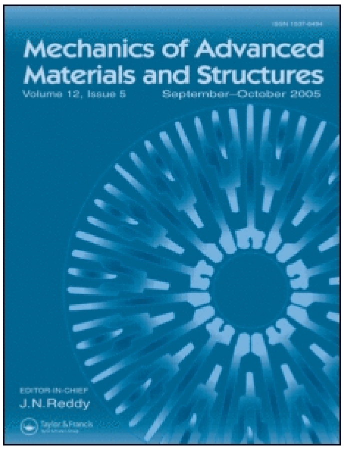

Mechanics of Advanced Materials and Structures
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713773278

Accurate Vibration Analysis of Multilayered Plates Made of Functionally Graded Materials

M. Cinefra[a b]; M. Soave[b]
[a] CRS H. Tudor, Luxembourg [b] Department of Aeronautics and Space Engineering, Politecnico di Torino, Turin, Italy

Online publication date: 24 January 2011

To cite this Article Cinefra, M. and Soave, M.(2011) 'Accurate Vibration Analysis of Multilayered Plates Made of Functionally Graded Materials', Mechanics of Advanced Materials and Structures, 18: 1, 3 — 13
To link to this Article: DOI: 10.1080/15376494.2010.519204
URL: http://dx.doi.org/10.1080/15376494.2010.519204

Mechanics of Advanced Materials and Structures, 18:3–13, 2011

ISSN: 1537-6494 print / 1537-6532 online
DOI: 10.1080/15376494.2010.519204

Accurate Vibration Analysis of Multilayered Plates Made of Functionally Graded Materials

M. Cinefra[1,2] and M. Soave[2]
[1]*CRS H. Tudor, Esch-sur-Alzette, Luxembourg*
[2]*Department of Aeronautics and Space Engineering, Politecnico di Torino, Turin, Italy*

Closed form solutions of free vibration problems in simply supported multilayered plates made of Functionally Graded Material (FGM) are dealt with in the present paper. Various refined plate theories are compared in the cases of both equivalent single-layer and layer-wise variable descriptions. Classical theories are compared to advanced ones, in which higher orders of expansion in the thickness direction are used for the transverse displacement. The plate governing equations are provided by applying Carrera's Unified Formulation (CUF) which, in this paper, is extended to the dynamics and free vibration response of plates embedding FGM layers. CUF permits one to handle the governing equations of the whole theory in a unified manner and a hierarchical evaluation of the natural frequencies of plates is therefore obtained. The results are given for one-layer and multi-layer plates embedding FGM layers. Conclusions are drawn regarding the accuracy of classical and advanced plate modelling for various vibration modes.

Keywords functionally graded materials, multilayered plates, Carrera Unified Formulation, free vibration

1. INTRODUCTION

Several studies concerning Functionally Graded Materials (FGM) have appeared in recent years. The improved properties of these materials, compared to conventional composites, make them a subject of interest for many applications, such as thermal barrier coatings, engine components or rocket nozzles. FGMs are composed of two or more phases of different materials and can have various types of behavior, which are dictated by the elasto-mechanical properties of the materials. These properties vary along one direction, normally the thickness, according to a power law which consists of a gradual variation of constituent phases, in terms of volume fraction. Various power laws have been used in the open literature, some of which have been provided by Zenkour [1], Kashtalyan [2], and Mori-Tanaka [3].

In this paper, attention is restricted to the dynamics and free vibration response of FGM plates. Many articles about free-vibration problems have recently been presented concerning various analysis methods. Batra and Jin [4] have used the first-order shear deformation theory (FSDT), coupled with the finite element method (FEM), to study free vibrations of a functionally graded anisotropic rectangular plate with the objective of maximizing one of its first five natural frequencies under different edge conditions. Rastgoftar et al. [5] have presented a solution to the boundary stabilization of an FGM plate in free transverse vibration. The composite laminated plate dynamics are presented by a linear fourth order partial differential equation. A linear control law is constructed to stabilize the plate. The control force consists of feedback of the velocity at the plate boundaries. The novelty of this work is that it is possible to asymptotically stabilize a free transversely vibrating composite plate with simply supported or clamped boundary conditions via boundary control, without resorting to truncation of the model. Kadoli and Ganesan [6] have proposed a linear thermal buckling and free vibration analysis of functionally graded cylindrical shells with clamped-clamped boundary conditions, based on temperature-dependent material properties. The material properties are assumed to vary smoothly and continuously across the thickness. The first-order shear deformation theory, along with Fourier series expansion of the displacement variables in the circumferential direction, are used to model the FGM shell. Free vibration studies of FGM shells under elevated temperatures show that the fall in natural frequency is very drastic for the mode corresponding to the lowest natural frequency, compared to the lowest buckling temperature mode. Li et al. [7] have studied free vibrations of rectangular functionally graded sandwich plates with simply supported and clamped edges. The study is based on the three-dimensional linear theory of elasticity. Two common types of FGM sandwich plates are considered: a sandwich plate with FGM facesheets and an homogeneous core; and a sandwich with homogeneous facesheets and an FGM core. The three displacements of the plates are expanded by a series of Chebyshev polynomials multiplied by appropriate functions to satisfy the essential boundary conditions. The natural frequencies are obtained using the Ritz method. The natural

Received 25 August 2010; accepted 2 September 2010.
Address correspondence to Maria Cinefra, Department of Aeronautics and Space Engineering, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy. E-mail: maria.cinefra@polito.it

frequencies of simply supported FGM sandwich plates are compared with the results from different two-dimensional plate theories. A parametric study has been performed for varying volume fractions, layer thickness ratios, thickness: length ratios and aspect ratios of the sandwich plates. Liew et al. [8] have performed linear and nonlinear vibration analysis of a three-layer coating-FGM-substrate cylindrical panel with general boundary conditions and subjected to a temperature gradient across the thickness due to steady heat conduction. The theoretical formulation is based on geometric nonlinearity, in the von-Karman sense, and on the first order shear deformation theory which accounts for the effects of transverse shear strains and rotary inertia. A nonlinear pre-vibration analysis has been conducted to determine the thermally induced deformation and pre-stress state of the cylindrical panel due to the bending and stretching coupling effect being introduced by the graded layer. The nonlinear governing differential equations of motion are obtained by adding an incremental dynamic state to the pre-vibration state. A numerical method, which makes use of differential quadrature approximation together with iterative algorithms, is employed to model the linear and nonlinear vibration behavior of the cylindrical panels. Sina et al. [9] have presented a new beam theory that is different from the traditional first-order shear deformation beam theory to analyze free vibration of functionally graded beams. The beam properties are assumed to be varied through the thickness according to a simple power law distribution in terms of volume fraction of the material constituents. It is assumed that the lateral normal stress of the beam is zero and the governing equations of motion are derived using Hamilton's principle. The resulting system of ordinary differential equations of free vibration analysis is solved using an analytical method. Different boundary conditions are considered and comparisons are made among different beam theories. Vel and Batra [10] have proposed a three-dimensional exact solution for free and forced vibrations of simply supported, functionally graded, rectangular plates. Suitable displacement functions that identically satisfy the boundary conditions are used to reduce the equations that govern the steady state vibrations of a plate to a set of coupled ordinary differential equations, which are then solved employing the power series method. The exact solution is valid for thick and thin plates, and for arbitrary variations of the material properties in the thickness direction. The results are presented for two-constituent metal-ceramic functionally graded rectangular plates that have a power-law through-the-thickness variation of the volume fractions of the constituents. The effective material properties are estimated at a point by either the Mori-Tanaka or the self-consistent schemes. Zenkour [11] has used a sinusoidal shear deformation plate theory to study the buckling and free vibration of a simply supported functionally graded sandwich plate. The influence of the transverse shear deformation, plate aspect ratio, side-to-thickness ratio and volume fraction distributions is studied. In addition, the effect of the core thickness, relative to the total thickness of the plate, on the critical buckling load and the eigenfrequencies, is investigated. An accurate kinematic description of the problem variables in the thickness direction appears to be a key point for the analysis of the behavior of FGM plates.

