Importance of Higher Order Modes and Refined Theories in Free Vibration Analysis of Composite Plates

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This paper evaluates frequencies of higher-order modes in the free vibration response of simply-supported multilayered orthotropic composite plates. Closed-form solutions in harmonic forms are given for the governing equations related to classical and refined plate theories. Typical cross-ply (0 deg/90 deg) laminated panels (10 and 20 layers) are considered in the numerical investigation (these were suggested by European Aeronautic Defence and Space Company (EADS) in the framework of the "Composites and Adaptive Structures: Simulation, Experimentation and Modeling" (CASSEM) European Union (EU) project. The Carrera unified formulation has been employed to implement the considered theories: the classical lamination theory, the first-order shear deformation theory, the equivalent single layer model with fourth-order of expansion in the thickness direction z, and the layerwise model with linear order of expansion in z for each layer. Higher-order frequencies and the related harmonic modes are computed by varying the number of wavelengths (m,n) in the two-plate directions and the degrees of freedom in the plate theories. It can be concluded above all that—refined plate models lead to higher-order frequencies, which cannot be computed by simplified plate theories—frequencies related to high values of wavelengths, even the fundamental ones, can be wrongly predicted when using classical plate theories, even though thin plate geometries are analyzed. [DOI: 10.1115/1.3173605]

1 Introduction

The application of composite materials in aircraft and space vehicles has increased rapidly over the past 3 decades due to their high strength and low weight. Composite materials offer many advantages, with respect to traditional metallic ones. However, a number of complicating effects arises in their design, analysis, modeling, and manufacturing [1]. In this work, attention is directed to the evaluation of frequencies for higher modes in the free vibration response. The behavior and design of vibrating shells and plates have been extensively discussed in the reports by Leissa [2,3], and more recently in the book by Werner [4], among others.

In the most general case of three-dimensional (3D) analyses, the number of frequencies for a free vibration problem is infinite: three displacement components (three degrees of freedom (DOF)) in each point (which are ∞ in the three directions x,y,z) leads to $3\times\infty^3$ vibration modes. Assumptions are made in the thickness direction z in the case of a two-dimensional plate/shell problem, that is, the three displacements in each point are expressed in terms of a given number of degrees of freedom (NDOF), which varies from theory to theory. As a result, the number of vibration modes is $\text{NDOF}\times\infty^2$. For beams it would be $\text{NDOF}\times\infty^1$. In the case of the application of computational models, such as the finite element method (FEM), the number of modes is a finite number; it coincides with the number of employed degrees of freedom: $\Sigma_1^{\text{Node}}\text{NDOF}_i$, where "Node" denotes the number of nodes used in the FE mathematical model, and NDOF_i is the NDOF in the i-node. Limitations of FE method are overcome by using the statistical energy analysis (SEA) method, which is usually preferred in case of higher modes such as acoustics problems [5]. It appears clear that some modes are tragically lost in simplified models (such as two-dimensional models) as well as in computational ones.

The case in which closed-form solutions are available for a given set of differential governing equations is of particular interest. Interest is herein focused on simply-supported multilayered plates made of orthotropic layers for which Navier type solutions exist. Harmonic forms are assumed for the unknown variables in the in-plane directions (x,y) for each couple of wave numbers (m,n). The number of vibration modes is in this case simply given by the NDOF of the adopted plate theory for each couple of integer (m,n) values. Since (m,n) can assume infinity values, the number of modes is $\text{NDOF}\times\infty^2$, as in the continuous case. In the above described scenario, the two following points are of interest as far as the evaluation of higher modes is concerned.

1. Different plate theories lead to different numbers of vibration modes, and higher modes related to higher NDOF can only be obtained by implementing refined plate theories.
2. Other higher vibration modes can be obtained by considering high values of m,n.

In conclusion, a complete investigation of higher vibration modes requires the use of refined theories and higher numbers of the couple (m,n). This investigation is the subject of the present paper. An overview on available works on this topic is made in the following.

The main features of free vibration analysis for composite plates have been discussed in the paper by Noor [6]; this paper summarizes results based on two-dimensional plate theories for a low-frequency free vibration analysis of simply-supported cross-ply laminated plates consisting of a large number of layers. These results show that the error in the predictions of the classical plate

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theories is highly dependent on the number and stacking of the layers, the degree of orthotropy of the individual layers, and the thickness ratio of the plate. Liew et al. [7] wrote a review of existing literature on the vibration analysis of thick plates (132 publications were cited) up to 1995.

The main aim of this paper is to understand the importance of advanced and refined two-dimensional models in the case of higher modes for the free vibration response of multilayered composite plates. Shuyu [8] wrote that shear deformation and the rotary inertia could be ignored for thin plates, but for higher frequencies, even if the plate is very thin, the shear deformation cannot be ignored, so classical theories are suitable only for thin plates and vibrational orders of low natural frequency. These conclusions refer to isotropic plates, and the situation drops away for composite plates. Thick isotropic plates are considered in Refs. [9,10] for low frequencies and the finite element method. Different boundary conditions, shapes of the considered plate, and thickness ratios are considered in Refs. [11–18], but they are restricted to isotropic shells and plates and to low and medium frequencies. Interesting conclusions were obtained for homogeneous plates, in the case of high-frequency investigations, by Zhao et al. [19] who introduced a discrete singular convolution (DSC) algorithm to investigate square isotropic plates with six distinct boundary conditions. The method was applied for high-frequency vibration analysis by providing extremely accurate frequency parameters for plates vibrating in the first 5000 modes, and an increase in mesh size is clearly shown for high mode numbers. Via new hierarchical functions, Beslin and Nicolas [20] investigated very high modes (up to 2048) for rectangular steel plates with different boundary conditions. Other interesting investigations for high frequencies, in the case of analytical or numerical solutions, are reported in Refs. [21–25]. In particular, Wei et al. [25] investigated high-frequency vibrations of structures for isotropic plates, by means of the DSC algorithm. The investigation considered the first 7100 modes of the beam and the first 4500 modes of the two-span plates.

The above cited works all concern homogeneous structures, however, the situation becomes more critical in the case of laminated structures. A substantial number of papers have been dedicated to the free vibrational response of multilayered plates, in particular, for low and medium frequencies ranges. Among these papers, Ref. [26] is of particular interest, where Kim and Hwang applied a modal parameter estimation technique to investigate natural frequencies and damping ratios of cross-ply and angle-ply composite laminates. Gorman and Ding [27] used the superposition method to obtain the first six modes for orthotropic plates and the first four modes for cross-ply plates. Comparisons, for low modes, between the traditional superposition method and the superposition-Galerkin method are reported in Ref. [28]. Cross-ply and angle-ply laminated plates with different boundary conditions are considered in Refs. [29,30]; the first paper is about a recursive solution based on a Lagrange multiplier method, and the second is about a finite element method (based on a third-order shear deformation theory), which, in the case of linear analysis, underlines the effects of wide-to-thickness ratio, material anisotropy, fiber orientation, aspect ratio, boundary conditions, and number of layers. In Refs. [31–35], for the case of analytical, semi-analytical, and finite element solutions, advanced theories such as mixed approaches, higher-order zig-zag theories, and higher-order discrete layer methods have been proposed for composite sandwiches and laminated plates, shells, beams and blades; all these investigations are restricted to low and natural frequencies. Other investigations about the free vibrations of composite structures, in the range of low and medium frequencies, are presented in Refs. [36–39].

