Hierarchic finite elements based on a unified formulation for the static analysis of shear actuated multilayered piezoelectric plates

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Abstract

Purpose – The purpose of this paper is to present several two-dimensional plate elements for the analysis of shear actuated laminate.

Design/methodology/approach – The limitations of the classical formulations based on the principle of virtual displacements in depicting the peculiar behavior of the transverse and normal stresses of multilayered structures have been easily overcome by using the mixed variational theorem proposed by Reissner (Reissner mixed variational theorem). In the framework of a unified formulation (UF), the assumptions of the unknowns is made through a common expansion leading both to global and layerwise description of the assumed unknowns. In addition, the possibility to choose the order of the expansion between one and four allows to be derived and compared 22 different plate models. The performances of the proposed elements have tested on application for whom an exact solution is available in open literature.

Findings – The obtained results complain quite well with the exact ones even if the need of advanced plate models come to evidence.

Originality/value – This paper describes how the capabilities of the UF to accurately analyze multilayered structures exploiting the shear mode actuation have been tested and states that in order to extend the capabilities of the UF, further efforts should be made toward the assumptions of discontinuous electric fields (potential and normal displacement). The paper confirms the need for advanced higher order plate models in modeling of adaptive laminate.

Keywords Structures, Piezoelectricity, Control technology, Shearing, Mechanical behaviour of materials

Paper type Research paper

1. Introduction

Laminated structures incorporating piezoelectric materials give the designers the unique possibility to build an adaptive structures combining the low density, superior mechanical, and thermal properties of composite materials along with the sensing, actuation, and control. Piezoelectric materials can be integrated in multilayered structures in form of layers and/or patches by bonding to the external surfaces or embedding into the host structure. Surface bonded actuators are generally poled in thickness direction and generate high-longitudinal forces that lead to bending moments applied to the structure's neutral line defining the so-called extension actuation mechanisms. During the last decades, significant efforts have been devoted to the study of such actuation mechanisms of piezoelectric materials and several three-dimensional exact (Heyliger, 1994; Xu *et al.*, 1997) and closed-form (Benjeddou and Deü, 2002; Ballhause *et al.*, 2005) solutions as well as finite elements (FE) (Oh and Cho, 2004) have been presented. Surface bonded actuators present several remarkable advantages in

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terms of ease in construction and maintainability nevertheless they generate high longitudinal and contact stresses that could be the primary cause of damage or debonding especially for brittle piezoceramics. A promising way to overcome such problems have been found in the shear actuation mechanism. Sun and Zhang (1995) presented a study in which axial poled piezoelectric patches have been embedded in a sandwich beam showing that when a through-thickness electric field is applied to the structure shear forces are generated on the core. Several analytical and numerical models are present in literature on the application of embedded shear-actuated patches to beams (Aldraihem and Kheider, 2000; Benjeddou *et al.*, 1997, 1999) and plates (Zhang and Sun, 1999). Exact three-dimensional solutions for the static (Benjeddou and Deü, 2001a, b; Vel and Batra, 2001a, b) and free-vibration (Deü and Benjeddou, 2005) analysis of transverse shear actuation and sensing of simply-supported piezoelectric laminated are also been presented in the recent past. It is well known that a good model for the analysis of multilayered structures should be able to reproduce what is know as zig-zag (ZZ) effect for the displacement fields and to satisfy the interlaminar continuity (IC) for transverse stresses. Most of the aforementioned models are based on the Kirchhoff-Love assumptions and generally result to be adequate only for thin laminates or at least for slightly thicker plates. In addition, for piezoelectric plates, lower order equivalent single layer (ESL) models such as those based on the classical lamination theory or the first-order shear deformation theory, lead to a compromised electromechanical coupling since the electric field is coupled to an oversimplified displacement field. In Carrera (1995, 2003), ZZ and IC requirements were summarized by introducing the acronyms C_z^0-requirements. That is, both displacement and transverse stresses should be C^0-continuous functions in the thickness z-direction in fact, the C^1 thickness continuity prevents the displacement field from presenting localized kinking at material interfaces where the mechanical properties change resulting in a loss of transverse stress equilibrium. The fulfillment of C_z^0-requirements is a crucial point of two-dimensional modeling of multilayered structures.

The present paper proposes a hierarchical family of plate FEs with four and nine nodes distinguished from one another by the laminate kinematic assumptions upon which each of them is based. The unified formulation (UF) introduced by Carrera and Demasi (2002) have been herein used to derive the discretized governing equations for the static analysis of multilayered shear-actuated piezoelectric plates. The UF result to be particularly suitable in treating multifield problems as all the unknowns are assumed using the same generalized expansion whose order is keep as a free parameter of the FE and ranges from one up to the four. In addition, the assumptions of the nodal unknowns is based on a set of functions herein indicated as thickness functions.

In addition, the impossibility to satisfy the IC in the recovering of transverse and normal stresses suggested the substitution of the classical displacement formulation based on the principle of virtual displacements (PVDs) with a mixed variational theorem such as that introduced by Reissner and already extended to piezoelectric (D'Ottavio and Kröplin, 2006) and thermopiezoelastic analyses (Robaldo, 2006). In doing so, it will be shown that the C_z^0-requirements could be completely satisfied a priori. The paper presents the derivation of the governing equations upon which the plate elements are built. Finally, several benchmarking cases taken from literature have been addressed to assess and compare the sensing and actuating capabilities of the various elements. The obtained results have shown the importance of refined higher order plate models in order to accurately predict the displacement and the interlaminar stress field of shear-actuated plates.

2. Geometry of multilayered plates

Consider a rectangular plate made of N-stacked layers until the desired thickness and stiffness are reached. Each layer can present a different material and/or fiber orientation (if made of fiber-reinforced composite) and/or thickness. Generally, the material and thickness of each layer can differ from one another. In particular, this work will focus on adaptive laminates whose peculiarity is the presence of embedded piezoelectric layers that could be used either as sensors or actuators. A Cartesian coordinate system $\langle x, y, z \rangle$ with plane xy laying on the middle surface of the plate has been defined as shown in Figure 1. Layers are numbered starting form the bottom surface and the total number of layers is identified by the symbol N. All the physical quantities and model entities related to the Kth layer are addressed by adding the k superscript. The dimensions of the plate are addressed with the symbols A and B, respectively, along x- and y-axis, while the global thickness with H. Dealing with fiber-reinforced composites, another reference system defined for each lamina should be introduced to take into account of different orientations of the fiber direction with respect to the laminate coordinate system. This system is generally indicated by the axes $\langle 1, 2, 3 \rangle$. The material coordinate 1-axis is taken to be parallel to the fiber direction, the 2-axis is transverse to the fiber direction in the plane of the lamina while the 3-axis is perpendicular to the lamina. All the material data such as Young moduli and piezoelectric coefficients, are given in the material reference system instead, the analysis of the structure is made in the laminate coordinate system. This implies that the governing equations are to be written first in the material coordinate system and then opportunely rotated in the laminate one.

3. Notations

Standard tensor notation is applied in a three-dimensional Euclidean space Ψ; at the same time, Einstein's summation convention is implied over repeated indices. A comma denotes partial differentiation with respect to the indicated space coordinate while a superposed dot stands for time differentiation. The symbol V indicates a regular, finite, and bounded region contained in the space Ψ and Γ is its boundary surface.

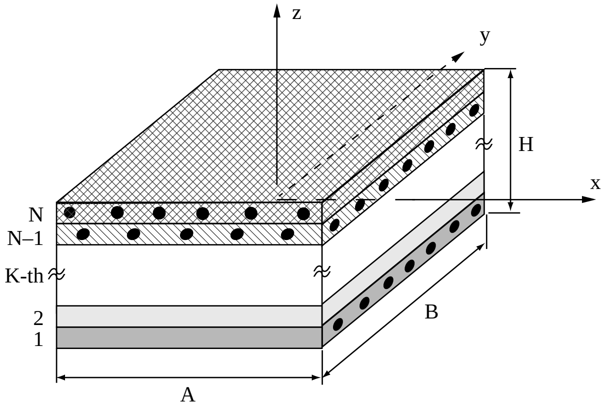


Figure 1.
Geometry and notation for multilayered plates

Single-subscript notation for stresses and strains will be applied according to the following scheme for indices: $xx \rightarrow 1$ $yy \rightarrow 2$ $zz \rightarrow 3$ $yz \rightarrow 4$ $xz \rightarrow 5$ $xy \rightarrow 6$. A superscript "T" indicates a transposition operation.

4. Electromechanical equations

The coupling between the mechanical and electrical fields can be established by the constitutive equations of an piezoelastic medium. The constitutive equations for a linear orthotropic lamina polarized along its first material axis in the laminate reference system can be written as:

$$\boldsymbol{\sigma}^k = \boldsymbol{C}^k \boldsymbol{\varepsilon}_k - \boldsymbol{e}^{kT} \boldsymbol{E}^k \tag{1}$$

$$\boldsymbol{D}^k = \boldsymbol{e}^k \boldsymbol{\varepsilon}^k + \boldsymbol{\epsilon}^k \boldsymbol{E}^k \tag{2}$$

where:

- $\boldsymbol{\sigma}$ the stress tensor.
- $\boldsymbol{\varepsilon}$ the engineering strain tensor.
- $\boldsymbol{E}$ the electric field vector.
- $\boldsymbol{D}$ the electric displacement vector.
- $\boldsymbol{C}$ the matrix of the elastic moduli.
- $\boldsymbol{e}$ the matrix of the piezoelectric constants.
- $\boldsymbol{\epsilon}$ the permittivity matrix.

The explicit form of the constitutive relations is given in Appendix 1.

From this point on stresses and strains are going to be separated in in-plane and normal components denoted, respectively, by the subscripts "p" and "n":

$$\boldsymbol{\sigma}_p^k = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}^k \quad \text{and} \quad \boldsymbol{\sigma}_n^k = \begin{bmatrix} \sigma_5 \\ \sigma_4 \\ \sigma_3 \end{bmatrix}^k \tag{3}$$

$$\boldsymbol{\varepsilon}_p^k = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}^k \quad \text{and} \quad \boldsymbol{\sigma}_n^k = \begin{bmatrix} \varepsilon_5 \\ \varepsilon_4 \\ \varepsilon_3 \end{bmatrix}^k \tag{4}$$

The same separation is made on the elastic and piezoelectric stiffnesses obtaining:

$$\boldsymbol{C}_{pp}^k = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^k \quad \boldsymbol{C}_{pn}^k = \begin{bmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ 0 & 0 & C_{36} \end{bmatrix}^k \quad \boldsymbol{C}_{nn}^k = \begin{bmatrix} C_{55} & C_{45} & 0 \\ C_{45} & C_{44} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^k \tag{5}$$

$$\boldsymbol{e}_p^k = \begin{bmatrix} e_{11} & e_{12} & 0 \\ 0 & 0 & e_{26} \\ 0 & 0 & 0 \end{bmatrix}^k \quad \text{and} \quad \boldsymbol{e}_n^k = \begin{bmatrix} 0 & 0 & e_{13} \\ 0 & 0 & 0 \\ e_{35} & 0 & 0 \end{bmatrix}^k \tag{6}$$

Using equations (3)-(6), the constitutive equations (1) and (2) can be rewritten as:

$$\boldsymbol{\sigma}_p^k = \boldsymbol{C}_{pp}^k \boldsymbol{\varepsilon}_p^k + \boldsymbol{C}_{pn}^k \boldsymbol{\varepsilon}_n^k - \boldsymbol{e}_p^{k\,T} \boldsymbol{E}^k \tag{7}$$

$$\boldsymbol{\sigma}_n^k = \boldsymbol{C}_{pn}^{k\;T} \boldsymbol{\varepsilon}_p^k + \boldsymbol{C}_{nn}^k \boldsymbol{\varepsilon}_n^k - \boldsymbol{e}_n^{k\,T} \boldsymbol{E}^k \tag{8}$$

$$\boldsymbol{D}^k = \boldsymbol{e}_p^k \boldsymbol{\varepsilon}_p^k + \boldsymbol{e}_n^k \boldsymbol{\varepsilon}_n^k + \boldsymbol{\epsilon}^k \boldsymbol{E}^k \tag{9}$$

The mechanical stresses are expressed via the strains and then they can be related to the displacement field u via geometric relations:

$$\boldsymbol{\epsilon}_{pG}^k = \boldsymbol{D}_p \boldsymbol{u}^k \tag{10}$$

$$\boldsymbol{\epsilon}_{nG}^k = (\boldsymbol{D}_{np} + \boldsymbol{D}_{nz}) \boldsymbol{u}^k \tag{11}$$

wherein the differential operator arrays are defined as follows:

$$\boldsymbol{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix} \quad \boldsymbol{D}_{np} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix} \quad \boldsymbol{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix} \tag{12}$$

A similar behavior can be observed for the electric field that is related to the electric potential Φ. The electric field $\boldsymbol{E}$ is defined as the gradient of the electric potential Φ:

$$\boldsymbol{E}^k = \begin{bmatrix} -\partial_x \\ -\partial_y \\ -\partial_z \end{bmatrix} \quad \Phi^k = (\boldsymbol{D}_{ep} + \boldsymbol{D}_{ez}) \Phi^k \tag{13}$$

where:

$$\boldsymbol{D}_{ep} = \begin{bmatrix} -\partial_x \\ -\partial_y \\ 0 \end{bmatrix} \quad \boldsymbol{D}_{ez} = \begin{bmatrix} 0 \\ 0 \\ -\partial_z \end{bmatrix} \tag{14}$$

5. Features of the UF

The UF introduced in Carrera (2000), has been already extensively applied for the mechanical (Carrera and Demasi, 2002), thermal (Carrera, 2000; Robaldo and Carrera, 2007), and piezoelastic (Robaldo *et al.*, 2006) analysis of multilayered plates. Through the use of the UF the same computational code can address not just one FE but a complete family of them with different order of the unknowns and different descriptions of the primary unknowns along the thickness of the structure.

