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THERMOMECHANICAL EFFECT IN VIBRATION ANALYSIS OF ONE-LAYERED AND TWO-LAYERED PLATES

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The free vibration problem of one-layered and two-layered metallic plates is investigated in this work. The thermomechanical effect is evaluated using a fully coupled thermomechanical model. The free frequency values of fully coupled problems are compared to the values of the pure mechanical problems. In pure mechanical models, the displacement is the only primary variable of the problem, while in fully coupled thermomechanical models, the temperature is also considered as a primary variable and the effect of the thermomechanical stiffness is evaluated. The thermoelastic coupling usually provides higher frequencies with respect to the pure mechanical case because it acts like a thermal source, which is proportional to the strain rate, which leads to a bigger global stiffness of the structure. Both thermomechanical and mechanical models are developed in the framework of Carrera's Unified Formulation (CUF). CUF permits several refined two-dimensional theories to be obtained with orders of expansion in the thickness direction, from linear to fourth-order, for both displacements and temperature. Both equivalent single layer and layer-wise approaches are considered for the multilayered plates. The thermomechanical effect is investigated, in terms of frequencies, for thick and thin one-layered and two-layered plates, and for lower and higher modes. It has mainly been concluded that the thermomechanical coupling: (a) Is correctly determined if both the thermal and mechanical parts are correctly approximated; (b) Is small for each investigated case; (c) Influences the various vibration modes in different ways; and (d) Has a limited dependence on the considered case, but this dependence vanishes if a global coupling is considered. Only fully coupled thermomechanical models allow to analyze this type of problem. The effect of the thermomechanical coupling on higher-order modes can only be investigated using refined two-dimensional theories.

Keywords: Multilayered plates; thermomechanical coupling; Carrera's Unified Formulation; free vibrations; refined theories; layerwise theories.

1. Introduction

The effect of the temperature field on the deformation field is not a one-way phenomenon: as suggested in Nowinski's book [Nowinski, 1978], a deformation of the

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body produces changes in its temperature, therefore the mechanical and thermal aspects are inseparable [Nowinski, 1978; Brischetto and Carrera, 2010a]. A fully coupled thermoelastic analysis, where both temperature and displacement fields are primary variables in the governing equations, has been proposed by Brischetto and Carrera [2010a]. The fully coupled thermomechanical governing equations directly give the displacements and the temperature through the thickness direction. This coupling considerably complicates the computational aspect of solving actual thermoelastic problems: as suggested by Nowinski [1978], it is generally possible to discount the coupling and to evaluate the temperature and deformation fields, in this order, separately. The thermoelastic problem, where the temperature and deformation fields are discounted, has been defined by Brischetto and Carrera [2010a] as a partially coupled thermomechanical problem. Partially coupled thermomechanical models are extensively employed in the analysis of typical aeronautical structures, such as one-layered isotropic or multilayered composite plates and shells, where the temperature variations are one of the most important factors, for the stress fields, that can cause failure of the structures [Librescu and Marzocca, 2003a, 2003b; Noor and Burton, 1992]. Several partially coupled thermomechanical models, where the temperature profile was *a priori* defined assuming it linear in the thickness direction or by solving the Fourier's heat conduction equation, have been developed. These models have been extended to plate geometries by Carrera [2000, 2002], and Brischetto *et al.* [2008] and to shell configurations by Brischetto and Carrera [2009] and Brischetto [2009]. However, partially coupled thermomechanical models cannot investigate the thermomechanical coupling effect, in terms of frequencies, because they do not consider the thermal part in the stiffness matrix.

The importance of refined two-dimensional theories for the free vibration analysis of multilayered plates, in particular for the cases of moderately thick plates, composite layers, and higher-order modes, has been dealt with in the study by Brischetto and Carrera [2010b]. In this paper, the effect of thermomechanical coupling is investigated in the free vibration analysis of one-layered and two-layered metallic plates in terms of frequency values. A preliminary study has been performed by Brischetto and Carrera [2010a], where only the fundamental frequency has been investigated, for wave numbers $m = n = 1$, in the case of pure mechanical and fully coupled thermomechanical models. This study gives a complete overview on the effects of the thickness ratio, layers stacking, higher-order modes (higher values of wave number m and n), and high frequency values (frequencies different from the fundamental one related to the degrees of freedom of the employed two-dimensional theory) on the thermomechanical coupling. In the open literature, a small amount of work has been devoted to the coupled thermomechanical analysis of structures, but only a few of these papers give numerical results. Altay and Dökmeci [1996a] have described the physical behavior of thermoelastic continuum by means of opportune variational principles. The stress equations of motion and the equation of heat conduction have been written as divergence equations. The strain–mechanical displacement relation and Fourier's heat conduction law have been written as gradient equations. The same authors have extended this method to thermopiezoelectric mediums in Altay and Dökmeci [1996b] by simply adding the electrostatics charge equation in the divergence equations and the electric field–electric potential relations in the gradient equations. Das *et al*. [1983] have avoided the use of the thermoelastic potential to solve the general problem of one-dimensional linearized simultaneous equations of thermoelasticity. Displacement and thermal fields have been obtained in the Laplace transformation domain. Cannarozzi and Ubertini [2001] have proposed a variational method for linear coupled quasi-static thermoelastic analysis. The variational support is a statement in terms of displacement, temperature, stress, and heat flux.

Examples of dynamic thermomechanical analysis have been collected from the following studies. Altay and Dökmeci [2001] have considered the thermal relaxation term in the dissipation function. A three-dimensional theory of coupled thermoelasticity has been given for a shell geometry. High-frequency vibrations of temperature-dependent materials have been investigated. The dynamic response of a thermally loaded elastic structure has been considered by Givoli and Rand [1994]. The thermoelastic coupling term in a heat equation acts like a thermal source that is proportional to the strain rate. The thermoelastic coupling effect on the temperature and displacement fields has been measured as a function of the normalized frequency. It has been concluded that the inclusion of dynamic thermoelastic coupling effects in the analysis of space structures should be considered whenever the applied thermal load contains a significant Fourier component with a frequency that is close to the critical frequency of the structure. Wilms and Cohen [1985] have studied wave propagation in a thermoelastically coupled half-space. Strain, temperature, stress, and particle velocity have been obtained. The dynamic interaction of magneto-thermoelastic waves has been investigated by Wauer [1996]. The thermally induced small thickness vibrations are of particular interest. The displacement and incremental temperature have been coupled and the thermoelastic modes have been given. Considering a thermomechanical coupling, the obtained results demonstrate that even though no mechanical damping has been considered, the eigenvalues have been characterized by their imaginary part (natural frequency) and also by their nonvanishing (negative) real part (dissipation). A modified model of thermoelasticity, with an extra thermal stress effect and wave-type heat conduction, has been given by Kosinski and Frischmuth [2001]. The model has been governed by a system of quasi-linear hyperbolic equations. All the essential material functions have been examined, and the impact of their nonlinearity on the solution of initial-boundary value problems has been studied. However, already at this temperature, the model solutions do not differ numerically from solutions obtained on the basis of Fourier's heat conduction law. This is not true around the critical temperature, where the error reaches about 1000% of the Fourier result. Thermoelastic vibrations of a free supported and clamped circular plate, caused by a thermal shock on the surface, have been analyzed by Trajkovski and Cukic [1999]. The partial differential equations of the coupled system have been reduced to Volterra's first and second kind of integral equations in the time domain. Very rapid thermal processes, under the action of a thermal shock, are interesting from the thermoelasticity point of view, since they require an analysis of the coupled temperature and deformation fields. This means that the temperature shock induces very rapid movements in the structure elements, thus causing the rise of very significant inertial forces, and, thereby, a rise in vibrations. Rapidly changeable contractions and expansions in oscillatory movements generate temperature changes in the material, which is susceptible to diffusion due to heat conduction. This means that, in the case of the exact solution, the plate behaves in a less rigid way. Yeh [2005] has presented an analysis of large amplitude, thermomechanically coupled vibrations of a simply supported orthotropic rectangular thin plate. The governing partial differential equation of thermomechanical coupling has been derived and simplified to a set of three, nonlinear, ordinary differential equations using the Galerkin method.

The fully coupled thermomechanical governing equations permit the frequencies of the free vibration analysis of simply supported plates to be evaluated. Such frequencies are compared with the pure mechanical ones to evaluate the thermomechanical effect. To obtain fully coupled governing equations, the Principle of Virtual Displacements (PVD) is modified by adding the internal thermal work. The employed refined two-dimensional theories are obtained as in Sec. 2; the displacement can be modeled in both Equivalent Single Layer (ESL) and Layer Wise (LW) form, but the temperature is always approximated in LW form because of its higher spatial gradient. Section 3 gives the relations between the strain components and the displacement vector, and Sec. 4 illustrates the thermomechanical constitutive equations. PVD is extended to the fully coupled thermomechanical analysis in Sec. 5. The results for simply supported metallic one-layered and two-layered plates are discussed in Sec. 6. Section 7 gives the main conclusions.

