Evaluation of various through the thickness and curvature approximations in free vibration analysis of cylindrical composites shells

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Abstract: This paper presents a comparison of various, significant shell theories to evaluate the free vibration response of multi-layered, orthotropic cylindrical shells. Carrera unified formulation for the modelling of composite spherical shell structures is adopted. Via this approach, higher order, zig-zag, layer-wise and mixed theories can be easily formulated. As a particular case, the equations related to Love's approximations and Donnell's approximations and as well as of the corresponding classical lamination and shear deformation theories (CLT and FSDT) are derived. The governing differential equations of the dynamic problem are presented in a compact general form. These equations are solved via a Navier-type, closed form solution. Thin and moderately thick as well as shallow and deep shells are investigated. Several parametric analyses are carried out depending on the stacking sequences of laminates, on the degree of orthotropic ratio and the thickness and on the curvature parameters. Conclusions are drawn with respect to the accuracy of the theories for the considered lay-outs and geometrical parameters.

Keywords: cylindrical shells; multilayer composites; free vibration; refined theories; Love's theory; Donnell's theory.

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1 Introduction

The development of appropriate shell theories is a well established topic in structural analysis of shell panels that are used to build significant structural parts of automotive, ship and aerospace vehicles, and shell-made civil constructions. The analysis of their vibration characteristics is a fundamental topic at the early design stage. Classical theories (classical lamination theory, CLT) developed for thin elastic shells are based on Love-Kirchhoff's (Love, 1888) assumptions:

1 the shell is thin
2 the deflections of the shell are small
3 normal stresses that are perpendicular to the middle surface can be neglected in comparison with other stresses
4 straight lines that are normal to the undeformed middle surface remain straight and normal to the deformed middle surface.

The last assumption leads to neglect the transverse shear strains. Over the last years, curved shell structures made of composite laminae have gained widespread acceptance for primary structural components due to high value of strength- and stiffness-to-weight ratios. Love-Kirchhoff's kinematic assumptions applied to layered anisotropic composite shells may not yield a correct prediction of displacement and stress fields. An elastic shell theory in which the thinness assumption is delayed has independently been derived by Flügge (1934), Lur'E (1940) and Byrne (1944). The introduction of transverse shear and normal stress represents an improvement to classical theories. The effects of transverse shear deformations (SDT – shear deformation theory) and normal stresses have been considered by Hildebrand et al. (1949) and Reissner (1952). Non-linear theories were considered by Sanders (1963). Moreover, other approximations on curvature terms for shallow shell analysis have been introduced by Donnell (1933) and Mushtari (1938). A survey of various classical shell theories can be found in the works of Naghdi (1956) and Leissa (1973).

The open literature offers several works concerning free vibration analysis of shell. Some of these are reviewed in the following text. A review of the recent researches (2000–2009) done on the dynamic behaviour of composites shells is presented by Qatu (2010). This review includes about 200 references that are organised according to the following criteria: 1 – shell theories, 2 – shell geometries, 3 – type of dynamic analysis, 4 – material complexity, and 5 – structural complexity. Laminated composite deep thick shells are investigated in the works of Qatu (1995, 1999). In Qatu (1999), accurate stress resultant equations are derived including initial pre-twist and, further, the term $(1 + z/R)$. Free vibration analysis yielded frequencies that are close to those obtained by 3-D theory of elasticity. Homogeneous and composite thick barrel shells were investigated by Qatu (2004a). Several natural frequencies analyses are carried out, taking into account the curvature and thickness effects. Reddy and Liu (1985) investigated the statics and the free vibrations of shallow thick cylindrical and spherical shells made of orthotropic layers via a cubic through-the-thickness approximation of the displacement components laying on shell surface and a constant transverse displacement. Governing equations were solved through a Navier-type solution. Chaudhuri and Kabir (1994) used four classical shallow shell theories – Donnel, Sanders, Reissner and modified Sanders – to obtain Fourier series solutions for cross-ply doubly curved panels with simply supported boundary conditions. They extended CLT-based analytical solutions for cross-ply curved panels to other types of boundary conditions. A boundary-discontinuous double Fourier series approach was used to solve a system of three partial differential equations (one fourth-order and two second-order, in terms of the transverse displacement). In the work of Ferreira et al. (2006), Reddy's higher-order shear deformation theory of laminated orthotropic elastic shells was implemented through a multiquadrics discretisation of equations of motion and boundary conditions. Static and free vibration analyses of doubly curved laminated elastic shells were carried out by the use of third-order theory in combination with a meshless technique based on the multiquadric radial basis function (RBF) method. Another mesh-free approach for vibration analysis of laminated composite cylindrical panels was presented by Zhao and Liew (Zhao et al., 2003).

Some studies of vibration response of composite shells included the presence of a cutout. In the work of Poore et al. (2008), a semi-analytical solution method is presented for determining the natural frequencies and mode shapes of laminated cylindrical shells containing a circular cutout. Ram and Babu (2002) investigated the free vibration of composite spherical shell caps with and without a cutout. The analysis is carried out using the finite element method based on a higher-order shear deformation theory that accounts for rotary inertia and parabolic variation of transverse shear strain across the thickness. The transverse displacement of the shell is assumed constant through the thickness. Narasimhan and Alwar (1992) studied the free vibration of orthotropic annular spherical shells with clamped boundary conditions at both the edges using Chebyshev-Galerkin spectral method. Fundamental frequencies were presented for cross-ply laminated shells with fibers oriented in circumferential and meridian directions. Xavier et al. (1995) modelled the vibration of thick orthotropic laminated composite shells using a simple higher-order layer-wise (LW) theory. The theory accounts for a cubic variation of both the in-plane displacements and the transverse shear stresses within each layer. They defined the general performance index (GPI) as a measure of the natural frequency and stress predicting capability of a theory. Composite cylindrical shells and their applications concerning free vibration analysis have been analysed by Yadav and Verma (2001), while Liew et al. presented a three-dimensional vibration analysis for spherical shells subjected to different boundary conditions in Liew et al. (2002). Matsunaga (2007) presented a two-dimensional global higher-order theory of cross-ply laminated composite circular cylindrical shells, able to accurately predict natural frequencies and buckling stresses. The effects of both shear deformations with thickness changes and rotatory inertia were considered.

Approximated three-dimensional solutions can be obtained assuming that the ratio between the panel thickness and its middle surface radii is negligible as compared to unity. Bhimaraddi (1991) analysed the free vibration of homogeneous and laminated doubly curved shells on rectangular planform and made of orthotropic material using the three-dimensional elasticity equations. Ye and Soldatos (1994) studied the three-dimensional flexural vibration response of laminated cylindrical shell panels of symmetric and antisymmetric cross-ply material. Recently, static and free vibration characteristics of anisotropic laminated cylindrical shell were analysed applying the differential quadrature method (DQM) by Alibeigloo (2009).

Nowadays, the use of composite materials is well established since these materials exhibit high transverse shear deformation and discontinuous material properties in the thickness direction. Both of these features require the development of refined theories (Carrera, 2001, 2003a for an accurate and effective design. Soldatos (1987) presented a good survey of the theories adopted in the dynamic analysis of composite laminated shells, while a review of equivalent-single-layer and layerwise laminate theories is presented by Reddy (1993). In the first survey, governing equations and numerical results are quoted for Donnell's, Love's, Sanders' and Flügge's theories based on CLT approximations, and for Donnell's, Love's and Sanders' theories based on SDT approximations. In Carrera (1991), the first author investigated of CLT and SDT assumptions and Donnell, Love and Flügge theories on buckling and vibrations of cross-ply laminated composite shells. More recent analyses are reported in the articles by Qatu (2002a, 2002b) and more extensively in the book by the same author (Qatu, 2004b) that documents some of the latest research in the field of vibration of composite shells and plates, presenting also deep thick shells. To the best of the authors' knowledge, no exhaustive results are known in which the approximation related to refined models, including transverse normal strain effects, are compared with those introduced by curvature (Love, Donnell).