The generalized expansion, upon which the UF is based, relies on a set of functions herein indicated as thickness functions. In doing so, the UF reduces a tri-dimensional problem to a bi-dimensional problem. Meanwhile, the order of the expansion along the thickness of the plate is taken as a free parameter of the FE and it can be changed in the interval ranging from one up to four.

By appropriately choosing the thickness functions either an ESL or a layer-wise (LW) description along the thickness of the plate is admissible. In an ESL model, a global assumption for the unknowns is considered along the thickness of the plate while in a LW model the expansion is made for each layer separately and then IC conditions are enforced. The latter leads generally to more accurate results but the number of the nodal degrees of freedom (DOFs) increases with the number of the layers coming out to a greater computational cost. Combining all the possible parameters, more than 20 different plate FEs are addressed in this work. In the following sections, the UF expansions for classical theory or displacement formulation based on PVD and for mixed problem based on the Reissner mixed variational theorem (RMVT) are introduced.

5.1 Displacement field

The unified assumption for the displacement field, in the most general case, states:

$$\boldsymbol{u}(x,y,z) = F_b(z)\boldsymbol{u}_b(x,y) + F_r(z)\boldsymbol{u}_r(x,y) + F_t(z)\boldsymbol{u}_t(x,y) = F_\tau \boldsymbol{u}_\tau \tag{15}$$

where N is the order of the expansion and F_τ are the so-called thickness functions.

In an ESL theory, the thickness functions F_τ take the form:

$$F_b = 1, \qquad F_r = z^r, \qquad F_t = z^N \quad r = 1,2,\ldots,N-1 \tag{16}$$

In this case, the ZZ form of the displacements can be reproduced by referring to Murakami's (1993) idea and the displacement model can be written in the following generalized form:

$$\boldsymbol{u} = \boldsymbol{u}_0 + (-1)^k \zeta_k \boldsymbol{u}_\zeta + z^r \boldsymbol{u}_r \quad r = 1,2,\ldots,N \tag{17}$$

where $\zeta_k = (2z_k/h_k)$ is a non-dimensional layer coordinate (z_k is the physical coordinate of the k-layer whose thickness is h_k). In generalized form, equation (17) can be written defining the thickness functions as:

$$F_b = 1, \qquad F_t = P_z(z) = (-1)^k \zeta_k, \qquad F_r = z^r \quad r = 1,2,\ldots,N \tag{18}$$

In a LW theory, the thickness functions are defined by:

$$F_t = \frac{P_0 + P_1}{2}, \qquad F_b = \frac{P_0 - P_1}{2}, \qquad F_r = P_r - P_{r-2} \quad r = 2,\ldots,N \tag{19}$$

where $P_i = P_i(\zeta_k)$ is the Legendre polynomial of ith-order defined in the domain $-1 \leq \zeta_k \leq 1$. The chosen thickness functions have the interesting properties:

$$\zeta_k = \begin{cases} 1: & F_t = 1, \quad F_b = 0, \quad F_r = 0 \\ -1: & F_t = 0, \quad F_b = 1, \quad F_r = 0 \end{cases} \tag{20}$$

Using these definitions, the generalized displacements assumptions of the k-th layer can be stated as:

$$\boldsymbol{u}^k(x,y,z) = F_b(z)\boldsymbol{u}_b^k(x,y) + F_r(z)\boldsymbol{u}_r^k(x,y) + F_t(z)\boldsymbol{u}_t^k(x,y) = F_\tau \boldsymbol{u}_\tau^k \tag{21}$$

$$\text{with} \quad r = 2, \ldots, N \quad k = 1, 2, \ldots, N_L$$

Thus, the displacement variables u_b and u_t are the actual displacements at the bottom and the top surfaces of the layer and the inter-laminar continuity can be easily linked:

$$\boldsymbol{u}_t^k = \boldsymbol{u}_b^{(k+1)}, \quad \text{with} \quad k = 1, \ldots, N_L - 1 \tag{22}$$

5.2 Transverse stress field

If the Reissner's mixed variational theorem is used, the transverse stress field $\boldsymbol{\sigma}_n^k$ will be written using the same notation of equation (21). An ESL assumption is not appropriate thus only a LW description will be employed:

$$\boldsymbol{\sigma}_n^k = F_t \boldsymbol{\sigma}_{nt}^k + F_b \boldsymbol{\sigma}_{nb}^k + F_r \boldsymbol{\sigma}_{nr}^k = F_\tau \boldsymbol{\sigma}_{n\tau}^k \tag{23}$$

$$\text{where} \quad r = 2, \ldots, N \quad k = 1, 2, \ldots, N_L$$

The same thickness functions F_τ are used as in the LW displacement case and the inter-laminar compatibility conditions can thus be imposed by:

$$\boldsymbol{\sigma}_{nt}^k = \boldsymbol{\sigma}_{nb}^{(k+1)}, \quad k = 1, \ldots, N_L - 1 \tag{24}$$

For convenience, the expansion order N of the transverse stress field assumption is chosen to be the same as the expansion of the displacement assumption.

5.3 Electric potential

The electric potential ϕ will be written using the same notation of equation (21). In this case, for the differences of the electric properties of each layer, ESL assumption is not appropriate:

$$\Phi^k = F_t \Phi_t^k + F_b \Phi_b^k + F_r \Phi_r^k = F_\tau \Phi_\tau^k \tag{25}$$

$$\text{where} \quad r = 2, \ldots, N \quad k = 1, 2, \ldots, N_L$$

The same thickness functions F_τ are used as in the LW displacement case and the inter-laminar compatibility conditions of the potential can thus be imposed by:

$$\Phi_t^k = \Phi_b^{(k+1)}, \quad k = 1, \ldots, N_L - 1 \tag{26}$$

For convenience, the expansion order N of the potential assumption is chosen to be the same as the expansion of the displacement assumption. The thickness description of each unknown can be choose independently in fact, also in global response models, the electrical potential has been limited to a LW description. This feature is particularly suitable since electric DOFs are often avoided for plates elements or a simple through-thickness linear variation (Heyliger *et al.*, 1994; Suleman and Venkayya, 1995; Saravanos *et al.*, 1997; Correia *et al.*, 2000) is assumed for the electric potential. One of the few plate elements that consider a quadratic assumption for the electric potential is the work by Carrera (1997). The linear through-thickness hypothesis for the electric potential neglects systematically the contribution of the induced potential leading to a partial electromechanical coupling. With the present formulation, the errors induced by the linear assumption of the electric potential can be easily pointed out.

6. Variational principles for piezoelasticity

The variational statements upon which the proposed plate elements are based are the PVDs and the Reissner's mixed variational theorem whose statements for a static analysis are reported below:

- PVD:

$$\int_V \left(\delta\varepsilon_{pG}^T \boldsymbol{\sigma}_{pC} + \delta\varepsilon_{nG}^T \boldsymbol{\sigma}_{nC} - \delta E_G^T \boldsymbol{D}_C\right) dV = \int_\Gamma (\delta u^T \bar{t} - \delta\Phi\bar{Q}) d\Gamma \tag{27}$$

- RMVT:

$$\int_V \left[\delta\varepsilon_{pG}^T \boldsymbol{\sigma}_{pC} - \delta\boldsymbol{E}_G^T \boldsymbol{D}_C + \delta\varepsilon_{nG}^T \boldsymbol{\sigma}_{nM} + \delta\boldsymbol{\sigma}_{nM}^T (\varepsilon_{nG} - \varepsilon_{nC})\right] dV = \int_\Gamma (\delta u^T \bar{t} - \delta\Phi\bar{Q}) d\Gamma \tag{28}$$

where the subscripts "G" and "C" indicate quantities defined, respectively, through the geometric relations and by the constitutive equations. The peculiarity of the Reissner's mixed variational theorem is the introduction of the transverse and normal stresses as primary variables (in the term $\boldsymbol{\sigma}_{nM}$) in addition to the displacement field as done in the PVD. The terms on the right side of the statements represent the external work done by imposed tractions ($\bar{t}$) and electric charge ($\bar{Q}$). A detailed derivation of the above mixed principle can be found in D'Ottavio and Kröplin (2006).

It is necessary to point out that the introduction of the additional primary variables in the RMVT requires to manipulate the constitutive equations in equation (7) in order to achieve a new set of mixed constitutive equations. The new form of the constitutive equations to use with the RMVT is:

$$\boldsymbol{\sigma}_{pC}^k = \hat{C}_{pp}^k \boldsymbol{\varepsilon}_{pG}^k + \hat{C}_{pn}^k \boldsymbol{\sigma}_{nM}^k + \hat{C}_{se}^k \boldsymbol{E}_G^k \tag{29}$$

$$\boldsymbol{\varepsilon}_{nC}^k = \hat{C}_{np}^k \boldsymbol{\varepsilon}_{pG}^k + \hat{C}_{nn}^k \boldsymbol{\sigma}_{nM}^k + \hat{C}_{de}^k \boldsymbol{E}_G^k \tag{30}$$

$$\boldsymbol{D}_C^k = \hat{C}_{ed}^k \boldsymbol{\varepsilon}_{pG}^k + \hat{C}_{es}^k \boldsymbol{\sigma}_{nM}^k + \hat{C}_{ee}^k \boldsymbol{E}_G^k \tag{31}$$

where the new stiffness matrices can be related to the old ones through the following relations:

$$\hat{C}_{pp}^k = (C_{pp} - C_{pn} C_{nn}^{-1} C_{np})^k \tag{32}$$

$$\hat{C}_{pn}^k = \left(C_{pn} C_{nn}^{-1}\right)^k \tag{33}$$

$$\hat{C}_{se}^k = \left(C_{pn} C_{nn}^{-1} \hat{e}_n^T - \hat{e}_p^T\right)^k = \left(\hat{C}_{pn} \hat{e}_n^T - \hat{e}_p^T\right)^k \tag{34}$$

$$\hat{C}_{np}^k = \left(-C_{nn}^{-1} C_{np}\right)^k \tag{35}$$

$$\hat{C}^k_{nn} = \left(C^{-1}_{nn}\right)^k \tag{36}$$

$$\hat{C}^k_{de} = \left(C^{-1}_{nn}\hat{e}^T_n\right)^k = \left(\hat{C}_{nn}\hat{e}^T_n\right)^k \tag{37}$$

$$\hat{C}^k_{ed} = \left(\hat{e}_p - \hat{e}_n C^{-1}_{nn} C_{np}\right)^k = (\hat{e}_p + \hat{e}_n \hat{C}_{np})^k \tag{38}$$

$$\hat{C}^k_{es} = \left(\hat{e}_n C^{-1}_{nn}\right)^k = \left(\hat{e}_n \hat{C}_{nn}\right)^k \tag{39}$$

$$\hat{C}^k_{ee} = \left(\boldsymbol{\epsilon} + \hat{e}_n C^{-1}_{nn}\hat{e}^T_n\right)^k = \left(\boldsymbol{\epsilon} + \hat{e}^k_n \hat{C}_{nn}\hat{e}^T_n\right)^k \tag{40}$$

7. Variables discretization

The discretization of the unknown along the thickness is carried on by the UF through the thickness functions. In the FE method, the in-plane discretization of the unknowns is performed in terms of their nodal values, via the shape functions N_i. Simple isoparametric Lagrangian elements with four and nine nodes have been developed in the present paper. Figure 2 shows sketches of four and nine node elements with element reference system (x_e, y_e, z_e) and node numeration. At this point, another important feature of the UF is noticeable: the possibility to build all the FE matrices involved in the analysis from simple arrays that are called the fundamental nuclei. These arrays have, in the most general case, (3×3) dimension and have therefore to be assembled in the opportune way depending on the used variable description.