Papers that consider higher mode investigations are not so numerous, and these are even less in the cases of thick plates and/or composite structures. Reference [40], which applies a displacement finite element based on C^0-continuous displacements to

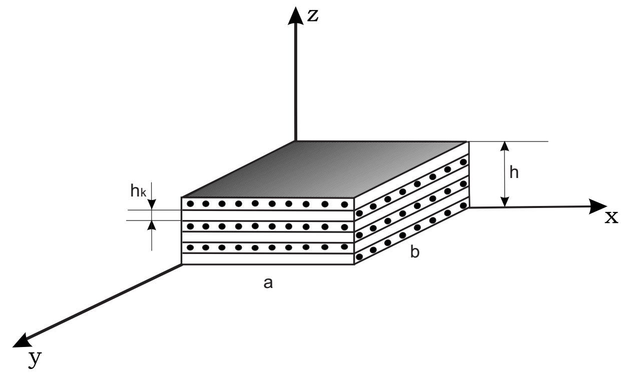

Fig. 1 Geometry and notations for the considered multilayered plates

simply-supported homogeneous and laminated plates in the case of thick structures and high frequencies modes, is of interest. A comparison with the exact solution of higher-order theories (HOTs), the first-order shear deformation theory (FSDT), and the classical lamination theory (CLT) is proposed for the first five antisymmetric modes and the first three symmetric modes. In Refs. [41,42], by means of the hierarchical finite element method (HFEM) and the Galerkin element method (GEM), respectively, the vibration analysis of laminated rectangular plates and damped sandwich plates is led over a wide frequency range.

In the recent past, the free vibration response of multilayered plates and shells has been investigated in various articles by employing the Carrera unified formulation (CUF): The transverse normal stress effects have been evaluated in Ref. [43], the benefits of mixed models based on the Reissner's mixed variational theorem have been considered in Ref. [44], and the advantages of layer-wise mixed models have been outlined in Ref. [45]. The present paper uses CUF to explore frequencies of higher-order free vibration modes, according to points 1 and 2. The aim is duplicate—to show the limitations of classical theories, such as the CLT and the FSDT—to predict the frequencies of the higher modes. Two refined plate theories are employed for this purpose: the first is based on an equivalent single layer (ESL) approach with fourth-order expansion of displacements in the thickness direction z; the second one is a layerwise (LW) model with linear expansion in the z direction in each layer. Numerical solutions are given in closed form in the case of two cross-ply laminated plates whose data have been provided by EADS in the framework of the EU FP7 project known as CASSEM.

The paper has been organized as follows: Sec. 2 illustrates the kinematics of plate theories; Sec. 3 outlines the governing equations; and the numerical analysis and conclusions are given in Secs. 4 and 5, respectively.

2 Kinematic Description of the Considered Plate Theories

Classical theories (such as the CLT and the FSDT), refined models (such as the ESL with fourth-order of expansion ($N=4$) and the LW with linear order of expansion in the z direction ($N=1$)) are employed for the dynamic analysis of laminated plates. These theories are developed in the framework of the CUF [46,47].

2.1 Classical Theories: CLT and FSDT. Multilayered plates are structures in which one dimension, the thickness, is one or two orders of magnitude less than the in-plane ones. This peculiarity leads to the possibility of modeling these structures as a two-dimensional continuum (see Fig. 1).

Classical two-dimensional theories such as CLT and FSDT are models in which the number of variables does not depend on the

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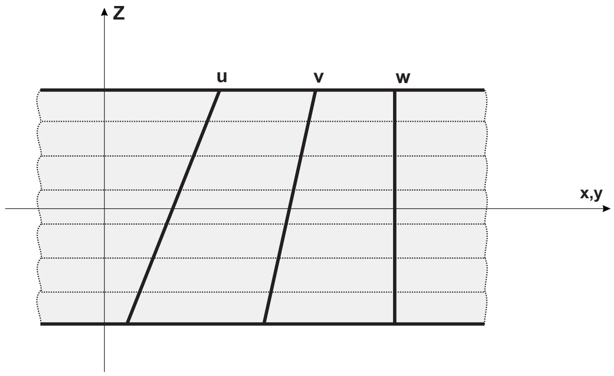

Fig. 2 Displacements distribution in thickness direction in case of CLT and FSDT

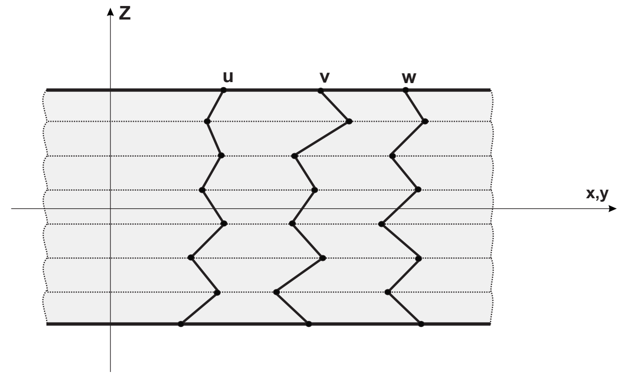

Fig. 4 Displacements distribution in thickness direction in case of LW(N=1)

number of layers (for a detailed overview see Ref. [48]). Kirchhoff hypotheses are considered in the CLT case as follows:

(1) Straight lines perpendicular to the midsurface before deformation remain straight after deformation.
(2) The transverse normals do not experience elongation.
(3) The transverse normals rotate in order to remain perpendicular to the midsurface after deformation.

The first two assumptions imply that the transverse displacement is independent of the transverse (or thickness) coordinate, and that the transverse normal strain ϵ_{zz} is neglected. The third assumption results in zero transverse shear strains ($\epsilon_{xz}=\epsilon_{yz}=0$). The displacement field (u,v,w) can then be described by using the following relations:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w_0}{\partial x} \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w_0}{\partial y} \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{1}$$

where the index 0 indicates midsurface displacement values.

The third of the above hypothesis is removed in the FSDT case, and the transverse normals do not remain perpendicular to the midsurface after deformation. The displacement field of FSDT is as follows:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + z\Phi_x(x,y) \\ v(x,y,z) &= v_0(x,y) + z\Phi_y(x,y) \\ w(x,y,z) &= w_0(x,y) \end{aligned} \tag{2}$$

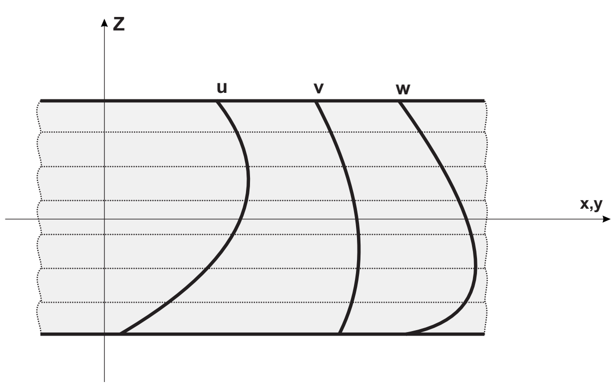

Fig. 3 Displacements distribution in thickness direction in case of ESL(N=4)

where

$$\frac{\partial u}{\partial z} = \Phi_x, \quad \frac{\partial v}{\partial z} = \Phi_y \tag{3}$$

denote the rotations of the $z-y$ and $z-x$ planes, respectively. Figure 2 shows the displacements field in z direction of a typical laminate, in the case of CLT and FSDT. For these two classical kinematics, plane-stress conditions are applied as in Ref. [49] to avoid the so called Poisson locking phenomena.

2.2 Carrera Unified Formulation and Refined Models. The key point of the CUF is the use of generalized assumptions for the primary variables of the problem; the generic variable $\boldsymbol{a}$ is expanded using a set of thickness functions that depend only on the thickness coordinate z (a detailed description of the CUF can be found in Refs. [46,47]):

$$\boldsymbol{a}(x,y,z) = F_\tau(z)\boldsymbol{a}_\tau(x,y) \tag{4}$$

where $\tau=0,1,\ldots,N$, and N is the desired order of the expansion. F_τ are the employed thickness functions. ESL and LW approaches can be both developed by appropriate choice of thickness functions.