Over the last decade the first author has developed a unified formulation (CUF, Carrera's unified formulation) (Carrera, 1998a, 2003b) that allows formulating several two-dimensional models on the basis of the choice of the a-priori main unknowns (displacements or mixed models), the approximation level (laminate or lamina level), the through-the-thickness polynomial approximation order. As a result, an exhaustive variable kinematic model has been obtained: models that account for the transverse normal and shear deformability, the continuity of the transverse stress components and the zig-zag variation along the thickness of displacement and transverse normal stresses can be formulated straightforwardly. The use of the refined theory has made it possible to conduct a quite comprehensive analysis of the thickness locking phenomenon (also known as Poisson locking) in the bending and vibration of metallic shells (Carrera and Brischetto, 2008a). The present work aims at evaluating the effect of various through the thickness and curvature approximations in multilayered orthotropic composite shells. It is a companion research of a previous one proposed in the same journal (Carrera et al., 2010) that was restricted to shells made by isotropic materials. Refined theories with up to fourth-order displacement field for both in-plane and transverse displacements, and two approaches for modelling variables (ESL, equivalent single layer and LW) are compared to evaluate the free vibration response of cylindrical and spherical shells. The CUF is employed to derive shell equations that are solved for the case of simply supported boundary conditions and doubly curved shells with constant curvatures. Navier-type closed form solution are obtained. Love's and Donnell's approximations are compared in the framework of higher order theories (HOTs) and classical ones.

2 Geometry

Shells are bi-dimensional structures with one dimension, in general the thickness along z direction, negligible with respect to the other two on the reference surface directions. The main features of shell geometry are shown in Figure 1. A laminated shell composed of N_l layers is considered. The integer k, used as superscript or subscript, indicates each layer starting from the shell bottom. The layer geometry is denoted by the same symbols as those used for the whole multilayered shell and vice-versa. α_k and β_k are the curvilinear orthogonal coordinates (coinciding with lines of principal curvature) on the layer reference surface Ω_k (middle surface of the k-layer). z_k denotes the rectilinear coordinate measured along the normal direction to Ω_k. The following relations hold in the orthogonal system of coordinates above described:

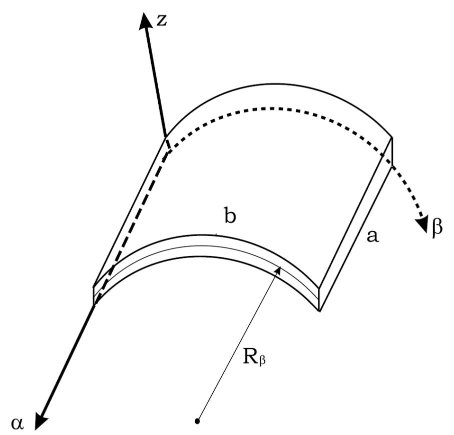
$$\begin{aligned} ds_k^2 &= H_\alpha^k d\alpha^2 + H_\beta^k d\beta^2 + H_z^k dz^2 \\ d\Omega_k &= H_\alpha^k H_\beta^k d\alpha_k d\beta_k \\ dV &= H_\alpha^k H_\beta^k H_z^k d\alpha_k d\beta_k dz_k \end{aligned} \tag{1}$$

where ds_k^2 is the square of line element, $d\Omega_k$ is the area of an infinitesimal rectangle on Ω_k and dV is an infinitesimal volume. Here

$$\begin{aligned} H_\alpha^k &= A^k \left(1 + z_k/R_\alpha^k\right) \\ H_\beta^k &= A^k \left(1 + z_k/R_\beta^k\right), H_z^k = 1. \end{aligned} \tag{2}$$

R_α^k and R_β^k are the radii of curvature along the two in-plane directions α_k and β_k respectively. A^k and B^k are the coefficients of the first fundamental form of Ω_k (Kraus, 1967). For shells with constant curvature these coefficients are equal to unity.

Figure 1 Geometry and reference system for cylindrical shell



3 Overview of the considered shell theories

Since a large variety of two-dimensional theories can be formulated on the basis of different kinematic assumptions, it may be useful to recall some details of the theories considered in this paper.

3.1 Classical theories

3.1.1 Classical lamination theory

Shells' CLT is based on Love-Kirchhoff's kinematic assumptions (Love, 1888). The relative displacements model can be written as follow:

$$\begin{aligned} u_q(\alpha,\beta,z) &= u_{q0}(\alpha,\beta) - z u_{z0,q} \quad q=\alpha,\beta \\ u_z(\alpha,\beta,z) &= u_{z0}(\alpha,\beta) \end{aligned} \tag{3}$$

that is, normals to the reference surface remain normal, straight and unstrained after deformation. Subscript '0' denotes the variable value in correspondence to the reference surface Ω. Subscripts preceded by comma represent spatial derivation. The transverse shear and the through-the-thickness deformations are discarded. The corresponding stresses can be obtained 'a posteriori' upon integration of the indefinite equilibrium equations, see Carrera (2000a). Poisson's locking is corrected via the assumption of reduced stiffness coefficients in Hooke's law as derived from the assumption of a plane stress state (see Carrera and Brischetto, 2008b).

3.1.2 First order shear deformation theory

Mindlin (1951) postulated a kinematic field that accounts for constant transverse shear strain components along the thickness, whereas the normal deformation is neglected. Such a model is known as first order shear deformation theory (FSDT):

$$\begin{aligned} u_q(\alpha,\beta,z) &= u_{q0}(\alpha,\beta) + z u_{q1}(\alpha,\beta) \quad q=\alpha,\ \beta \\ u_z(\alpha,\beta,z) &= u_{z0}(\alpha,\beta) \end{aligned} \tag{4}$$

As for CLT, more accurate results can be obtained upon 'a posteriori' integration of the indefinite equilibrium equations.

3.2 Higher order theories

According to Koiter (1959) statement: "... a refinement of Love's first approximation theory is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time", HOTs can be formulated adopting the following expansion for displacements variables **u**:

$$\begin{aligned} & u_q(\alpha,\beta,z) = u_{q0}(\alpha,\beta) + z^r u_{qr}(\alpha,\beta) \\ & q=\alpha,\beta,z \ \ r=1,2,\dots,N \end{aligned} \tag{5}$$

The summing convention for repeated indexes has been adopted. N is the order of expansion and it is a free parameter. In this paper, an approximation order as high as four is considered. A N-order theory based upon equation (5) is addressed as 'EDN'. Letter

'E' denotes that the kinematic is preserved for the whole layers of the shell, as in the so-called ESL approach. 'D' indicates that only displacement unknowns are used. 'N' stands for the expansion order of the through-the-thickness polynomial approximation. For instance, the displacement field of an ED3 model is:

$$\begin{aligned} u_\alpha &= u_{\alpha 0} + z u_{\alpha 1} + z^2 u_{\alpha 2} + z^3 u_{\alpha 3} \\ u_\beta &= u_{\beta 0} + z u_{\beta 1} + z^2 u_{\beta 2} + z^3 u_{\beta 3} \\ u_z &= u_{z0} + z u_{z1} + z^2 u_{z2} + z^3 u_{z3} \end{aligned} \tag{6}$$

This theory accounts for a parabolic and cubic variation along the thickness of transverse normal and shear strains, respectively. If transverse normal strain is discarded the previous theory becomes:

$$\begin{aligned} &u_q(\alpha,\beta,z) = u_{q0}(\alpha,\beta) + z^r u_{qr}(\alpha,\beta) \\ &q = \alpha,\beta \;\; r = 1,2,\ldots,N \\ &u_z(\alpha,\beta,z) = u_{z0}(\alpha,\beta) \end{aligned} \tag{7}$$

and is denoted as 'EDNd'.

3.2.1 HOTs including zig-zag effect

Laminate structures are characterised by a change in slope of displacements and transverse normal and shear stresses at layer interfaces. These quantities are C^0 class function of the transverse coordinate. EDN and EDNd models are based on C^∞ functions in z and, therefore, are intrinsically incapable to describe this zig-zag variation. This latter can be accounted for within an ESL approach via Murakami's function $M(\zeta_k)$ (see Murakami, 1986; Carrera, 2004):

$$M(\zeta_k) = (-1)^k \zeta_k \tag{8}$$

where k counts the laminae, h_k stands for the thickness of a k-layer and $\zeta_k = z_k/2h_k$ such that $-1 \le \zeta_k \le 1$, being z_k a k-layer local coordinate. The following properties hold:

a $M(\zeta_k)$ is a piece-wise linear function of the local coordinate z_k

b its slope assumes opposite sign between two adjacent layers.

The kinematic field including Murakami's function is:

$$\begin{aligned} &u_q(\alpha,\beta,z) = u_{q0}(\alpha,\beta) + z^r u_{qr}(\alpha,\beta) + (-1)^k \zeta_k u_{qN+1}(\alpha,\beta) \\ &q = \alpha,\beta,z \;\; r = 1,\ 2,\ \ldots,\ N \end{aligned} \tag{9}$$

A model based on Murakami's function is addressed as 'EDZN'. Other possible manners of modelling the zig-zag variation are reported in Carrera (2003).