The present work deals with Carrera's Unified Formulation (CUF) for the analysis of FGM plates. CUF, proposed in [12, 13], has been extended to FGM structure in [14, 15]. The generalized expansion, upon which the CUF is based, relies on a set of functions indicated as thickness functions. In this manner, CUF reduces a three-dimensional problem to a bi-dimensional one and the order of expansion along the thickness of the plate is taken as a free parameter of the problem. As a result, an exhaustive variable kinematic model is obtained. The principle of virtual displacements (PVD) has been used in [14], while in [15], Reissner's mixed variational theorem (RMVT) has been employed to model both displacements and transverse shear/normal stresses. The classical and mixed advanced hierarchical models presented in these two papers have been developed for multilayer FGM plates.

In this paper, CUF is extended to the dynamic field, and the natural frequencies of single-layered and sandwich plates are compared to the previous 3D solution provided in [7, 10, 16].

The article has been organized as follows: the geometry of plates is given in section 2 ; the strain-displacement relations and constitutive equations are given in sections 2 and 3, respectively; models contained in CUF, used variational statements and governing equations are described in sections 4, 5 and 6; the closed form solution for the considered free-vibration problem is proposed in section 7; the numerical discussion is conducted in section 8 and the conclusions are then given.

2. GEOMETRICAL RELATIONS

Plates are bi-dimensional structures in which a dimension, normally the thickness z, is negligible with respect to the others. The geometry and the reference system for a plate are proposed in Figure 1, where (x, y, z) are the cartesian coordinates and k is indicative of the layer.

The stresses and strains are conveniently split into in-plane and normal components, which are denoted by the subscripts p and n, respectively. The strains of the kth layer can be related to the displacement field $u^k = \{u^k, v^k, w^k\}$ via the geometric relations:

$$\varepsilon_p^k = D_p u^k \quad \varepsilon_n^k = (D_{np} + D_{nz})u^k \tag{1}$$

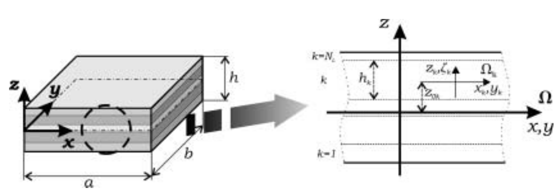

FIG. 1. Geometry and reference system for multilayer plate.

wherein the differential operator arrays are defined as follows:

$$D_P = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix} \quad D_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix} \quad D_{np} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix} \tag{2}$$

with $\varepsilon_p = (\varepsilon_{xx},\ \varepsilon_{yy}, \varepsilon_{xy})$ and $\varepsilon_n = (\varepsilon_{xz}, \varepsilon_{yz}, \varepsilon_{zz})$.

3. HOOKE'S LAW

In Hooke's standard law the matrix of elasticity that relates the stresses to the deformations refers to the coordinates system (x, y, z) for plates. Therefore, the constitutive equations are the following:

$$\sigma_p^k = \tilde{C}_{pp}^k \varepsilon_p^k + \tilde{C}_{pn}^k \varepsilon_n^k \tag{3}$$

$$\sigma_n^k = \tilde{C}_{np}^k \varepsilon_p^k + \tilde{C}_{nn}^k \varepsilon_n^k \tag{4}$$

where the in-plane and out-of-plane stresses are:

$$\sigma_p^k = \begin{bmatrix} \sigma_{xx}^k & \sigma_{yy}^k & \sigma_{xy}^k \end{bmatrix}^T; \quad \sigma_n^k = \begin{bmatrix} \sigma_{xz}^k & \sigma_{yz}^k & \sigma_{zz}^k \end{bmatrix}^T \tag{5}$$

In the case of FGM layers, the coefficients in Eqs. (3) and (4) vary in the thickness direction, z, according to a given "grading" law:

$$C(z) = C_0 * g(z), \tag{6}$$

where C_0 is the reference stiffness matrix and $g(z)$ gives the variation along z. For the sake of convenience, $C(z)$ is rewritten as a weighted summation on the constants C_r and the weights are given by the thickness functions F_r:

$$C(z) = F_b(z)C_b + F_t(z)C_t + F_\gamma(z)C_\gamma = F_r C_r \quad with \quad r = 1, \ldots, N_r \tag{7}$$

F_r are a combination of Legendre polynomials and they coincide with those used in Carrera Unified Formulation for Layer-Wise expansion cases (see Sec. 4). The order of expansion can be freely chosen as can the displacements. In this paper, the maximum value of N_r is 10. It is mandatory to choose such a high order of expansion to ensure the necessary accuracy.

The procedure to calculate C_r can be accomplished by solving a simple algebraic system of order N_r for each component C_{ijr}, since the actual values of C_{ij} are known at N_r different locations along the thickness $(z_1, \ldots, zN_r)$:

$$\begin{bmatrix} C_{ij}(z_1) \\ \vdots \\ C_{ij}(z_{N_r}) \end{bmatrix} = \begin{bmatrix} F_b(z_1) & \cdots & F_\gamma(z_1) & \cdots & F_t(z_1) \\ \vdots & & \vdots & & \vdots \\ F_b(z_{N_r}) & \cdots & F_\gamma(z_{N_r}) & \cdots & F_t(z_{N_r}) \end{bmatrix} \begin{bmatrix} C_{ijb} \\ \vdots \\ C_{ijr} \\ \vdots \\ C_{ijt} \end{bmatrix} \tag{8}$$

4. CARRERA UNIFIED FORMULATION

Carrera Unified Formulation permits several two dimensional models to be obtained for plates, thanks to the separation of the unknown variables into a set of thickness functions only depending on the thickness coordinate z, and the correspondent unknowns depending on the in-plane coordinates (x, y). The governing equations are written in terms of few fundamental nuclei which do not formally depend on:

- the order of expansion N that is used in the z-direction;
- the variables description in the multilayered structure (Layer Wise [LW] or Equivalent Single Layer [ESL]).

The generic variable $\boldsymbol{a}(x, y, z)$ and its variation $\delta\boldsymbol{a}(x, y, z)$ can be written according to the following general expansion:

$$\boldsymbol{a}(x, y, z) = F_\tau(z)\boldsymbol{a}_\tau(x, y) \quad \delta\boldsymbol{a}(x, y, z) = F_s(z)\delta\boldsymbol{a}_s(x, y) \quad with \quad \tau, s = 1, \ldots, N \tag{9}$$

Bold letters denote arrays; the summing convention with repeated indexes τ and s is assumed; and the order of expansion N goes from 1 to 4. Depending on the used thickness functions, a model can be ESL when the variable is assumed for the whole multilayer or LW when the variable is considered independent in each layer. In this way any configuration embedding FGM layers can be considered a single layer FGM plate or a sandwich structure with a functionally graded core. The proposed two-dimensional models have been coded according to the CUF. Details can be found in previous authors' works (see [14, 15, 17, 18]).