Equivalent single layer models are based on the assumption of a global description of the displacement field along the whole plate thickness; a Taylor expansion is used as follows:

$$\boldsymbol{u}(x,y,z) = F_\tau(z)\boldsymbol{u}_\tau(x,y) = z^r\boldsymbol{u}_r, \quad r=0,1,2,\ldots,N \tag{5}$$

N is a free parameter that can assume values from 1 (linear) to 4 (fourth order). The case N=4 is used in this work as follows:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) + zu_1(x,y) + z^2u_2(x,y) + z^3u_3(x,y) + z^4u_4(x,y) \\ v(x,y,z) &= v_0(x,y) + zv_1(x,y) + z^2v_2(x,y) + z^3v_3(x,y) + z^4v_4(x,y) \\ w(x,y,z) &= w_0(x,y) + zw_1(x,y) + z^2w_2(x,y) + z^3w_3(x,y) + z^4w_4(x,y) \end{aligned} \tag{6}$$

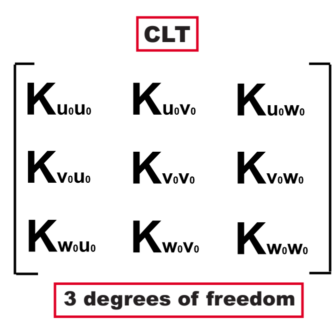

Fig. 5 Multilayer stiffness matrix for CLT

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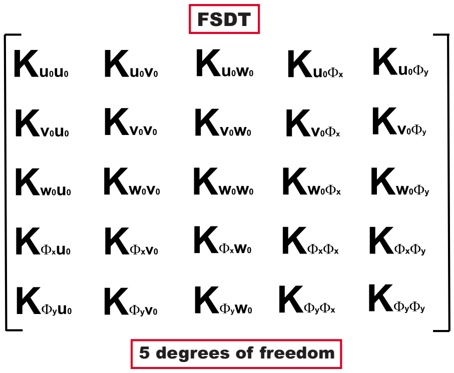

Fig. 6 Multilayer stiffness matrix for FSDT

Related displacement fields in the z direction are drawn in Fig. 3.

A layerwise description includes an independent expansion for each layer k as follows:

$$\boldsymbol{u}^k(x,y,z) = F_t(z)\boldsymbol{u}_t^k(x,y) + F_b(z)\boldsymbol{u}_b^k(x,y) + F_r(z)\boldsymbol{u}_r^k(x,y) = F_\tau \boldsymbol{u}_\tau^k \tag{7}$$

with

$$\tau = t,b,r, \quad r = 2,\ldots,N, \quad k = 1,2,\ldots,N_L$$

where t and b mean top and bottom of the considered layer, and r represents the higher-order terms of expansion up to the fourth (N=4); N_L is the number of layers. F_τ are appropriate combinations of Legendre polynomials (see Refs. [46,47]). In the present investigation a LW model with N=1 will be used as follows:

$$u^k(x,y,z) = F_t(z)u_t^k(x,y) + F_b(z)u_b^k(x,y)$$
$$v^k(x,y,z) = F_t(z)v_t^k(x,y) + F_b(z)v_b^k(x,y) \tag{8}$$
$$w^k(x,y,z) = F_t(z)w_t^k(x,y) + F_b(z)w_b^k(x,y)$$

The displacement forms in the z direction are considered in Fig. 4.

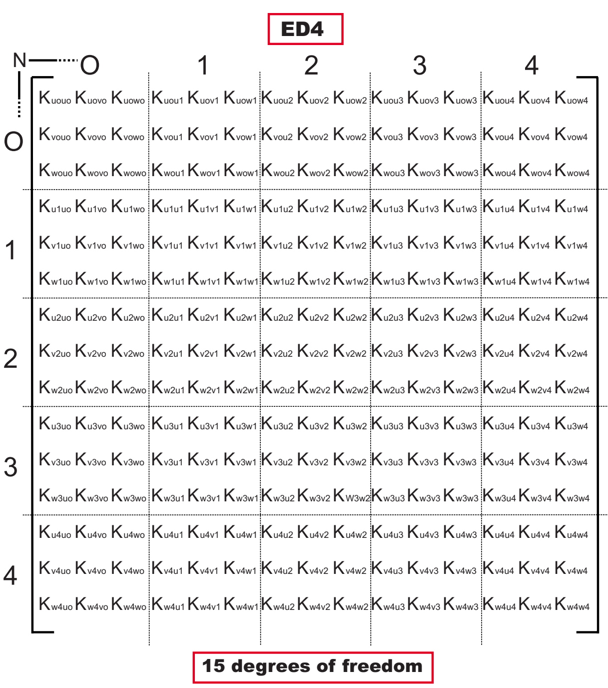

Fig. 7 Multilayer stiffness matrix for ESL(*N*=4)

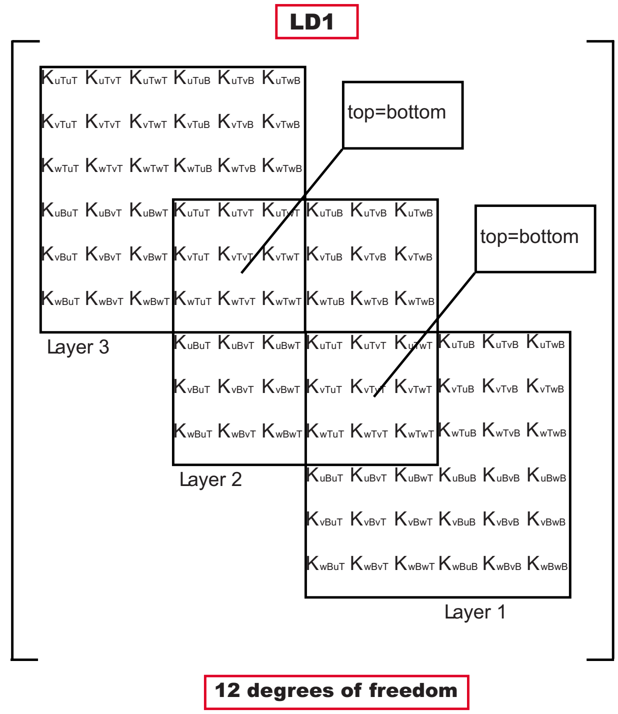

Fig. 8 Multilayer stiffness matrix for LW(*N*=1) in case of a three-layered plate

CLT and FSDT can be obtained as particular cases of ESL models (N=1), by using a penalty technique on. CLT is obtained by forcing constant w and infinite transverse shear correction factor; FSDT has constant w and the transverse shear correction factor is equal to 1 (see Refs. [46,47]).

3 Governing Equations and Fundamental Nuclei

For a dynamic problem related to multilayered plates, the principle of virtual displacements (PVDs) states [47],

$$\int_V (\delta\boldsymbol{\epsilon}_{pG}^T \boldsymbol{\sigma}_{pC} + \delta\boldsymbol{\epsilon}_{nG}^T \boldsymbol{\sigma}_{nC}) dV = \delta L^{\mathrm{in}} + \delta L^e \tag{9}$$

where subscripts p and n indicate in-plane and out-plane components, respectively; $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the stress and strain vectors, δL^{in} is the virtual variation of inertial work, δL^e is the virtual variation of external work, and V is the considered volume.

The governing equations of the proposed models can be obtained upon substitution of geometrical relations, constitutive equations, and by applying the unified formulation by Carrera [46,47].