3.3 LW theories

Multi-layered shells can be analysed by independent kinematic assumptions for each layer. According to Reddy (2004), this approach is stated as LW. In order to satisfy the compatibility of the displacement field, a MacLaurin's expansion across the thickness, typical of ESL models, is not convenient. Interface values should be rather assumed as unknown variables. The following expansion is, therefore, adopted:

$$u_q^k = F_t u_{qt}^k + F_b u_{qb}^k + F_r u_{qr}^k \quad q = \alpha, \beta, z \;\; r = 2, 3, \ldots, N \;\; k = 1, 2, \ldots, N_l \tag{10}$$

N_l represents the total number of layers. Subscripts 't' and 'b' denote values evaluated at top and bottom surface of a k-layer, respectively. The thickness functions F_t, F_b and F_r depend on ζ_k. They are defined as follows:

$$F_t = \frac{P_0 + P_1}{2} \quad F_b = \frac{P_0 - P_1}{2} \quad F_r = P_r - P_{r-2} \quad r = 2, 3, .., N \tag{11}$$

$P_j = P_j(\zeta_k)$ is a Legendre's polynomials of order j. The first four Legendre's polynomials are:

$$P_0 = 1 \quad P_1 = \zeta_k \quad P_2 = \frac{3\zeta_k^2 - 1}{2} \quad P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2} \quad P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8} \tag{12}$$

The following properties hold:

$$\zeta_k = 1: \quad F_t = 1,\; F_b = 0,\; F_r = 0 \qquad \zeta_k = -1: F_t = 0,\; F_b = 1,\; F_r = 0 \tag{13}$$

Top and bottom displacements of each lamina are assumed as unknown variable. Interlaminar compatibility of displacements can be easily linked:

$$u_{qt}^k = u_{qb}^{(k+1)} \quad q = \alpha, \beta, z \quad k = 1, 2, \ldots, N_l - 1 \tag{14}$$

The acronym used for these theories is 'LDN', where 'L' stands for the LW approach. For all the models that has been previously described, the governing equations and the boundary conditions are derived via the principle of virtual displacement (PVD) in Section 4.

3.4 Mixed theories based on Reissner's mixed variational theorem

The kinematics described previously does not satisfy the interlaminar continuity of transverse shear and normal stresses. It can be fulfilled 'a priori' assuming transverse shear and normal stresses together with displacements as primary variables by means of Reissner's mixed variational theorem (Reissner, 1984, 1986) (RMVT). Transverse stresses $\sigma_{\alpha z}^k$, $\sigma_{\beta z}^k$ and $\sigma_{\alpha z}^k$ are approximated via the same model as that addressed in equation (10):

$$\sigma_{qz}^k = F_t \sigma_{qzt}^k + F_b \sigma_{qzb}^k + F_r \sigma_{qzr}^k \quad q = \alpha, \beta, z \quad r = 2, 3, \ldots, N \quad k = 1, 2, \ldots, N_l \tag{15}$$

The interlaminar continuity is imposed straightforwardly:

$$\sigma_{qzt}^{k} = \sigma_{qzb}^{(k+1)} \quad q = \alpha, \beta, z \quad k = 1, 2, \ldots, N_l - 1 \tag{16}$$

This group of models is denoted by 'LMN'. 'M' means mixed models based on RMVT.

3.5 Carrera unified formulation

CUF permits several two dimensional models to be obtained for shells, thanks to the separation of the unknown variables into a set of thickness functions only depending on the thickness coordinate z, and the correspondent unknowns depending on the in-plane coordinates (α, β). In force of that the considered theories can be all unified considering that CLT and FSDT are a peculiar case of ESL higher order models. These latter models can be regarded as a particular case of LW models in which the number of layers is equal to one and the through-the-thickness polynomial approximation is performed via the classical base $\{z^r : \ r = 0, \ 1, \ \ldots, N\}$. In the case of EDZN models, Murakami's function is also considered. equations (3), (4), (5), (9) and (10) can be unified into the following compact notation:

$$\begin{array}{l} u_q^k = F_\tau u_{q\tau}^k \quad q = \alpha, \beta, z \quad \tau = t, \ b, \ r, \\ \sigma_{qz}^k = F_\tau \sigma_{qz\tau}^k \ r = 2, \ 3, \ \ldots, \ N \\ k = 1, \ 2, \ \ldots, \ N_l \end{array} \tag{17}$$

The governing equations are derived according to the chosen variational statement (either PVD or RMVT) in a general way that does not depend upon the approximation approach (ESL or LW) and the polynomial expansion order.

4 Governing equations

The displacement approach is formulated in terms of $\mathbf{u}^k$ by variational imposing the equilibrium via PVD. In the dynamic case, this establishes:

$$\begin{array}{l} \sum\limits_{k=1}^{N_l} \int\limits_{\Omega_k} \int\limits_{h_k} \left(\delta \boldsymbol{\epsilon}_p^{kT} \boldsymbol{\sigma}_p^k + \delta \boldsymbol{\epsilon}_n^{kT} \boldsymbol{\sigma}_n^k \right) \ dz_k \ d\Omega_k \\ = \sum\limits_{k=1}^{N_l} \int\limits_{\Omega_k} \int\limits_{h_k} \rho^k \delta \mathbf{u}^k \ddot{\mathbf{u}}^k dV \end{array} \tag{18}$$

$$\begin{array}{ll} \boldsymbol{\epsilon}_p = \begin{Bmatrix} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \epsilon_{\alpha\beta} \end{Bmatrix} & \boldsymbol{\epsilon}_n = \begin{Bmatrix} \epsilon_{\alpha z} \\ \epsilon_{\beta z} \\ \epsilon_{zz} \end{Bmatrix} \\ \boldsymbol{\sigma}_p = \begin{Bmatrix} \sigma_{\alpha\alpha} \\ \sigma_{\beta\beta} \\ \sigma_{\alpha\beta} \end{Bmatrix} & \boldsymbol{\sigma}_n = \begin{Bmatrix} \sigma_{\alpha z} \\ \sigma_{\beta z} \\ \sigma_{zz} \end{Bmatrix} \end{array} \tag{19}$$

'T' as superscript stands for the transposition operator. δ signifies virtual variations and ρ^k denotes mass density. The variation of the internal work has been split into in-plane and out-of-plane parts and involves the stress obtained from Hooke's Law and the strain from the geometrical relations. Geometrical relations link strains $\boldsymbol{\epsilon}$ and displacements **u**. Strains are conveniently grouped into in-plane and normal components denoted by the subscripts p and n, respectively. The geometric relations are:

$$\begin{aligned}\boldsymbol{\epsilon}_p^k &= \mathbf{D}_p\mathbf{u}^k + \mathbf{A}_p\mathbf{u}^k \\ \boldsymbol{\epsilon}_n^k &= \mathbf{D}_{n\Omega}\mathbf{u}^k + \lambda_D\mathbf{A}_n\mathbf{u}^k + \mathbf{D}_{nz}\mathbf{u}^k\end{aligned} \tag{20}$$

in which $\mathbf{D}_p$, $\mathbf{D}_{n\Omega}$, and $\mathbf{D}_{nz}$ are differential matrix operators and $\mathbf{A}_p$ and $\mathbf{A}_n$ are geometrical terms accounting for the through-the-thickness variation of the curvature:

$$\mathbf{D}_p = \begin{bmatrix} \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} & 0 & 0 \\ 0 & \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} & 0 \\ \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} & \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} & 0 \end{bmatrix}, \quad \mathbf{A}_p = \begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k R_\alpha^k} \\ 0 & 0 & \frac{1}{H_\beta^k R_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \frac{1}{H_\alpha^k}\frac{\partial}{\partial\alpha} \\ 0 & 0 & \frac{1}{H_\beta^k}\frac{\partial}{\partial\beta} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_n = \begin{bmatrix} -\frac{1}{H_\alpha^k R_\alpha^k} & 0 & 0 \\ 0 & -\frac{1}{H_\beta^k R_\beta^k} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$\mathbf{D}_{nz} = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix}$$

H_α^k and H_β^k account for the change in length of a k-layer segment due to the curvature. λ_D is a trace operator, it has been introduced to identify terms that are neglected in the Donnell-type shallow shell theories.

The aim of this work is to evaluate the effect of a second class of shell theories derived from approximations on curvature terms in the strain-displacement relations. It can be shown, see Kraus (1967) that by putting $\lambda_D = 0$, the shell curvature becomes that of the corresponding plate as it is in Donnell- type approximation theory. Donnell's approximation is strongly related to the geometrical parameter a/R_α and b/R_β.