4.1. Layer Wise Theories

Layer Wise (LW) approach describes each layer as an independent structure. This description can be applied to the displacement components $\boldsymbol{u} = (u, v, w)$ and transverse shear/normal stresses $\boldsymbol{\sigma}_n = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$. The stresses are always modelled via LW approach to ensure the interlaminar continuity, while the displacements can be modelled via both ESL and LW approach. The Layer Wise description is introduced according to the following expansion:

$$(\boldsymbol{u}^k, \boldsymbol{\sigma}_n^k) = F_t(\boldsymbol{u}_t^k, \boldsymbol{\sigma}_{nt}^k) + F_b(\boldsymbol{u}_b^k, \boldsymbol{\sigma}_{nb}^k) + F_l(\boldsymbol{u}_l^k, \boldsymbol{\sigma}_{nl}^k) = F_\tau(\boldsymbol{u}_\tau^k, \boldsymbol{\sigma}_\tau^k) \tag{10}$$

where

$$\tau = t, b, l \quad with \quad l = 2, \ldots, N$$

In the considered plates, some layers can be functionally graded and in this case a further assembling on the index r is necessary. t and b are the top and bottom values, and l terms denote the higher order terms of expansion. The thickness functions $F_\tau(\zeta_k)$ are defined at the k-layer level and they are a linear combination of Legendre polynomials $P_j = P_j(\zeta_k$ of jth)-order, defined in ζ_k-domain ($\zeta_k = \frac{2z_k}{h_k}$ with z_k local coordinate and h_k thickness, both referred to kth layer, so $-1 \leq \zeta_k \leq 1$). The first five Legendre polynomials are:

$$P_0 = 1, \quad P_1 = \zeta_k, \quad P_2 = \frac{(3\zeta_k^2 - 1)}{2}, \quad P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2},$$
$$P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{8}{3} \tag{11}$$

their combinations for the thickness functions are:

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_l = P_l - P_{l-2} \quad with \quad l = 2, \ldots, N \tag{12}$$

The chosen functions have the following interesting properties:

$$\zeta_k = 1 \quad : \quad F_t = 1; \quad F_b = 0; \quad F_l = 0 \quad at\ top \tag{13}$$
$$\zeta_k = -1 \quad : \quad F_t = 0; \quad F_b = 1; \quad F_l = 0 \quad at\ bottom \tag{14}$$

that is, the interface values of the variables are considered as unknowns.

The LW models can be conveniently used in case of a single-layer plate, via the introduction of mathematical interfaces and using an N order of expansion in the each fictitious layers. Accuracy of the solution increases by increasing the number of the mathematical interfaces. This procedure can be used to obtain exact solutions.

4.2. Equivalent Single Layer Theories

The displacement is described according to the Equivalent Single Layer model if the unknowns are the same for the whole structure. The stiffness matrices in the case of FGM layers have a further assembling procedure that considers the elastic coefficients depending on z coordinate.

The z expansion is obtained via Taylor polynomials, that is:

$$\boldsymbol{u} = F_0\boldsymbol{u}_0 + F_1\boldsymbol{u}_1 + \ldots + F_N\boldsymbol{u}_N = F_\tau \boldsymbol{u}_\tau \quad with \quad \tau = 0, 1, \ldots, N \tag{15}$$

N is the order of expansion that ranges from linear (1) to fourth (4) order:

$$F_0 = z^0 = 1, \quad F_1 = z^1 = z, \ldots, F_N = z^N \tag{16}$$

Higher-order theories (HOTs) consider the same order of expansion in the thickness direction for the three displacement components (including the transverse one). These theories are obtained in the ESL overview. In HOTs including transverse normal strains, higher-order terms can be introduced in the kinematic assumptions in order to obtain refinements of the classical lamination theories.

According to the acronym system developed within the CUF, the letter E denotes that the kinematic is assumed for the whole multilayer, as in the ESL approach; D denotes that only displacement unknowns are considered; and the last number N states the through-the-thickness order of expansion. It is important to note that the EDN models are not able to describe the discontinuity of the first derivative of displacements in correspondence of the layer interfaces. If the approach is LW, the first letter is L and the second one is M if the transverse stresses are modelled a priori. Resuming, one can have EDN, LDN, EMN and LMN theories. The FSDT model [19, 20] can be obtained from the ED1 theory by considering a constant transverse displacement through the thickness. An appropriate application of penalty technique to shear correction factor leads to CLT [21].

5. VARIATIONAL STATEMENTS

This section presents the variational statements Principle of Virtual Displacements (PVD) and Reissner's Mixed Variational Theorem (RMVT) in the case of free vibration analysis of plates embedding FGM layers.

For a multilayered plate, in the case of pure mechanical problem, the PVD states:

$$\int_V (\delta \boldsymbol{\varepsilon}_{pG}^T \boldsymbol{\sigma}_{pH} + \delta \boldsymbol{\varepsilon}_{nG}^T \boldsymbol{\sigma}_{nH}) dV = \int_V \rho \delta \boldsymbol{u} \ddot{\boldsymbol{u}} dV + \delta L_e \tag{17}$$

in this equation ρ indicates mass density. T indicates the array transposition and V is the volume of the body. The subscripts p and n indicate the in-plane and out-of-plane components of stresses and strains, respectively. The subscript H highlights that the strains are written according to Hooke's law, while the subscript G indicates that the strains are derived from geometrical relations. δL_e is the virtual external work.

The RMVT, compared to PVD, assumes both transverse shear/normal stresses and displacement as independent variables [22]:

$$\int_V (\delta \boldsymbol{\varepsilon}_{pG}^T \boldsymbol{\sigma}_{pH} + \delta \boldsymbol{\varepsilon}_{nG}^T \boldsymbol{\sigma}_{nM} + \delta \boldsymbol{\sigma}_{nM}^T (\boldsymbol{\varepsilon}_{nG} - \boldsymbol{\varepsilon}_{nH})) dV = \int_V \rho \delta \boldsymbol{u} \ddot{\boldsymbol{u}} dV + \delta L_e \tag{18}$$

a Lagrange multiplier $\delta \boldsymbol{\sigma}_{nM}$ is added to permit to assume a priori interlaminar continuous transverse stresses $\boldsymbol{\sigma}_{nM}$ (subscript M means modelled variables). Therefore, RMVT permits the fulfillment a priori of C_z^0-requirements for transverse shear/normal stresses.

6. GOVERNING EQUATIONS

This section presents the derivation of the governing equations based on the variational statements in the previous sections. A dynamic solution will be developed. The procedure is finalized to obtain the so-called *fundamental nuclei*. These consist of [3×3] arrays that represent the basic items from which the stiffness matrix of the whole structure can be computed.

For a laminate with N_l layers, the PVD for pure mechanical analysis, neglecting any body forces and considering only applied traction loads, is formulated as:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\varepsilon}_{pG}^{kT}\boldsymbol{\sigma}_{pC}^{k}+\delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nC}^{k}\right\}d\Omega_k dz=\sum_{k=1}^{N_L}\delta L_e^k-\delta L_{in}^k \tag{19}$$

where the integration domains Ω_k and A_k indicate respectively the reference plane of the lamina and its thickness. Subscript G and C indicate geometrical and constitutive equations respectively. δL_e^k is the expression of the external work that takes into account the external loads for a generic layer k, while δL_{in}^k is the expression of the virtual variation of inertial load.