Thus, at element level the primary variables can be written using the following relations:

- Displacement field:

$$\boldsymbol{u}^k_\tau(x_e, y_e) = N_i(x_e, y_e)\boldsymbol{q}^k_{\tau i} \quad i = 1, 2, \ldots, N_n \tag{41}$$

- Transverse and normal stresses:

$$\boldsymbol{\sigma}^k_{n\tau}(x_e, y_e) = N_i(x_e, y_e)\boldsymbol{f}^k_{\tau i} \quad i = 1, 2, \ldots, N_n \tag{42}$$

- Electric potential:

$$\Phi^k_\tau(x_e, y_e) = N_i(x_e, y_e) g^k_{\tau i} \quad i = 1, 2, \ldots, N_n \tag{43}$$

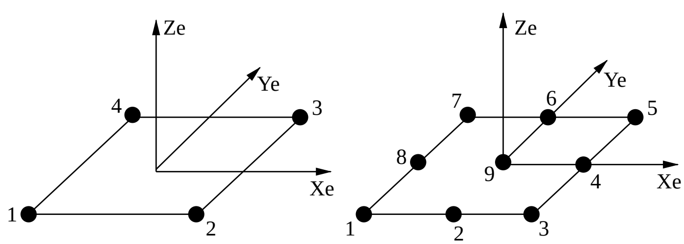


Figure 2. Sketch of four and nine node elements with element reference system and node numeration

Substituting equations (41)-(43), in equations (21), (23), and (25), the final expressions of the discretized primary variables for the kth layer can be obtained:

$$\boldsymbol{u}^k(x_e, y_e, z_e) = F_\tau(z_e)N_i(x_e, y_e)\boldsymbol{q}^k_{\tau i} \tag{44}$$

$$\boldsymbol{\sigma}^k_n(x_e, y_e, z_e) = F_\tau(z_e)N_i(x_e, y_e)\boldsymbol{f}^k_{\tau i} \tag{45}$$

$$\Phi^k(x_e, y_e, z_e) = F_\tau(z_e)N_i(x_e, y_e)g^k_{\tau i} \tag{46}$$

8. Derivation of FE matrices

For a multilayered plate, the variational statements from which the element stiffness matrices can be derived take the following form:

- PVD:

$$\sum_{k=1}^{N} \int_{A_k} \int_{h_k} \left\{ \delta\boldsymbol{\epsilon}^{k\ T}_{pG} \boldsymbol{\sigma}^k_{pC} + \delta\boldsymbol{\epsilon}^{k\ T}_{nG} \boldsymbol{\sigma}^k_{nC} - \delta E^T_G \boldsymbol{D}_C \right\} dA_k dz = \delta L^k_e \tag{47}$$

- RMVT:

$$\sum_{k=1}^{N_l} \int_{A_k} \int_{h_k} \left[\delta\varepsilon^{kT}_{pG} \boldsymbol{\sigma}^k_{pC} + \delta\varepsilon^{kT}_{nG} \boldsymbol{\sigma}^k_{nM} - \delta\boldsymbol{E}^{kT}_G \boldsymbol{D}^k_C + \delta\boldsymbol{\sigma}^{kT}_{nM} \left(\varepsilon^k_{nG} - \varepsilon^k_{nC} \right) \right] dA_k dz = \delta L^k_e \tag{48}$$

where A_k and h_k are the planar surface and the thickness of the k-th layer. The expression of the external work can be written as:

$$\delta L^k_e = \int_{A_k} \left(\delta\boldsymbol{u}^{k^T}\left(x, y, \zeta^k_1\right) \bar{\boldsymbol{t}}^k\left(x, y, \zeta^k_1\right) - \delta\Phi^{k^T}\left(x, y, \zeta^k_1\right) \bar{Q}^k\left(x, y, \zeta^k_1\right) \right) dA_k \tag{49}$$

that takes into account of the external loads for a generic layer k on which a surface traction $\bar{\boldsymbol{t}}$ and a electric charge $\bar{Q}$ are applied on a plane $\zeta^k = \zeta^k_1$ where ζ^k is the adimensionalized thickness coordinate of the lamina:

$$\zeta^k = \frac{2z^k}{h^k} \tag{50}$$

The calculation of the stiffness matrices goes through the computation of the fundamental nuclei and can be carry out in six steps:

(1) Substituting of the constitutive relations in the variational statement equation (47) or equation (48).
(2) Introducing the geometric relations.
(3) Substituting the through-thickness modeling.
(4) Substituting the unknown discretization: equations (41) and (43).
(5) Developing of the matrix product.
(6) Apply the definition of virtual variation.

The following sections will show the above procedure for both the PVD and the RMVT.

8.1 PVD fundamental nuclei derivation

- Starting from equation (47) and substituting the constitutive relations (equations (7)-(9)) for the k-layer we obtain:

$$\begin{aligned}\int_{A_k}\int_{h_k}\Big\{&\delta\boldsymbol{\varepsilon}_{pG}^{k\,T}\Big(\boldsymbol{C}_{pp}^{k}\boldsymbol{\varepsilon}_{pG}^{k}+\boldsymbol{C}_{pn}^{k}\boldsymbol{\varepsilon}_{nG}^{k}-e_{p}^{k\,T}\boldsymbol{E}_{G}^{k}\Big)\\ &+\delta\boldsymbol{\varepsilon}_{nG}^{k\,T}\Big(\boldsymbol{C}_{pn}^{k\,T}\boldsymbol{\varepsilon}_{pG}^{k}+\boldsymbol{C}_{nn}^{k}\boldsymbol{\varepsilon}_{nG}^{k}-e_{n}^{k\,T}\boldsymbol{E}_{G}^{k}\Big)\\ &-\delta\boldsymbol{E}_{G}^{k\,T}\Big(e_{p}^{k}\boldsymbol{\varepsilon}_{pG}^{k}+e_{n}^{k}\boldsymbol{\varepsilon}_{nG}^{k}+\boldsymbol{\epsilon}^{k}\boldsymbol{E}_{G}^{k}\Big)\Big\}dA_k dz=\delta L_e^k\end{aligned} \tag{51}$$

- Introducing the geometric relations (10)-(12) the equation takes the following form:

$$\begin{aligned}\int_{A_k}\int_{h_k}\Big\{&(\boldsymbol{D}_p\delta u^k)^T\Big[\Big(\boldsymbol{C}_{pp}^{k}\boldsymbol{D}_p+\boldsymbol{C}_{pn}^{k}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)u^k-e_p^{k\,T}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Phi^k\Big]\\ &+((\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\delta\boldsymbol{u}^k)^T\Big[\Big(\boldsymbol{C}_{pn}^{k\,T}\boldsymbol{D}_p+\boldsymbol{C}_{nn}^{k}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\boldsymbol{u}^k-e_n^{k\,T}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Phi^k\Big]\\ &-((\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\delta\Phi^k)^T\Big[\Big(e_p^k\boldsymbol{D}_p+e_n^k(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\boldsymbol{u}^k\\ &+\boldsymbol{\epsilon}^k(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Phi^k\Big]\Big\}dA_k dz=\delta L_e^k\end{aligned} \tag{52}$$

- Introducing the UF through equations (21) and (25):

$$\begin{aligned}\int_{A_k}\int_{h_k}\Big\{&\big(\boldsymbol{D}_pF_\tau^u\delta\boldsymbol{u}_\tau^k\big)^T\Big[\Big(\boldsymbol{C}_{pp}^{k}\boldsymbol{D}_p+\boldsymbol{C}_{pn}^{k}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\big(F_s^u\boldsymbol{u}_s^k\big)\\ &-e_p^{k\,T}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F_s^e\Phi_s^k\Big)\Big]\\ &+\big((\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})(F_\tau^u\delta\boldsymbol{u}_\tau^k)\big)^T\Big[\Big(\boldsymbol{C}_{pn}^{k\,T}\boldsymbol{D}_p+\boldsymbol{C}_{nn}^{k}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\big(F_s^u\boldsymbol{u}_s^k\big)\\ &-e_n^{k\,T}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F_s^e\Phi_s^k\Big)\Big]\\ &-\Big((\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F_\tau^e\delta\Phi_\tau^k\Big)\Big)^T\Big[\Big(e_p^k\boldsymbol{D}_p+e_n^k(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\big(F_s^u\boldsymbol{u}_s^k\big)\\ &+\boldsymbol{\epsilon}^k(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F_s^e\Phi_s^k\Big)\Big]\Big\}dA_k dz=\delta L_e^k\end{aligned} \tag{53}$$

  where the superscripts u and e have been applied on the thickness functions to keep memory of their origin.

- At last introducing the FE discretization through equations (41) and (43) the following statement of the PVD can be obtain:

$$
\begin{aligned}
&\int_{A_k}\int_{h_k}\Big\{\left(\boldsymbol{D}_p F^u_\tau N_i \delta\boldsymbol{q}^k_{\tau i}\right)^T\Big[\Big(\boldsymbol{C}^k_{pp}\boldsymbol{D}_p+\boldsymbol{C}^k_{pn}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\Big(F^u_s N_j \boldsymbol{q}^k_{sj}\Big)\\
&\quad -e^{k\,T}_p(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F^e_s N_j g^k_{sj}\Big)\Big]\\
&\quad +\left((\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\left(F^u_\tau N_i \delta\boldsymbol{q}^k_{\tau i}\right)\right)^T\Big[\Big(\boldsymbol{C}^{k\,T}_{pn}\boldsymbol{D}_p+\boldsymbol{C}^k_{nn}(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\Big(F^u_s N_j \boldsymbol{q}^k_{sj}\Big)\\
&\quad -e^{k\,T}_n(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F^e_s N_j g^k_{sj}\Big)\\
&\quad -\left((\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\left(F^e_\tau N_i \delta g^k_{\tau i}\right)\right)^T\Big[\Big(e^k_p\boldsymbol{D}_p+e^k_n(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\Big)\Big(F^u_s N_j \boldsymbol{q}^k_{sj}\Big)\\
&\quad +\boldsymbol{\epsilon}^k(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\Big(F^e_s N_j g^k_{sj}\Big)+\Big]\Big]\Big\}dA_k dz=\delta L^k_e
\end{aligned}
\tag{54}
$$

- Separating each term and making the following assumptions:

$$
\boldsymbol{I}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad \boldsymbol{I}^*=\begin{bmatrix}0\\0\\-1\end{bmatrix}
\tag{55}
$$

The following expression can be found:

$$
\begin{aligned}
&\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_p N_i\right)\boldsymbol{C}^k_{pp}\left[\int_{h_k}F^u_\tau F^u_s dz\right](\boldsymbol{D}_p N_j)dA_k\boldsymbol{q}^k_{sj}\\
&\quad+\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_p N_i\right)\boldsymbol{C}^k_{pn}\left[\int_{h_k}F^u_\tau F^u_s dz\right](\boldsymbol{D}_{np} N_j)dA_k\boldsymbol{q}^k_{sj}\\
&\quad+\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_p N_i\right)\boldsymbol{C}^k_{pn}\left[\int_{h_k}F^u_\tau F^u_{s,z} dz\right]N_j dA_k\boldsymbol{q}^k_{sj}\\
&\quad-\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_p N_i\right)e^{k\,T}_p\left[\int_{h_k}F^u_\tau F^e_s dz\right](\boldsymbol{D}_{ep} N_j)dA_k g^k_{sj}\\
&\quad-\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_p N_i\right)e^{k\,T}_p\left[\int_{h_k}F^u_\tau F^e_{s,z} dz\right](\boldsymbol{I}^* N_j)dA_k g^k_{sj}\\
&\quad+\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_{np} N_i\right)\boldsymbol{C}^{k\,T}_{pn}\left[\int_{h_k}F^u_\tau F^u_s dz\right](\boldsymbol{D}_p N_j)dA_k\boldsymbol{q}^k_{sj}\\
&\quad+\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_{np} N_i\right)\boldsymbol{C}^k_{nn}\left[\int_{h_k}F^u_\tau F^u_s dz\right](\boldsymbol{D}_{np} N_j)dA_k\boldsymbol{q}^k_{sj}\\
&\quad+\delta\boldsymbol{q}^{k\,T}_{\tau i}\int_{A_k}\left(\boldsymbol{D}^T_{np} N_i\right)\boldsymbol{C}^k_{nn}\left[\int_{h_k}F^u_\tau F^u_{s,z} dz\right]N_j dA_k\boldsymbol{q}^k_{sj}
\end{aligned}
$$

$$
\begin{aligned}
& - \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{np}^T N_i\right) e_n^{k\,T} \left[\int_{h_k} F_\tau^u F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
& - \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{np}^T N_i\right) e_n^{k\,T} \left[\int_{h_k} F_\tau^u F_{s,z}^e dz\right] (\boldsymbol{I}^* N_j) dA_k g_{sj}^k \\
& + \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} N_i \boldsymbol{C}_{pn}^{k\,T} \left[\int_{h_k} F_{\tau,z}^u F_s^u dz\right] (\boldsymbol{D}_p N_j) dA_k \boldsymbol{q}_{sj}^k \\
& + \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} N_i \boldsymbol{C}_{nn}^k \left[\int_{h_k} F_{\tau,z}^u F_s^u dz\right] (\boldsymbol{D}_{np} N_j) dA_k \boldsymbol{q}_{sj}^k \\
& + \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} N_i \boldsymbol{C}_{nn}^k \left[\int_{h_k} F_{\tau,z}^u F_{s,z}^u dz\right] N_j dA_k \boldsymbol{q}_{sj}^k \\
& - \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} N_i e_n^{k\,T} \left[\int_{h_k} F_{\tau,z}^u F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
& - \delta \boldsymbol{q}_{\tau i}^{k\,T} \int_{A_k} N_i e_n^{k\,T} \left[\int_{h_k} F_{\tau,z}^u F_{s,z}^e dz\right] (N_j \boldsymbol{I}^*) dA_k g_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) e_p^k \left[\int_{h_k} F_\tau^e F_s^u dz\right] (\boldsymbol{D}_p N_j) dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) e_n^k \left[\int_{h_k} F_\tau^e F_s^u dz\right] (\boldsymbol{D}_{np} N_j) dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) e_n^k \left[\int_{h_k} F_\tau^e F_{s,z}^u dz\right] N_j dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \boldsymbol{\epsilon}^k \left[\int_{h_k} F_\tau^e F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \boldsymbol{\epsilon}^k \left[\int_{h_k} F_\tau^e F_{s,z}^e dz\right] (N_j \boldsymbol{I}^*) dA_k g_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} (N_i \boldsymbol{I}^*)^T e_p^k \left[\int_{h_k} F_{\tau,z}^e F_s^u dz\right] (\boldsymbol{D}_p N_j) dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} (N_i \boldsymbol{I}^*)^T e_n^k \left[\int_{h_k} F_{\tau,z}^e F_s^u dz\right] (\boldsymbol{D}_{np} N_j) dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} (N_i \boldsymbol{I}^*)^T e_n^k \left[\int_{h_k} F_{\tau,z}^e F_{s,z}^u dz\right] N_j dA_k \boldsymbol{q}_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} (N_i \boldsymbol{I}^*)^T \boldsymbol{\epsilon}^k \left[\int_{h_k} F_{\tau,z}^e F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
& - \delta g_{\tau i}^{k\,T} \int_{A_k} (N_i \boldsymbol{I}^*)^T \boldsymbol{\epsilon}^k \left[\int_{h_k} F_{\tau,z}^e F_{s,z}^e dz\right] (\boldsymbol{I}^* N_j) dA_k g_{sj}^k \\
& = \delta L_e^k
\end{aligned}
\tag{56}
$$