2. Carrera's Unified Formulation (CUF)

Carrera's Unified Formulation (CUF) permits a large variety of two-dimensional plate theories [Carrera, 1995, 2003a] to be obtained in a unified manner. According to CUF, the governing equations are written in terms of a few fundamental nuclei, which do not formally depend on the order of expansion N used in the thickness direction and on the description of variables (equivalent single layer (ESL) or layer-wise (LW)). The application of a two-dimensional method for plates permits the unknown variables to be expressed as a set of thickness functions that only depend on the thickness coordinate z and the correspondent variable, which depends on the in-plane coordinates x and y. Therefore, the generic variable array $\mathbf{f}(x, y, z)$, for example, the displacement vector, and its variation $\delta\mathbf{f}(x, y, z)$ are written according to the following general expansion:

$$\mathbf{f}(x, y, z) = F_\tau(z)\mathbf{f}_\tau(x, y), \quad \delta\mathbf{f}(x, y, z) = F_s(z)\delta\mathbf{f}_s(x, y), \quad \tau, s = 1, \dots, N \tag{1}$$

where the bold letters denote arrays, (x, y) are the in-plane coordinates, and z is the thickness one. The summing convention, with repeated indexes τ and s, is assumed. The order of expansion N goes from first to fourth-order, and depending on the used thickness functions, a model can be ESL or LW. In the thermomechanical models, displacements can be modeled in both ESL or LW form, temperature is always considered in LW form. Therefore, a two-dimensional thermomechanical model is defined as ESL or LW, depending on the choice made for the displacement vector [Carrera and Brischetto, 2009].

2.1. *Equivalent single layer approach*

For the displacement $\mathbf{u} = (u, v, w)$ in equivalent single layer (ESL) form, the unknowns are the same for the whole plate [Reddy, 2004] (see Fig. 1). The z expansion is obtained via Taylor polynomials, that is

$$\begin{aligned} u &= F_0 u_0 + F_1 u_1 + \cdots + F_N u_N = F_\tau u_\tau \\ v &= F_0 v_0 + F_1 v_1 + \cdots + F_N v_N = F_\tau v_\tau \\ w &= F_0 w_0 + F_1 w_1 + \cdots + F_N w_N = F_\tau w_\tau \end{aligned} \tag{2}$$

with $\tau = 0, 1, \ldots, N$; N is the order of expansion that ranges from 1 (linear) to 4:

$$F_0 = z^0 = 1, \quad F_1 = z^1 = z, \ldots, \quad F_N = z^N \tag{3}$$

Equation (2) can be written in a vectorial form:

$$\mathbf{u}(x, y, z) = F_\tau(z)\mathbf{u}_\tau(x, y), \quad \delta\mathbf{u}(x, y, z) = F_s(z)\delta\mathbf{u}_s(x, y), \quad \tau, s = 1, \ldots, N \tag{4}$$

Simpler theories, such as those which discard the ϵ_{zz} effect, are obtained from refined ESL models by imposing that the transverse displacement w is constant in z. First-order Shear Deformation Theory (FSDT) [Mindlin, 1951] is obtained from an ESL model with linear expansion in the thickness direction z, imposing a constant transverse displacement w in z. Classical Lamination Theory (CLT) [Cauchy, 1828; Poisson, 1829; Kirchhoff, 1850] is obtained from FSDT imposing an infinite

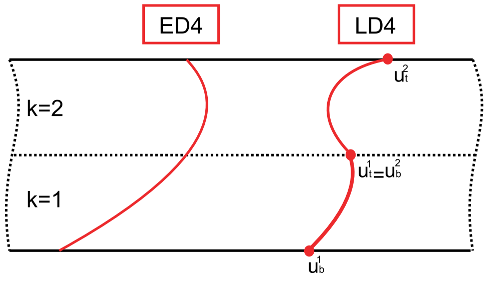


Fig. 1. Equivalent single layer and layerwise theories for a two-layered plate.

transverse shear rigidity. All the ESL theories, with constant or linear transverse displacement w, which means zero or constant transverse normal strain ϵ_{zz}, show Poisson's locking phenomena; this can be overcome via plane-stress conditions in constitutive equations [Carrera and Brischetto, 2008a, 2008b].

2.2. *Layerwise approach*

In a layerwise (LW) approach, each layer of a multilayered structure is described as an independent plate [Reddy, 2004]. The displacement $\mathbf{u}^k = (u, v, w)^k$ is described for each layer k: the zigzag form of displacement [Murakami, 1986; Carrera, 2003b], in multilayered transverse-anisotropy structures, is easily obtained as indicated in Fig. 1. The z expansion for displacement components is made for each layer k:

$$\begin{aligned} u^k &= F_0 u_0^k + F_1 u_1^k + \cdots + F_N u_N^k = F_\tau u_\tau^k \\ v^k &= F_0 v_0^k + F_1 v_1^k + \cdots + F_N v_N^k = F_\tau v_\tau^k \\ w^k &= F_0 w_0^k + F_1 w_1^k + \cdots + F_N w_N^k = F_\tau w_\tau^k \end{aligned} \tag{5}$$

with $\tau = 0, 1, \ldots, N, N$ is the order of expansion that ranges from 1 (linear) to 4. $k = 1, \ldots, N_l$, where N_l indicates the number of layers. Equation (5) written in a vectorial form is

$$\begin{gathered} \mathbf{u}^k(x, y, z) = F_\tau(z)\mathbf{u}_\tau^k(x, y), \quad \delta\mathbf{u}^k(x, y, z) = F_s(z)\delta\mathbf{u}_s^k(x, y) \\ \tau, s = t, b, r, \quad k = 1, \ldots, N_l \end{gathered} \tag{6}$$

where t and b indicate the top and bottom of each layer k, respectively; r indicates the higher-orders of expansion in the thickness direction: $r = 2, \ldots, N$. The thickness functions $F_\tau(\zeta_k)$ and $F_s(\zeta_k)$ have now been defined at the k-layer level; they are a linear combination of Legendre polynomials $P_j = P_j(\zeta_k)$ of the jth-order defined in ζ_k-domain ($\zeta_k = \frac{2z_k}{h_k}$ with z_k local coordinate and h_k thickness, both referred to kth layer, so $-1 \leq \zeta_k \leq 1$). The first five Legendre polynomials are

$$\begin{gathered} P_0 = 1, \quad P_1 = \zeta_k, \quad P_2 = \frac{(3{\zeta_k}^2 - 1)}{2}, \quad P_3 = \frac{5{\zeta_k}^3}{2} - \frac{3\zeta_k}{2}, \\ P_4 = \frac{35{\zeta_k}^4}{8} - \frac{15{\zeta_k}^2}{4} + \frac{3}{8} \end{gathered} \tag{7}$$

Their combinations for the thickness functions are

$$F_t = F_0 = \frac{P_0 + P_1}{2}, \quad F_b = F_1 = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2}, \quad r = 2, \ldots, N \tag{8}$$

The chosen functions have the following interesting properties:

$$\zeta_k = 1 : F_t = 1; \quad F_b = 0; \quad F_r = 0 \quad \text{at top} \tag{9}$$

$$\zeta_k = -1 : F_t = 0; \quad F_b = 1; \quad F_r = 0 \quad \text{at bottom} \tag{10}$$

that is interface values of the variables are considered as variable unknowns (see Fig. 1). This feature permits to easily impose the compatibility conditions for displacements at each layer interface. In LW theories, even when a linear expansion in z is considered for the transverse displacement w, Poisson's locking phenomena does not appear: the transverse normal strain ϵ_{zz} is piecewise constant in the thickness direction of a multilayered plate [Carrera and Brischetto, 2008a, 2008b].

In case of thermomechanical problems, the primary variables are the displacement vector $\mathbf{u} = (u, v, w)$ and the scalar sovra-temperature θ (temperature T_1 referred to the reference external room temperature T_0, $\theta = T_1 - T_0$). By considering the higher spatial gradient of the temperature field and the different thermal properties of the embedded layers, the variable θ^k is always modeled as LW [Brischetto and Carrera, 2010a; Carrera *et al.*, 2008]:

$$\begin{aligned} \theta^k(x,y,z) = F_\tau(z)\theta^k_\tau(x,y), \quad \delta\theta^k(x,y,z) = F_s(z)\delta\theta^k_s(x,y) \\ \tau, s = t, b, r, \quad k = 1, \ldots, N_l \end{aligned} \tag{11}$$

The thickness functions are a combination of Legendre polynomials as indicated in Eqs. (7) and (8).