Geometrical relations link strain components to displacement vector, they are introduced in Eq. (9) where the subscript G is considered. The following relations hold in case of a plate:

$$\boldsymbol{\epsilon}_{pG}^k = [\epsilon_{xx}, \epsilon_{yy}, \gamma_{xy}]^{kT} = \boldsymbol{D}_p \boldsymbol{u}^k, \quad \boldsymbol{\epsilon}_{nG}^k = [\gamma_{xz}, \gamma_{yz}, \epsilon_{zz}]^{kT} = (\boldsymbol{D}_{np} + \boldsymbol{D}_{nz})\boldsymbol{u}^k \tag{10}$$

where $\boldsymbol{\epsilon}_{pG}^k$ and $\boldsymbol{\epsilon}_{nG}^k$ are the in-plane and out-plane strain components, respectively. $\boldsymbol{u}^k = [u^k, v^k, w^k]$ is the displacement vector, and k indicates the considered layer. The explicit form of the introduced arrays are as follows:

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Table 1 Number of frequencies for the two considered plates

	$m=n=500$	
	$N_L=10$ (10-layered plate)	$N_L=20$ (20-layered plate)
CLT ($m\times n\times$NDOF)	$500\times500\times3=750{,}000$	$500\times500\times3=750{,}000$
FSDT ($m\times n\times$NDOF)	$500\times500\times5=1{,}250{,}000$	$500\times500\times5=1{,}250{,}000$
ESL($N=4$) ($m\times n\times$NDOF)	$500\times500\times15=3{,}750{,}000$	$500\times500\times15=3{,}750{,}000$
LW($N=1$) ($m\times n\times((N_L+1)\times3)$)	$500\times500\times33=8{,}250{,}000$	$500\times500\times63=15{,}750{,}000$

Table 2 Geometry and materials of multilayered plates in Refs. [50,51]

	Assessment [50]
$\frac{G_{12}}{E_T}=\frac{G_{13}}{E_T}$	0.50
$\frac{G_{23}}{E_T}$	0.35
$\nu_{12}=\nu_{13}$	0.30
ν_{23}	0.49
N_L(No. of layers)	10
θ(orientation)	0 deg/90 deg/…
$a=b$ (m)	-
$\frac{a}{h}$	5

	Assessment [51]
$\frac{E_L}{E_T}$	40
$\frac{G_{12}}{E_T}=\frac{G_{13}}{E_T}$	0.50
$\frac{G_{23}}{E_T}$	0.60
$\nu_{12}=\nu_{13}=\nu_{23}$	0.25
N_L(No. of layers)	4
θ(orientation)	0 deg/90 deg/90 deg /0 deg
$a=b$ (m)	-

Table 3 Free vibration response of a simply-supported square plate. Fundamental circular frequency $\bar{\omega}=\omega h\sqrt{\rho/E_T}$. Wavelengths $m=n=1$.

E_L/E_T	3	30
3D [50]	0.2530	0.4027
CLT	0.2858	0.6322
FSDT	0.2576	0.4306
ESL($N=1$)	0.2576	0.4306
LW($N=1$)	0.2534	0.4042
ESL($N=4$)	0.2539	0.4078
LW($N=4$)	0.2530	0.4027

$$\boldsymbol{D}_p=\begin{bmatrix}\partial_x & 0 & 0\\ 0 & \partial_y & 0\\ \partial_y & \partial_x & 0\end{bmatrix},\quad \boldsymbol{D}_{np}=\begin{bmatrix}0 & 0 & \partial_x\\ 0 & 0 & \partial_y\\ 0 & 0 & 0\end{bmatrix},\quad \boldsymbol{D}_{nz}=\begin{bmatrix}\partial_z & 0 & 0\\ 0 & \partial_z & 0\\ 0 & 0 & \partial_z\end{bmatrix} \tag{11}$$

the symbol ∂ indicates the partial derivatives.

Constitutive equations link stress components to strain components. They are introduced in Eq. (9) where the subscript C is considered. The following relations hold:

$$\begin{aligned}\boldsymbol{\sigma}_{pC}^k&=[\sigma_{xx},\sigma_{yy},\sigma_{xy}]^{kT}=\boldsymbol{C}_{pp}^k\boldsymbol{\epsilon}_{pG}^k+\boldsymbol{C}_{pn}^k\boldsymbol{\epsilon}_{nG}^k,\\ \boldsymbol{\sigma}_{nC}^k&=[\sigma_{xz},\sigma_{yz},\sigma_{zz}]^{kT}=\boldsymbol{C}_{np}^k\boldsymbol{\epsilon}_{pG}^k+\boldsymbol{C}_{nn}^k\boldsymbol{\epsilon}_{nG}^k\end{aligned} \tag{12}$$

The explicit form of the introduced arrays are as follows:

$$\begin{aligned}\boldsymbol{C}_{pp}^k&=\begin{bmatrix}C_{11} & C_{12} & C_{16}\\ C_{12} & C_{22} & C_{26}\\ C_{16} & C_{26} & C_{66}\end{bmatrix}^k,\quad \boldsymbol{C}_{pn}^k=\begin{bmatrix}0 & 0 & C_{13}\\ 0 & 0 & C_{23}\\ 0 & 0 & C_{36}\end{bmatrix}^k\\ \boldsymbol{C}_{np}^k&=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ C_{13} & C_{23} & C_{36}\end{bmatrix}^k,\quad \boldsymbol{C}_{nn}^k=\begin{bmatrix}C_{55} & C_{45} & 0\\ C_{45} & C_{44} & 0\\ 0 & 0 & C_{33}\end{bmatrix}^k\end{aligned} \tag{13}$$

Closed form solutions are obtained for particular cases such as simply-supported plates, and particular conditions for the material properties. In this case the governing equations can be exactly solved. The following harmonic assumptions can be made for the field variables (see Refs. [43,47]):

$$\begin{aligned}u_\tau^k&=\sum_{m,n}U_\tau^k\cos\frac{m\pi x_k}{a_k}\sin\frac{n\pi y_k}{b_k}e^{i\omega_{mn}t}\quad k=1,N_L\\ v_\tau^k&=\sum_{m,n}V_\tau^k\sin\frac{m\pi x_k}{a_k}\cos\frac{n\pi y_k}{b_k}e^{i\omega_{mn}t}\quad \tau=t,b,r\\ w_\tau^k&=\sum_{m,n}W_\tau^k\sin\frac{m\pi x_k}{a_k}\sin\frac{n\pi y_k}{b_k}e^{i\omega_{mn}t}\quad r=2,\ldots,N\end{aligned} \tag{14}$$

in which a_k and b_k are the plate lengths in the x_k and y_k directions, respectively; U_τ^k, V_τ^k, and W_τ^k are the displacement amplitudes, while m and n are the correspondent wave numbers; $i=\sqrt{-1}$, t is the time, and ω_{mn} is the circular frequency. These assumptions correspond to the simply-supported boundary conditions. The sys-

Table 4 Free vibration response of a simply-supported square plate. Fundamental circular frequency $\bar{\omega}=\omega\sqrt{\frac{a^4\rho}{E_T h^2}}$. Wavelengths $m=n=1$.

a/h	2	4	10	20	100
CLT [51]	15.830	17.907	18.652	18.767	18.804
FSDT [51]	5.492	9.369	15.083	17.583	18.751
ESL($N=1$)	5.919	9.936	15.524	17.763	18.759
LW($N=1$)	5.414	9.473	15.334	17.703	18.761
ESL($N=4$)	5.380	9.384	15.232	17.655	18.754
LW($N=4$)	5.260	9.224	15.148	17.626	18.753

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Table 5 Geometry and materials of multilayered plates for the two proposed benchmarks

	Material
E_{11} (GPa)	310
E_{22} (GPa)	6.50
E_{33} (GPa)	6.50
ν_{12}	0.365
ν_{13}	0.365
ν_{23}	0.365
G_{12} (GPa)	4.30
G_{13} (GPa)	4.30
G_{23} (GPa)	4.30
ρ (kg/m^3)	1500
	Benchmark 1
h_k(kth layer) (m)	0.0001
N_L(No. of layers)	10
θ(orientation)	0 deg/90 deg/…
a (m)	1.5
b (m)	1.5
	Benchmark 2
h_k(kth layer) (m)	0.0001
N_L(No. of layers)	20
θ(orientation)	0 deg/90 deg/…
a (m)	0.75
b (m)	0.75