Love's shell theory is instead related to the following approximation for the coefficients of the second fundamental form [see equation (1)] of the shell:

$$H_\alpha = H_\beta = 1 \tag{22}$$

It appears evident that Love's approximation is related to the shell parameter h/R_α and h/R_β, where h is the shell thickness.

In the case of linear elastic material, stresses and strains are related via Hooke's generalised law:

$$\begin{aligned}\boldsymbol{\sigma}_p^k &= \widetilde{\mathbf{C}}_{pp}^k\boldsymbol{\epsilon}_p^k + \widetilde{\mathbf{C}}_{pn}^k\boldsymbol{\epsilon}_n^k \\ \boldsymbol{\sigma}_n^k &= \widetilde{\mathbf{C}}_{np}^k\boldsymbol{\epsilon}_p^k + \widetilde{\mathbf{C}}_{nn}^k\boldsymbol{\epsilon}_n^k\end{aligned} \tag{23}$$

Terms $\widetilde{\mathbf{C}}^k_{pp}$, $\widetilde{\mathbf{C}}^k_{pn}$, $\widetilde{\mathbf{C}}^k_{np}$ and $\widetilde{\mathbf{C}}^k_{nn}$ are the material stiffness matrices for a k-layer in the global reference system, see Carrera (2003). By replacing equations (23), (20) and the unified displacement field in equation (17). into equation (18), PVD reads:

$$\begin{aligned}
&\sum_{k=1}^{N_l}\int_{\Omega_k}\delta\mathbf{u}_\tau^{kT}\int_{h_k}\{(-F_\tau\mathbf{D}_p^T+F_\tau\mathbf{A}_p^T)[\widetilde{\mathbf{C}}^k_{pp}(F_s\mathbf{D}_p\\
&+F_s\mathbf{A}_p)+\widetilde{\mathbf{C}}^k_{pn}(F_s\mathbf{D}_{n\Omega}+F_s\mathbf{A}_n+F_{s,z})]\\
&+(-F_\tau\mathbf{D}_{n\Omega}^T+F_\tau\mathbf{A}_n^T+F_{\tau,z})[\widetilde{\mathbf{C}}^k_{np}(F_s\mathbf{D}_p+F_s\mathbf{A}_p)\\
&+\widetilde{\mathbf{C}}^k_{nn}(F_s\mathbf{D}_{n\Omega}+F_s\mathbf{A}_n+F_{s,z})]\}H_\alpha^k H_\beta^k dz_k\mathbf{u}_s^k\ d\Omega_k\\
&+\sum_{k=1}^{N_l}\int_{\Gamma_k}\delta\mathbf{u}_\tau^{kT}\int_{h_k}\{F_\tau\mathbf{I}_p^T[\widetilde{\mathbf{C}}^k_{pp}(F_s\mathbf{D}_p+F_s\mathbf{A}_p)\\
&+\widetilde{\mathbf{C}}^k_{pn}(F_s\mathbf{D}_{n\Omega}+F_s\mathbf{A}_n+F_{s,z})]\\
&+F_\tau\mathbf{I}_{n\Omega}^T[\widetilde{\mathbf{C}}^k_{np}(F_s\mathbf{D}_p+F_s\mathbf{A}_p)+\ \widetilde{\mathbf{C}}^k_{nn}(F_s\mathbf{D}_{n\Omega}\\
&+F_s\mathbf{A}_n+F_{s,z})]\}H_\alpha^k H_\beta^k dz_k\mathbf{u}_s^k\ d\Gamma_k\\
&=\sum_{k=1}^{N_l}\int_{\Omega_k}\delta\mathbf{u}_\tau^{kT}\rho^k F_\tau F_s\ddot{\mathbf{u}}^k
\end{aligned}\tag{24}$$

being:

$$\mathbf{I}_p=\begin{bmatrix}\frac{1}{H_\alpha^k} & 0 & 0\\ 0 & \frac{1}{H_\beta^k} & 0\\ \frac{1}{H_\beta^k} & \frac{1}{H_\alpha^k} & 0\end{bmatrix}\qquad \mathbf{I}_{n\Omega}=\begin{bmatrix}0 & 0 & \frac{1}{H_\alpha^k}\\ 0 & 0 & \frac{1}{H_\beta^k}\\ 0 & 0 & 0\end{bmatrix}\tag{25}$$

By assigning the definition of virtual variations for the unknown displacement variables, the differential system of governing equations and related boundary conditions for the N_l k-layers in each Ω_k domain are found. The equilibrium and compatibility equations are:

$$\mathbf{K}_d^{k\tau s}\mathbf{u}_s^k=\mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_s^k\tag{26}$$

with boundary conditions

$$\begin{aligned}
&\mathbf{u}_\tau^k=\overline{\mathbf{u}}_\tau^k && \text{geometrical on } \Gamma_k^g\\
&\textstyle\prod_d^{k\tau s}\mathbf{u}_s^k=\prod_d^{k\tau s}\overline{\mathbf{u}}_s^k && \text{mechanical on } \Gamma_k^m
\end{aligned}\tag{27}$$

Differential stiffness, inertia and mechanical boundary conditions matrices are:

$$\begin{aligned}
\mathbf{K}_d^{k\tau s}=&\int_{h_k}\{(-F_\tau\mathbf{D}_p^T+F_\tau\mathbf{A}_p^T)[\widetilde{\mathbf{C}}^k_{pp}(F_s\mathbf{D}_p\\
&+F_s\mathbf{A}_p)+\widetilde{\mathbf{C}}^k_{pn}(F_s\mathbf{D}_{n\Omega}+F_s\mathbf{A}_n+\ F_{s,z})]\\
&+(-F_\tau\mathbf{D}_{n\Omega}^T+F_\tau\mathbf{A}_n^T+F_{\tau,z})[\widetilde{\mathbf{C}}^k_{np}(F_s\mathbf{D}_p\\
&+F_s\mathbf{A}_p)+\widetilde{\mathbf{C}}^k_{nn}(F_s\mathbf{D}_{n\Omega}+\ F_s\mathbf{A}_n+F_{s,z})]\}H_\alpha^k H_\beta^k dz_k
\end{aligned}\tag{28}$$

$$\mathbf{M}^{k\tau s} = \int_{h_k} \rho^k F_\tau F_s \mathbf{I} H_\alpha^k H_\beta^k dz_k \tag{29}$$

$$\begin{aligned} \textstyle\prod_d^{k\tau s} = & \int_{h_k} \{F_\tau \mathbf{I}_p^T [\widetilde{\mathbf{C}}_{pp}^k (F_s \mathbf{D}_p + F_s \mathbf{A}_p) \\ & + \widetilde{\mathbf{C}}_{pn}^k (F_s \mathbf{D}_{n\Omega} + F_s \mathbf{A}_n + F_{s,z})] \\ & + F_\tau \mathbf{I}_{n\Omega}^T [\widetilde{\mathbf{C}}_{np}^k (F_s \mathbf{D}_p + F_s \mathbf{A}_p) \\ & + \widetilde{\mathbf{C}}_{nn}^k (F_s \mathbf{D}_{n\Omega} + F_s \mathbf{A}_n + F_{s,z})]\} H_\alpha^k H_\beta^k dz_k \end{aligned} \tag{30}$$