3. Geometrical Relations

A thin plate is a three-dimensional body bounded by two closely spaced surfaces; the distance between the two surfaces must be small in comparison with the other dimensions. The middle surface of the plate is the locus of points which lie midway between these surfaces. The distance between the surfaces measured along the normal to the middle surface is the thickness of the plate at that point [Leissa, 1969]. The reference system, indicated in Fig. 2, is a rectilinear cartesian one (x, y, z) [Reddy, 2004]. The in-plane dimensions are indicated with a and b in x- and y-directions, respectively. The thickness value in z-direction is indicated with h.

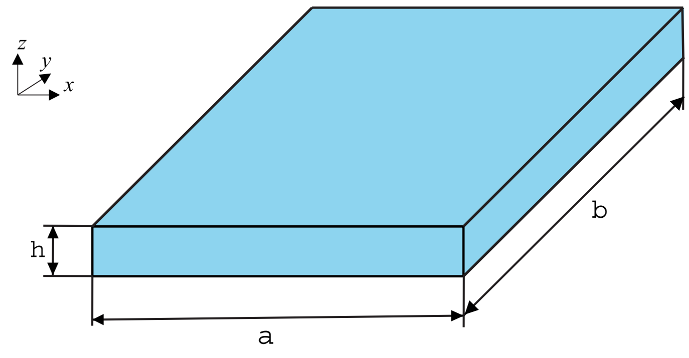


Fig. 2. Reference system and geometry for the considered plates.

The geometrical relations for plates link the mechanical strains with the displacement vector:

$$\boldsymbol{\epsilon}_{pG}^k = [\epsilon_{xx}\epsilon_{yy}\gamma_{xy}]^{kT} = \mathbf{D}_p\mathbf{u}^k \tag{12}$$

$$\boldsymbol{\epsilon}_{nG}^k = [\gamma_{xz}\gamma_{yz}\epsilon_{zz}]^{kT} = (\mathbf{D}_{np} + \mathbf{D}_{nz})\mathbf{u}^k \tag{13}$$

$\boldsymbol{\epsilon}_{pG}^k$ and $\boldsymbol{\epsilon}_{nG}^k$ are the in-plane and transverse strains, respectively. $\mathbf{u}^k = (u, v, w)^k$ is the displacement vector. T means the transpose of a vector. The differential operators do not depend on the layer k:

$$\mathbf{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}, \quad \mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix} \tag{14}$$

The symbols in differential operators matrices indicate the partial derivatives $\partial_x = \frac{\partial}{\partial x}$, $\partial_y = \frac{\partial}{\partial y}$ and $\partial_z = \frac{\partial}{\partial z}$. Further details for geometrical relations of plates, in case of thermomechanical problems, can be found in the study by Brischetto and Carrera [2010a].

4. Constitutive Equations

Constitutive equations for the fully coupled thermomechanical problems have been obtained by Brischetto and Carrera [2010a] by using thermodynamical principles and Maxwell's relations [Altay and Dökmeci, 1996a, 1996b; Cannarozzi and Ubertini, 2001; Altay and Dökmeci, 2001]. A Gibbs free energy function and a thermomechanical enthalpy density [Nowinski, 1978; Ikeda, 1990] have been employed. Constitutive equations for the fully coupled thermomechanical vibrations are a particular case of the more general ones obtained by Brischetto and Carrera [2010a]:

$$\boldsymbol{\sigma}_{pC}^k = \mathbf{Q}_{pp}^k\boldsymbol{\epsilon}_{pG}^k + \mathbf{Q}_{pn}^k\boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_p^k\theta^k \tag{15}$$

$$\boldsymbol{\sigma}_{nC}^k = \mathbf{Q}_{np}^k\boldsymbol{\epsilon}_{pG}^k + \mathbf{Q}_{nn}^k\boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\lambda}_n^k\theta^k \tag{16}$$

$$\eta_C^k = \boldsymbol{\lambda}_p^{kT}\boldsymbol{\epsilon}_{pG}^k + \boldsymbol{\lambda}_n^{kT}\boldsymbol{\epsilon}_{nG}^k + \chi^k\theta^k \tag{17}$$

with in-plane (p) and out-of-plane (n) stress and strain components for each layer k:

$$\boldsymbol{\sigma}_{pC}^k = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}^k, \quad \boldsymbol{\sigma}_{nC}^k = \begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix}^k, \quad \boldsymbol{\epsilon}_{pG}^k = \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}^k, \quad \boldsymbol{\epsilon}_{nG}^k = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \\ \epsilon_{zz} \end{Bmatrix}^k \tag{18}$$

The sovra-temperature θ^k (effective temperature referred to the reference external room temperature T_0), the term $\chi^k = \frac{\rho^k C_v^k}{T_0}$, and the entropy for unit volume η^k are scalar variables in each layer k. ρ^k is the material density, C_v^k is the specific heat per unit mass, and T_0 is the reference temperature [Brischetto and Carrera, 2010a]. The explicit forms of the split matrices in Eqs. (15)–(17) are

- Elastic coefficient matrices:

$$\mathbf{Q}_{pp}^k = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix}^k, \quad \mathbf{Q}_{pn}^k = \begin{bmatrix} 0 & 0 & Q_{13} \\ 0 & 0 & Q_{23} \\ 0 & 0 & Q_{36} \end{bmatrix}^k$$
$$\mathbf{Q}_{np}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{13} & Q_{23} & Q_{36} \end{bmatrix}^k, \quad \mathbf{Q}_{nn}^k = \begin{bmatrix} Q_{55} & Q_{45} & 0 \\ Q_{45} & Q_{44} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}^k \tag{19}$$

- Thermomechanical coupling coefficients:

$$\boldsymbol{\lambda}_p^k = \mathbf{Q}_{pp}^k \boldsymbol{\alpha}_p^k + \mathbf{Q}_{pn}^k \boldsymbol{\alpha}_n^k = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_6 \end{bmatrix}^k, \quad \boldsymbol{\lambda}_n^k = \mathbf{Q}_{np}^k \boldsymbol{\alpha}_p^k + \mathbf{Q}_{nn}^k \boldsymbol{\alpha}_n^k = \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix}^k \tag{20}$$

where $\boldsymbol{\alpha}_p^{kT} = (\alpha_1, \alpha_2, 0)$ and $\boldsymbol{\alpha}_n^{kT} = (0, 0, \alpha_3)$ are the thermal expansion coefficients split in in-plane and out-of-plane components. In Eqs. (15)–(17) subscript C means constitutive equations, G means the use of geometrical relations of Sec. 3.

5. Governing Equations

The variational statement for the thermomechanical vibrations includes the internal thermal work ($\delta\theta^k \eta_C^k$) in the classical principle of virtual displacements:

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \{\delta \boldsymbol{\epsilon}_{pG}^{k}{}^T \boldsymbol{\sigma}_{pC}^k + \delta \boldsymbol{\epsilon}_{nG}^{k}{}^T \boldsymbol{\sigma}_{nC}^k - \delta\theta^k \eta_C^k\} d\Omega_k dz = \sum_{k=1}^{N_l} \delta L_{in}^k \tag{21}$$

where N_l are the layers of the considered laminate, Ω_k is the in-plane surface domain, and A_k is the thickness direction domain. δL_{in}^k is the inertial virtual work at the k-layer level.

By considering the constitutive equations as obtained in Eqs. (15)–(17), the geometrical relations for plates as obtained in Sec. 3 and CUF as described in Sec. 2, Eq. (21) is rewritten in the following form for a generic layer k:

$$\begin{aligned} \int_{\Omega_k} \int_{A_k} & [(\mathbf{D}_p F_s \delta \mathbf{u}_s^k)^T ((\mathbf{Q}_{pp}^k \mathbf{D}_p + \mathbf{Q}_{pn}^k (\mathbf{D}_{np} + \mathbf{D}_{nz})) F_\tau \mathbf{u}_\tau^k - \boldsymbol{\lambda}_p^k F_\tau \theta_\tau^k) \\ & + ((\mathbf{D}_{np} + \mathbf{D}_{nz}) F_s \delta \mathbf{u}_s^k)^T ((\mathbf{Q}_{np}^k \mathbf{D}_p + \mathbf{Q}_{nn}^k (\mathbf{D}_{np} + \mathbf{D}_{nz})) F_\tau \mathbf{u}_\tau^k - \boldsymbol{\lambda}_n^k F_\tau \theta_\tau^k) \\ & - F_s \delta\theta_s^{kT} ((\boldsymbol{\lambda}_p^{kT} \mathbf{D}_p + \boldsymbol{\lambda}_n^{kT} (\mathbf{D}_{np} + \mathbf{D}_{nz})) F_\tau \mathbf{u}_\tau^k + \chi^k F_\tau \theta_\tau^k)] d\Omega_k dz = \delta L_{in}^k \end{aligned} \tag{22}$$