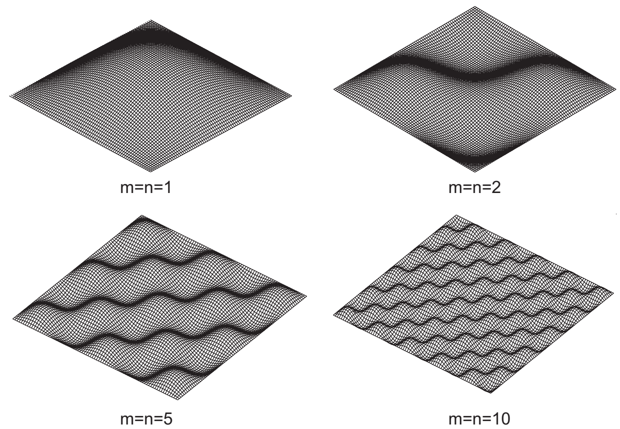

Fig. 9 Examples of in-plane modes

tem of governing equations, at multilayered plate level could be formally written as in the following:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{M}\ddot{\boldsymbol{u}} \tag{15}$$

The stiffness matrix $\boldsymbol{K}$ and the inertial matrix $\boldsymbol{M}$ are assembled via indices τ and s for the order of expansion in the thickness direction z, and via the index k for the multilayer assembling (ESL or

Table 6 Benchmark 1, 10-layered plate, and $a/h=1500$. Comparison between classical, HOT, and LW theories. Circular frequencies parameter $\bar{\omega}=\omega\sqrt{a^4\rho/10^6 E_T h^2}$.

					$m=1$, $n=1$						
CLT	0.0203	23.108	24.076								
FSDT	0.0203	23.108	24.076	6339.3	6339.3						
ESL(N=4)	0.0203	23.113	24.076	7617.3	5750.7	5750.7	15,285	47,267	11,540	31,616
	11,540	35,684	23,869	23,869	35,684						
LW(N=1)	0.0203	23.108	24.076	5772.8	5772.8	7646.5	11,688	11,688	15,482	17,890
	17,890	23,698	24,521	24,521	32,480	31,696	31,696	39,439	39,439	41,987
	47,530	47,530	83,970	80,981	52,244	73,175	55,246	55,245	62,947	61,134
	63,392	61,134	63,392								
					$m=5$, $n=5$						
CLT	0.5078	115.54	120.38								
FSDT	0.5078	115.54	120.38	6340.4	6340.3						
ESL(N=4)	0.5077	115.56	120.38	5752.0	5751.8	7617.2	……	……	……	……
LW(N=1)	0.5078	115.54	120.38	5773.9	5774.0	7646.5	……	……	……	……
					$m=10$, $n=10$						
CLT	2.0313	231.08	240.76								
FSDT	2.0299	231.08	240.75	6343.5	6344.0						
ESL(N=4)	2.0295	231.12	240.75	5755.8	5755.3	7617.1	……	……	……	……
LW(N=1)	2.0298	231.08	240.75	5777.4	5777.9	7646.3	……	……	……	……
					$m=50$, $n=50$						
CLT	50.737	1155.4	1203.8								
FSDT	49.873	1154.9	1203.3	6443.8	6458.1						
ESL(N=4)	49.684	1155.1	1203.0	5878.5	5865.6	7613.4	……	……	……	……
LW(N=1)	49.696	1154.9	1203.0	5887.2	5900.2	7642.8	……	……	……	……
					$m=100$, $n=100$						
CLT	202.38	2311.1	2407.6								
FSDT	189.67	2306.8	2403.3	6801.4	6748.8						
ESL(N=4)	187.13	2307.2	2401.5	6244.1	6197.7	7604.1	……	……	……	……
LW(N=1)	187.25	2306.8	2401.5	6218.0	6264.8	7634.1	……	……	……	……
					$m=500$, $n=500$						
CLT	4657.4	11,584	12,038								
FSDT	2379.6	11,196	11,514	13,639	13,961						
ESL(N=4)	2216.2	7576.2	11,116	11,433	12,981	13,145	……	……	……	……
LW(N=1)	2220.2	7609.8	11,111	11,431	13,011	13,146	……	……	……	……

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Table 7 Benchmark 2, 20-layered plate, and $a/h=375$. Comparison between classical, HOT, and LW theories. Circular frequencies parameter $\bar{\omega}=\omega\sqrt{a^4\rho/10^6E_Th^2}$.

					$m=1$, $n=1$					
CLT	0.0205	5.7765	6.0183							
FSDT	0.0205	5.7765	6.0182	396.21	396.20					
ESL(N=4)	0.0205	5.7776	6.0182	359.43	359.42	476.03	955.24	2953.8	721.18	721.18
	1975.8	2230.0	1491.7	1491.6	2230.0					
LW(N=1)	0.0205	5.7765	6.0182	359.69	359.70	476.38	721.54	721.54	955.71	1087.8
	1087.8	1440.9	1460.9	1460.9	1842.8	1842.8	1935.0	2236.1	2236.1	2441.0
	2642.7	2642.7	2961.9	3064.8	3064.8	3504.1	3504.1	3500.5	3961.6	3961.6
	4059.6	4437.3	4437.3	4641.5	4929.3	4929.3	5247.5	5433.3	5433.3	10495
	10,399	5877.6	10,121	5940.6	5940.6	9690.6	9146.1	6529.4	6437.8	6437.8
	8527.4	6904.9	6904.9	7196.8	7315.9	7315.9	7640.9	7640.9	7868.9	7923.2
	7923.2	7850.6	7850.6							
					$m=5$, $n=5$					
CLT	0.5128	28.882	30.091							
FSDT	0.5113	28.882	30.091	397.21	397.36					
ESL(N=4)	0.5111	28.888	30.090	360.67	360.54	475.99	……	……	……	……
LW(N=1)	0.5111	28.882	30.090	360.80	360.94	476.34	……	……	……	……
					$m=10$, $n=10$					
CLT	2.0504	57.765	60.183							
FSDT	2.0273	57.761	60.178	400.35	400.93					
ESL(N=4)	2.0228	57.772	60.172	364.51	363.99	475.87	……	……	……	……
LW(N=1)	2.0229	57.761	60.172	364.26	364.78	476.23	……	……	……	……
					$m=50$, $n=50$					
CLT	50.550	288.85	300.91							
FSDT	40.649	288.33	300.36	500.71	490.60					
ESL(N=4)	39.321	288.39	299.03	475.24	460.75	469.45	……	……	……	……
LW(N=1)	39.336	288.33	299.03	460.93	469.66	475.60	……	……	……	……
					$m=100$, $n=100$					
CLT	194.03	601.83	577.89							
FSDT	112.44	574.19	596.91	727.21	704.64					
ESL(N=4)	106.26	466.69	574.18	605.33	682.55	694.41	……	……	……	……
LW(N=1)	106.27	467.08	574.06	605.33	682.58	694.54	……	……	……	……
					$m=500$, $n=500$					
CLT	2557.8	2923.5	3009.1							
FSDT	671.32	2820.0	3087.6	3100.0	2832.1					
ESL(N=4)	654.92	771.34	1107.4	2015.5	2495.8	2508.4	……	……	……	……
LW(N=1)	640.60	759.49	1101.4	1507.8	1943.4	2394.3	……	……	……	……

LW). Upon substitution of Eq. (14) in Eq. (15), the free vibration response leads to an eigenvalue problem as follows:

$$\|\boldsymbol{K} - \omega_{mn}^2\boldsymbol{M}\| = 0 \tag{16}$$

3.1 Numbers of Higher Modes for the Considered Problems. By giving the in-plane wave numbers m and n, the number of vibration modes is equal to the number of degrees of freedom of the considered two-dimensional model. In other words, the number of eigenvalues (circular frequencies) depends on the dimension of the multilayer level matrix $\boldsymbol{K}$. CLT, FSDT, and ESL models have a number of degrees of freedom that does not depend on the number of layers. CLT has three total degrees of freedom and the matrix $\boldsymbol{K}$ has dimension 3×3, see Fig. 5. FSDT has five degrees of freedom and $\boldsymbol{K}$ is 5×5 (Fig. 6). The ESL model with N=4 has 15 DOFs for the whole plate and the stiffness matrix has dimension 15×15, as illustrated in Fig. 7. The above dimensions do not change when the number of layers is changed. This is not true for the case of layer-wise analyses, which have a number of degrees of freedom that depends on the number of the considered layers. The LW theory with N=1 has 6 DOFs for each layer k; this leads, for the two considered laminates, 10 and 20 layers, to 33 DOFs and 63 DOFs, respectively. The corresponding stiffness matrices $\boldsymbol{K}$ have dimension 33×33 in the 10-layer case and 63×63 in the 20-layer case. Figure 8 shows the case of the LW model (N=1) for a three layer plate as an example. In this case, the matrix dimension is 12×12.