Similarly, the RMVT for a laminate becomes:

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\left\{\delta\boldsymbol{\varepsilon}_{pG}^{T}\boldsymbol{\sigma}_{pC}+\delta\boldsymbol{\varepsilon}_{nG}^{T}\boldsymbol{\sigma}_{nM}+\delta\boldsymbol{\sigma}_{nM}^{T}(\boldsymbol{\varepsilon}_{nG}-\boldsymbol{\varepsilon}_{nC})\right\}$$
$$d\Omega_k dz=\sum_{k=1}^{N_L}\delta L_e^k-\delta L_{in}^k \tag{20}$$

The steps to obtain the consistent governing equations are: choice of the opportune variational statement (PVD or RMVT); substitutions of the consistent constitutive equations; use of the geometrical relations; introduction of CUF for the two-dimensional approximation.

In the RMVT case, the constitutive equations are consistent if the Eqs. (3) and (4) are rearranged so that $\boldsymbol{\sigma}_p$ and $\boldsymbol{\varepsilon}_n$ are derived from $\boldsymbol{\varepsilon}_p$ and $\boldsymbol{\sigma}_n$.

In order to obtain a strong form of differential equations on the domain Ω_k, as well as the correspondence boundary conditions on edge Γ_k, the integration by parts must be employed after the substitutions. Further details on this integration procedure are reported in [14].

The governing equation on the domain Ω_k, in the PVD case, is:

$$\delta\boldsymbol{u}_s^{k^T}:\boldsymbol{K}_{uu}^{k\tau sr}\boldsymbol{u}_\tau^k=\boldsymbol{P}_{us}^k-\boldsymbol{M}_{uu}^{k\tau sr}\ddot{\boldsymbol{u}}_\tau^k \tag{21}$$

Boundary conditions of Neumann type are:

$$\prod_{uu}^{k\tau sr}\boldsymbol{u}_\tau^k=\prod_{uu}^{k\tau sr}\bar{\boldsymbol{u}}_\tau^k \tag{22}$$

In Eq. (21), $P_{u\tau}^k$ is the external mechanical load and the fundamental nucleus $K_{uu}^{k\tau sr}$ has to be assembled through the depicted indexes: the internal loop is on index r; τ and s consider the order of expansion in z for the displacements; superscript k indicates the assembling on the number of layers. $\ddot{\boldsymbol{u}}_\tau^k$ denotes the second derivative with respect to the time of the displacement components and matrix $M_{uu}^{k\tau sr}$ represents the fundamental nucleus for the inertial array:

$$\boldsymbol{M}^{k\tau sr}=\int_{Ak}\boldsymbol{I}\rho_r^k F_r F_s F_r dz, \tag{23}$$

The governing equations in the RMVT case, after integration by parts, are:

$$\begin{aligned}\delta\boldsymbol{u}_s^k&:\boldsymbol{K}_{uu}^{k\tau sr}\boldsymbol{u}_\tau^k+\boldsymbol{K}_{u\sigma}^{k\tau sr}\boldsymbol{\sigma}_{n\tau}^k=\boldsymbol{P}_{us}^k-\boldsymbol{M}^{k\tau sr}\ddot{u}_\tau^k\\ \delta\boldsymbol{\sigma}_{ns}^k&:\boldsymbol{K}_{\sigma u}^{k\tau sr}\boldsymbol{u}_\tau^k+\boldsymbol{K}_{\sigma\sigma}^{k\tau sr}\boldsymbol{\sigma}_{n\tau}^k=0\end{aligned} \tag{24}$$

four fundamental nuclei relative to stiffness array are obtained. These are completely different from those obtained in the PVD case while the inertial array does not change.

Corresponding boundary conditions of Neumann type are:

$$\begin{aligned}&\prod_{uu}^{k\tau sr}\boldsymbol{u}_\tau^k+\prod_{u\sigma}^{k\tau sr}\boldsymbol{\sigma}_{n\tau}^k=\prod_{uu}^{k\tau sr}\bar{\boldsymbol{u}}_\tau^k+\prod_{u\sigma}^{k\tau sr}\bar{\boldsymbol{\sigma}}_{n\tau}^k\\ &\prod_{\phi u}^{k\tau sr}\boldsymbol{u}_\tau^k+\prod_{\phi\sigma}^{k\tau sr}\boldsymbol{\sigma}_{n\tau}^k=\prod_{\phi u}^{k\tau sr}\bar{\boldsymbol{u}}_\tau^k+\prod_{\phi\sigma}^{k\tau sr}\bar{\boldsymbol{\sigma}}_{n\tau}^k\end{aligned} \tag{25}$$

The explicit form of fundamental nuclei involved in this analysis is reported in [14, 15].

TABLE 1
Compared theories for an Isotropic Al plate for thickness ratio $a/h=10$ -Adim: $\bar{\omega}=\frac{\omega a}{h}\sqrt{\frac{\rho_m}{E_m}}$

	Natural frequency					
$\bar{\omega}$	Ref. [10]	LD4	ED4	ED1	FSDT	CLT
$\bar{\omega}_1$	5.7769	5.7769	5.7769	5.7944	5.7944	5.9248
$\bar{\omega}_2$	27.554	27.553	27.554	27.554	27.553	27.553
$\bar{\omega}_3$	46.503	46.502	46.502	44.408	46.574	46.574
$\bar{\omega}_4$	196.77	196.82	196.82	216.59	216.59	–
$\bar{\omega}_5$	201.34	201.39	201.39	221.46	221.46	–
$\bar{\omega}_6$	357.42	357.75	357.75	364.45	–	–
$\bar{\omega}_7$	390.64	392.04	392.04	–	–	–
$\bar{\omega}_8$	399.77	401.11	401.11	–	–	–

7. CLOSED FORM SOLUTION FOR FREE-VIBRATION PROBLEM

For the derived boundary value problem, for particular geometry, material symmetry and boundary conditions, an analytical solution can be derived. For simply supported plates, a Navier-type closed-form solution may be found with the following

TABLE 2
Compared theories for an FGM Al/ZrO_2 plate for thickness ratio $a/h = 10$ -Adim: $\bar{\omega} = \frac{\omega a}{h}\sqrt{\frac{\rho_m}{E_m}}$