Integrating by parts the Eq. (22), as suggested by Carrera [2002], and Brischetto [2009], the fundamental nuclei $\mathbf{K}_{uu}^{k\tau s}, \mathbf{K}_{u\theta}^{k\tau s}, \mathbf{K}_{\theta u}^{k\tau s}, \mathbf{K}_{\theta\theta}^{k\tau s}$, and that for the inertial matrix $\mathbf{M}_{uu}^{k\tau s}$ are obtained by using the governing equations in the following form:

$$\begin{aligned} \delta\mathbf{u}_s^k &: \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_\tau^k + \mathbf{K}_{u\theta}^{k\tau s}\theta_\tau^k = \mathbf{M}_{uu}^{k\tau s}\ddot{\mathbf{u}}_\tau^k \\ \delta\theta_s^k &: \mathbf{K}_{\theta u}^{k\tau s}\mathbf{u}_\tau^k + \mathbf{K}_{\theta\theta}^{k\tau s}\theta_\tau^k = 0 \end{aligned} \tag{23}$$

$\mathbf{u}_\tau^k$ is the vector for the degrees of freedom of the displacement, θ_τ^k is the vector for the degrees of freedom of the sovra-temperature, and $\ddot{\mathbf{u}}_\tau^k$ is the second temporal derivative of $\mathbf{u}_\tau^k$. Along with these governing equations, the following boundary conditions on the edge Γ_k of the in-plane integration domain Ω_k hold:

$$\begin{aligned} \prod_{uu}^{k\tau s}\mathbf{u}_\tau^k + \prod_{u\theta}^{k\tau s}\theta_\tau^k &= \prod_{uu}^{k\tau s}\bar{\mathbf{u}}_\tau^k + \prod_{u\theta}^{k\tau s}\bar{\theta}_\tau^k \\ \prod_{\theta u}^{k\tau s}\mathbf{u}_\tau^k + \prod_{\theta\theta}^{k\tau s}\theta_\tau^k &= \prod_{\theta u}^{k\tau s}\bar{\mathbf{u}}_\tau^k + \prod_{\theta\theta}^{k\tau s}\bar{\theta}_\tau^k \end{aligned} \tag{24}$$

As indicated by Brischetto and Carrera [2010a], the sovra-temperature θ^k (effective temperature referred to the external room temperature T_0) is a variable of the problem (always in LW form). The displacements $\mathbf{u}^k$ can be seen in ESL or LW form.

Governing equations for the pure mechanical problem are obtained from Eq. (23) by simply discarding the thermal part:

$$\delta\mathbf{u}_s^k : \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_\tau^k = \mathbf{M}_{uu}^{k\tau s}\ddot{\mathbf{u}}_\tau^k \tag{25}$$

with

$$\prod_{uu}^{k\tau s}\mathbf{u}_\tau^k = \prod_{uu}^{k\tau s}\bar{\mathbf{u}}_\tau^k$$

Fundamental nuclei for the thermomechanical vibration problem are

$$\begin{aligned} \mathbf{K}_{uu}^{k\tau s} &= \int_{A_k} [(-\mathbf{D}_p)^T(\mathbf{Q}_{pp}^k\mathbf{D}_p + \mathbf{Q}_{pn}^k(\mathbf{D}_{np} + \mathbf{D}_{nz})) \\ &\quad + (-\mathbf{D}_{np} + \mathbf{D}_{nz})^T(\mathbf{Q}_{np}^k\mathbf{D}_p + \mathbf{Q}_{nn}^k(\mathbf{D}_{np} + \mathbf{D}_{nz}))]F_sF_\tau dz \end{aligned} \tag{26}$$

$$\mathbf{K}_{u\theta}^{k\tau s} = \int_{A_k} [(-\mathbf{D}_p)^T(-\lambda_p^k) + (-\mathbf{D}_{np} + \mathbf{D}_{nz})^T(-\lambda_n^k)]F_sF_\tau dz \tag{27}$$

$$\mathbf{K}_{\theta u}^{k\tau s} = \int_{A_k} [-\boldsymbol{\lambda}_p^{kT}\mathbf{D}_p - \boldsymbol{\lambda}_n^{kT}(\mathbf{D}_{np} + \mathbf{D}_{nz})]F_sF_\tau dz \tag{28}$$

$$\mathbf{K}_{\theta\theta}^{k\tau s} = \int_{A_k} [-\chi^k]F_sF_\tau dz \tag{29}$$

$$\mathbf{M}_{uu}^{k\tau s} = \int_{A_k} (\rho^k\mathbf{I})F_sF_\tau dz \tag{30}$$

$\mathbf{I}$ is the (3×3) identity matrix. Nuclei for boundary conditions on the edge Γ_k state:

$$\prod_{uu}^{k\tau s} = \int_{A_k} [\mathbf{I}_p^T(\mathbf{Q}_{pp}^k \mathbf{D}_p + \mathbf{Q}_{pn}^k(\mathbf{D}_{np} + \mathbf{D}_{nz})) + \mathbf{I}_{np}^T(\mathbf{Q}_{np}^k \mathbf{D}_p + \mathbf{Q}_{nn}^k(\mathbf{D}_{np} + \mathbf{D}_{nz}))] F_s F_\tau dz \tag{31}$$

$$\prod_{u\theta}^{k\tau s} = \int_{A_k} [\mathbf{I}_p^T(-\lambda_p^k) + \mathbf{I}_{np}^T(-\lambda_n^k)] F_s F_\tau dz \tag{32}$$

$$\prod_{\theta u}^{k\tau s} = \prod_{\theta\theta}^{k\tau s} = 0 \tag{33}$$

Matrices $\mathbf{I}_p$ and $\mathbf{I}_{np}$ are obtained from matrices $\mathbf{D}_p$ and $\mathbf{D}_{np}$ by simply replacing the differential operators with 1. In the case of pure mechanical vibrations, only nuclei $\mathbf{K}_{uu}^{k\tau s}, \mathbf{M}_{uu}^{k\tau s}$, and $\mathbf{\Pi}_{uu}^{k\tau s}$ in Eqs. (26), (30), and (31), respectively, are considered.

To write the explicit form of fundamental nuclei obtained in this section, the following integrals in the z-thickness direction must be defined:

$$(J^{k\tau s}, J^{k\tau_z s}, J^{k\tau s_z}, J^{k\tau_z s_z}) = \int_{A_k} \left(F_\tau F_s, \frac{\partial F_\tau}{\partial z} F_s, F_\tau \frac{\partial F_s}{\partial z}, \frac{\partial F_\tau}{\partial z}\frac{\partial F_s}{\partial z} \right) dz \tag{34}$$

By using Eq. (34) and developing the matrices products, the explicit forms of fundamental nuclei are obtained.

Navier-type closed form solution is obtained via substitution of harmonic expressions for the displacements and temperature as well as considering the following material coefficients equal to zero: $Q_{16} = Q_{26} = Q_{36} = Q_{45} = 0$ and $\lambda_6 = 0$. The following harmonic assumptions can be made for the variables, which correspond to simply supported boundary conditions:

$$\begin{aligned} u_\tau^k &= \sum_{m,n} (\hat{U}_\tau^k) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad k = 1, N_l \\ v_\tau^k &= \sum_{m,n} (\hat{V}_\tau^k) \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \quad \tau = t, b, r \\ (w_\tau^k, \theta_\tau^k) &= \sum_{m,n} (\hat{W}_\tau^k, \hat{\theta}_\tau^k) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \quad r = 2, N \end{aligned} \tag{35}$$

where $\hat{U}_\tau^k, \hat{V}_\tau^k, \hat{W}_\tau^k, \hat{\theta}_\tau^k$ are the amplitudes, m and n are the wave numbers in x- and y-directions, respectively, a and b are the plate dimensions, k is the indicative of the layer, τ is the index for the thickness expansion where r indicates the higher-order terms. The explicit algebraic closed-form of nuclei in Eqs. (26)–(30) is not given here for sake of brevity, but they can be found in the study of Brischetto and Carrera [2010a].

5.1. *Acronyms*

A system of acronyms is given here in order to define the refined two-dimensional models developed in this work. The displacements can be in ESL or LW form, but the temperature is always considered in LW form. Therefore, a two-dimensional model is defined as ESL or LW, depending on the choice made for the displacement. ESL models are indicated as ED1–ED4, where E means the ESL approach, D means that the Principle of Virtual Displacements extended to thermomechanical vibrations analysis has been employed; the last digit, from 1 to 4, indicates the order of expansion in the thickness direction for both displacement and temperature. In the case of LW models, the letter E is replaced by a letter L, therefore the relative models are indicated as LD1–LD4. In the case of a thermomechanical analysis, additional parentheses are introduced in the acronyms: (TM) means a fully coupled thermo(T)-mechanical(M) analysis. No parentheses are added in the case of a pure mechanical model.