8.2 RMVT fundamental nuclei derivation

The derivation of the RMVT nuclei is performed following the same procedure of the PVD:

- Substituting the constitutive equations (29)-(31) for the k-layer states:

$$\begin{aligned}&\int_{A_k}\int_{h_k}\Big\{\delta\boldsymbol{\varepsilon}_{pG}^{kT}\Big(\hat{\boldsymbol{C}}_{pp}^{k}\boldsymbol{\varepsilon}_{pG}^{k}+\hat{\boldsymbol{C}}_{pn}^{k}\boldsymbol{\sigma}_{nM}^{k}+\hat{\boldsymbol{C}}_{pn}^{k}\boldsymbol{E}_{G}^{k}\Big)+\delta\boldsymbol{\varepsilon}_{nG}^{kT}\boldsymbol{\sigma}_{nM}^{k}\\&+\delta\boldsymbol{\sigma}_{nM}^{kT}\Big(\boldsymbol{\varepsilon}_{nG}^{K}-\Big(\hat{\boldsymbol{C}}_{np}^{k}\boldsymbol{\varepsilon}_{pG}^{k}+\hat{\boldsymbol{C}}_{nn}^{k}\boldsymbol{\sigma}_{nM}^{k}+\hat{\boldsymbol{C}}_{de}^{k}\boldsymbol{E}_{G}^{k}\Big)\Big)\\&-\delta\boldsymbol{E}_{G}^{kT}\Big(\hat{\boldsymbol{C}}_{ed}^{k}\boldsymbol{\varepsilon}_{pG}^{k}+\hat{\boldsymbol{C}}_{es}^{k}\boldsymbol{\sigma}_{nM}^{k}+\hat{\boldsymbol{C}}_{ee}^{k}\boldsymbol{E}_{G}^{k}\Big)\Big\}dA_k dz=\delta L_e^k\end{aligned}\tag{57}$$

- Introducing the geometric relations (10)-(12) the equation takes the following form:

$$\begin{aligned}&\int_{A_k}\int_{h_k}\Big\{\delta\boldsymbol{u}^{kT}\boldsymbol{D}_p^T\Big[\hat{\boldsymbol{C}}_{pp}^{k}\boldsymbol{D}_p u^k+\hat{\boldsymbol{C}}_{pn}^{k}\boldsymbol{\sigma}_{nM}^{k}+\hat{\boldsymbol{C}}_{se}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\boldsymbol{\phi}^k\Big]\\&+\delta\boldsymbol{u}^{kT}\Big(\boldsymbol{D}_{np}^T+\boldsymbol{D}_{nz}^T\Big)\boldsymbol{\sigma}_{nM}^{k}\\&+\delta\boldsymbol{\sigma}_{nM}^{T}\Big[(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})\boldsymbol{u}^k-\hat{\boldsymbol{C}}_{np}^{k}\boldsymbol{D}_p\boldsymbol{u}^k-\hat{\boldsymbol{C}}_{nn}^{k}\boldsymbol{\sigma}_{nM}^{k}-\hat{\boldsymbol{C}}_{de}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})\boldsymbol{\phi}^k\Big]\\&-\delta\boldsymbol{\phi}^{kT}\Big(\boldsymbol{D}_{ep}^T+\boldsymbol{D}_{ez}^T\Big)\Big[\hat{\boldsymbol{C}}_{ed}^{k}\boldsymbol{D}_p\boldsymbol{u}^k+\hat{\boldsymbol{C}}_{es}^{k}\boldsymbol{\sigma}_{nM}^{k}+\hat{\boldsymbol{C}}_{ee}^{k}(\boldsymbol{D}_{ep}\\&+\boldsymbol{D}_{ez})\boldsymbol{\phi}^k\Big]\Big\}dA_k dz=\delta L_e^k\end{aligned}\tag{58}$$

- Introducing the UF through equations (21), (25), and (23):

$$\begin{aligned}&\int_{A_k}\int_{h_k}\Big\{\delta\boldsymbol{u}_\tau^{kT}F_\tau^u\boldsymbol{D}_p^T\Big[\hat{\boldsymbol{C}}_{pp}^{k}\boldsymbol{D}_pF_s^u\boldsymbol{u}_s^k+\hat{\boldsymbol{C}}_{pn}^{k}F_s^\sigma\boldsymbol{\sigma}_{ns}^k+\hat{\boldsymbol{C}}_{se}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^e\phi_s^k\Big]\\&+\delta\boldsymbol{u}_\tau^{kT}F_\tau^u\Big(\boldsymbol{D}_{np}^T+\boldsymbol{D}_{nz}^T\Big)F_s^\sigma\boldsymbol{\sigma}_{ns}^k+\delta\boldsymbol{\sigma}_{n\tau}^{kT}F_\tau^\sigma\Big[(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})F_s^u\boldsymbol{u}_s^k-\hat{\boldsymbol{C}}_{np}^{k}\boldsymbol{D}_pF_s^u\boldsymbol{u}_s^k\\&-\hat{\boldsymbol{C}}_{nn}^{k}F_s^\sigma\boldsymbol{\sigma}_{ns}^k-\hat{\boldsymbol{C}}_{de}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^e\phi_s^k]-F_\tau^e\delta\phi_\tau^k\Big(\boldsymbol{D}_{ep}^T+\boldsymbol{D}_{ez}^T\Big)\Big[\hat{\boldsymbol{C}}_{ed}^{k}\boldsymbol{D}_pF_s^u\boldsymbol{u}_s^k\\&+\hat{\boldsymbol{C}}_{es}^{k}F_s^\sigma\boldsymbol{\sigma}_s^k+\hat{\boldsymbol{C}}_{ee}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^e\phi_s^k\Big]\Big\}dA_k dz=\delta L_e^k\end{aligned}\tag{59}$$

- At last introducing the shape functions through equations (41), (43), and (42) the following statement can be obtained:

$$\begin{aligned}&\int_{A_k}\int_{h_k}\Big\{\Big(\delta\boldsymbol{q}_{\tau i}^{Tk}N_iF_\tau^u\boldsymbol{D}_p^T\Big)\Big[\hat{\boldsymbol{C}}_{pp}^{k}\boldsymbol{D}_pF_s^uN_j\boldsymbol{q}_{sj}^k+\hat{\boldsymbol{C}}_{pn}^{k}F_s^\sigma N_j\boldsymbol{f}_{sj}^k\\&+\hat{\boldsymbol{C}}_{se}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^eN_jg_{sj}^k\Big]+\big(\delta\boldsymbol{q}_{\tau i}^{Tk}N_iF_\tau^u\big)\Big(\boldsymbol{D}_{np}^T+\boldsymbol{D}_{nz}^T\Big)F_s^\sigma N_j\boldsymbol{f}_{sj}^k\\&+\Big(\delta\boldsymbol{f}_{\tau i}^kN_iF_\tau^\sigma\Big)\Big[(\boldsymbol{D}_{np}+\boldsymbol{D}_{nz})F_s^uN_j\boldsymbol{q}_{sj}^k-\hat{\boldsymbol{C}}_{np}^{k}\boldsymbol{D}_pF_s^uN_j\boldsymbol{q}_{sj}^k\\&-\hat{\boldsymbol{C}}_{nn}^{k}F_s^\sigma N_j\boldsymbol{f}_{sj}^k-\hat{\boldsymbol{C}}_{de}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^eN_jg_{sj}^k-\Big]\\&-\delta g_{\tau i}^kN_iF_\tau^e\Big(\boldsymbol{D}_{ep}^T+\boldsymbol{D}_{ez}^T\Big)\Big[\hat{\boldsymbol{C}}_{ed}^{k}\boldsymbol{D}_pF_s^uN_j\boldsymbol{q}_{sj}^k+\hat{\boldsymbol{C}}_{es}^{k}F_s^\sigma N_j\boldsymbol{f}_{sj}^k\\&+\hat{\boldsymbol{C}}_{ee}^{k}(\boldsymbol{D}_{ep}+\boldsymbol{D}_{ez})F_s^eN_jg_{sj}^k\Big]\Big\}dA_k dz=\delta L_e^k\end{aligned}\tag{60}$$

- Separating each term and making the following assumptions:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{I}^* = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \tag{61}$$

The following expression can be found:

$$\begin{aligned}
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}\left(\boldsymbol{D}_p^T N_i\right)\hat{\boldsymbol{C}}_{pp}^k\left[\int_{h_k}F_\tau^u F_s^u dz\right](\boldsymbol{D}_p N_j)dA_k\boldsymbol{q}_{sj}^k\\
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}\left(\boldsymbol{D}_p^T N_i\right)\hat{\boldsymbol{C}}_{pn}^k\left[\int_{h_k}F_\tau^u F_s^\sigma dz\right]N_j dA_k\boldsymbol{f}_{sj}^k\\
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}\left(\boldsymbol{D}_p^T N_i\right)\hat{\boldsymbol{C}}_{se}^k\left[\int_{h_k}F_\tau^u F_s^e dz\right]\boldsymbol{D}_{ep}N_j dA_k g_{sj}^k\\
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}\left(\boldsymbol{D}_p^T N_i\right)\hat{\boldsymbol{C}}_{se}^k\left[\int_{h_k}F_\tau^u F_{s,z}^e dz\right]\boldsymbol{I}^* N_j dA_k g_{sj}^k\\
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}\left(\boldsymbol{D}_{np}^T N_i\right)\left[\int_{h_k}F_\tau^u F_s^\sigma dz\right]N_j dA_k\boldsymbol{f}_{sj}^k\\
&+\delta\boldsymbol{q}_{\tau i}^{kT}\int_{A_k}(N_i\boldsymbol{I}^T)\left[\int_{h_k}F_{\tau,z}^u F_s^\sigma dz\right]N_j dA_k\boldsymbol{f}_{sj}^k\\
&+\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\left[\int_{h_k}F_\tau^\sigma F_s^u dz\right](\boldsymbol{D}_{np}N_j)dA_k\boldsymbol{q}_{sj}^k\\
&+\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\left[\int_{h_k}F_\tau^\sigma F_{s,z}^u dz\right](\boldsymbol{I}N_j)dA_k\boldsymbol{q}_{sj}^k\\
&-\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\hat{\boldsymbol{C}}_{np}^k\left[\int_{h_k}F_\tau^\sigma F_s^u dz\right](\boldsymbol{D}_p N_j)dA_k\boldsymbol{q}_{sj}^k\\
&-\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\hat{\boldsymbol{C}}_{nn}^k\left[\int_{h_k}F_\tau^\sigma F_s^\sigma dz\right]N_j dA_k\boldsymbol{f}_{sj}^k\\
&-\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\hat{\boldsymbol{C}}_{de}^k\left[\int_{h_k}F_\tau^\sigma F_s^e dz\right](\boldsymbol{D}_{ep}N_j)dA_k g_{sj}^k\\
&-\delta\boldsymbol{f}_{\tau i}^{kT}\int_{A_k}N_i\hat{\boldsymbol{C}}_{de}^k\left[\int_{h_k}F_\tau^\sigma F_{s,z}^e dz\right](\boldsymbol{I}^* N_j)dA_k g_{sj}^k
\end{aligned} \tag{62}$$

$$
\begin{aligned}
&- \delta g_{\tau i}^{kT} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \hat{\boldsymbol{C}}_{ed}^k \left[\int_{h_k} F_\tau^e F_s^u dz\right] \boldsymbol{D}_p N_j dA_k \boldsymbol{q}_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} (N_i \boldsymbol{I}^{*T}) \hat{\boldsymbol{C}}_{ed}^k \left[\int_{h_k} F_{\tau,z}^e F_s^u dz\right] (\boldsymbol{D}_p N_j) dA_k \boldsymbol{q}_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \hat{\boldsymbol{C}}_{es}^k \left[\int_{h_k} F_\tau^e F_s^\sigma dz\right] N_j dA_k \boldsymbol{f}_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} (N_i \boldsymbol{I}^{*T}) \hat{\boldsymbol{C}}_{es}^k \left[\int_{h_k} F_{\tau,z}^e F_s^\sigma dz\right] N_j dA_k \boldsymbol{f}_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \hat{\boldsymbol{C}}_{ee}^k \left[\int_{h_k} F_\tau^e F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} (N_i \boldsymbol{I}^{*T}) \hat{\boldsymbol{C}}_{ee}^k \left[\int_{h_k} F_{\tau,z}^e F_s^e dz\right] (\boldsymbol{D}_{ep} N_j) dA_k g_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} \left(\boldsymbol{D}_{ep}^T N_i\right) \hat{\boldsymbol{C}}_{ee}^k \left[\int_{h_k} F_\tau^e F_{s,z}^e dz\right] (\boldsymbol{I}^* N_j) dA_k g_{sj}^k \\
&- \delta g_{\tau i}^{kT} \int_{A_k} (N_i \boldsymbol{I}^{*T}) \hat{\boldsymbol{C}}_{ee}^k \left[\int_{h_k} F_{\tau,z}^e F_{s,z}^e dz\right] \boldsymbol{I}^* N_j dA_k g_{sj}^k = \delta L_e^k
\end{aligned}
$$