In conclusion, for each couple (m,n), we obtain 3 eigenvalues for the CLT, 5 for the FSDT, 15 for the ESL(N=4), and 33 and 63 eigenvalues for the LW(N=1). These eigenvalues represent the number of modes related to the same in-plane harmonic mode. It is clear that only refined theories permit higher-order modes to be investigated. Table 1 shows the number of frequencies for the four considered theories for the two cases of 10- and 20-layered plates. The case in which the couple of positive integers (m,n) varies from 1 to 500 is considered. It is evident that some modes are tragically lost when using simplified plate model analyses.

4 Results and Discussion

Two assessments are first made to demonstrate the effectiveness of the considered refined models. A higher vibration modes analysis is then made by referring to two benchmarks.

4.1 Assessments. The problem proposed by Noor and Burton in Ref. [50] is reconsidered. A 3D solution was given in Ref. [50] for a simply-supported 10-layered plate with a thickness ratio a/h=5; the geometry and material are reported in Table 2, and the

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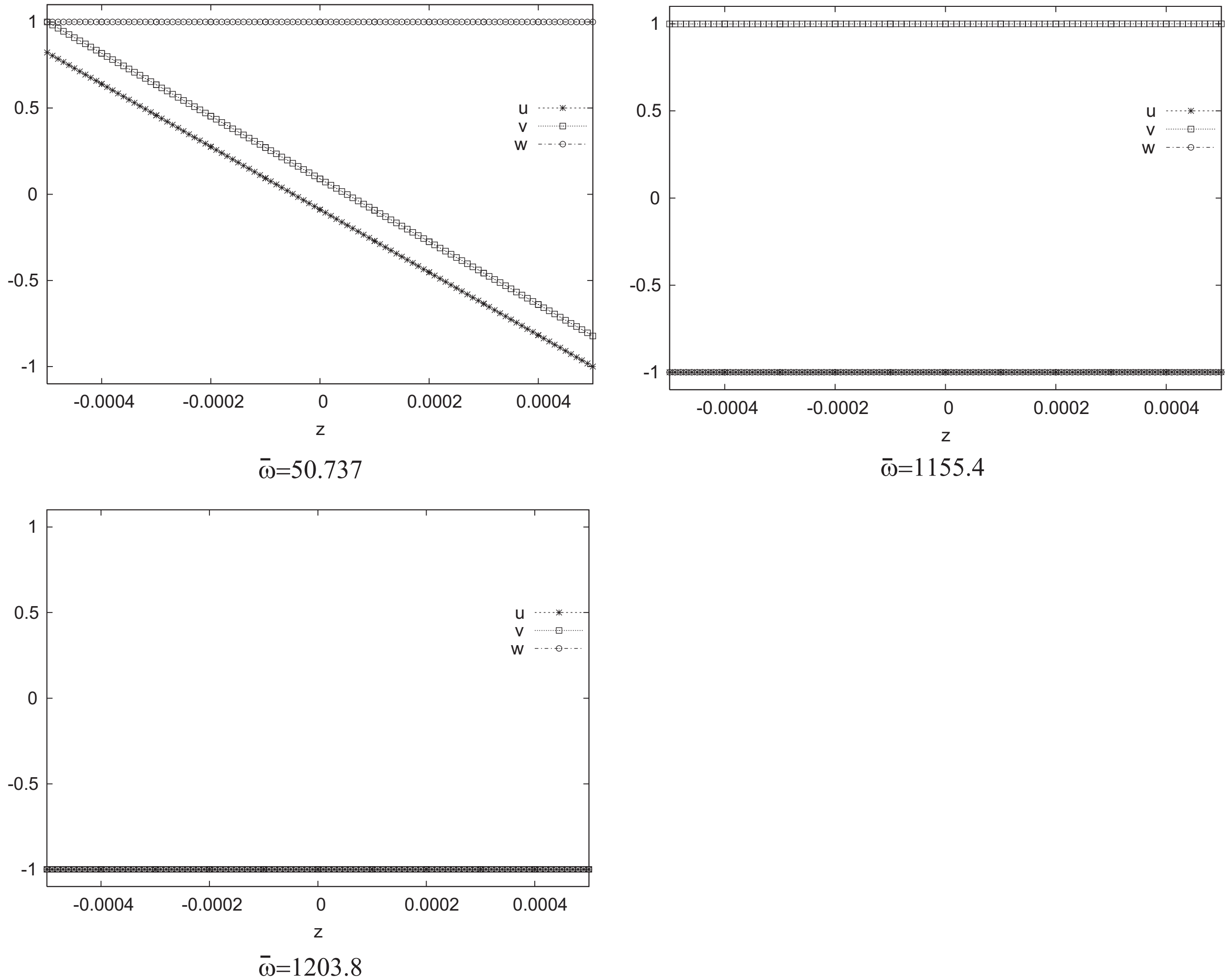

Fig. 10 Benchmark 1, ten-layered plate, $m=n=50$. Through the thickness z distribution of CLT modes.

results are given in Table 3. Two orthotropic ratio values (E_L/E_T) are investigated for a very thick plate. These results demonstrate the effectiveness of the implemented plate models; in particular, the LW analysis coincides with the 3D results. The CLT and FSDT results by Reddy and Phan [51] are compared with the present analyses in Table 4. Very thick and thin geometries have been considered, and the material data and geometry are indicated in Table 2. Table 4 shows that, for a thin plate $(a/h=100)$, the differences between ESL and LW models decrease, and a lower order of expansion can be applied. CLT and FSDT are inadequate because they do not consider typical effects connected to multilayered composite plates such as in-plane and transverse anisotropy. In these two assessments, attention has been restricted to the fundamental frequency parameter.

4.2 Benchmarks Discussion. Frequencies related to higher vibration modes have been discussed for two-plate problems. These consist of two cross-ply laminated plates with the material data and geometry indicated in Table 5. These plates were suggested by EADS in the framework of the CASSEM EU project. Benchmark 1 considers a 10-layered plate with a thickness ratio $a/h=1500$, while benchmark 2 coincides with a 20-layered plate with a thickness ratio $a/h=375$. Fiber orientation 0 deg/90 deg are only considered for the sake of brevity. Further orientations are not investigated because they do not introduce more remarks.