6. Results

The results given in this section refer to a simply supported square ($a = b$) plate with total thickness $h = 1\,\text{m}$ (see Fig. 2). The investigated thickness ratios are $a/h = 5, 10, 50$, and 100. Two different layered configurations are investigated for this free vibration problem. The first is a one-layered isotropic plate in $Al2024$ with Young's modulus $E = 73\,\text{GPa}$, Poisson's ratio $\nu = 0.3$, and mass density $\rho = 2800\,\text{kg/m}^3$. The thermal properties are the specific heat per unit mass $C_v = 897\,\text{J/kg K}$ and the thermal expansion coefficient $\alpha = 25 \times 10^{-6}\,1/\text{K}$. The two-layered isotropic plate has two layers of thickness $h_1 = h_2 = h/2 = 0.5\,\text{m}$, the bottom layer is in $Al2024$ and the top layer is in Ti22. The mechanical properties of the Ti22 layer are $E = 110\,\text{GPa}$, $\nu = 0.32$, and $\rho = 4420\,\text{kg/m}^3$. Its thermal properties are $C_v = 560\,\text{J/kg K}$ and $\alpha = 8.6 \times 10^{-6}\,1/\text{K}$. By imposing the wave numbers m and n in the x- and y-directions, respectively, the considered two-dimensional theory gives a number of frequencies equal to the number of degrees of freedom through the thickness direction. The degrees of freedom, and consequently the number of frequencies for the fixed wave numbers are given in Table 1 for CLT, FSDT, and refined ESL and LW models for the case of one-layered and two-layered plates. By increasing the wave numbers, it is possible to investigate the higher-order modes [Brischetto and Carrera, 2010b].

The fundamental frequency in Hz for the one-layered plate is given in Tables 2–4 for fixed wave numbers $m = n = 1$, $m = n = 10$, and $m = n = 50$, respectively. A comparison between the pure mechanical frequency and the thermomechanical frequency is conducted for several thickness ratios a/h and for several two-dimensional theories (refined LW theories, FSDT and CLT). The plate is one-layered, therefore there are no differences between the ESL and LW theories for the same order of expansion N in the thickness direction. The difference between the pure mechanical

Table 1. Number of frequencies (depending on $NDOF$) for fixed wave numbers (m, n) in the in-plane directions.

	NDOF	
Theory	$N_l = 1$	$N_l = 2$
CLT	3	3
FSDT	5	5
ESL ($N = 4$)	15	15
LW ($N = 2$)	9	15
LW ($N = 4$)	15	27

Note: In the thickness direction, $NDOF = ((N \times N_l)+1)\times 3$ in the case of LW theories, where N_l is the number of layers.

Table 2. Free vibrations of one-layered isotropic plate.

a/h	5	10	50	100
LD4	172.40	46.946	1.9390	0.4852
LD4(TM)	173.10(0.406%)	47.158(0.452%)	1.9481(0.469%)	0.4875(0.474%)
LD2	174.15	47.093	1.9392	0.4853
LD2(TM)	174.86(0.408%)	47.306(0.452%)	1.9484(0.474%)	0.4876(0.474%)
FSDT	174.10	47.088	1.9392	0.4853
FSDT(TM)	175.52(0.816%)	47.517(0.911%)	1.9577(0.954%)	0.4899(0.948%)
CLT	188.08	48.148	1.9411	0.4854
CLT(TM)	189.87(0.952%)	48.607(0.953%)	1.9596(0.953%)	0.4900(0.948%)

Note: Fundamental frequency f in Hz for $m = n = 1$. $\Delta(\%) = \frac{f_{\mathrm{TM}}-f}{f} \times 100$ is put in brackets.

Table 3. Free vibrations of one-layered isotropic plate.

a/h	5	10	50	100
LD4	4102.1	1886.6	172.40	46.946
LD4(TM)	4106.5(0.107%)	1889.8(0.170%)	173.10(0.406%)	47.158(0.452%)
LD2	4313.1	1971.3	174.15	47.093
LD2(TM)	4315.2(0.049%)	1974.0(0.137%)	174.86(0.408%)	47.306(0.452%)
FSDT	4311.7	1970.0	174.10	47.088
FSDT(TM)	4315.9(0.097%)	1975.3(0.268%)	175.52(0.816%)	47.517(0.911%)
CLT	4478.3	2239.1	188.08	48.148
CLT(TM)	4478.3(0.000%)	2239.1(0.000%)	189.87(0.952%)	48.607(0.953%)

Note: Fundamental frequency f in Hz for $m = n = 10$. $\Delta(\%) = \frac{f_{\mathrm{TM}}-f}{f} \times 100$ is put in brackets.

frequency f and the thermomechanical frequency f_{TM} is evaluated by means of the parameter $\Delta(\%) = \frac{f_{\mathrm{TM}}-f}{f}\times 100$. For lower-order modes ($m = n = 1$), the thermomechanical coupling does not depend on the thickness ratio and it is about $0.4 \div 0.5\%$. Classical theories, such as FSDT and CLT, give larger differences because such theories are not suitable to evaluate the thermal part. Even if the FSDT includes transverse shear strains, the hypotheses of constant transverse displacement and linear through-the-thickness temperature profile are a big limitation in thermomechanical analysis, in particular for thick and/or multilayered plates. When the wave

Table 4. Free vibrations of one-layered isotropic plate.

a/h	5	10	50	100
LD4	22150	10822	1886.6	763.94
LD4(TM)	22155(0.023%)	10828(0.055%)	1889.8(0.170%)	766.03(0.274%)
LD2	22355	11124	1971.3	786.64
LD2(TM)	22356(0.004%)	11125(0.009%)	1974.0(0.137%)	788.72(0.264%)
FSDT	22355	11124	1970.0	786.06
FSDT(TM)	22356(0.004%)	11126(0.018%)	1975.3(0.269%)	790.18(0.524%)
CLT	22391	11196	2239.1	1021.5
CLT(TM)	22391(0.000%)	11196(0.000%)	2239.1(0.000%)	1031.3(0.959%)

Note: Fundamental frequency f in Hz for $m = n = 50$. $\Delta(\%) = \frac{f_{\mathrm{TM}} - f}{f} \times 100$ is put in brackets.

numbers increase, as in Tables 3 and 4, the thermomechanical coupling decreases and depends on the thickness ratio ($\Delta(\%)$ is larger for thin plates). The use of the LD4 theory is mandatory. The LD2 theory in fact exhibits some difficulties for $m = n = 10$ and $m = n = 50$, in particular for thick plates. CLT and FSDT are totally inadequate. The same results are also proposed for the two-layered plate in Tables 5–7, where the investigated wave numbers m and n are the same as those

Table 5. Free vibrations of two-layered isotropic plate.

a/h	5	10	50	100
LD4	167.89	45.716	1.8881	0.4725
LD4(TM)	168.31(0.250%)	45.847(0.286%)	1.8938(0.302%)	0.4739(0.296%)
ED4	168.04	45.729	1.8881	0.4725
ED4(TM)	168.46(0.250%)	45.860(0.286%)	1.8938(0.302%)	0.4739(0.296%)
FSDT	169.87	45.881	1.8884	0.4725
FSDT(TM)	170.76(0.524%)	46.149(0.584%)	1.8999(0.609%)	0.4754(0.614%)
CLT	183.36	46.899	1.8902	0.4726
CLT(TM)	184.48(0.611%)	47.185(0.610%)	1.9017(0.608%)	0.4755(0.614%)

Note: Fundamental frequency f in Hz for $m = n = 1$. $\Delta(\%) = \frac{f_{\mathrm{TM}} - f}{f} \times 100$ is put in brackets.

Table 6. Free vibrations of two-layered isotropic plate.

a/h	5	10	50	100
LD4	3976.2	1835.3	167.89	45.716
LD4(TM)	3977.4(0.030%)	1836.8(0.082%)	168.31(0.250%)	45.847(0.286%)
ED4	4005.6	1842.3	168.04	45.729
ED4(TM)	4006.8(0.030%)	1843.8(0.081%)	168.46(0.250%)	45.860(0.286%)
FSDT	4231.2	1932.0	169.87	45.881
FSDT(TM)	4233.8(0.061%)	1935.4(0.176%)	170.76(0.524%)	46.149(0.584%)
CLT	4395.4	2197.7	183.36	46.899
CLT(TM)	4395.4(0.000%)	2197.7(0.000%)	184.48(0.611%)	47.185(0.610%)

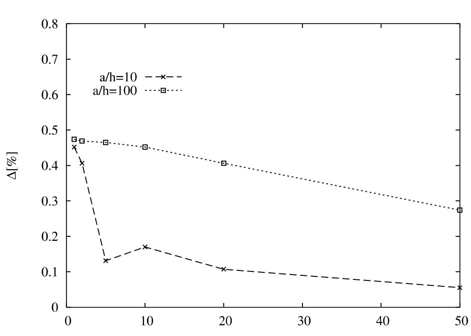
Note: Fundamental frequency f in Hz for $m = n = 10$. $\Delta(\%) = \frac{f_{\mathrm{TM}} - f}{f} \times 100$ is put in brackets.