8.3 Derivation of FE mechanical and electric load vector

The aim of this section is obtaining the nodal values equivalent to the external loads for a generic layer k on which a surface traction $\bar{\boldsymbol{t}}_j$ and a electric charge $\bar{Q}$ are applied on a plane $\zeta^k = \zeta_1^k$ where ζ^k is the adimensionalized thickness coordinate of the lamina:

$$\zeta^k = \frac{2z^k}{h^k} \tag{63}$$

The expression of the external work for our formulation can be written as:

$$\delta L_e^k = \int_{A_k} \left(\delta \boldsymbol{u}^{k^T}(x, y, \zeta_1^k) \bar{\boldsymbol{t}}^k(x, y, \zeta_1^k) - \delta \Phi^{k^T}(x, y, \zeta_1^k) \bar{Q}^k(x, y, \zeta_1^k)\right) dA_k \tag{64}$$

Where:

$$\bar{\boldsymbol{t}}^k = \begin{bmatrix} \bar{t}_x^k \\ \bar{t}_y^k \\ \bar{t}_z^k \end{bmatrix} \qquad \bar{Q}^k = \bar{Q}^k(x, y, \zeta_1^k) \tag{65}$$

Now, using the unified expansion for the traction and the electric charge, they can be rewritten as:

$$\bar{\boldsymbol{t}}^k = F_b(\zeta_1^k)\bar{\boldsymbol{t}}_b^k(x,y) + F_r(\zeta_1^k)\bar{\boldsymbol{t}}_r^k(x,y) + F_t(\zeta_1^k)\bar{\boldsymbol{t}}_t^k(x,y) = F_\tau^1\bar{\boldsymbol{t}}_\tau^k$$
$$\text{con} \quad r = 1,2,\ldots,N \quad F_\tau^1 = F_\tau(\zeta_1^k) \tag{66}$$

$$\bar{Q}^k = F_b(\zeta_1^k)\bar{Q}_b^k(x,y) + F_r(\zeta_1^k)\bar{Q}_r^k(x,y) + F_t(\zeta_1^k)\bar{Q}_t^k(x,y) = F_\tau^1\bar{Q}_\tau^k$$
$$\text{con} \quad r = 1,2,\ldots,N \quad F_\tau^1 = F_\tau(\zeta_1^k) \tag{67}$$

At this point introducing the shape functions N_i, the generic surface traction $\bar{\boldsymbol{t}}_\tau^k$ and the generic electric charge can be written as:

$$\bar{\boldsymbol{t}}_\tau^k = \boldsymbol{m}_\tau^k N_i \boldsymbol{p}_{\tau i} \tag{68}$$

$$\bar{Q}_\tau^k = n_\tau^k N_i \Psi_{\tau i} \tag{69}$$

where $\boldsymbol{m}_\tau^k$ is the diagonal matrix that takes into account of possible in-plane shape of the pressure load and n is a scalar quantity that takes into account of possible in-plane shape of the electric load while $\boldsymbol{p}_{\tau i}^k$ is the nodal amplitude:

$$\boldsymbol{m}_\tau^k = \begin{bmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & m_z \end{bmatrix} \qquad \boldsymbol{p}_{\tau i}^k = \begin{bmatrix} p_{x\tau i} \\ p_{y\tau i} \\ p_{z\tau i} \end{bmatrix} \tag{70}$$

Substituting equations (68) and (69), respectively, into equations (66) and (67) we obtain:

$$\bar{\boldsymbol{t}}^k = F_\tau^1 \boldsymbol{m}_\tau^k N_i \boldsymbol{p}_{\tau i} \tag{71}$$

$$\bar{Q}^k = F_\tau^1 n_\tau^k N_i \Psi_{\tau i} \tag{72}$$

Substituting equations (71), (72) into equation (64) we obtain:

$$\delta L_e^k = \delta \boldsymbol{q}_{\tau i}^{k\,T}\left[\int_{A_k} F_\tau^1 (N_i N_j \boldsymbol{m}_s^k) F_s^1 dA_k\right]\boldsymbol{p}_{sj}^k - \delta g_{\tau i}^{k\,T}\left[\int_{A_k} F_\tau^1 (N_i N_j n_s^k) F_s^1 dA_k\right]\Psi_{sj}^k \tag{73}$$

Setting:

$$\boldsymbol{K}_{up}^{k\tau sij} = \int_{A_k} F_\tau^1 (N_i N_j \boldsymbol{m}_s^k) F_s^1 dA_k \tag{74}$$

$$\boldsymbol{K}_{ef}^{k\tau sij} = \int_{A_k} F_\tau^1 (N_i N_j n_s^k) F_s^1 dA_k \tag{75}$$

Equation (73) takes the following form:

$$\delta L_e^k = \delta \boldsymbol{q}_{\tau i}^{k\,T} \boldsymbol{K}_{up}^{k\tau sij} \boldsymbol{p}_{sj} - \delta g_{\tau i}^{k\,T} \boldsymbol{K}_{ef}^{k\tau sij} \Psi_{sj}^k \tag{76}$$

8.4 Governing equations through fundamental nuclei

The final step in the derivation of the FE governing equations for the two variational statements is to collect the terms in equations (56) and (62) in a more compact way considering that each virtual variation is independent and arbitrary. The governing equations can be then written as:

- PVD:

$$\delta \boldsymbol{q}_{\tau i}^{k\ T} : \boldsymbol{K}_{uu}^{k\tau sij}\boldsymbol{q}_{sj}^{k} + \boldsymbol{K}_{ue}^{k\tau sij} g_{sj}^{k} = \boldsymbol{P}_{u\tau}^{k} \qquad \delta g_{\tau i}^{k\ T} : \boldsymbol{K}_{eu}^{k\tau sij}\boldsymbol{q}_{sj}^{k} + \boldsymbol{K}_{ee}^{k\tau sij} g_{sj}^{k} = \boldsymbol{P}_{e\tau}^{k} \quad (77)$$

- RMVT:

$$\begin{aligned} \delta \boldsymbol{q}_{\tau i}^{k\ T} &: \quad \boldsymbol{K}_{muu}^{k\tau sij}\boldsymbol{q}_{sj}^{k} + \boldsymbol{K}_{mu\sigma}^{k\tau sij}\boldsymbol{f}_{sj}^{k} + \boldsymbol{K}_{mue}^{k\tau sij} g_{sj}^{k} = \boldsymbol{P}_{u\tau}^{k} \\ \delta \boldsymbol{f}_{\tau i}^{kT} &: \quad \boldsymbol{K}_{m\sigma u}^{k\tau sij}\boldsymbol{q}_{sj}^{k} + \boldsymbol{K}_{m\sigma\sigma}^{k\tau sij}\boldsymbol{f}_{sj}^{k} + \boldsymbol{K}_{m\sigma e}^{k\tau sij} g_{sj}^{k} = 0 \\ \delta g_{\tau i}^{k\ T} &: \quad \boldsymbol{K}_{meu}^{k\tau sij}\boldsymbol{q}_{sj}^{k} + \boldsymbol{K}_{me\sigma}^{k\tau sij}\boldsymbol{f}_{sj}^{k} + \boldsymbol{K}_{mee}^{k\tau sij} g_{sj}^{k} = \boldsymbol{P}_{e\tau}^{k} \end{aligned} \quad (78)$$

Where:

$$\boldsymbol{P}_{u\tau}^{k} = \boldsymbol{K}_{up}^{k\tau sij}\boldsymbol{p}_{sj} \quad (79)$$

$$\boldsymbol{P}_{e\tau}^{k} = -\boldsymbol{K}_{ef}^{k\tau sij}\Psi_{sj} \quad (80)$$

The new terms $\boldsymbol{K}$ in the systems of equations (77) and (78) are the fundamental nuclei. The fundamental nuclei are simple vectors or matrices that represent the basic component of any element stiffness matrix. Depending on the variables they are connecting, the shape of the generic fundamental nucleus $\boldsymbol{K}_{\alpha,\beta}$ varies accordingly to Figure 3 where subscripts α, β can take the values u, σ, ϕ.

The following relations on the fundamental nuclei can be easily found:

$$\boldsymbol{K}_{eu} = \boldsymbol{K}_{ue}^{T} \quad \boldsymbol{K}_{meu} = \boldsymbol{K}_{mue}^{T}$$
$$\boldsymbol{K}_{msu} = \boldsymbol{K}_{mus}^{T} \quad \boldsymbol{K}_{mse} = \boldsymbol{K}_{mes}^{T}$$

The explicit form of the fundamental nuclei can be found in Appendix 2. The above systems are valid just at nodal level of a one layer structure; in order to obtain the

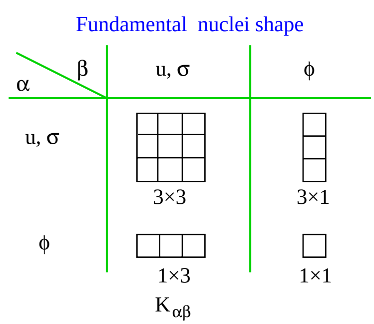


Figure 3.
Fundamental nuclei shape table

governing equations for all the structure, the fundamental nuclei should be opportunely assembled expanding the indices i, j, τ, s up to their extents. For the sake of brevity, the assembling procedure is not herein reported but a detailed explanation can be found in Carrera and Demasi (2002).

9. Numerical results and remarks

The capability of the present formulation to accurately modelize piezoelectric plates has been already verified in Robaldo *et al.* (2006) where the analytical and numerical results have been compared to three-dimensional exact ones. For the shear actuation .mechanism the cases proposed in Benjeddou and Deü (2001) has been taken as a benchmark. Both sensor and actuator, configurations have been considered as shown in Figure 4(a) and (b).

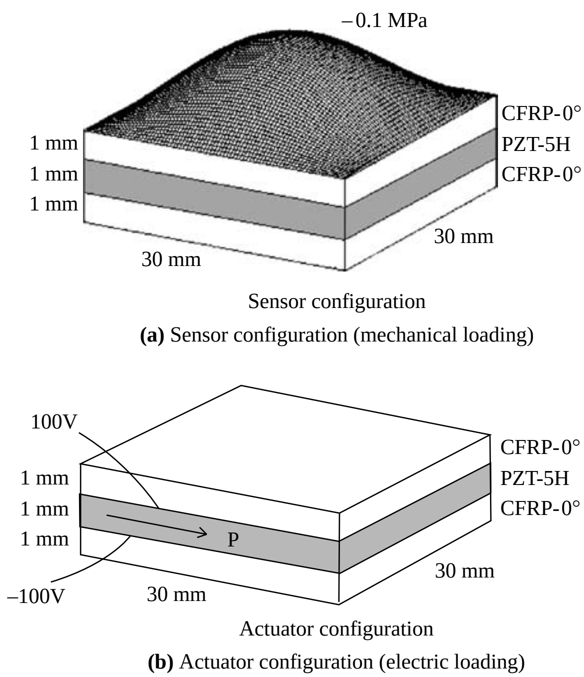
The properties of the considered materials are listed in Table I.

9.1 Sensor configuration

In the sensor configuration, the plate of Figure 4(a) is subjected to a bi-sinusoidal pressure of amplitude $t_0 = -0.1$ MPa applied on its top surface:

$$t = t_0 sin\left(\frac{\pi x}{A}\right) sin\left(\frac{\pi y}{B}\right) \tag{81}$$

The displacement results for the four (Q4) and nine (Q9) node elements are given in Tables II and III and in graphical form in Figure 5 for the various plate models. In-plane and transverse stresses are shown in Figure 6.