By assigning the number of waves (m,n) in the two in-plane directions (Fig. 9), a number of modes, equal to the number of degrees of freedom of the considered two-dimensional models, are obtained. Tables 6 and 7 show the obtained results for the following wave number values (m,n): (1,1), (5,5), (10,10), (50,50), (100,100), and (500,500). These cases cover a large range of frequencies, from low to higher modes. For the sake of brevity, the complete set of frequencies is only given once, in correspondence to the $m=n=1$ case. Since the number of frequencies differs from theory to theory, a check is required to recognize the mode. This can be done by plotting the three displacement components in the thickness direction z. Figures 10–13 plot some modes for the four considered theories in the case of a 10-layered plate with $m=n=50$. For refined models, such as the ESL($N=4$) and LW($N=1$), there are other modes that have not been indicated in Figs. 12 and 13, which are typical of these refined theories. These modes are not considered in CLT and FSDT analyses. Frequencies can be compared if and only if these refer to the same mode. Comparing Fig. 12 (ESL($N=4$)) and Fig. 13 (LW($N=1$)), it is clear that the corresponding modes are: $\bar{\omega}=49.684$ (ESL($N=4$)) and $\bar{\omega}=49.696$ (LW($N=1$)), $\bar{\omega}=1155.1$ (ESL($N=4$)) and $\bar{\omega}=1154.9$ (LW($N=1$)), $\bar{\omega}=1203.0$ (ESL($N=4$)) and $\bar{\omega}=1203.0$ (LW($N=1$)), $\bar{\omega}=5878.5$ (ESL($N=4$)) and $\bar{\omega}=5900.2$ (LW($N=1$)), $\bar{\omega}=5865.6$ (ESL($N=4$)) and $\bar{\omega}=5887.2$ (LW($N=1$)), and $\bar{\omega}=7613.4$ (ESL($N=4$)) and $\bar{\omega}=7642.8$ (LW($N=1$)). The fundamental frequencies for each pair of wave numbers are underlined in Tables 6 and 7. For low values of (m,n), there are no differences between CLT, FSDT, ESL($N=4$), and LW($N=1$) for either benchmark. If the in-plane waves number is increased, the differ-

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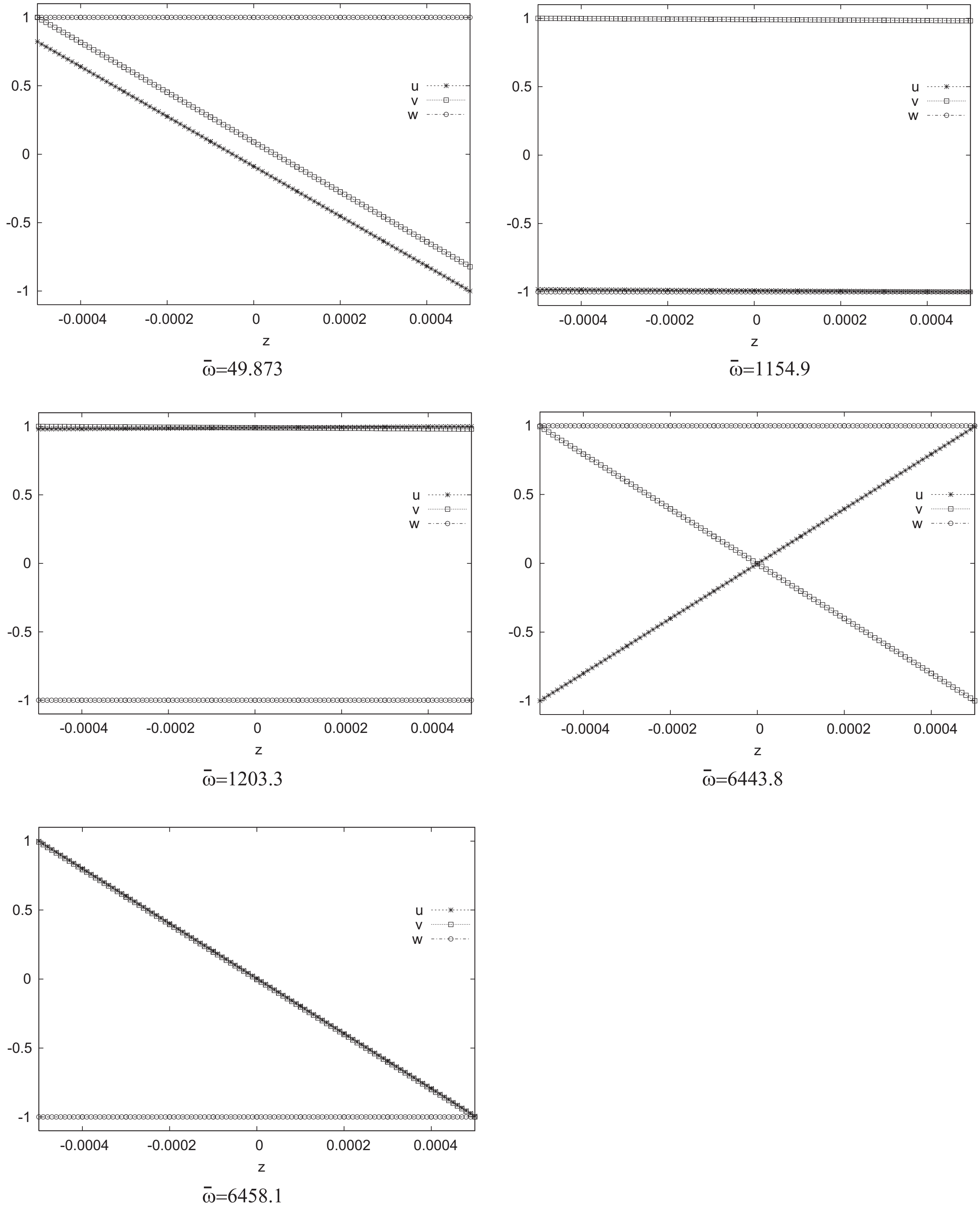

Fig. 11 Benchmark 1, ten-layered plate, *m*=*n*=50. Through the thickness *z* distribution of FSDT modes.

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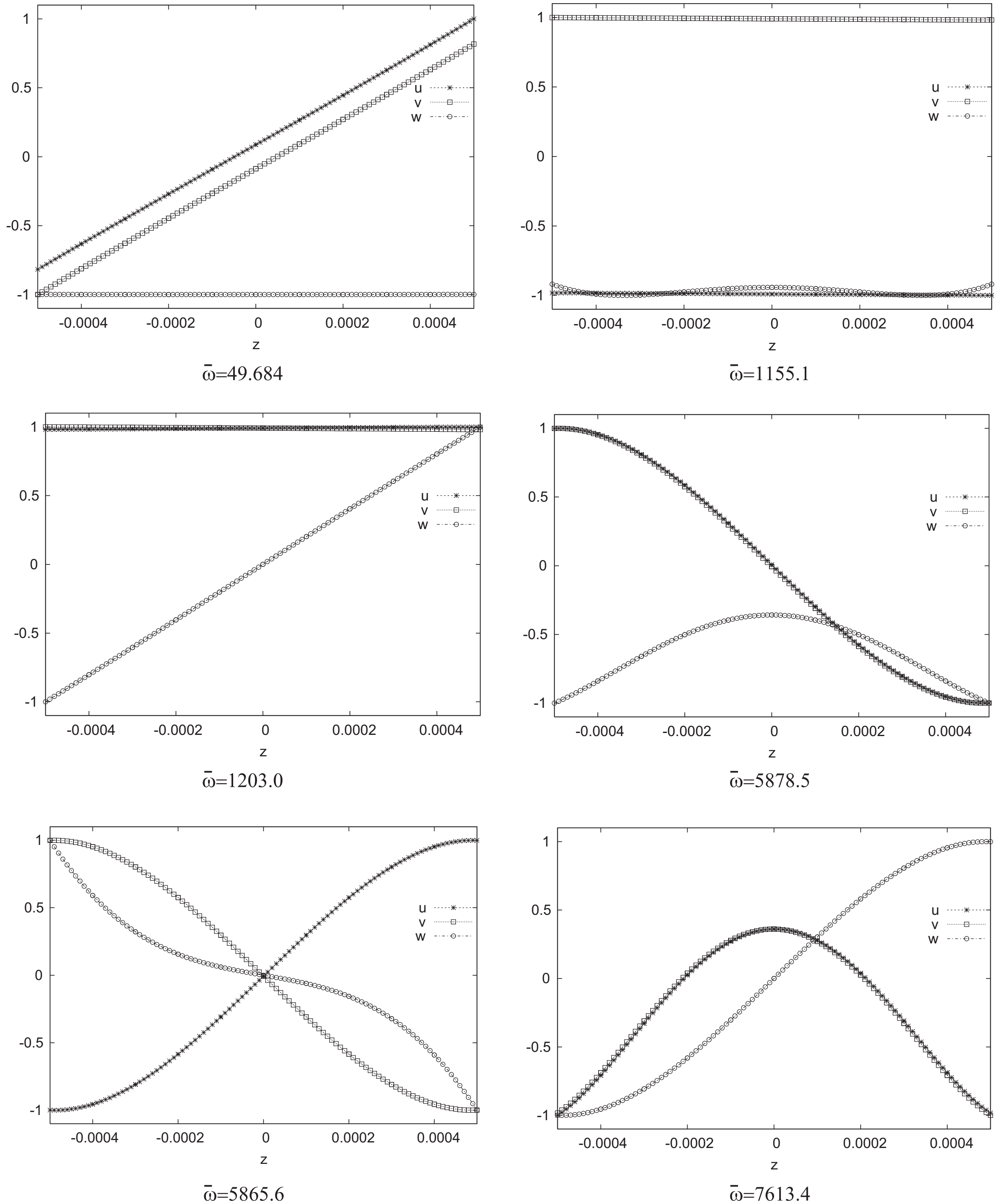