I is the unit array. Equation (26) is solved via a Navier-type solution upon assumption of the following harmonic form for the unknown displacements:

$$\begin{aligned} \left(u_\alpha^k\right) &= \sum_{m=1}^{\overline{m}} \sum_{n=1}^{\overline{n}} \left(U_\alpha^k(z)\right) \cos\left(\frac{m\pi}{a}\alpha\right) \sin\left(\frac{n\pi}{b}\beta\right) e^{i\omega_{mn}t} \\ \left(u_\beta^k\right) &= \sum_{m=1}^{\overline{m}} \sum_{n=1}^{\overline{n}} \left(U_\beta^k(z)\right) \sin\left(\frac{m\pi}{a}\alpha\right) \cos\left(\frac{n\pi}{b}\beta\right) e^{i\omega_{mn}t} \\ \left(u_z^k\right) &= \sum_{m=1}^{\overline{m}} \sum_{n=1}^{\overline{n}} \left(U_z^k(z)\right) \sin\left(\frac{m\pi}{a}\alpha\right) \sin\left(\frac{n\pi}{b}\beta\right) e^{i\omega_{mn}t} \end{aligned} \tag{31}$$

where a and b are the lengths of the shell along the two coordinates α and β. m and n represent the number of half-waves in α and β direction, respectively. These numbers characterise the vibration mode associated to the circular frequency ω_{mn}. $i = \sqrt{-1}$ is the imaginary unit and t the time. Capital letters indicate maximal amplitudes. These assumptions correspond to the simply-supported boundary conditions. Upon substitution of equation (31), the governing equations assume the form of a linear system of algebraic equations in the time domain:

$$\boldsymbol{K}^* \hat{\boldsymbol{U}} = -\omega_{mn}^2 \boldsymbol{M} \hat{\boldsymbol{U}} \tag{32}$$

where $\boldsymbol{K}^*$ is the equivalent stiffness matrix obtained by means of static condensation (for further details see Carrera (1998b, 2000), $\boldsymbol{M}$ is the inertial matrix and $\hat{\boldsymbol{U}}$ is the vector of unknown variables. By defining $\lambda_{mn} = -\omega_{mn}^2$, the solution of the associated eigenvalue problem becomes:

$$||\mathbf{K}^* - \lambda_{mn}\mathbf{M}|| = 0 \tag{33}$$

The eigenvectors $\hat{\boldsymbol{U}}$ associated to the eigenvalues λ_{mn} (or to circular frequencies ω_{mn}) define the vibration modes of the structure in terms of primary variables. Once the wave numbers (m, n) have been defined in the in-plane directions, the number of obtained frequencies becomes equal to the degrees of freedom of the employed two-dimensional model. It is possible to obtain the relative eigenvector, in terms of primary variables, for each value of frequency, in order to plot the modes in the thickness direction.

5 Results

The two-dimensional theories described above have been applied to the free vibration analysis of multilayered, simply supported, composites shell. Fundamental frequencies are considered and results are compared in the framework of the various theories as well as Love's and Donnell's approximations. Parametric analysis have been carried out since the behaviour of laminated composite shells made of high-modulus and low-density materials are strongly dependent of the degree of orthotropy of the individual layer, the stacking sequence of laminates and the thickness parameter of the shells. Unless otherwise specified, the mechanical material properties of the lamina are those used by Pagano (1969): $E_L = 25E_T,\ G_{LT} = 0.5E_T,\ G_{TT} = 0.2E_T,\ \nu_{LT} = \nu_{TT} = 0.25$, $\rho = 1.0$, $\chi = 1.0$. Subscript 'L' stands for direction parallel to the fibers, 'T' identifies the transverse direction, ν_{LT} is the major Poisson ratio, ρ denotes the density, and χ the shear correction factor. $\chi = 5/6$ is adopted in $FSDT^{5/6}$ theory. The fiber orientation of the different laminae alternate between 0 and 90 deg with respect to the α-axis. Both symmetric and antisymmetric laminations with respect to the middle surface are considered. Cylindrical shell geometry have been studied using the side-to-thickness parameter $a/h = 10$, whereas the other geometric features are specified in each case. All the numerical results are shown as the dimensionless quantity:

$$\bar{\omega} = \omega \times a^2 \sqrt{\rho/h^2 E_T} \tag{34}$$

The analysis were carried out in order to investigate the effects on the natural frequencies and the accuracy of the CUF two-dimensional models with respect to Love's and Donnell's approximation theories.

5.1 Assessment

Firstly, an assessment of the present solutions with results from literature is presented. A comparison with the exact solution by Ye and Soldatos (1994) has been provided in Table 1. The superscript accompanying exact numerical results in Table 1 indicates the circumferential wave number, n, for which the fundamental frequency was detected. In the case that corresponding results values were obtained for the same value of n such a superscript is omitted. A three-layered, moderately thick cylindrical ringed shell $[0/90_{\mathtt{2t}}/0]$ has been considered, where the subscript $\mathtt{2t}$ means $h_1 = h_3 = h_2/2$ (h_i is the thickness of each lamina, $i = 1, 2, 3$). Mixed theories solutions match the exact one and accordance is still verified even for the lowest order theory LM1. Furthermore, LW4 accuracy is confirmed. Concerning ESLM, it should be noted that the higher-order theory ED4 can lead to poorer results than EDZ3 and EDZ2. Because of the different continuity conditions of displacement components at the interface between the layers the LW theories results always more accurate then the equivalent single-layer theories. Fine results with respect to standard classical displacement formulation are found. Refined theories lead to lower values of $\overline{\omega}$ according to the related reduction in stiffness. A second assessment analysis is carried out considering a two layer [90/0] cylindrical shell. The fundamental frequency parameter $\overline{\omega}$ is shown in Table 2 and compared with corresponding results by Ye and Soldatos (1994), Bhimaraddi (1991) and Qatu (2004a). The shell has equal axial and circumferential lengths and 0 deg outer layer. Similar consideration to that made for Table 1 about the accuracy of the proposed theories are valid also in this case.

Table 1 Comparison of present analysis with available reference solution for [0/90$_{2t}$/0] cylindrical shell

R_β/a	*5*	*10*	*50*	*100*
Exact	10.305^{14}	10.027^{22}	9.834^{24}	9.815^{2}
Present analysis				
$LM4$	10.305	10.027	9.834	9.815
$LM3$	10.305	10.027	9.834	9.815
$LM2$	10.306	10.027	9.835	9.816
$LM1$	10.324	10.046	9.855	9.836
$LD4$	10.305	10.027	9.834	9.815
$LD3$	10.305	10.027	9.834	9.815
$LD2$	10.307	10.028	9.835	9.816
$LD1$	10.368	10.091	9.899	9.880
$ED4$	10.453	10.178	9.987	9.969
$ED3$	10.453	10.179	9.988	9.969
$ED2$	11.291	11.040	10.86	10.84
$ED1$	11.294	11.043	10.86	10.85
$EDZ3$	10.307	10.030	9.837	9.819
$EDZ2$	10.367	10.090	9.898	9.879
$EDZ1$	10.383	10.104	9.908	9.890
$FSDT^{5/6}$	10.958	10.698	10.51	10.50
$FSDT$	11.295	11.044	10.86	10.85
CLT	13.708	13.507	13.35	13.33

Notes: Values of $\bar{\omega}$ for $a/h = 10$, $m = 1$, n given in superscripts

5.2 Influence of the degree of orthotropy

Fundamental frequencies parameters computed via the CUF two-dimensional models are presented in Table 3 for several values of the degree of orthotropy of the single layer. E_L/E_T is considered to be as low as 3 and as high as 40. A symmetric [0/90/0] cross-ply laminated composite cylindrical shell is considered, with equal thickness of each lamina and $\gamma = \pi/3$. A deep moderately thick shell and a shallow thin one are considered, depending on the radii-to-thickness parameter $R_\beta/h = 4$ or $R_\beta/h = 100$, respectively. Considering the CUF-two dimensional models, similar consideration made for the assessment can be done also for each analysis once the parameters are fixed. Results for $R_\beta/h = 100$ and $E_L/E_T = 3$ show that in this case classical theories lead to a good accuracy respect to the refined CUF-two dimensional theories. The hypotheses at the base of classical theories are quite verified in this case. As E_L/E_T increases, the difference between classical theories and higher-order ones becomes more significant. Analogous consideration can be done for the case $R_\beta/h = 4$, but here the error between classical theories and HOTs increases, as the thickness assumption is not verified. CLT gives higher values of $\overline{\omega}$ then $FSDTs$, whereas these last theories agree with HOTs, and this effect is more evident when the ratio E_L/E_T is low. The extreme case of highest degree of orthotropy and deep moderately thick shell yields to the most significant differences between classical theories and refined models.

