 269.5 | 45.3 |
| | $e_{31}$ | $e_{33}$ | $e_{15}$ | $\kappa_{11}$ | $\kappa_{33}$ |
| $BaTiO_3$ | 11.6 | -4.4 | 18.6 | $11.2\times10^{-9}$ | $12.6\times10^{-9}$ |
| $CoFe_2O_4$ | 0 | 0 | 0 | $0.08\times10^{-9}$ | $0.093\times10^{-9}$ |
| | $q_{31}$ | $q_{33}$ | $q_{15}$ | $\xi_{11}$ | $\xi_{33}$ |
| $BaTiO_3$ | 0 | 0 | 0 | $5\times10^{-6}$ | $10\times10^{-6}$ |
| $CoFe_2O_4$ | 550 | 580.3 | 699.7 | $-590\times10^{-6}$ | $157\times10^{-6}$ |



shaped inhomogeneities. The semi-axis of the inhomogeneity are denoted $a$, $b$, and $c$. The results are obtained for the case where $a = b$ and $c/a = 1000$. The results are in good agreement with the experimental results from [17]. The effective $d_{33}$ value with 40% volume fraction of inclusions is around 85% of that of the pure piezoelectric material. This saturation effect is characteristic of high aspect ratios $c/a$ of the ellipsoidal inclusion. In the case of unitary or low aspect ratios the saturation effect is absent: with high volume fraction of inclusions, the effective piezoelectric properties are much lower than that of the pure piezoelectric material.

The second set of material properties takes into account the magneto-electro-elastic coupling. A two-phase composite material is addressed. The magneto-electro-elastic properties of the constituent materials are given in Table 2.

The magneto-electric coefficients, $\lambda_{ij}$, of the phases are not reported in Table 2 because they are null. This is not the case for the composite itself [24]. Figure 3 shows the effective magneto-electric coefficient, $\lambda_{33}$, of $CoFe_2O_4$ matrix containing $BaTiO_3$ ellipsoidal-shaped inhomogeneities. The results are obtained for $a = b$ and $c/a = 1000$. The peak of the curve is around 50% volume fraction, i.e., in the case in which the piezoelectric material ($BaTiO_3$) and piezomagnetic material ($CoFe_2O_4$) have the same volume fraction, in accordance with results presented in [24]. $\lambda_{11}$ and $\lambda_{22}$ are not shown because their value is negligible with respect to $\lambda_{33}$ for this particular geometry.

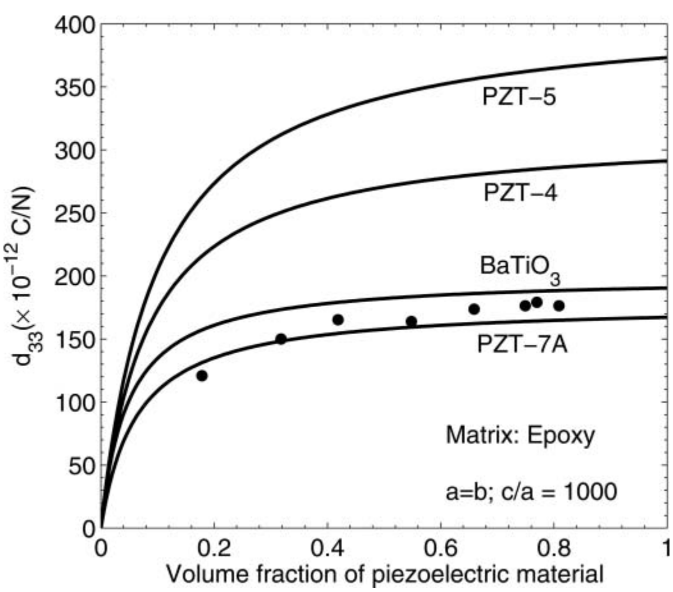


FIG. 2. Effective $d_{33}$ calculated with different inhomogeneity material properties. Results obtained with the proposed micromechanical model in solid lines (-), experimental results from [17] for PZT-7A inclusions in dots (•).

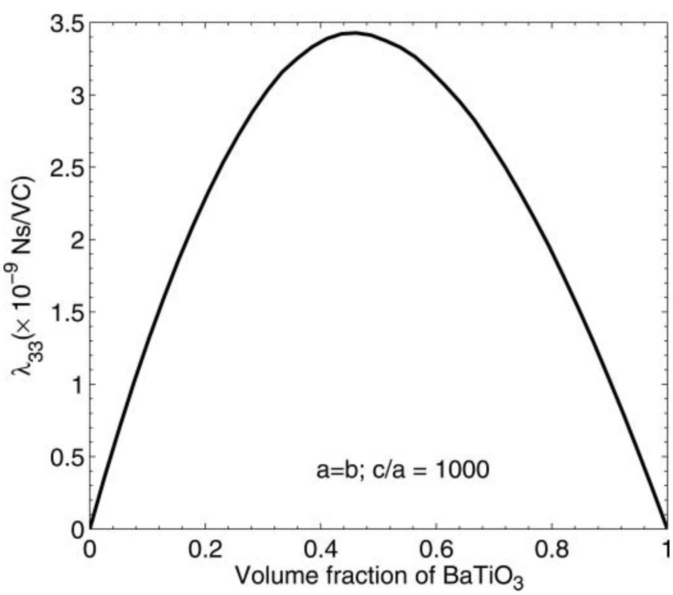


FIG. 3. Effective magneto-electric coupling coefficient, $\lambda_{33}$. $CoFe_2O_4$ matrix containing $BaTiO_3$ inhomogeneities.

## 4. CONCLUSIONS

This work is an attempt to generalize the multi-coating approach for the effective properties of coupled fields composite materials. The results obtained with this micromechanics model for electro-elastic coupling are in good agreement with experimental data. Application to magneto-electro-elastic coupling correctly predicts the existence in the composite of the magneto-electric moduli, which are null in the constituent materials. The model proposed here has a wide range of applications thanks to its multi-coating capability and thanks to the wide set of properties considered. The next step will be the application of this formalism to compute the effective properties of composite materials containing ellipsoidal-shaped functionally graded inhomogeneities. Further works may address design issues like problems of optimization of the characteristics of the inhomogeneities to reach some desired effective properties for materials by design purposes.

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