	Natural frequency						
	$\bar{\omega}$	Ref. [16]	LD4	ED4	ED1	FSDT	CLT
$k = 1$	$\bar{\omega}_1$	6.1932	6.1932	6.1932	6.2112	6.2112	6.3405
	$\bar{\omega}_2$	30.685	30.685	30.685	30.686	30.686	30.687
	$\bar{\omega}_3$	51.795	51.795	51.795	49.455	51.867	51.873
	$\bar{\omega}_4$	222.43	222.43	222.43	246.37	246.37	–
	$\bar{\omega}_5$	227.29	227.30	227.30	251.59	251.59	–
	$\bar{\omega}_6$	403.06	403.73	403.73	414.70	–	–
	$\bar{\omega}_7$	433.49	436.90	436.90	–	–	–
	$\bar{\omega}_8$	444.99	447.93	447.93	–	–	–

harmonic assumptions for the field variables:

$$(u^k_{x_\tau}, \sigma^k_{xz_\tau}) = \sum_{m,n} (\hat{U}^k_{x_\tau}, \hat{S}^k_{xz_\tau}) \cos\frac{m\pi x_k}{a_k} \sin\frac{n\pi y_k}{b_k} e^{i w_{mn} t}, \quad k = 1, N_l,$$
$$(u^k_{y\tau}, \sigma^k_{yz_\tau}) = \sum_{m,n} (\hat{U}^k_{y_\tau}, \hat{S}^k_{yz_\tau}) \sin\frac{m\pi x_k}{a_k} \cos\frac{n\pi y_k}{b_k} e^{i w_{mn} t}, \quad \tau = t, b, r,$$
$$(u^k_{z_\tau}, \sigma^k_{zz_\tau}) = \sum_{m,n} (\hat{U}^k_{z_\tau}, \hat{S}^k_{zz_\tau}) \sin\frac{m\pi x_k}{a_k} \sin\frac{n\pi y_k}{b_k} e^{i w_{mn} t}, \quad r = 2, N, \tag{26}$$

where a_k and b_k are the lengths of the plate along the two coordinates x and y. m and n represent the number of half-waves in x and y direction, respectively. These numbers characterize the vibration mode associated to the circular frequency ω_{mn}. $i = \sqrt{-1}$ is the imaginary unit and t the time. The quantities with indicate the amplitudes. These assumptions correspond to the simply supported boundary conditions. Upon substitution of Eq. (26), the governing equations on Ω^k assume the form of a linear system of algebraic equations in the domain, while the boundary conditions are exactly fulfilled.

Only the free-vibration analysis is addressed in this article. Therefore, the external mechanical loading is set to zero and the linear system of algebraic equations is:

$$\boldsymbol{K}^*\hat{\boldsymbol{U}} = \omega^2_{mn}\boldsymbol{M}\hat{\boldsymbol{U}}, \tag{27}$$

where $\boldsymbol{K}^*$ is the equivalent stiffness matrix obtained by means of static condensation (for further details see [23, 24]), $\boldsymbol{M}$ is the inertial matrix and $\hat{\boldsymbol{U}}$ is the vector of unknown variables. By defining $\lambda_{mn} = \omega^2_{mn}$, the solution of the associated eigenvalue problem becomes:

$$||\boldsymbol{K}^* - \lambda_{mn}\hat{\boldsymbol{M}}|| = 0. \tag{28}$$

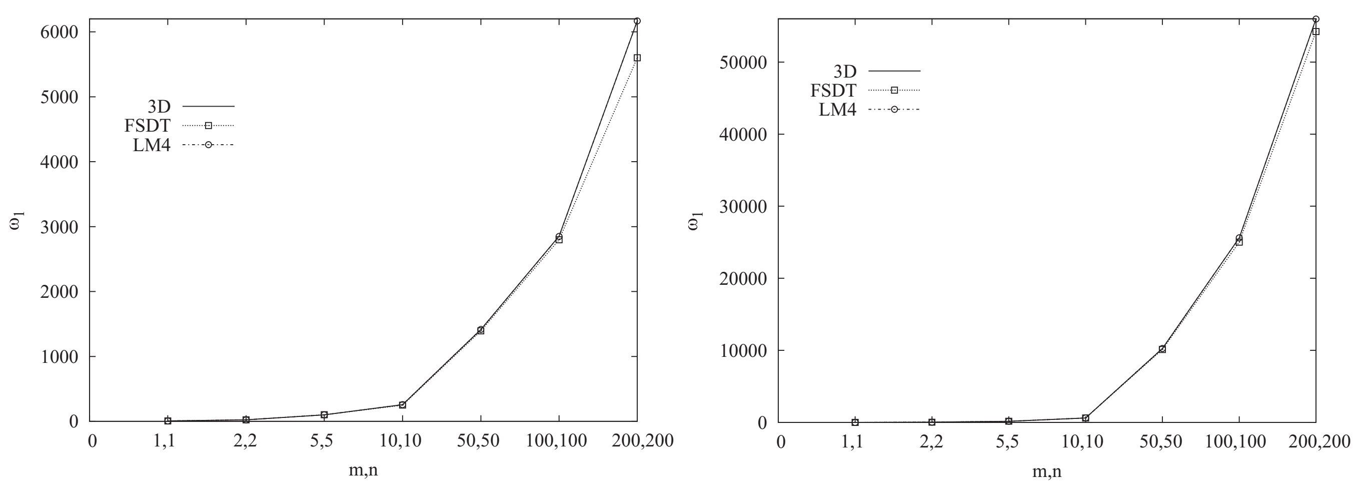

FIG. 2. Simply supported FGM Al/ZrO_2 plate. Fundamental frequency for $K = 1$ and several m, n. Compared theories for thickness ratio: (left) $a/h = 10$; (right) $a/h = 100$.

TABLE 3
Compared theories for an FGM Al/Al_2O_3 sandwich plate for a/h =4 -Adim: $\bar{\omega} = \frac{\omega a}{h}\sqrt{\frac{\rho_0}{E_0}}$

	Natural frequency						
	$\bar{w}$	LD4	ED4	LM3	ED2	FSDT	CLT
K = 1	$\bar{w}_1$	1.1964	1.1965	1.1964	1.2045	1.1864	1.3133
	$\bar{w}_2$	2.8716	2.8718	2.8716	2.8735	2.8755	2.8983
	$\bar{w}_3$	4.7622	4.7634	4.7624	4.7731	4.8215	4.9300
	$\bar{w}_4$	8.1708	8.2040	8.1762	8.4166	8.7656	–
	$\bar{w}_5$	9.0592	9.1023	9.0665	9.3589	9.8675	–
	$\bar{w}_6$	13.608	13.785	13.630	14.850	–	–
K = 5	$\bar{w}_1$	1.0357	1.0365	1.0351	1.0654	1.0628	1.1955
	$\bar{w}_2$	2.3620	2.3628	2.3616	2.3779	2.3762	2.4159
	$\bar{w}_3$	3.8287	3.8323	3.8276	3.9061	3.9546	4.1396
	$\bar{w}_4$	5.8725	5.8887	5.9047	6.6376	7.2248	–
	$\bar{w}_5$	6.8237	6.8438	6.8829	7.6085	8.3316	–
	$\bar{w}_6$	9.6932	9.7987	9.7674	11.628	–	–
K = 10	$\bar{w}_1$	1.0095	1.0123	1.0098	1.0535	1.0484	1.1951
	$\bar{w}_2$	2.2269	2.2283	2.2275	2.2447	2.2408	2.2765
	$\bar{w}_3$	3.6058	3.6122	3.6086	3.6896	3.7331	3.8992
	$\bar{w}_4$	5.5612	5.5964	5.5541	6.4245	6.8148	–
	$\bar{w}_5$	6.5135	6.5449	6.5238	7.3763	7.9046	–
	$\bar{w}_6$	9.0712	9.2246	9.1161	11.260	–	–

TABLE 4
Compared theories for an FGM Al/Al_2O_3 sandwich plate for a/h =10 -Adim: $\bar{\omega} = \frac{\omega a}{h}\sqrt{\frac{\rho_0}{E_0}}$