Table 2 Comparison of present analysis with available reference solution for [90/0] cylindrical shell

R_β/a	*1*	*2*	*5*	*10*	*20*	∞
YS (Ye and Soldatos, 1994)	10.697	9.4951	9.4951	8.9778	8.9778	8.9248
B (Bhimaraddi, 1991)	10.409	9.3627	9.0200	8.9564	8.9341	8.9179
Q (Qatu, 2004)	10.666	9.4577	9.0286	8.9479	8.9199	8.9001
Present analysis						
$LM4$	10.698	9.4936	9.0598	8.9759	8.9460	8.9241
$LM3$	10.698	9.4936	9.0598	8.9759	8.9460	8.9241
$LM2$	10.714	9.5123	9.0791	8.9953	8.9653	8.9434
$LM1$	10.734	9.5230	9.0841	8.9986	8.9679	8.9452
$LD4$	10.698	9.4936	9.0598	8.9759	8.9460	8.9241
$LD3$	10.698	9.4937	9.0598	8.9759	8.9460	8.9241
$LD2$	10.735	9.5384	9.1066	9.0229	8.9929	8.9709
$LD1$	10.785	9.5920	9.1612	9.0774	9.0474	9.0252
$ED4$	10.721	9.5219	9.0892	9.0054	8.9755	8.9536
$ED3$	10.765	9.5656	9.1322	9.0479	9.0177	8.9955
$ED2$	10.785	9.5920	9.1612	9.0774	9.0474	9.0252
$ED1$	10.721	9.5190	9.0857	9.0019	8.9719	8.9500
$EDZ3$	10.729	9.5275	9.0946	9.0109	8.9810	8.9592
$EDZ2$	10.769	9.5705	9.1379	9.0540	9.0239	9.0018
$EDZ1$	10.744	9.5529	9.1302	9.0500	9.0219	9.0017
$FSDT^{5/6}$	10.668	9.4582	9.0287	8.9479	8.9199	8.9001
$FSDT$	10.751	9.5564	9.1309	9.0502	9.0219	9.0017
CLT	11.225	10.105	9.6992	9.6184	9.5887	9.5661

Notes: Values of $\bar{\omega}$ for $a/h = 10$, $a/b = 1$, $m = n = 1$, $\nu_{12} = 0.25$, $\nu_{13} = 0.03$, $\nu_{23} = 0.4$

5.3 *Effect of the lamination sequence*

The influence of the lamination sequence on the fundamental frequencies parameter $\overline{\omega}$ of cross-ply laminated composite shells with simply supported edges is presented in Tables 4 and 5. All the layers have the same thickness. Both symmetric and anti-symmetric laminations with respect to the middle plane are considered. Compact notation is adopted to indicate the lamination sequence (Jones, 1999). The first value in the sequence refers to the first layer of the laminate, starting from the bottom of the shell. In the symmetric laminates having an odd number of layers, the $0\ deg$ layers are at the outer surfaces of the laminate. As the number of layer increases, results obtained via HOTs theories move away from those that come by classical theories. To remark the result obtained with $FSDT^{5/6}$ and $[0/90]$, that is practically the same achieved using refined theories, as LD4 and LM4. In fact a shear correction factor $\chi = 5/6$ means to correct the deficiencies of $FSDT$ theory (non-parabolic variation of shear stresses and non-vanishing of the shear stresses at the top and bottom surfaces of the shell), but still violates continuity requirement of the inter-laminar shear stresses. Usually a comparison between $EDZ3$ and $ED4$ theories, bring to better results for the first one. This trend is not verified in the case of [0/90] lamination. When the number of layers is not high, for instance $N_L = 2$, $ED4$ model yields more accurate results then $EDZ3$ do. On the contrary, when the number of layer is high, the results achieved with $EDZ3$ theory are definitively finer than the same obtained with higher-order theory $ED4$.

Table 3 Effect of degree of orthotropy of the individual layers E_L/E_T on $\overline{\omega}$ of three layers [0/90/0] symmetric cylindrical shells

E_L/E_T	*3*	*10*	*25*	*40*
		$R_\beta/h=4$		
CLT	18.095	20.896	25.447	29.202
$FSDT$	16.680	18.841	21.741	23.729
$FSDT^{5/6}$	16.437	18.505	21.201	23.009
$ED1$	17.167	18.792	21.691	23.670
$ED4$	16.520	18.494	20.974	22.575
$EDZ1$	16.906	18.765	21.134	22.649
$EDZ3$	16.348	18.335	20.779	22.323
$LD2$	16.352	18.345	20.799	22.351
$LD4$	16.345	18.317	20.690	22.151
$LM2$	16.350	18.329	20.718	22.193
$LM4$	16.345	18.317	20.690	22.151
		$R_\beta/h=100$		
CLT	5.0841	9.0638	14.205	17.864
$FSDT$	4.9462	8.3217	11.748	13.576
$FSDT^{5/6}$	4.9200	8.1951	11.400	13.051
$ED1$	4.9387	8.3177	11.746	13.574
$ED4$	4.8851	8.0322	10.989	12.471
$EDZ1$	4.9394	7.9805	10.819	12.240
$EDZ3$	4.8548	7.9065	10.700	12.084
$LD2$	4.8548	7.9070	10.702	12.089
$LD4$	4.8546	7.9059	10.698	12.081
$LM2$	4.8548	7.9065	10.700	12.085
$LM4$	4.8546	7.9059	10.698	12.081

Notes: $\gamma=\pi/3$, $m=1$, $n=1$

Table 4 Effect of the stacking sequence of laminates on $\overline{\omega}$ of cross-ply anti-symmetric cylindrical shells

	$[0/90]$	$[0/90]_2$	$[0/90]_3$	$[0/90]_4$	$[0/90]_5$
CLT	6.984	9.922	10.376	10.530	10.60
$FSDT$	6.714	8.987	9.299	9.402	9.449
$FSDT^{5/6}$	6.665	8.834	9.127	9.223	9.266
$ED2$	6.707	8.956	9.283	9.393	9.443
$ED3$	6.686	8.687	9.050	9.181	9.242
$ED4$	6.675	8.571	9.038	9.179	9.241
$EDZ2$	6.703	8.758	9.076	9.184	9.233
$EDZ3$	6.686	8.441	8.831	8.972	9.037
$LD1$	6.714	8.448	8.812	8.944	9.005
$LD2$	6.694	8.397	8.785	8.928	8.995
$LD4$	6.665	8.395	8.785	8.927	8.995
$LM1$	6.671	8.407	8.791	8.931	8.997
$LM3$	6.665	8.395	8.785	8.927	8.995
$LM4$	6.665	8.395	8.785	8.927	8.995

Notes: $\gamma=\pi/3$, $R_\beta/h=100$, $m=1$, $n=1$

Table 5 Effect of the stacking sequence of laminates on $\overline{\omega}$ of cross-ply symmetric cylindrical shells

	$[0/\overline{90}]_s$	$[0/90/\overline{0}]_s$	$[0/90/0/\overline{90}]_s$	$[0/90/0/90/\overline{0}]_s$
CLT	14.205	13.030	12.437	12.084
$FSDT$	11.748	11.028	10.655	10.430
$FSDT^{5/6}$	11.400	10.735	10.389	10.180
$ED1$	11.746	11.026	10.653	10.427
$ED2$	11.746	11.027	10.654	10.429
$ED3$	10.990	10.709	10.413	10.214
$ED4$	10.989	10.709	10.413	10.214
$EDZ1$	10.819	10.422	10.182	10.027
$EDZ2$	10.808	10.408	10.167	10.011
$EDZ3$	10.700	10.328	10.075	9.9084
$LD1$	10.810	10.362	10.087	9.9072
$LD2$	10.702	10.318	10.063	9.8930
$LD3$	10.698	10.317	10.063	9.8930
$LD4$	10.698	10.317	10.063	9.8930
$LM1$	10.751	10.334	10.071	9.8973
$LM2$	10.700	10.318	10.063	9.8930
$LM3$	10.698	10.317	10.063	9.8930
$LM4$	10.698	10.317	10.063	9.8930

Notes: $\gamma = \pi/3$, $R_\beta/h = 100$, $m = 1$, $n = 1$

5.4 Influence of the radius-to-thickness ratio and Love's approximation

A three layers $[0/90_{2t}/0]$ composite shell with variable thickness parameter R_β/h is considered. An effective comparison of the effect of shear deformation, higher order shear deformation, through-the-thickness strain and Love's approximation could be conveniently built for such shell geometry. Fundamental frequency parameter $\overline{\omega}$ are compared in Tables 6 and 7. R_β/h is considered to be as low as 2 and as high as 1000, going from a deep moderately thick shell to a shallow thin one. Two values of the number of waves n along the circumferential β direction are considered, whereas the axial wave number m is fixed to 1. It is confirmed that the error of CLT increases as the thickness increases. Larger absolutes errors are obtained from an increase of the circumferential wave number n. All the theories match in the case of thin shells. Slight differences between LD4 and LM4 theories can be noticed for lower values of R_β/h, while the results agree when the radius-to-thickness parameter rises. This effect becomes more evident when the number of waves n increases.

Love type approximations are significantly subordinate to the thickness parameter R_β/h. Tables 6 and 7 compares Love's approximation results with the exact (Flügge type ones). CLT, FSDT and fourth-order shell theories are considered for $n = 1$ and $n = 10$ waves numbers. Love's results match to exact solutions for high values of the thickness parameter. In fact in such cases the term H_β is almost equal to the unit.