(a) Sensor configuration (mechanical loading)

(b) Actuator configuration (electric loading)

Figure 4.
Proposed test-cases

Table I.
Elastic, piezoelectric, and dielectric properties of used materials

Property	PZT-5H	Gr/Ep
E_1 (GPa)	48.164	180.999
E_2 (GPa)	60.013	10.299
E_3 (GPa)	60.013	10.299
ν_{12}	0.409	0.280
ν_{13}	0.409	0.280
ν_{23}	0.290	0.330
G_{23} (GPa)	23.000	2.870
G_{13} (GPa)	23.000	7.170
G_{12} (GPa)	23.000	7.170
e_{11} (C/m^2)	23.3	0
e_{12} (C/m^2)	-6.50	0
e_{13} (C/m^2)	-6.50	0
e_{26} (C/m^2)	17.0	0
e_{35} (C/m^2)	17.0	0
ϵ_{11} (10^{-8} F/m)	1.3	1.53
ϵ_{22} (10^{-8} F/m)	1.503	1.53
ϵ_{33} (10^{-8} F/m)	1.503	1.53

Table II.
Sensor configuration: elements displacement results (μm)

Sensor configuration: Q4 element mesh: (10 × 10)								
	LD4	*LM4*	*LD3*	*LM3*	*LD2*	*LM2*	*LD1*	*LM1*
u (0.0,B/2,H/2)	−0.2507	−0.2502	−0.2507	−0.2502	−0.2506	−0.2502	−0.2369	−0.2502
v (A/2,0.0,H/2)	−0.2764	−0.2751	−0.2764	−0.2751	−0.2763	−0.2752	−0.2617	−0.2753
w (A/2,B/2,H/2)	−1.7914	−1.7903	−1.7914	−1.7902	−1.7909	−1.7898	−1.6991	−1.7904
	ED4	*EM4*	*ED3*	*EM3*	*ED2*	*EM2*	*ED1*	*EM1*
u (0.0,B/2,H/2)	−0.2472	−0.2466	−0.2464	−0.2461	−0.2493	−0.2499	−0.2388	−0.2442
v (A/2,0.0,H/2)	−0.2682	−0.2674	−0.2674	−0.2671	−0.2691	−0.2707	−0.2576	−0.2677
w (A/2,B/2,H/2)	−1.7122	−1.7143	−1.7048	−1.7133	−1.7006	−1.7242	−1.6437	−1.7628
	EDz3	*EMz3*	*EDz2*	*EMz2*	*EDz1*	*EMz1*		
u (0.0,B/2,H/2)	−0.2487	−0.2488	−0.2490	−0.2490	−0.2380	−0.2444		
v (A/2,0.0,H/2)	−0.2744	−0.2736	−0.2716	−0.2708	−0.2592	−0.2657		
w (A/2,B/2,H/2)	−1.7738	−1.7768	−1.7446	−1.7480	−1.6814	−1.7286		

Several remarks can be pointed out:

- The obtained results are generally in good compliance with the exact solution.
- In-plane displacements (Figure 5(a) and (b)) are accurately described by every model; on the contrary, out-of-plane ones (Figure 5(c)-(f)) require higher order LW models.
- Mixed models improve drastically the performances of ESL models.
- The description of the in-plane stresses given by every LW models could be considered satisfactory (Figure 6(a)-(c)).
- Higher order mixed models are mandatory to modelize transverse and normal stresses (Figure 6(d)-(f)) in order to satisfy the top-bottom conditions.

Sensor configuration: Q9 element mesh: (6 × 6)								
	LD4	*LM4*	*LD3*	*LM3*	*LD2*	*LM2*	*LD1*	*LM1*
u (0.0,B/2,H/2)	−0.2508	−0.2509	−0.2506	−0.2509	−0.2507	−0.2508	−0.2370	−0.2508
v (A/2,0.0,H/2)	−0.2751	−0.2750	−0.2751	−0.2750	−0.2750	−0.2750	−0.2604	−0.2751
w (A/2,B/2,H/2)	−1.7953	−1.7959	−1.7953	−1.7959	−1.7948	−1.7954	−1.7030	−1.7963
	ED4	*EM4*	*ED3*	*EM3*	*ED2*	*EM2*	*ED1*	*EM1*
u (0.0,B/2,H/2)	−0.2472	−0.2478	−0.2464	−0.2472	−0.2490	−0.2504	−0.2385	−0.2446
v (A/2,0.0,H/2)	−0.2668	−0.2674	−0.2660	−0.2670	−0.2675	−0.2706	−0.2560	−0.2673
w (A/2,B/2,H/2)	−1.7157	−1.7220	−1.7082	−1.7208	−1.7033	−1.7317	−1.6460	−1.7680
	EDz3	*EMz3*	*EDz2*	*EMz2*	*EDz1*	*EMz1*		
u (0.0,B/2,H/2)	−0.2489	−0.2493	−0.2492	−0.2496	−0.2381	−0.2448		
v (A/2,0.0,H/2)	−0.2730	−0.2734	−0.2703	−0.2708	−0.2579	−0.2654		
w (A/2,B/2,H/2)	−1.7776	−1.7821	−1.7490	−1.7544	−1.6852	−1.7344		

Table III.
Sensor configuration: elements displacement results (μm)

- The exact solution for the electric potential for the sensor configuration shows a jump at the interfaces between elastic and piezoceramic layers. At the moment, the present FE formulation is unable to include in the modelized variables such discontinuous fields thus it does not lead to a satisfactory description of the through-the-thickness behavior of the electric quantities.

9.2 Actuator configuration

In the actuator configuration, the piezoceramic core of the plate of Figure 4(b) is subjected to $\Phi^- = 100\,\mathrm{V}$ and a $\Phi^+ = -100\,\mathrm{V}$ on its lower and upper surface electrodes with an in-plane shape given by:

$$\Phi_{TOP} = \Phi^+ cos\left(\frac{\pi x}{A}\right) sin\left(\frac{\pi y}{B}\right) \tag{82}$$

$$\Phi_{BOT} = \Phi^- cos\left(\frac{\pi x}{A}\right) sin\left(\frac{\pi y}{B}\right) \tag{83}$$

The displacement results for the four (Q4) and nine (Q9) node elements are given in Tables IV and V and in graphical form in Figure 7 for the various plate models. In-plane and transverse stresses as well the electric potential are shown in Figure 8.

Several remarks can be pointed out:

- In general, the obtained results could be considered satisfactory and they matches quite well the exact ones.
- The applications of the mixed principle improves noticeably the results for lower order theories. This effect comes to evidence for the less fine mesh (Q4 (10 × 10)). Compare lower order LW models on Tables IV and V.
- Lower order ESL models are completely inadequate, even if the use of the Murakami's function improves the through-thickness description capabilities of such models.
- Only LW models lead to accurate description of the displacement fields (Figure 7).

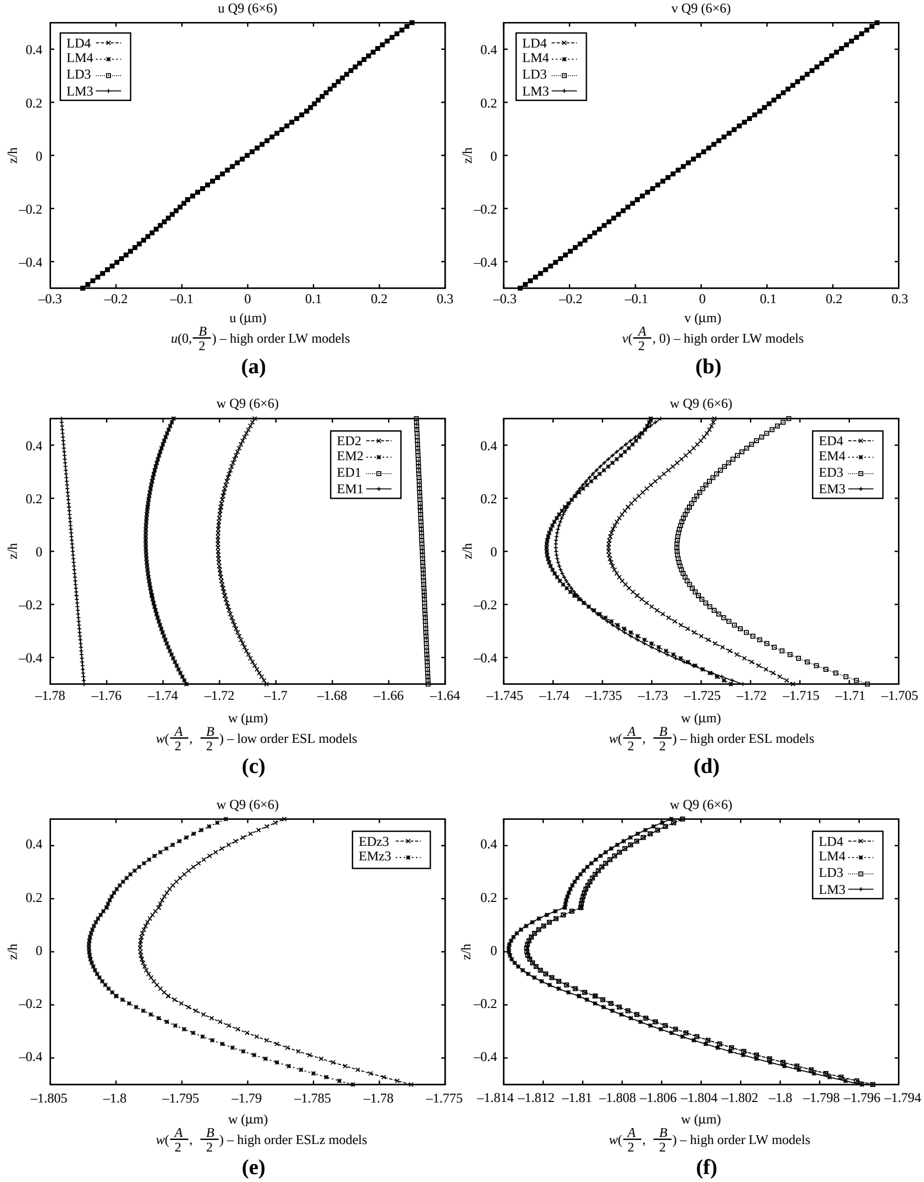


Figure 5.
Sensor configuration: displacements

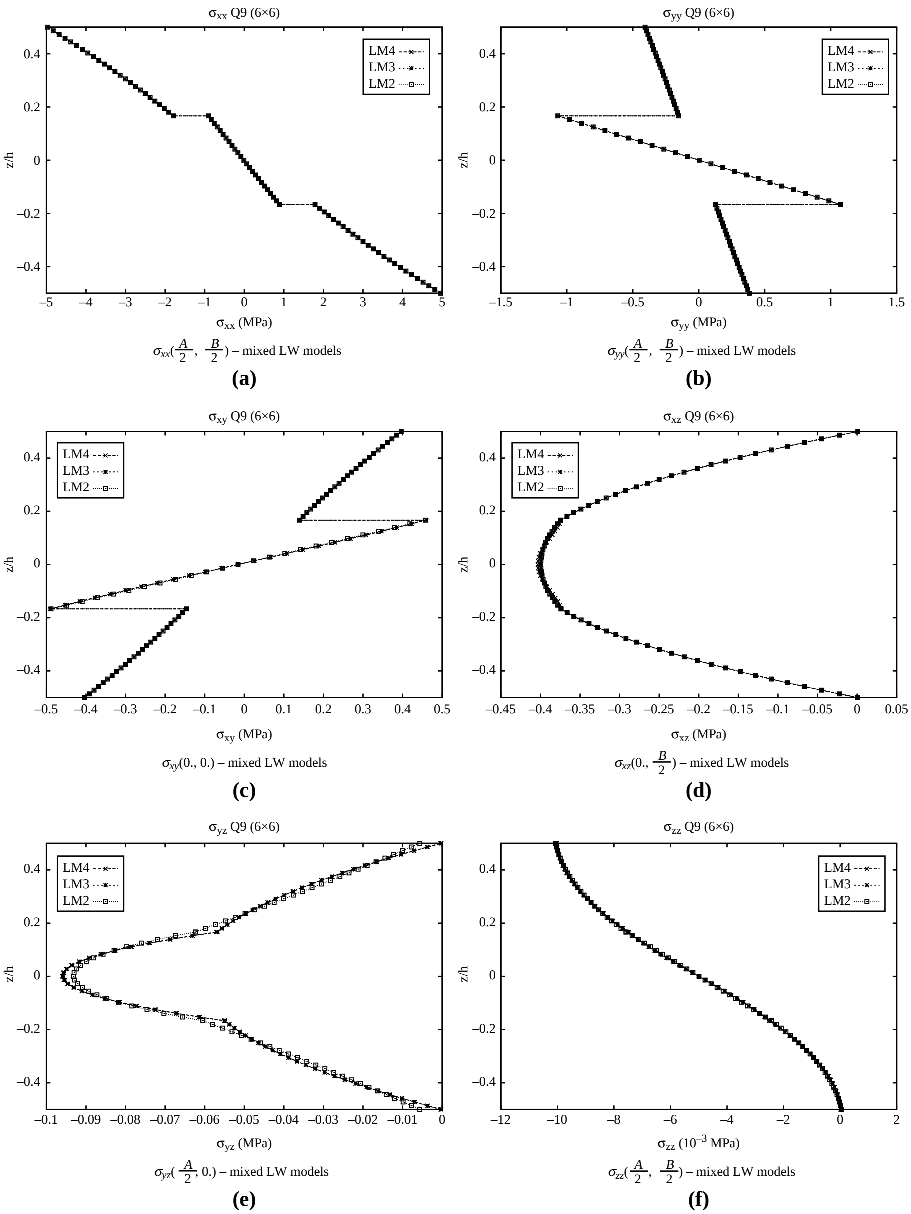


Figure 6.
Sensor configuration: stresses

Table IV.
Actuator configuration: elements displacement results (μm)