Table 7. Free vibrations of two-layered isotropic plate.

a/h	5	10	50	100
LD4	21090	10230	1835.3	743.64
LD4(TM)	21093(0.014%)	10232(0.019%)	1836.8(0.082%)	744.80(0.156%)
ED4	21507	10531	1842.3	745.53
ED4(TM)	21508(0.005%)	10533(0.019%)	1843.8(0.081%)	746.71(0.158%)
FSDT	21767	10913	1932.0	769.53
FSDT(TM)	21767(0.000%)	10913(0.000%)	1935.4(0.176%)	772.12(0.337%)
CLT	21977	10988	2197.7	1000.2
CLT(TM)	21977(0.000%)	10988(0.000%)	2197.7(0.000%)	1006.3(0.610%)

Note: Fundamental frequency f in Hz for $m = n = 50$. $\Delta(\%) = \frac{f_{\mathrm{TM}} - f}{f} \times 100$ is put in brackets.

imposed for the one-layered case. The thermomechanical coupling is less pronounced for this two-layered plate; about $0.2 \div 0.3\%$ for the $m = n = 1$ case. All the considerations already made for the one-layered plates are confirmed for the two-layered case: the thermomechanical coupling depends on the imposed wave numbers; it does not depend on the thickness ratio for lower values of m and n, but the thermomechanical coupling increases with the thickness ratio for higher values of m and n. The plate in Tables 5–7 is two-layered, therefore there are some differences between the ESL and LW theories for the same order of expansion in the thickness direction. CLT and FSDT are totally inadequate for each thickness ratio a/h and for lower and higher values of the wave numbers. All the considerations made for Tables 2–7 are confirmed and summarized in Figs. 3 and 4. Figure 3 gives the variation in the thermomechanical coupling with the wave numbers for thick ($a/h = 10$) and thin ($a/h = 100$) one-layered and two-layered plates. Figure 4 discusses the variation in the thermomechanical coupling with the thickness ratio for wave numbers $m = n = 1$ and $m = n = 10$, in the case of one-layered and two-layered plates.



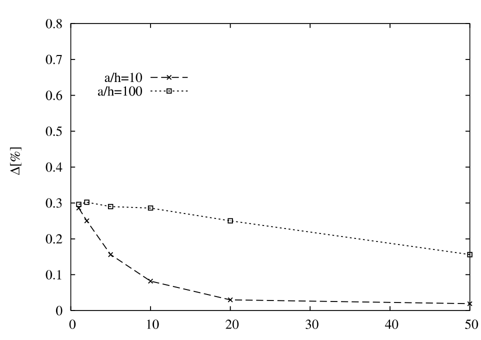


Fig. 3. Thermomechanical coupling in one-layered (left) and two-layered (right) isotropic plates for different values of wave number. $\Delta(\%) = \frac{f_{\mathrm{TM}} - f}{f} \times 100$ calculated for the fundamental frequency by using LD4 and LD4 (TM) theories.

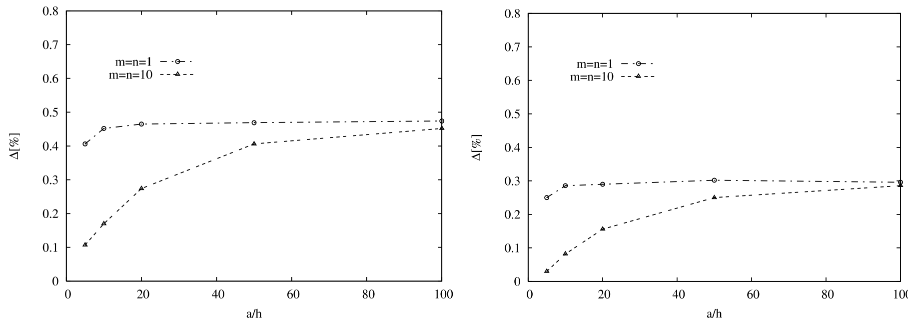


Fig. 4. Thermomechanical coupling in one-layered (left) and two-layered (right) isotropic plates for different values of thickness ratio. $\Delta(\%) = \frac{f_{\mathrm{TM}}-f}{f} \times 100$ calculated for the fundamental frequency by using LD4 and LD4 (TM) theories.

The main conclusions are: $\Delta(\%)$ decreases with (m,n) and increases with a/h for the fundamental frequency, while $\Delta(\%)$ is larger for the one-layered plate.

Tables 8 and 9 give all the frequencies related to the degrees of freedom of the employed two-dimensional theory (see Table 1), for three different values of the wave numbers ($m = n = 1$, $m = n = 10$, and $m = n = 10{,}000$). Table 8 is for the one-layered plate: CLT gives three frequencies, FSDT gives five frequencies, and LD4 gives 15 frequencies. Table 9 is for the two-layered plate: only the LD4 theory has been investigated and it gives 27 frequencies for each pair of m and n. There are some frequencies which correspond to a particular vibration mode, where there is no thermomechanical coupling. This fact is clearly seen in Figs. 5 and 6 for the case of one-layered and two-layered plates, respectively. Figures 5 and 6 give the first three vibration modes in terms of displacement components and sovra-temperature. The first three frequencies are given for thickness ratio $a/h = 10$ and $m = n = 1$. For bending modes (related to f_1 and f_3), there is the mode in terms of sovra-temperature different from zero. The second frequency is an in-plane mode: the mode in terms of transverse displacement w is zero and consequently the mode in terms of sovra-temperature is also zero; this feature explains why this mode does not have any thermomechanical coupling. From Tables 8 and 9, it is clear that when all the frequencies are considered (not only the fundamental one), the thermomechanical coupling does not depend on the considered wave numbers (m,n) and on the layers stacking. The coupling effect depends on the considered mode, in terms of displacement and sovra-temperature. The use of refined theories, such as LD4, is mandatory for such an investigation. CLT and FSDT unfortunately lose some modes because of their reduced number of degrees of freedom.

The considerations about the vibration modes in Figs. 5 and 6, and in Tables 8 and 9, create the necessity of defining a global energetic parameter to evaluate the thermomechanical coupling in such plates. A possible parameter is $\Delta^* = \sqrt{\sum_i \left(\frac{f_{\mathrm{TM}i}^2 - f_i^2}{f_i^2}\right)}$, which evaluates the thermomechanical coupling of all the obtained frequencies when the wave number is fixed and the theory is chosen. When a plate

Table 8. Free vibrations of one-layered isotropic plate.

			$m = 1, n = 1$		
LD4	46.946	223.91	377.90	1599.5	1636.6
	2907.3	3186.0	3259.6	5918.6	6577.4
	6616.7	9830.0	9835.4	12303	18388
LD4(TM)	47.158(0.452%)	223.91(0.000%)	379.65(0.463%)	1599.5(0.000%)	1636.9(0.018%)
	2935.4(0.966%)	3186.0(0.000%)	3267.8(0.252%)	5986.1(1.140%)	6577.4(0.000%)
	6621.3(0.069%)	9830.0(0.000%)	9835.8(0.004%)	12452(1.211%)	18610(1.207%)
FSDT	47.088	223.91	378.48	1760.1	1799.8
FSDT(TM)	47.517(0.911%)	223.91(0.000%)	382.09(0.954%)	1760.1(0.000%)	1800.5(0.039%)
CLT	48.148	223.91	378.48		
CLT(TM)	48.607(0.953%)	223.91(0.000%)	382.09(0.954%)		
			$m = 10, n = 10$		
LD4	1886.6	2239.1	2621.5	2742.6	3749.1
	3887.7	4136.5	5368.0	5848.5	6944.4
	8440.5	10079	10580	12831	18606
LD4(TM)	1889.8(0.170%)	2239.1(0.000%)	2623.4(0.072%)	2742.6(0.000%)	3751.4(0.061%)
	3887.7(0.000%)	4171.1(0.836%)	5408.4(0.753%)	5899.5(0.872%)	6944.4(0.000%)
	8486.7(0.547%)	10079(0.000%)	10618(0.359%)	12971(1.091%)	18833(1.220%)
FSDT	1970.0	2239.1	2839.3	3784.8	4301.8
FSDT(TM)	1975.3(0.269%)	2239.1(0.000%)	2839.3(0.000%)	3820.9(0.954%)	4331.2(0.683%)
CLT	2239.1	2984.8	3784.8		
CLT(TM)	2239.1(0.000%)	3013.2(0.951%)	3820.9(0.954%)		
			$m = 10\,000, n = 10\,000$		
LD4	2239135	2239135	2239137	2239138	2239138
	2239140	2239147	2239157	2239159	2239197
	4189043	4189046	4189049	4189058	4189058
LD4(TM)	2239135(0.000%)	2239135(0.000%)	2239137(0.000%)	2239138(0.000%)	2239138(0.000%)
	2239140(0.000%)	2239147(0.000%)	2239157(0.000%)	2239159(0.000%)	2239197(0.000%)
	4239640(1.208%)	4239644(1.208%)	4239646(1.208%)	4239655(1.208%)	4239655(1.208%)
FSDT	2239137	2239137	2239138	3784833	3784834
FSDT(TM)	2239137(0.000%)	2239137(0.000%)	2239138(0.000%)	3820916(0.953%)	3820917(0.953%)
CLT	2239137	3784832	3784832		
CLT(TM)	2239137(0.000%)	3784833(0.000%)	3820915(0.953%)		