	Natural frequency							
	$\bar{\omega}$	Ref. [7]	LD4	ED4	LM3	ED2	FSDT	CLT
K = 1	$\bar{\omega}_1$	1.3485	1.3485	1.3485	1.3485	1.3503	1.3509	1.3746
	$\bar{\omega}_2$	-	7.2355	7.2355	7.2355	7.2362	7.2384	7.2459
	$\bar{\omega}_3$	-	12.195	12.195	12.195	12.199	12.224	12.260
	$\bar{\omega}_4$	-	48.833	49.021	48.868	50.470	57.293	-
	$\bar{\omega}_5$	-	49.867	50.062	49.903	51.547	58.507	-
	$\bar{\omega}_6$	-	88.302	89.312	88.373	93.423	–	-
K = 5	$\bar{\omega}_1$	-	1.2225	1.2227	1.2223	1.2307	1.2306	1.2618
	$\bar{\omega}_2$	-	6.0181	6.0184	6.0180	6.0247	6.0238	6.0399
	$\bar{\omega}_3$	-	10.104	10.105	10.103	10.136	10.154	10.232
	$\bar{\omega}_4$	-	33.923	34.037	34.077	39.099	43.787	-
	$\bar{\omega}_5$	-	35.068	35.184	35.247	40.236	42.527	-
	$\bar{\omega}_6$	-	61.885	62.254	62.234	72.299	–	-
K = 10	$\bar{\omega}_1$	-	1.2150	1.2159	1.2150	1.2275	1.2265	1.2615
	$\bar{\omega}_2$	-	5.6707	5.6713	5.6710	5.6783	5.6766	5.6912
	$\bar{\omega}_3$	-	9.5172	9.5201	9.5186	9.5540	9.5703	9.6406
	$\bar{\omega}_4$	-	31.869	32.132	31.809	37.796	39.925	-
	$\bar{\omega}_5$	-	33.053	33.306	33.001	38.929	41.182	-
	$\bar{\omega}_6$	-	57.928	58.615	57.996	69.889	–	-

TABLE 5
Compared theories for an FGM Al/Al_2O_3 sandwich plate for a/h =100 -Adim: $\bar{\omega} = \frac{\omega a}{h}\sqrt{\frac{\rho_0}{E_0}}$

	Natural frequency							
	$\bar{\omega}$	Ref. [7]	LD4	ED4	LM3	ED2	FSDT	CLT
K = 1	$\bar{\omega}_1$	1.3867	1.3867	1.3867	1.3867	1.3867	1.3787	1.3787
	$\bar{\omega}_2$	–	72.458	72.458	72.458	72.458	72.458	72.458
	$\bar{\omega}_3$	–	122.47	122.47	122.47	122.47	116.83	116.83
	$\bar{\omega}_4$	–	4840.8	4859.2	4844.2	5006.4	5687.1	–
	$\bar{\omega}_5$	–	4841.9	4860.2	4845.3	5007.5	5688.3	–
	$\bar{\omega}_6$	–	9051.1	9087.7	9057.4	9365.1	–	–
K = 5	$\bar{\omega}_1$	–	1.2748	1.2748	1.2748	1.2749	1.2749	1.2753
	$\bar{\omega}_2$	–	60.397	60.397	60.397	60.398	60.398	60.399
	$\bar{\omega}_3$	–	102.08	102.08	102.08	102.08	102.08	102.09
	$\bar{\omega}_4$	–	3339.0	3350.8	3353.7	3864.5	4202.6	–
	$\bar{\omega}_5$	–	3340.2	3351.9	3354.9	3865.7	4203.9	–
	$\bar{\omega}_6$	–	6245.1	6267.3	6272.6	7229.0	–	–
K = 10	$\bar{\omega}_1$	–	1.2745	1.2745	1.2745	1.2746	1.2746	1.2750
	$\bar{\omega}_2$	–	56.910	56.910	56.910	56.911	56.911	56.912
	$\bar{\omega}_3$	–	961.89	961.89	961.89	961.93	961.94	962.01
	$\bar{\omega}_4$	–	3130.9	3158.0	3124.7	3734.4	3941.4	–
	$\bar{\omega}_5$	–	3132.1	3159.2	3125.9	3735.6	3942.7	–
	$\bar{\omega}_6$	–	5855.6	5906.6	5844.2	9619.3	–	–

The eigenvectors $\hat{\boldsymbol{U}}$ associated to the eigenvalues λ_{mn} (or to circular frequencies ω_{mn}) define the vibration modes of the structure in terms of primary variables. Once the wave numbers (m, n) have been defined in the in-plane directions, the number of obtained frequencies becomes equal to the degrees of freedom of the employed two-dimensional model. It is possible to obtain the relative eigenvector, in terms of primary variables, for each value of frequency, in order to plot the modes in the thickness direction.

8. NUMERICAL RESULTS

In the same manner as in the previous CUF works and with the available reference solutions, the power law distribution given in [1] is here employed. The variation of properties in terms of

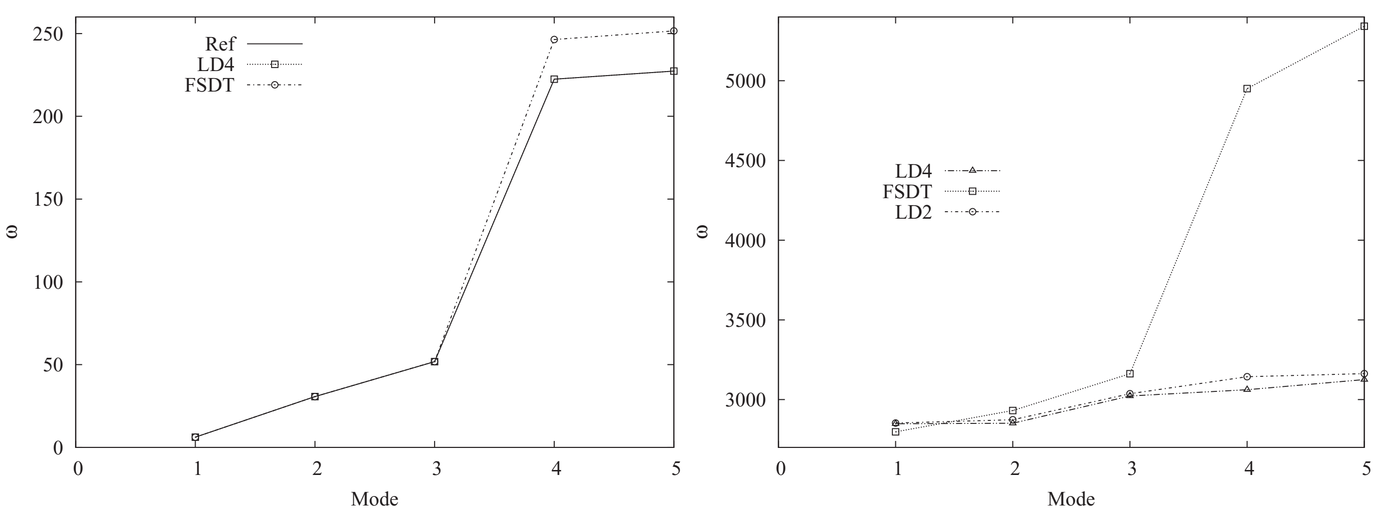

FIG. 3. Simply supported FGM *Al/ZrO*$_2$ plate. Fundamental frequency for $K = 1$: (left) $m, n = 1$; (right) $m, n = 100$. Compared theories for thickness ratio $a/h = 10$. *Ref* solution by Matsunaga [16].