Actuator configuration: Q4 element mesh: (10 × 10)								
	LD4	*LM4*	*LD3*	*LM3*	*LD2*	*LM2*	*LD1*	*LM1*
u (0.0,B/2,H/2)	0.0067	0.0138	0.0067	0.0138	0.0067	0.0139	0.0084	0.0137
v (A/2,0.0,H/2)	0.0779	0.0850	0.0779	0.0851	0.0778	0.0851	0.0797	0.0849
w (A/2,B/2,H/2)	0.5041	0.5560	0.5041	0.5560	0.5039	0.5558	0.5161	0.5552
	ED4	*EM4*	*ED3*	*EM3*	*ED2*	*EM2*	*ED1*	*EM1*
u (0.0,B/2,H/2)	0.0125	0.0187	0.0126	0.0190	−0.0277	−0.0197	−0.0308	−0.0169
v (A/2,0.0,H/2)	0.1068	0.1116	0.1069	0.1099	0.1072	0.1072	0.1037	0.0758
w (A/2,B/2,H/2)	0.6865	0.7234	0.6877	0.7131	0.6835	0.6902	0.6663	0.4998
	EDz3	*EMz3*	*EDz2*	*EMz2*	*EDz1*	*EMz1*		
U (0.0,B/2,H/2)	0.0074	0.0148	−0.0297	−0.0227	−0.0295	−0.0237		
V (A/2,0.0,H/2)	0.0785	0.0856	0.0793	0.0862	0.0796	0.0851		
W (A/2,B/2,H/2)	0.5101	0.5605	0.5125	0.5617	0.5138	0.5553		

Table V.
Actuator configuration: elements displacement results (μm)

Actuator configuration: Q9 element mesh: (6 × 6)								
	LD4	*LM4*	*LD3*	*LM3*	*LD2*	*LM2*	*LD1*	*LM1*
u (0.0,B/2,H/2)	0.0139	0.0137	0.0135	0.0136	0.0143	0.0137	0.0136	0.0136
v (A/2,0.0,H/2)	0.0847	0.0845	0.0847	0.0845	0.0846	0.0845	0.0859	0.0843
w (A/2,B/2,H/2)	0.5520	0.5556	0.5521	0.5555	0.5518	0.5552	0.5614	0.5540
	ED4	*EM4*	*ED3*	*EM3*	*ED2*	*EM2*	*ED1*	*EM1*
u (0.0,B/2,H/2)	0.0182	0.0193	0.0183	0.0196	−0.0179	−0.0192	−0.0213	−0.0166
v (A/2,0.0,H/2)	0.1130	0.1107	0.1130	0.1090	0.1134	0.1065	0.1095	0.0752
w (A/2,B/2,H/2)	0.7315	0.7218	0.7326	0.7115	0.7284	0.6890	0.7092	0.4990
	EDz3	*EMz3*	*EDz2*	*EMz2*	*EDz1*	*EMz1*		
u (0.0,B/2,H/2)	0.0152	0.0143	−0.0202	−0.0221	−0.0204	−0.0232		
v (A/2,0.0,H/2)	0.0853	0.0850	0.0859	0.0856	0.0858	0.0845		
w (A/2,B/2,H/2)	0.5576	0.5601	0.5592	0.5613	0.5583	0.5548		

- Mixed models slightly overestimated the normal displacement with respect to classical theories. A relative average error of less than 1 percent has been found between LD4 and LM4 results.
- Every LW model depicts correctly the in-plane stresses (Figure 8(a)-(c)).
- Transverse and normal stresses complain with the exact ones only for LM models (Figure 8(d)-(e)).
- The continuous electric potential is computed exactly by every model (Figure 8(f)).

10. Conclusions

At the present level of the development, the obtained results are quite satisfactory and promising. The capabilities of the UF to accurately analyze multilayered structures exploiting the shear mode actuation has been tested. In order to extend once more the capabilities of the UF, further efforts should be made toward the assumptions of discontinuous electric fields (potential and normal displacement). Finally, the present paper confirms the need for advanced higher order plate models in modeling of adaptive laminate.

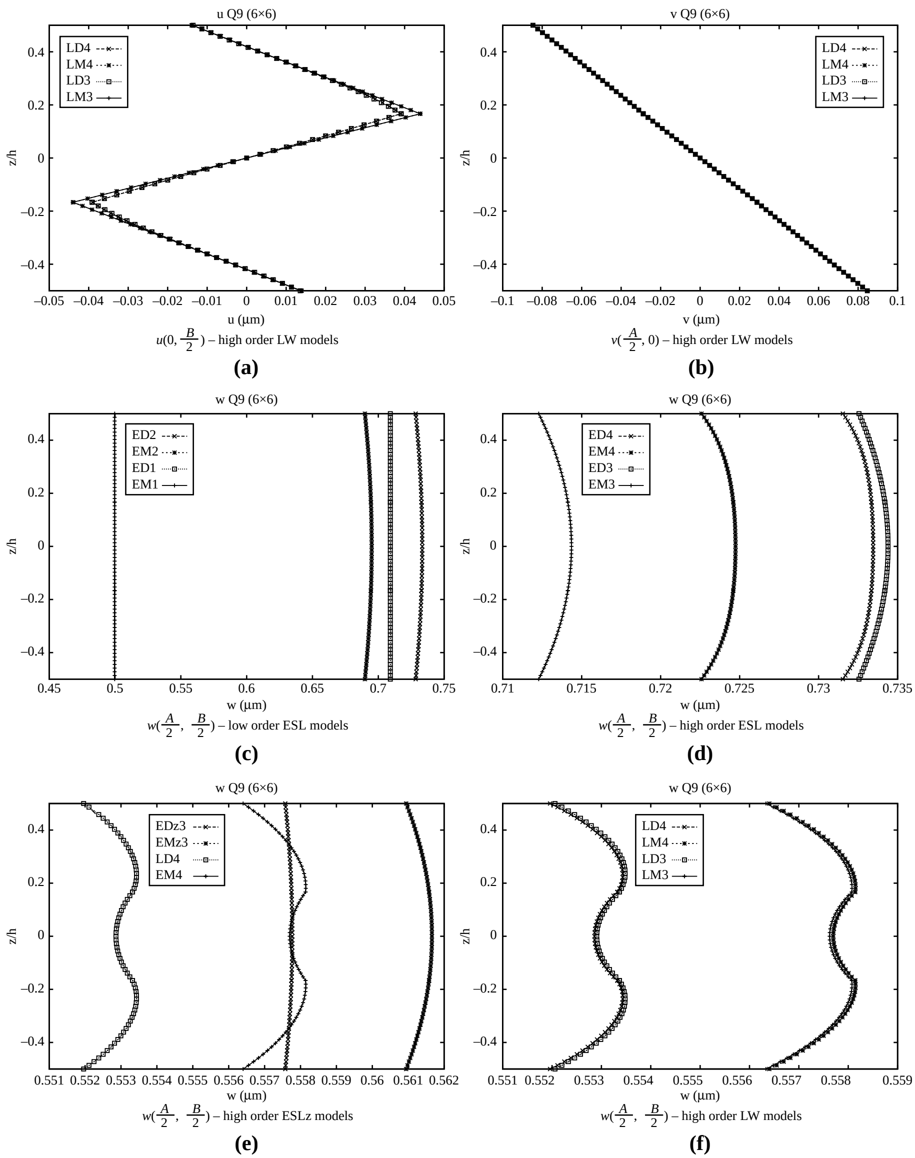
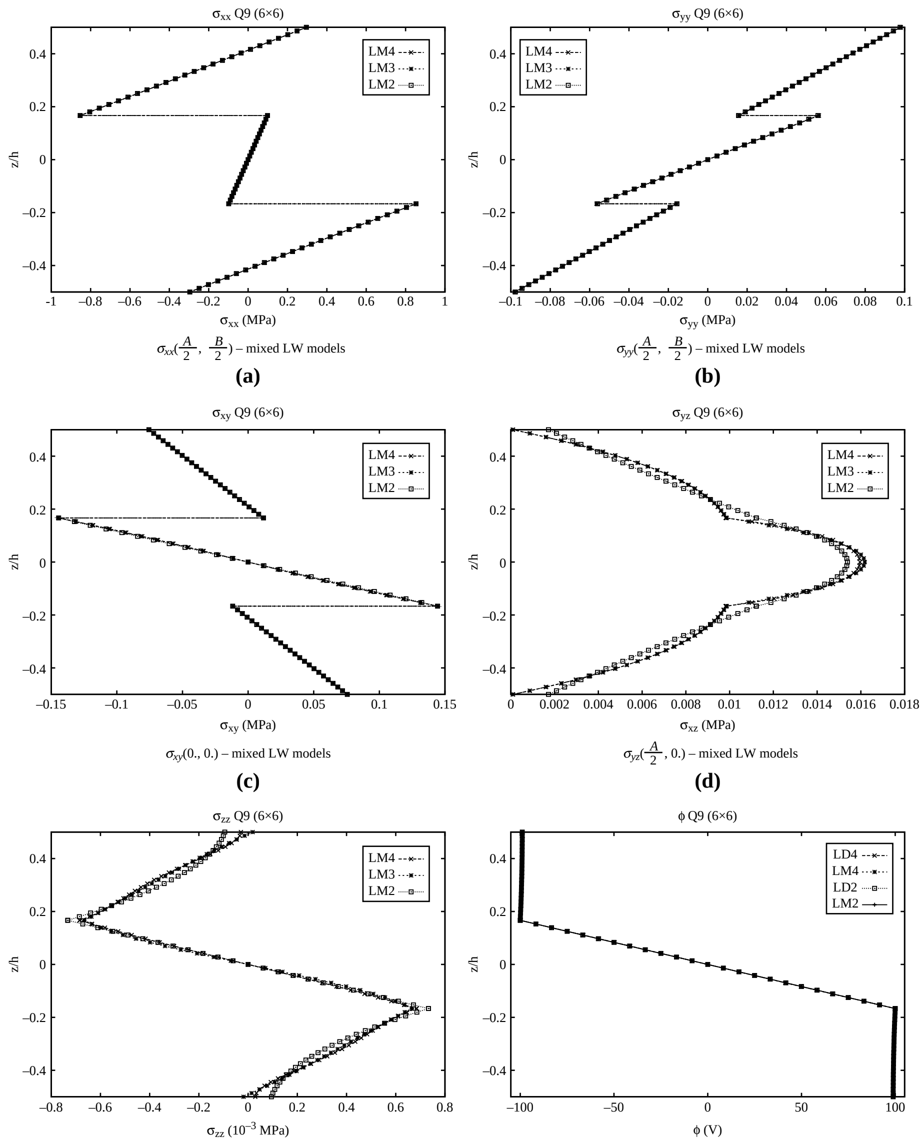


Figure 7.
Actuator configuration: displacements



$\sigma_{xx}(\frac{A}{2}, \frac{B}{2})$ – mixed LW models
(a)

$\sigma_{yy}(\frac{A}{2}, \frac{B}{2})$ – mixed LW models
(b)

$\sigma_{xy}(0., 0.)$ – mixed LW models
(c)

$\sigma_{yz}(\frac{A}{2}, 0.)$ – mixed LW models
(d)

$\sigma_{zz}(\frac{A}{2}, \frac{B}{2})$ – mixed LW models
(e)

$\phi(0., \frac{B}{2})$ – pvd and mixed LW models
(f)

Figure 8.
Actuator Configuration: in-plane stresses and electric potential

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Appendix 1

The explicit form of the constitutive relations in the laminate reference system is herein given. Taking into account of possible rotation of the lamina with respect to the laminate reference system, the constitutive relations take the following form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}^k = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}^k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}^k - \begin{bmatrix} e_{11} & 0 & 0 \\ e_{12} & 0 & 0 \\ e_{13} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_{35} \\ 0 & e_{26} & 0 \end{bmatrix}^k \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^k \quad \text{(A1)}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix}^k = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & 0 \end{bmatrix}^k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}^k + \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}^k \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}^k \tag{A2}$$

Appendix 2

The explicit expressions of the fundamental nuclei are listed below. The following notations are introduced:

$$E_{\tau s}^{\alpha\beta} = \int_{h_k} F_\tau^\alpha F_s^\beta dz \tag{A3}$$

$$E_{\tau_{,z}s}^{\alpha\beta} = \int_{h_k} F_{\tau_{,z}}^\alpha F_s^\beta dz \tag{A4}$$

$$E_{\tau s_{,z}}^{\alpha\beta} = \int_{h_k} F_\tau^\alpha F_{s_{,z}}^\beta dz \tag{A5}$$

$$E_{\tau_{,z}s_{,z}}^{\alpha\beta} = \int_{h_k} F_{\tau_{,z}}^\alpha F_{s_{,z}}^\beta dz \tag{A6}$$

with α and β that can assume the values: u, e, σ.

The symbol $\triangleleft \ldots \triangleright_{A_k}$ indicates integration over the element domain.