Table 6 Evaluation of Love effects in refined theories

R_β/h	*2*	*4*	*10*	*100*	*1,000*
CLT	107.22	31.879	15.493	13.678	13.330
CLT_{LOVE}	155.75	124.96	57.377	13.695	13.330
$FSDT$	63.693	25.841	13.367	11.289	10.846
$FSDT_{\text{LOVE}}$	63.384	25.818	13.372	11.289	10.846
$FSDT^{5/6}$	59.855	25.011	13.077	10.956	10.496
$FSDT^{5/6}_{\text{LOVE}}$	59.584	24.989	13.082	10.956	10.496
$ED4$	58.534	24.435	12.660	10.455	9.9688
$ED4_{\text{LOVE}}$	58.197	24.409	12.663	10.455	9.9688
$EDZ3$	58.658	24.362	12.545	10.313	9.8187
$EDZ3_{\text{LOVE}}$	58.261	24.334	12.548	10.313	9.8187
$LD4$	57.713	24.104	12.535	10.309	9.8154
$LD4_{\text{LOVE}}$	57.340	24.063	12.536	10.309	9.8154
$LM4$	57.688	24.103	12.535	10.309	9.8154
$LM4_{\text{LOVE}}$	57.312	24.062	12.536	10.309	9.8154

Notes: Comparison on $\overline{\omega}$ of 0/90/0 multilayer composite shell. $E_L/E_T = 25$, $\gamma = \pi/3$, $m = 1$, $n = 1$.

Table 7 Evaluation of Love effects in Refined theories

R_β/h	*2*	*4*	*10*	*100*	*1,000*
CLT	1078.0	543.64	240.53	14.877	13.336
CLT_{LOVE}	1076.2	543.43	240.52	16.491	13.336
$FSDT$	890.37	439.66	167.71	12.366	10.850
$FSDT_{\text{LOVE}}$	888.47	439.41	167.68	12.366	10.850
$FSDT^{5/6}$	813.17	402.09	154.66	12.014	10.500
$FSDT^{5/6}_{\text{LOVE}}$	811.44	401.86	154.65	12.014	10.500
$ED4$	686.84	402.27	154.46	11.519	9.9724
$ED4_{\text{LOVE}}$	701.12	404.69	154.44	11.519	9.9724
$EDZ3$	705.85	410.38	155.54	11.377	9.8222
$EDZ3_{\text{LOVE}}$	722.30	412.92	155.52	11.377	9.8222
$LD4$	640.58	394.39	153.67	11.367	9.8190
$LD4_{\text{LOVE}}$	651.53	396.45	153.65	11.367	9.8190
$LM4$	639.66	393.23	153.37	11.367	9.8190
$LM4_{\text{LOVE}}$	649.27	394.97	153.35	11.367	9.8190

Notes: Comparison on $\overline{\omega}$ of 0/90/0 multilayer composite shell. $E_L/E_T = 25$, $\gamma = \pi/3$, $m = 1$, $n = 10$.

5.5 Analysis of curvature parameter and Donnell effect

A three layer [0/90/0] deep moderately thick cylindrical shell with a variable span angle γ is considered. A comparison on $\overline{\omega}$ of curvature approximation through CUF two-dimensional models is presented in Table 8. As the span angle γ ranges from $\pi/12$ to $\pi/2$ the frequency parameter decreases, depending on the reduction in stiffness and the augment in mass of the structure analysed. Lower errors are found in the case of higher span angle γ, comparing classical theories with refined ones. In the case of $\gamma = \pi/12$, the aspect ratio of the shell is $a/b \approx 10$ and, therefore, the structure can be considered nearly one-dimensional. For this reason CLT, leads to relative errors of about 80% respect the higher-order theories reported in Table 8. Using a FSDT this error falls to about 6%. Rather it cannot be notice any improvement from FSDT when a ED1 model is adopted. When $\gamma = \pi/2$ the aspect ratio of the shell is about 1.6 and therefore the error between classical theories and refined ones decrease significantly, due to the best two-dimensional approximation of the problem. LW second-order models provide good results, which are even better in the case of mixed formulation. Notable results are obtained adopting a EDZ3 model. In Table 9, a comparison between Donnell type approximations and exact solutions (Flügge type ones), for classical and higher-order shells theories is presented. Two values of the radius-to-thickness parameter are reported considering a thin shallow shell when $R_\beta/h = 100$ and a deep moderately thick shell when $R_\beta/h = 4$. In the first case the curvature parameter γ is irrelevant on the values of $\overline{\omega}$. The stiffness and the mass of the structure do not vary significantly and the frequency parameter remains almost the same. The aspect ratio a/b ranges from 0.4 to 0.01, then the shell is more like a ring. Donnell's approximation results in such case are nearly identical to the exact ones and are not influenced by the variation of γ. This is due to the radius-to-thickness parameter $R_\beta/h = 100$, and, in such case, the structure correspond to a thin shallow shell.

On the contrary, when $R_\beta/h = 4$, it can be noticed that the Donnell's approximation is meaningful respect to the curvature parameter γ. As γ increases, the error between exact solution and Donnell's one is becoming more important. To conclude, considering high-order effect in shell theories can result meaningless unless an accurate description of the curvatures term related to γ is made at the same time.

Table 8 Evaluation of curvature approximation through classical vs. refined theories comparison on $\overline{\omega}$ of [0/90/0] cylindrical shells

γ	CLT	$FSDT$	$ED1$	$ED4$	$EDZ1$	$EDZ3$	$LD2$	$LD4$	$LM2$	$LM4$
$\pi/12$	248.90	147.36	147.36	140.36	144.40	140.81	140.61	138.72	138.95	138.66
$\pi/10$	202.31	117.94	117.93	112.46	115.42	112.76	112.79	111.15	111.40	111.12
$\pi/9$	169.54	103.16	103.14	98.476	100.93	98.689	98.784	97.307	97.555	97.290
$\pi/8$	138.38	88.387	88.352	84.509	86.488	84.631	84.764	83.483	83.715	83.473
$\pi/6$	82.756	59.240	59.174	56.921	58.056	56.882	57.010	56.209	56.374	56.206
$\pi/5$	59.380	45.292	45.219	43.650	44.423	43.548	43.645	43.105	43.224	43.105
$\pi/4$	39.898	32.400	32.332	31.294	31.744	31.142	31.199	30.903	30.972	30.902
$\pi/3$	25.447	21.741	21.691	20.974	21.134	20.779	20.799	20.690	20.718	20.690
$\pi/2$	17.963	15.891	15.871	15.320	15.270	15.121	15.120	15.104	15.109	15.104

Notes: $R_\beta/h = 4$, $m = 1$, $n = 1$

5.6 *Natural frequencies versus wave mode number curves*

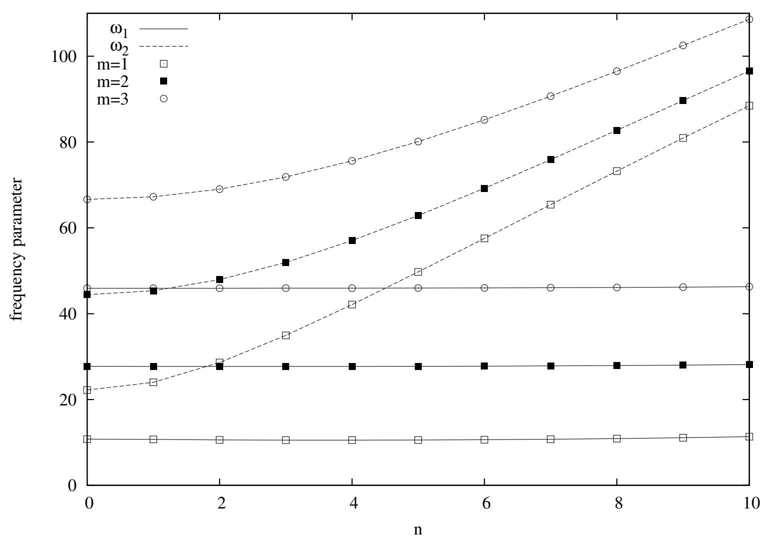
The first two natural frequencies without axial stress are plotted in Figure 2. A symmetric [0/90/0] cross-ply laminated composite cylindrical shell is considered, with equal thickness of each lamina and $\gamma = \pi/3$. A shallow thin shell is considered, since the radius-to-thickness parameter $R_\beta/h = 100$. Only results obtained using a LD4 model are plotted, as the same trend can be found also for all higher-order theories. The dominant first two eigenvalues that correspond to the lowest two natural frequencies are of most concern. The lower natural frequency $\overline{\omega}_1$ is a flexural mode with some shear deformations, whereas the upper frequency $\overline{\omega}_2$ is an extensional mode with thickness changes. Figures 3 to 5 show the variation of the first two natural frequencies for $m = 1 - 3$ with respect to $n = 0 - 10$. Although, in general, the natural frequencies increase as the circumferential wave number n grows, the lowest frequencies occur at specific higher modes in the case of the first natural frequencies for $m = 1, 2$, as shown in detail in Figures 3 and 4. When $m = 3$ also the first natural frequency increases as n goes from 0–10 (see Figure 5).