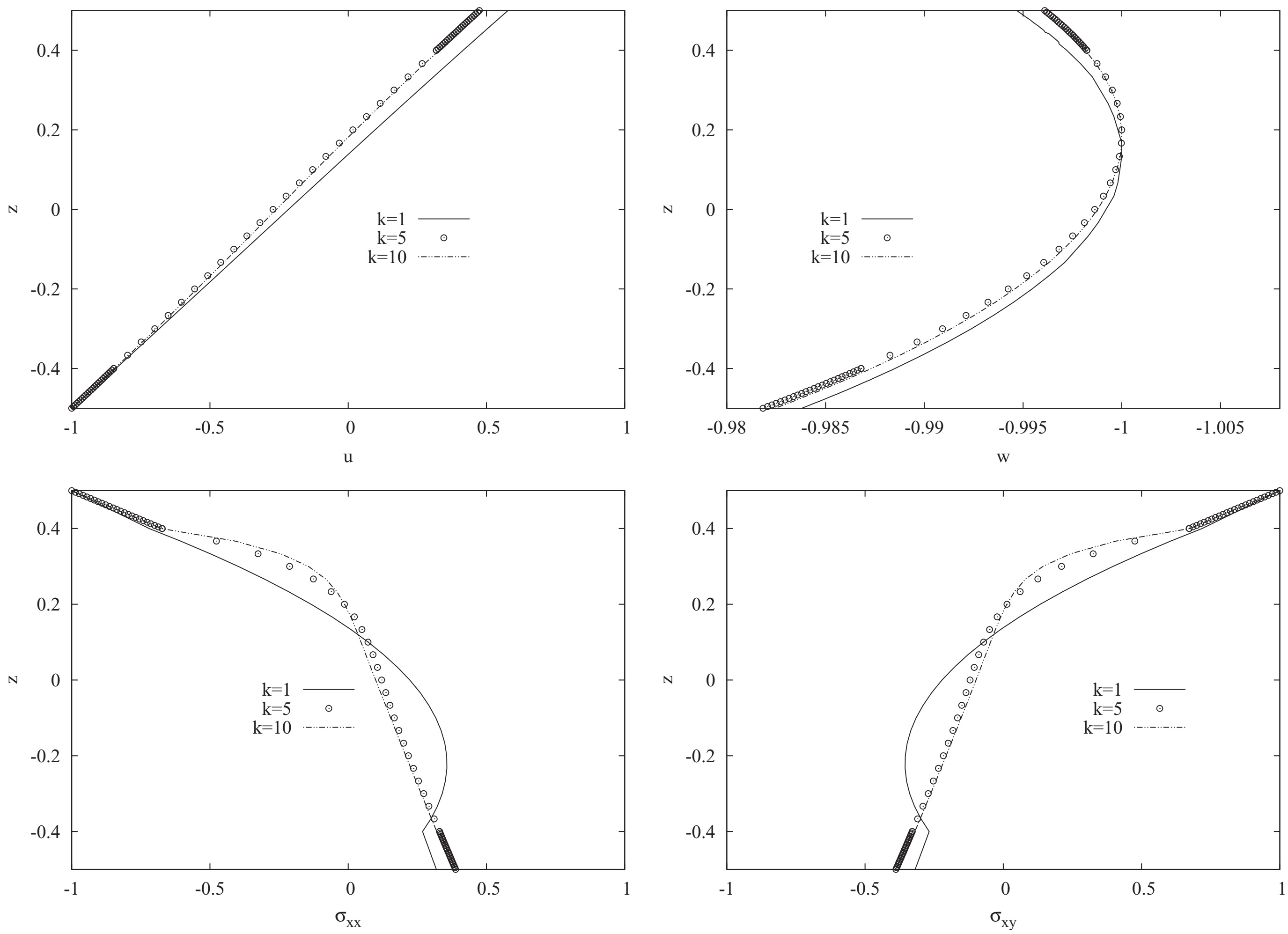

FIG. 4. FGM Al/Al_2O_3 sandwich plate. Stresses and displacements for first fundamental natural frequency through the thickness using *LD*4 theory for $a/h = 10$. Several k compared.

Young's Modulus, shear modulus and density, is related to the change in volume fractions of the materials in agreement with the law:

$$\wp = \wp_m + (\wp_c - \wp_m)\left(\frac{1}{2} + \frac{z}{H}\right)^k \tag{29}$$

where $\wp_m$ is the property of metallic phase, $\wp_c$ is the property of the ceramic phase, H is the global thickness and z is the variation in the thickness coordinate, which varies between $-H/2$ and $H/2$, and k is the power law exponent.

In order to validate the accuracy of CUF for free vibration analysis of FGM plates, the following problems have been examined:

- Case 1. Isotropic Aluminium plate ($E_m = 70\ Gpa$, $\rho = 2702$ Kg/m^3, $\nu = 0.3$). The reference solution has been provided by Vel and Batra [10].
- Case 2. Functionally graded plate with varying properties from Alluminium ($E_m = 70\ Gpa$, $\rho = 2702$ Kg/m^3, $\nu = 0.3$) at the top to Zirconia ZrO_2 ($E_c = 200\ Gpa$, $\rho = 5700$ Kg/m^3, $\nu = 0.3$) at the bottom surface. The reference solution has been provided by Matsunaga [16].
- Case 3. Functionally graded sandwich plate: isotropic Al layer (h $= 0.1h_{tot}$, $E_m = 70\ Gpa$, $\rho = 2702$ Kg/m^3, $\nu = 0.3$) at the bottom; FGM core ($h = 0.8h_{tot}$); isotropic Al_2O_3 layer (h $= 0.1h_{tot}$, $E_c = 380\ Gpa$, $\rho = 3800$ Kg/m^3, $\nu = 0.3$) at the top. The values 4, 10, 100 have been considered for the thickness ratio a$/h$. The reference solution for $k = 1$ is given by Li [7].

For cases 1 and 2, the first eight natural frequencies $\bar{\omega} = \frac{\omega a}{h}\sqrt{\frac{\rho_m}{E_m}}$ are compared for various models: hEm LD4, ED4, ED1, FSDT and CLT. The last three modes lead to less than eight frequencies: the CLT theory has only three unknowns u, v, w and therefore there are three natural frequencies; FSDT has five degrees of freedom u, v, w,ϕ_x,ϕ_y; ED1 has six unknowns, two for each displacement component; the ED4 and LD4 theories have (4 + 1) degrees of freedom for each displacement component, therefore there are 15 frequencies. Tables 1 and 2 show the results obtained for cases 1 and 2, respectively. Both plates have a thickness ratio a/h equal to 10 and can therefore be considered thick. It is possible to notice that the LD4 and ED4 theories provide the same values of frequencies: in the case of a single layer plate, there is no difference between the LW and ESL approaches. LD4 and ED4 are more accurate than classical plate theories (CLT, FSDT); in particular, the use of high-order theories permits the exact value of lower order frequencies to be obtained. The ED1 theory also provides good results. Figure 2 shows the first fundamental frequency of the FGM Al/ZrO_2 plate obtained by varying the wave numbers m, n ($K = 1$ and $a/h = 10$, 100). FSDT and LM4 (this is a layer-wise mixed theory with fourth order expansion for both displacements and transverse stresses) theories are compared with the reference solution. It is possible to note that, for low values of m, n, both models provide good results; increasing the wave numbers, the FSDT theory leads to a larger error in the thick plate case ($a/h = 10$). The natural frequencies corresponding to various modes are shown in Figure 3, for the same plate. Again in this case, it is possible to see that the FSDT theory gives erroneous results compared to the LD4 and LD2 models, especially for higher modes and and higher wave numbers (m, $n = 100$). These results demonstrate that the use of CUF is quite convenient to establish the accuracy of various theories for both isotropic and functionally graded structures.