A2.1. PVD fundamental nuclei

- $\boldsymbol{K}_{uu}$ $\boldsymbol{K}_{uu}^{k\tau sij}$ nucleo is a matrix:

$$\boldsymbol{K}_{uu}^{k\tau sij} = \begin{bmatrix} K_{uu_{11}} & K_{uu_{12}} & K_{uu_{13}} \\ K_{uu_{21}} & K_{uu_{22}} & K_{uu_{23}} \\ K_{uu_{31}} & K_{uu_{32}} & K_{uu_{33}} \end{bmatrix}$$

Its elements are:

$$K_{uu_{11}} = C_{55} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau_{,z}s_{,z}}^{uu} + C_{11} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + C_{16} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$
$$+ C_{16} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + C_{66} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{uu_{12}} = C_{45} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau_{,z}s_{,z}}^{uu} + C_{16} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + C_{12} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$
$$+ C_{66} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + C_{26} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{uu_{13}} = C_{13} \triangleleft N_{i,x} N_j \triangleright_{A_k} E_{\tau s_{,z}}^{uu} + C_{36} \triangleleft N_{i,y} N_j \triangleright_{A_k} E_{\tau s_{,z}}^{uu} + C_{55} \triangleleft N_i N_{j,x} \triangleright_{A_k} E_{\tau_{,z}s}^{uu}$$
$$+ C_{45} \triangleleft N_i N_{j,y} \triangleright_{A_k} E_{\tau_{,z}s}^{uu}$$

$$K_{uu_{21}} = C_{45}\lhd N_i N_j \rhd_{A_k} E^{uu}_{\tau_{,z}s_{,z}} + C_{12}\lhd N_{i,y} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{26}\lhd N_{i,y} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s} + C_{16}\lhd N_{i,x} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{66}\lhd N_{i,x} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s}$$

$$K_{uu_{22}} = C_{44}\lhd N_i N_j \rhd_{A_k} E^{uu}_{\tau_{,z}s_{,z}} + C_{26}\lhd N_{i,y} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{22}\lhd N_{i,y} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s} + C_{66}\lhd N_{i,x} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{26}\lhd N_{i,x} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s}$$

$$K_{uu_{23}} = C_{36}\lhd N_{i,x} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{23}\lhd N_{i,y} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{45}\lhd N_i N_{j,x} \rhd_{A_k} E^{uu}_{\tau_{,z}s} + C_{44}\lhd N_i N_{j,y} \rhd_{A_k} E^{uu}_{\tau_{,z}s}$$

$$K_{uu_{31}} = C_{55}\lhd N_{i,x} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{45}\lhd N_{i,y} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{13}\lhd N_i N_{j,x} \rhd_{A_k} E^{uu}_{\tau_{,z}s} + C_{36}\lhd N_i N_{j,y} \rhd_{A_k} E^{uu}_{\tau_{,z}s}$$

$$K_{uu_{32}} = C_{45}\lhd N_{i,x} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{44}\lhd N_{i,y} N_j \rhd_{A_k} E^{uu}_{\tau s_{,z}} + C_{36}\lhd N_i N_{j,x} \rhd_{A_k} E^{uu}_{\tau_{,z}s} + C_{23}\lhd N_i N_{j,y} \rhd_{A_k} E^{uu}_{\tau_{,z}s}$$

$$K_{uu_{33}} = C_{33}\lhd N_i N_j \rhd_{A_k} E^{uu}_{\tau_{,z}s_{,z}} + C_{45}\lhd N_{i,y} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{44}\lhd N_{i,y} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s} + C_{55}\lhd N_{i,x} N_{j,x} \rhd_{A_k} E^{uu}_{\tau s} + C_{45}\lhd N_{i,x} N_{j,y} \rhd_{A_k} E^{uu}_{\tau s}$$

$\boldsymbol{K}_{ue}$

- $\boldsymbol{K}^{k\tau sij}_{ue}$ nucleo is a matrix:

$$\boldsymbol{K}^{k\tau sij}_{ue} = \begin{bmatrix} K_{ue_{11}} \\ K_{ue_{21}} \\ K_{ue_{31}} \end{bmatrix}$$

Its elements are:

$$K_{ue_{11}} = e_{35}\lhd N_i N_j \rhd_{A_k} E^{ue}_{\tau_{,z}s_{,z}} + e_{21}\lhd N_{i,x} N_{j,y} \rhd_{A_k} E^{ue}_{\tau s}$$
$$K_{ue_{21}} = e_{26}\lhd N_{i,y} N_{j,y} \rhd_{A_k} E^{ue}_{\tau s} + e_{12}\lhd N_{i,y} N_{j,x} \rhd_{A_k} E^{ue}_{\tau s}$$
$$K_{ue_{31}} = e_{35}\lhd N_{i,x} N_j \rhd_{A_k} E^{ue}_{\tau s_{,z}} + e_{13}\lhd N_i N_{j,x} \rhd_{A_k} E^{ue}_{\tau_{,z}s}$$

$\boldsymbol{K}_{ee}$

- $\boldsymbol{K}^{k\tau sij}_{ee}$ nucleo is a scalar quantity:

$$K_{ee_{11}} = -\epsilon_{11}\lhd N_{i,x} N_{j,x} \rhd_{A_k} E^{ee}_{\tau s} - \epsilon_{12}\lhd N_{i,x} N_{j,y} \rhd_{A_k} E^{ee}_{\tau s} - \epsilon_{21}\lhd N_{i,y} N_{j,x} \rhd_{A_k} E^{ee}_{\tau s} - \epsilon_{22}\lhd N_{i,y} N_{j,y} \rhd_{A_k} E^{ee}_{\tau s} - \epsilon_{33}\lhd N_i N_j \rhd_{A_k} E^{ee}_{\tau_{,z}s_{,z}}$$

$\boldsymbol{K}_{up}$

- $\boldsymbol{K}^{k\tau sij}_{up}$ nucleo is a matrix:

$$\boldsymbol{K}_{up}^{k\tau sij} = \begin{bmatrix} K_{up_{11}} & 0 & 0 \\ 0 & K_{up_{22}} & 0 \\ 0 & 0 & K_{up_{33}} \end{bmatrix}$$

Its elements are:

$$K_{up_{11}} = m_{x_s}^{k} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{1} \qquad K_{up_{21}} = m_{y_s}^{k} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{1}$$
$$K_{up_{31}} = m_{z_s}^{k} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{1}$$

- $\boldsymbol{K}_{uf}$ $\boldsymbol{K}_{uf}^{k\tau sij}$ nucleo is a scalar quantity:

$$K_{ef_{11}} = n_{s}^{k} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{1}$$

A2.2. RMVT fundamental nuclei

- $\boldsymbol{K}_{muu}$ $\boldsymbol{K}_{muu}^{k\tau sij}$ nucleo is a matrix:

$$\boldsymbol{K}_{muu}^{k\tau sij} = \begin{bmatrix} K_{muu_{11}} & K_{muu_{12}} & K_{muu_{13}} \\ K_{muu_{21}} & K_{muu_{22}} & K_{muu_{23}} \\ K_{muu_{31}} & K_{muu_{32}} & K_{muu_{33}} \end{bmatrix}$$

Its elements are:

$$K_{muu_{11}} = \hat{C}_{11} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{16} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{16} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{66} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{muu_{12}} = \hat{C}_{16} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{66} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{12} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{26} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{muu_{13}} = 0$$

$$K_{muu_{21}} = \hat{C}_{16} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{12} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{66} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{26} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{muu_{22}} = \hat{C}_{66} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{26} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{26} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu} + \hat{C}_{22} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{uu}$$

$$K_{muu_{23}} = 0$$

$$K_{muu_{31}} = 0$$

$K_{muu_{32}} = 0$

$K_{muu_{33}} = 0$

- $\boldsymbol{K}_{mu\sigma}$ $\boldsymbol{K}_{mu\sigma}^{k\tau sij}$ nucleo is a matrix:

$$\boldsymbol{K}_{mu\sigma}^{k\tau sij} = \begin{bmatrix} K_{mu\sigma_{11}} & K_{mu\sigma_{12}} & K_{mu\sigma_{13}} \\ K_{mu\sigma_{21}} & K_{mu\sigma_{22}} & K_{mu\sigma_{23}} \\ K_{mu\sigma_{31}} & K_{mu\sigma_{32}} & K_{mu\sigma_{33}} \end{bmatrix}$$

Its elements are:

$$K_{mu\sigma_{11}} = \lhd N_i N_j \rhd_{A_k} E_{\tau,zs}^{u\sigma}$$

$$K_{mu\sigma_{12}} = 0$$

$$K_{mu\sigma_{13}} = \hat{C}_{13} \lhd N_{i,x} N_j \rhd_{A_k} E_{\tau s}^{u\sigma} + \hat{C}_{36} \lhd N_{i,y} N_j \rhd_{A_k} E_{\tau s}^{u\sigma}$$

$$K_{mu\sigma_{21}} = 0$$

$$K_{mu\sigma_{22}} = \lhd N_i N_j \rhd_{A_k} E_{\tau,zs}^{u\sigma}$$

$$K_{mu\sigma_{23}} = \hat{C}_{36} \lhd N_{i,x} N_j \rhd_{A_k} E_{\tau s}^{u\sigma} + \hat{C}_{23} \lhd N_{i,y} N_j \rhd_{A_k} E_{\tau s}^{u\sigma}$$

$$K_{mu\sigma_{31}} = \lhd N_{i,x} N_j \rhd_{A_k} E_{\tau s}^{u\sigma}$$

$$K_{mu\sigma_{32}} = \lhd N_{i,y} N_j \rhd_{A_k} E_{\tau s}^{u\sigma}$$

$$K_{mu\sigma_{33}} = \lhd N_i N_j \rhd_{A_k} E_{\tau,zs}^{u\sigma}$$

- $\boldsymbol{K}_{mue}$ $\boldsymbol{K}_{mue}^{k\tau sij}$ nucleo is a matrix:

$$\boldsymbol{K}_{mue}^{k\tau sij} = \begin{bmatrix} K_{mue_{11}} \\ K_{mue_{21}} \\ K_{mue_{31}} \end{bmatrix}$$

Its elements are:

$$K_{mue_{11}} = -\hat{C}_{se_{11}} \lhd N_{i,x} N_{j,x} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{31}} \lhd N_{i,y} N_{j,x} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{12}} \lhd N_{i,x} N_{j,y} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{32}} \lhd N_{i,y} N_{j,y} \rhd_{A_k} E_{\tau s}^{ue}$$

$$K_{mue_{21}} = -\hat{C}_{se_{31}} \lhd N_{i,x} N_{j,x} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{21}} \lhd N_{i,y} N_{j,x} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{32}} \lhd N_{i,x} N_{j,y} \rhd_{A_k} E_{\tau s}^{ue} - \hat{C}_{se_{22}} \lhd N_{i,y} N_{j,y} \rhd_{A_k} E_{\tau s}^{ue}$$

$$K_{mue_{31}} = 0$$

- $\boldsymbol{K}_{m\sigma\sigma}$ $\boldsymbol{K}_{m\sigma\sigma}^{k\tau sij}$ nucleo is a matrix:

$$
\boldsymbol{K}_{m\sigma\sigma}^{k\tau sij} = \begin{bmatrix} K_{m\sigma\sigma_{11}} & K_{m\sigma\sigma_{12}} & K_{m\sigma\sigma_{13}} \\ K_{m\sigma\sigma_{21}} & K_{m\sigma\sigma_{22}} & K_{m\sigma\sigma_{23}} \\ K_{m\sigma\sigma_{31}} & K_{m\sigma\sigma_{32}} & K_{m\sigma\sigma_{33}} \end{bmatrix}
$$

Its elements are:

$$
\begin{aligned}
K_{m\sigma\sigma_{11}} &= -\hat{C}_{55} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{\sigma\sigma} \\
K_{m\sigma\sigma_{12}} &= -\hat{C}_{45} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{\sigma\sigma} \\
K_{m\sigma\sigma_{13}} &= 0 \\
K_{m\sigma\sigma_{21}} &= -\hat{C}_{45} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{\sigma\sigma} \\
K_{m\sigma\sigma_{22}} &= -\hat{C}_{44} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{\sigma\sigma} \\
K_{m\sigma\sigma_{23}} &= 0 \\
K_{m\sigma\sigma_{31}} &= 0 \\
K_{m\sigma\sigma_{32}} &= 0 \\
K_{m\sigma\sigma_{33}} &= -\hat{C}_{33} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s}^{\sigma\sigma}
\end{aligned}
$$

- $\boldsymbol{K}_{m\sigma e}$ $\boldsymbol{K}_{m\sigma e}^{k\tau sij}$ nucleo is a matrix:

$$
\boldsymbol{K}_{m\sigma e}^{k\tau sij} = \begin{bmatrix} K_{m\sigma e_{11}} \\ K_{m\sigma e_{21}} \\ K_{m\sigma e_{31}} \end{bmatrix}
$$

Its elements are:

$$
\begin{aligned}
K_{m\sigma e_{11}} &= \hat{C}_{de_{13}} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s,z}^{\sigma e} \\
K_{m\sigma e_{21}} &= \hat{C}_{de_{23}} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau s,z}^{\sigma e} \\
K_{m\sigma e_{31}} &= \hat{C}_{de_{31}} \triangleleft N_i N_{j,x} \triangleright_{A_k} E_{\tau s}^{\sigma e} + \hat{C}_{de_{32}} \triangleleft N_i N_{j,y} \triangleright_{A_k} E_{\tau s}^{\sigma e}
\end{aligned}
$$

- $\boldsymbol{K}_{mee}$ $\boldsymbol{K}_{mee}^{k\tau sij}$ nucleo is a scalar quantity:

$$
\begin{aligned}
\boldsymbol{K}_{mee}^{k\tau sij} = &-\hat{C}_{ee11} \triangleleft N_{i,x} N_{j,x} \triangleright_{A_k} E_{\tau s}^{ee} - \hat{C}_{ee12} \triangleleft N_{i,x} N_{j,y} \triangleright_{A_k} E_{\tau s}^{ee} - \hat{C}_{ee21} \triangleleft N_{i,y} N_{j,x} \triangleright_{A_k} E_{\tau s}^{ee} \\
&- \hat{C}_{ee22} \triangleleft N_{i,y} N_{j,y} \triangleright_{A_k} E_{\tau s}^{ee} - \hat{C}_{ee33} \triangleleft N_i N_j \triangleright_{A_k} E_{\tau,z s,z}^{ee}
\end{aligned}
$$

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