Note: Higher-order frequency values f in Hz. Thickness ratio $a/h = 10$. $\Delta_i(\%) = \frac{f_{\mathrm{TM}i} - f_i}{f_i} \times 100$ is put in brackets.

Table 9. Free vibrations of two-layered isotropic plate.

			$m = 1, n = 1$		
LD4	45.716	219.77	374.11	1579.5	1616.2
	2901.4	3116.3	3202.0	4704.1	4709.0
	5966.9	6242.3	6245.4	8299.8	8299.9
	8943.9	11949	12902	12903	12902
	16229	16235	19308	19311	24709
	30832	36939			
LD4(TM)	45.847(0.286%)	219.77(0.000%)	374.99(0.235%)	1579.5(0.000%)	1616.4(0.012%)
	2918.9(0.603%)	3116.3(0.000%)	3209.4(0.231%)	4704.3(0.004%)	4709.0(0.000%)
	6003.5(0.613%)	6242.3(0.000%)	6245.9(0.008%)	8299.9(0.001%)	8300.0(0.001%)
	9017.5(0.823%)	12023(0.619%)	12902(0.000%)	12903(0.000%)	15884(0.780%)
	16229(0.000%)	16236(0.006%)	19308(0.000%)	19311(0.000%)	24828(0.482%)
	31088(0.830%)	37082(0.387%)			
			$m = 10, n = 10$		
LD4	1835.3	2196.1	2610.5	2710.6	3659.5
	3806.6	4095.2	5052.2	5198.2	5279.5
	6614.2	6764.0	7204.2	8587.3	8727.1
	9741.9	12382	13064	13087	16378
	16413	16933	19431	19770	24946
	30989	37035			

Table 9. (*Continued*)

LD4(TM)	1836.8(0.082%)	2196.1(0.000%)	2612.5(0.077%)	2710.6(0.000%)	3660.2(0.019%)
	3806.6(0.000%)	4114.4(0.469%)	5072.6(0.404%)	5198.2(0.000%)	5290.4(0.206%)
	6614.2(0.000%)	6775.7(0.173%)	7242.2(0.527%)	8587.3(0.000%)	8736.3(0.105%)
	9810.4(0.703%)	12446(0.517%)	13087(0.176%)	13092(0.038%)	16378(0.000%)
	16487(0.451%)	16983(0.295%)	19431(0.000%)	19777(0.035%)	25067(0.485%)
	31247(0.832%)	37180(0.391%)			
			$m = 10\,000, n = 10\,000$		
LD4	2171033	2171043	2171045	2171047	2171057
	2171062	2171101	2171195	2197754	2197853
	2239128	2239138	2239141	2239142	2239153
	2239160	2239198	2239295	4189046	4189055
	4189077	4189096	4207913	4219751	4219762
	4219785	4219803			
LD4(TM)	2171033(0.000%)	2171043(0.000%)	2171045(0.000%)	2171047(0.000%)	2171057(0.000%)
	2171062(0.000%)	2171101(0.000%)	2171195(0.000%)	2197754(0.000%)	2197853(0.000%)
	2239128(0.000%)	2239138(0.000%)	2239141(0.000%)	2239142(0.000%)	2239153(0.000%)
	2239160(0.000%)	2239198(0.000%)	2239297(0.000%)	4230512(0.990%)	4230523(0.990%)
	4230546(0.990%)	4230564(0.990%)	4232425(0.582%)	4239643(0.471%)	4239653(0.471%)
	4239675(0.471%)	4239693(0.471%)			

Note: Higher-order frequency values f in Hz. Thickness ratio $a/h = 10$. $\Delta_i(\%) = \frac{f_{\mathrm{TM}i} - f_i}{f_i} \times 100$ is put in brackets.

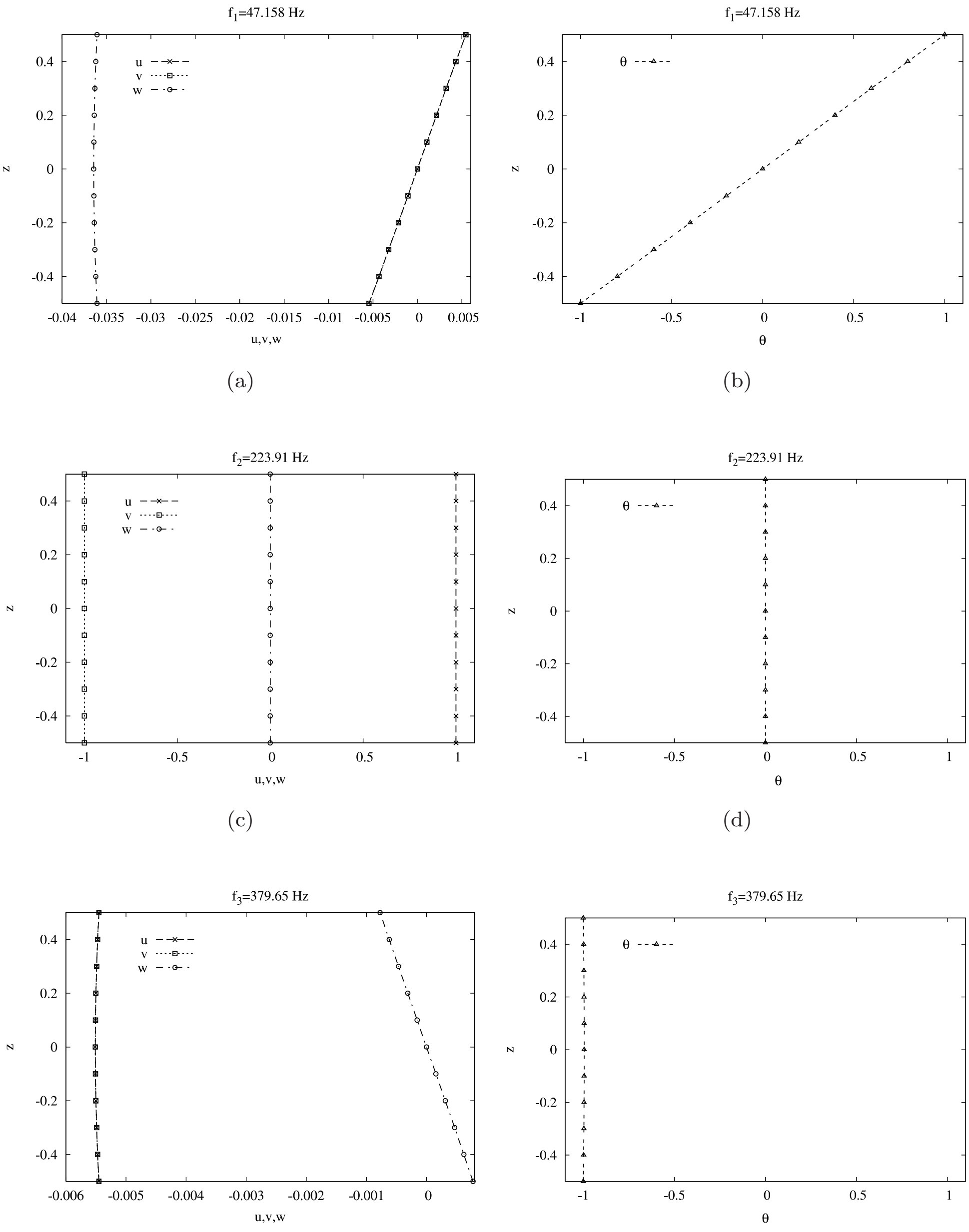


Fig. 5. One-layered isotropic plate, $m = n = 1$ and $a/h = 10$. First three modes in terms of displacements and sovra-temperature. LD4 (TM) model.