Case 3 proposes a benchmark for a square and simply supported sandwich plate with a functionally graded core embedded between two isotropic skins: Aluminum at the bottom and Alumina at the top. This configuration ensures the continuity of the mechanical properties at the layer interfaces. A dynamic analysis of the plate is proposed for various thickness ratios ($a/h = 4$, 10, 100), material exponents (k =1, 5, 10) and selected models (LD4, ED4, LM3, ED2, FSDT, CLT). For the cases $k = 1$ and $a/h = 10$, 100, the first fundamental frequencies are compared with a reference solution, that is provided in [7]. Tables 3, 4 and 5 confirm all the conclusions concerning cases 1 and 2. The CUF results are in good agreement with the reference solution and the differences become smaller as the plate thickness decreases. In particular, it is possible to note that for thickness ratio $a/h = 100$, the classic theories CLT and FSDT provide an acceptable precision to calculate the frequencies, while for $a/h = 4$, it is mandatory to use high-order theories in order to obtain the correct natural frequencies. In terms of frequencies, there are no significant differences between the PVD and RMVT models. The latter, however, permits the continuity of transverse shear/normal stresses to be obtained at the layer interfaces, if the vibrational modes are plotted in the thickness direction. Figure 4 plots the modes for the fundamental frequency, in terms of displacements and in-plane stresses, obtained by varying the exponent k. It is important to note that the transverse displacement is not constant even though the material gradient is low ($k = 1$). This permits one to conclude that it is not appropriate to use classical theories to model structures embedding FGM layers.

9. CONCLUSIONS

In this paper, Carrera's Unified Formulation has been extended to free-vibration analysis of multilayered plates embedding functionally graded layers. The use of CUF permits results that are in good agreement with the reference solution to be obtained for the case of single layer functionally graded plates. In particular, the analysis shows that high-order theories are necessary to provide an adequate accuracy for thick plates, while classical theories give erroneous results, especially for high wave numbers and high-order frequencies.

REFERENCES

1. A.M. Zenkour, Generalized Shear Deformation Theory for Bending Analysis of Functionally Graded Plates, Appl. Math. Model., vol. 30, pp. 67–84, 2006.
2. M. Kashtalyan, Three-Dimensional Elasticity Solution for Bending of Functionally Graded Rectangular Plates, European J. Mech. A Solids, vol. 23, pp. 853–864, 2004.
3. T. Mori and K. Tanaka, Average Stress in Matrix and Average Elastic Energy of Materials with Misfitting Inclusions, Acta Metallurgica, vol. 21, pp. 571–574, 1973.
4. R.C. Batra and J. Jin, Natural Frequencies of a Functionally Graded Anisotropic Rectangular Plate, J. Sound Vib, vol. 282, pp. 509–516, 2005.
5. H. Rastogoftar, M. Eghtesad, A. Khayatian, et al., Boundary Control of Large Vibration of Fgm Rectangular Plates. Automation Congress, 2008. WAC 2008: 1–6.
6. R. Kadoli and L Ganesan, Buckling and Free Vibration Analysis of Functionally Graded Cylindrical Shells Subjected to a Temperature-Specified Boundary Condition, J. Sound Vib., vol. 289, pp. 450–480, 2006.
7. Q. Li, V.P. Iu, and K.P. Kou, Three-Dimensional Vibration Analysis of Functionally Graded Material Sandwich Plates, J. Sound Vib., vol. 311, pp. 498–515, 2008.
8. K.M. Liew, J. Yang, and Y.F. Wu, Nonlinear Vibration of a Coating-FGM-Substrate Cylindrical Panel Subjected to a Temperature Gradient, Comput. Methods Appl. Mech. Engrg., vol. 195, pp. 1007–1026, 2006.
9. S.A. Sina, H.M. Navazi, and H. Haddadpour, An Analytical Method for Free Vibration Analysis of Functionally Graded Beams, Materials and Design, vol. 30, pp. 741–747, 2009.
10. S.S. Vel and R.C. Batra, Three-Dimensional Vibration Analysis of Functionally Graded Material Sandwich Plates, J. Sound Vib., vol. 272, pp. 703–730, 2004.
11. A.M. Zenkour, A Comprehensive Analysis of Functionally Graded Sandwich Plates: Part 2-Buckling and Free Vibration, Int. J. Solids Struct., vol. 42, pp. 5243–5258, 2005.
12. E. Carrera, Evaluation of Layer-Wise Mixed Theories for Laminated Plates Analysis, AIAA Journal, vol. 26, pp. 830–839, 1998.
13. E. Carrera, Theories and Finite Elements for Multilayered Plates and Shells: A Unified Compact Formulation with Numerical Assessments and Benchmarks, Arch. Comput. Meth. Eng., vol. 10, pp. 215–296, 2003.
14. E. Carrera, S. Brischetto, and A. Robaldo, Variable Kinematic Model for the Analysis of Functionally Graded Material Plates, AIAA Journal, vol. 46, pp. 194–203, 2008.
15. S. Brischetto and E. Carrera, Advanced Mixed Theories for Bending Analysis of Functionally Graded Plates, Comput. Struct., in press, available on line on May 2008.
16. H. Matsunaga, Free Vibration and Stability of Functionally Graded Plates According to a 2-D Higher-Order Deformation Theory, Compos. Struct., vol. 82, pp. 499–512, 2008.

17. E. Carrera, A Class of Two Dimensional Theories for Multilayered Plates Analysis, Atti Accademia delle Scienze di Torino. Memorie Scienze Fisiche. vol. 19–20, pp. 49–87, 1995.
18. E. Carrera, Theories and Finite Elements for Multilayered Anisotropic, Composite Plates and Shells, Arch. Comput. Meth. Eng., vol. 9, pp. 87–140, 2002.
19. E. Reissner, The Effect of Transverse Shear Deformation on the Bending of Elastic Plates, J. Appl. Mech., vol. 12, pp. 69–76, 1945.
20. R.D. Mindlin, Influence of Rotatory Inertia and Shear in Flexural Motions of Isotropic Elastic Plates, J. Appl. Mech., vol. 18, pp. 31–28, 1951.
21. G. Kirchho, Uber das Gleichgewicht und die Bewegung Einer Elastishen Scheibe, Crelles Journal, vol. 40, pp. 51–88, 1850.
22. E. Reissner, On a Certain Mixed Variational Theory and a Proposed Application, Int. J. Numer. Meth. Eng., vol. 20, no. 7, pp. 1366–1368, 1984.
23. E. Carrera, Layer-Wise Mixed Models for Accurate Vibration Analysis of Multilayered Plates, J. Appl. Mech., vol. 65, pp. 820–828, 1998.
24. E. Carrera, An Assessment of Mixed and Classical Theories on Global and Local Response of Multilayered, Orthotropic Plates, Comput. Struct., vol. 50, pp. 183–198, 2000.