Table 9 Evaluation of Donnell effect in classical vs. refined theories comparison on $\overline{\omega}$ of [0/90/0] cylindrical shells

γ	CLT		$FSDT$		$EDZ3$		$LD4$		$LM4$	
	$Exact$	$Donnell$	$Exact$	$Donnell$	$Exact$	$Donnell$	$Exact$	$Donnell$	$Exact$	$Donnell$
					$R_\beta/h = 100$					
$\pi/12$	14.116	14.975	11.605	11.605	10.530	10.531	10.528	10.529	10.528	10.529
$\pi/10$	14.101	14.703	11.600	11.601	10.530	10.530	10.528	10.529	10.528	10.529
$\pi/9$	14.099	14.589	11.605	11.605	10.537	10.538	10.535	10.536	10.535	10.536
$\pi/8$	14.102	14.491	11.615	11.615	10.550	10.551	10.548	10.549	10.548	10.549
$\pi/6$	14.126	14.345	11.652	11.652	10.594	10.595	10.592	10.593	10.592	10.593
$\pi/5$	14.147	14.299	11.680	11.680	10.625	10.626	10.624	10.624	10.624	10.624
$\pi/4$	14.174	14.271	11.713	11.713	10.662	10.662	10.660	10.660	10.660	10.660
$\pi/3$	14.205	14.260	11.748	11.748	10.700	10.700	10.698	10.698	10.698	10.698
					$R_\beta/h = 4$					
$\pi/12$	248.90	248.98	147.36	148.48	140.81	141.88	138.72	139.72	138.66	139.67
$\pi/10$	202.31	219.47	117.94	119.20	112.76	113.98	111.15	112.31	111.12	112.28
$\pi/9$	169.54	205.42	103.16	104.52	98.689	99.995	97.307	98.558	97.290	98.540
$\pi/8$	138.38	191.98	88.387	89.844	84.631	86.031	83.483	84.832	83.473	84.821
$\pi/6$	82.756	167.51	59.240	60.925	56.882	58.489	56.209	57.776	56.206	57.773
$\pi/5$	59.380	156.84	45.292	47.090	43.548	45.261	43.105	44.785	43.105	44.784
$\pi/4$	39.898	147.53	32.400	34.272	31.142	32.929	30.903	32.664	30.902	32.663
$\pi/3$	25.447	139.82	21.741	23.510	20.779	22.486	20.690	22.379	20.690	22.379

Notes: $m = 1$, $n = 1$

Figure 2 Natural frequency parameter versus circumferential mode number curves of a [0/90/0] cylindrical shell



Notes: $a/h = 10$, $\gamma = \pi/3$, $R_\beta/h = 100$

Figure 3 Detail for $\overline{\omega}_1$ with axial wave number m = 1

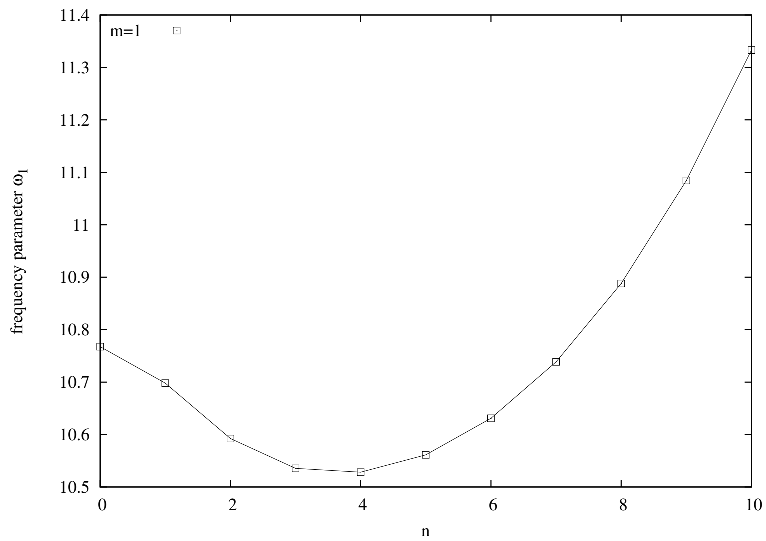


Figure 4 Detail for $\overline{\omega}_1$ with axial wave number m = 2

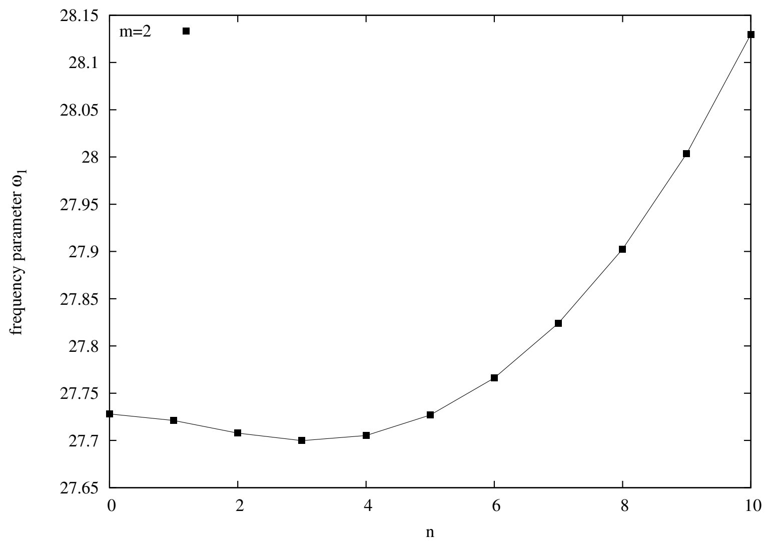
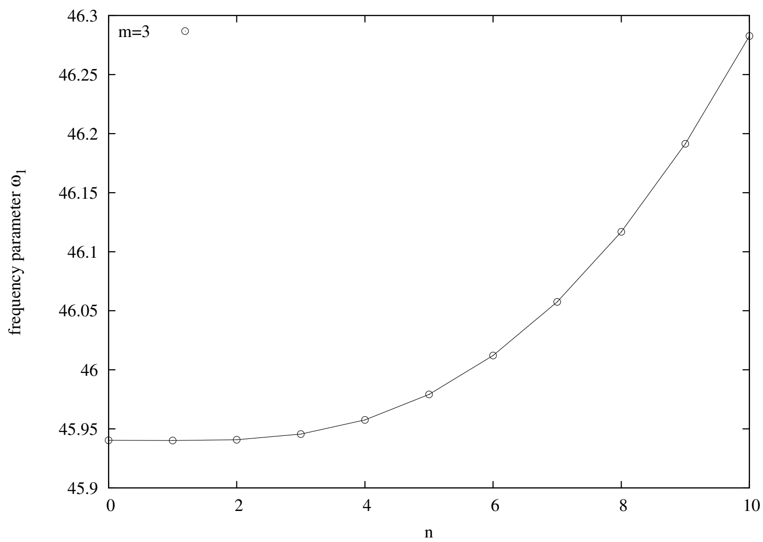


Figure 5 Detail for $\overline{\omega}_1$ with axial wave number $m = 3$



6 Conclusions

A unified approach to formulate two-dimensional shell theories has been here addressed to evaluate the free vibration response of cylindrical multi-layered shells made of composite materials. Both thin shallow shells and moderately thick deep ones are considered, as well as lower and higher vibration modes. Analyses are carried out considering the influence of the stacking sequences of laminate, the degree of orthotropy, the thickness and the curvature parameters. Higher-order theories are considered in the framework of Love- and Donnell-type approximations. Classical theories have been considered for comparison purpose. The following main remarks can be made:

1 classical models yield good results only for thin shallow shells

2 zig-zag function increases the results accuracy of ESL models

3 a very accurate description of the vibration response of highly anisotropic thick shells requires a LW approach.

Above all it must be highlighted that inclusion of shear deformation and higher-order effects could result meaningless unless curvatures terms are correctly included in a given theories.

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