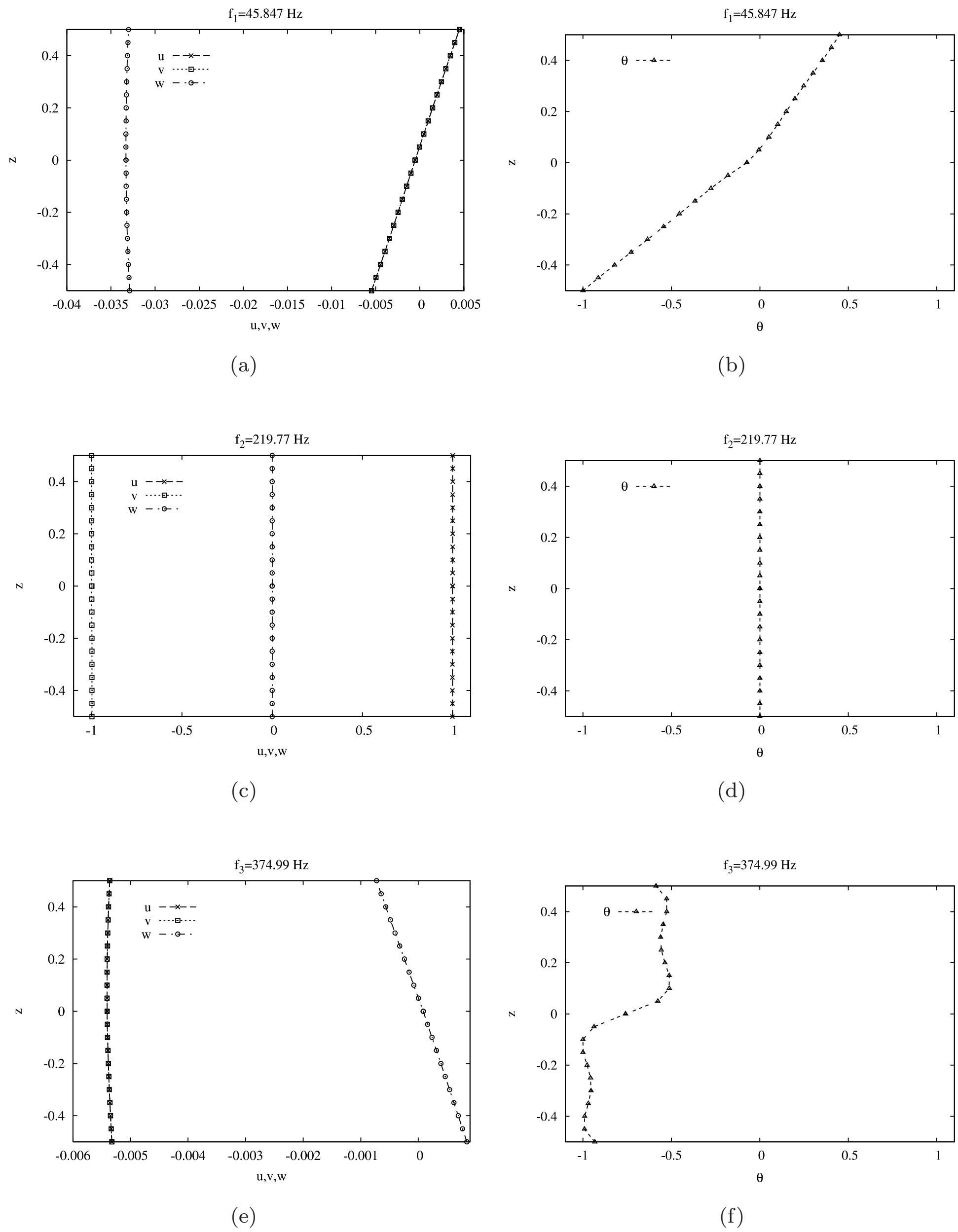


Fig. 6. Two-layered isotropic plate, $m = n = 1$ and $a/h = 10$. First three modes in terms of displacements and sovra-temperature. LD4 (TM) model.

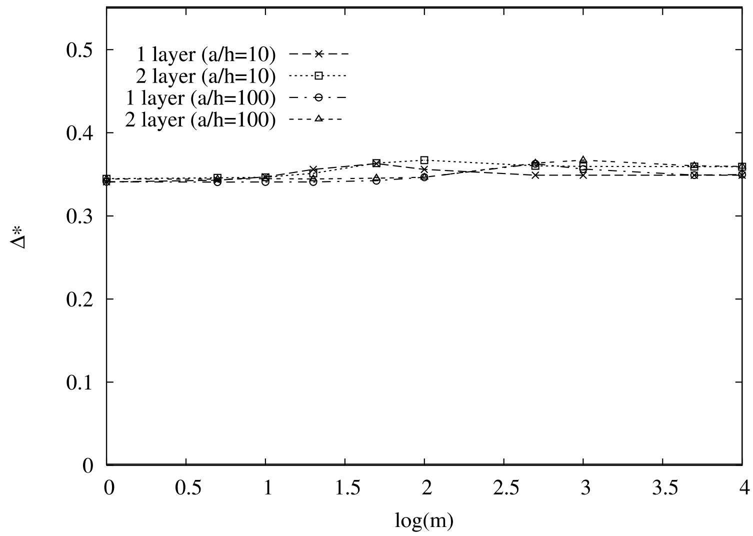


Fig. 7. Global energetic thermomechanical coupling in one-layered and two-layered isotropic plates for different values of wave number (from $m = n = 1$ to $m = n = 10{,}000$) and thickness ratio ($a/h = 10$ and $a/h = 100$). $\Delta^* = \sqrt{\sum_i \left(\frac{f^2_{\mathrm{TM}i} - f^2_i}{f^2_i}\right)}$ calculated by using LD4 and LD4 (TM) theories.

configuration is given (in particular its layers stacking), a two-dimensional theory is chosen and the wave numbers m, n are imposed, the parameter Δ^* gives the global thermomechanical coupling. From Fig. 7, it is clear that the global thermomechanical coupling does not depend on the thickness ratio, on the layers stacking, and on the imposed wave numbers.

The cases presented in this paper demonstrate how thermomechanical coupling is very small in the free vibration analysis of one-layered and two-layered metallic plates, and it can therefore be discarded: pure mechanical refined two-dimensional theories are sufficient to obtain a complete investigation of lower- and higher-order vibration modes for both thin and thick plates [Brischetto and Carrera, 2010b]. However, the fully coupled thermomechanical models could be a valid application for thermography investigations, as suggested by Spiessberger *et al.* [2008], Fantoni *et al.* [2008], and Ibarra-Castanedo *et al.* [2008]: the increment in temperature is experimentally measured to determine the strains and stresses which have generated it.

7. Conclusions

This paper has investigated the free vibration of one-layered and two-layered metallic plates through a comparison between pure mechanical frequencies and coupled thermomechanical frequencies. The thermomechanical frequencies have been obtained using opportune fully coupled thermomechanical theories where both the displacements and temperature are primary variables of the considered problem. Classical partially coupled thermomechanical models, given in literature, are not able to accomplish this type of analysis. Some conclusions have already been partially discussed by Brischetto and Carrera [2010a]:

- The use of refined two-dimensional theories is mandatory to correctly determine the thermal and mechanical parts in thermomechanical investigations;
- The thermomechanical coupling is very small, less than the 0.5%, and it can be discarded in free vibration analyses. However, it usually provides higher frequencies with respect to the pure mechanical case because it acts like a thermal source which leads to a bigger global stiffness of the structure. Such an analysis has never been performed in the open literature and it could play a fundamental role in thermography investigations.

Additional conclusions are summarized as follows:

- The thermomechanical coupling is zero for in-plane vibration modes, where the modes in terms of transverse displacement and sovra-temperature are zero;
- When the fundamental frequency is considered, the thermomechanical coupling has a limited dependence on the thickness ratio of the plate, its configuration (in particular the layer stacking), and the imposed wave numbers;
- When frequencies different from the fundamental one (related to the degrees of freedom of the employed two-dimensional theory) are considered, the thermomechanical coupling effect cannot be predicted "*a priori*";
- If a global parameter for the thermomechanical coupling, which considers all the frequencies related to the degrees of freedom of the employed two-dimensional theory, is opportunely defined, the thermomechanical coupling effect does not depend on the investigated case (layers stacking, thickness ratio, and imposed wave numbers).

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