Fig. 12 Benchmark 1, ten-layered plate, $m=n=50$. Through the thickness z distribution of first six ESL($N=4$) modes (the total number of modes is 15).

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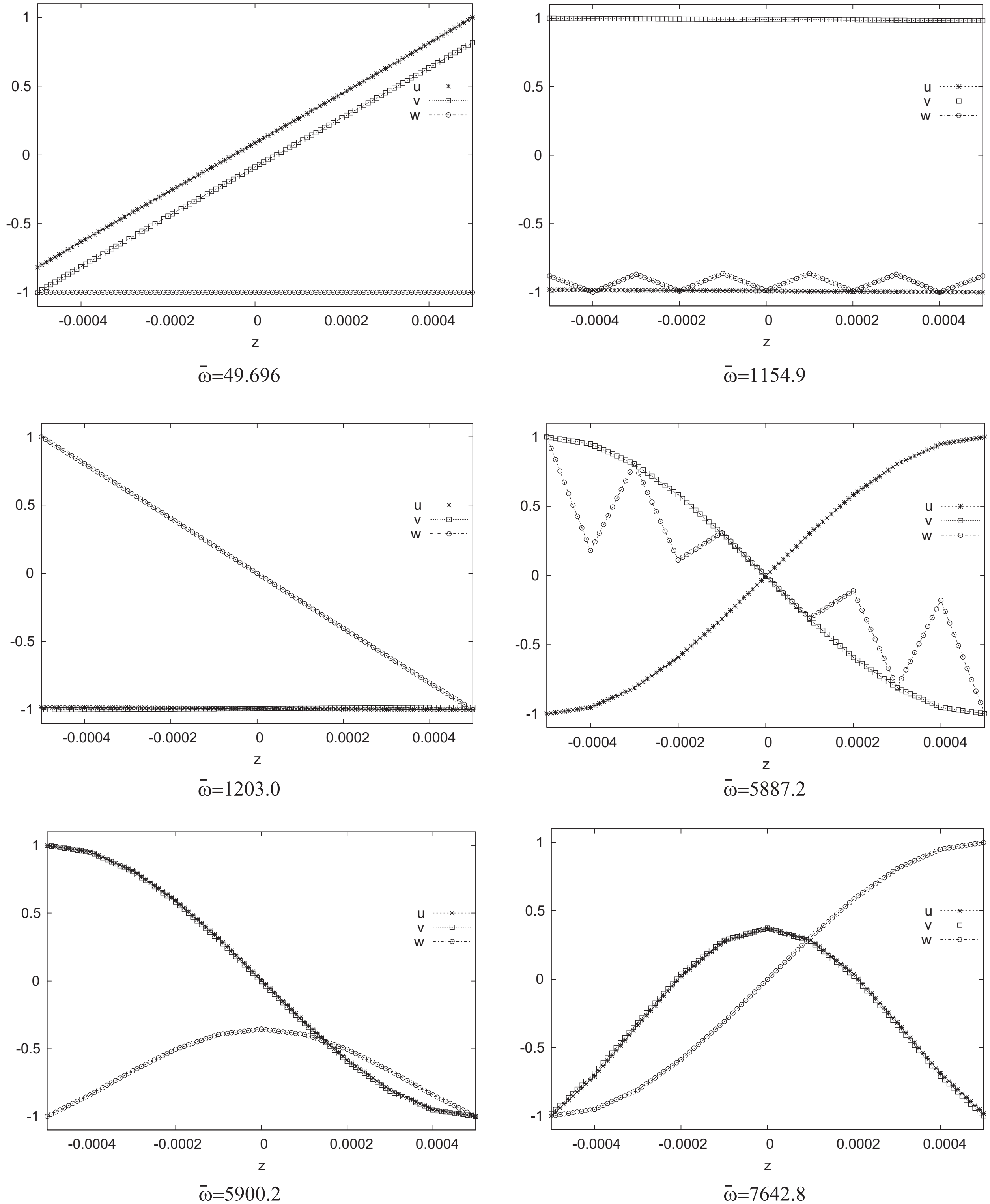

Fig. 13 Benchmark 1, ten-layered plate, *m*=*n*=50. Through the thickness *z* distribution of first six LW(*N*=1) modes (the total number of modes is 33).

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Table 8 Benchmark 1, 10-layered plate, and $a/h=1500$. Fundamental frequency f in Hz for different waves number. $\Delta\%=|f_{CLT}-f_i/f_{CLT}\cdot 100|$ is the difference between CLT and other theories.

m,n	CLT	FSDT	Δ%	ESL(N=4)	Δ%	LW(N=1)	Δ%
1,1	2.98913	2.98913	(0.00)	2.98913	(0.00)	2.98913	(0.00)
1,2	8.56982	8.56982	(0.00)	8.56982	(0.00)	8.56982	(0.00)
2,1	8.56982	8.56982	(0.00)	8.56982	(0.00)	8.56982	(0.00)
2,2	11.9565	11.9565	(0.00)	11.9565	(0.00)	11.9712	(0.12)
5,5	74.7724	74.7724	(0.00)	74.7724	(0.00)	74.7724	(0.00)
5,10	214.172	214.039	(0.06)	214.010	(0.08)	214.010	(0.08)
10,5	214.172	214.039	(0.06)	214.010	(0.08)	214.010	(0.08)
10,10	299.104	298.898	(0.07)	298.839	(0.09)	298.883	(0.07)
10,50	5126.14	5044.27	(1.60)	5026.30	(1.95)	5026.89	(1.94)
50,10	5126.14	5044.27	(1.60)	5026.30	(1.95)	5026.89	(1.94)
50,50	7470.91	7343.69	(1.70)	7315.86	(2.07)	7317.62	(2.05)
100,100	29,800.0	27,928.5	(6.28)	27,554.5	(7.53)	27,572.1	(7.48)
500,500	685,792	350,391	(48.9)	326,330	(52.4)	326,890	(52.3)

ences between classical theories (CLT and FSDT) and refined models become more evident; these differences are larger for the 20-layered plate than the 10-layered one. These results are summarized in Tables 8 and 9 and in Figs. 14 and 15, where these differences are clearly shown. Table 8 shows that the differences become critical ($>1\%$) for a value of waves number ($m=10$, $n=50$) that is larger than the case of Table 9 ($m=5$, $n=10$); this means that both the thickness ratio of the plate, and the number and stacking of the layers are fundamental parameters of the considered problems. These considerations are confirmed by Figs. 14 and 15. It should be noticed that the error can be larger than 50% for the higher modes related to high values of m and n.

5 Conclusions

A comparison between classical theories (classical lamination theory and first-order shear deformation theory) and refined ones (equivalent single layer with fourth-order of expansion and layer

Table 9 Benchmark 2, 20 layered plate, $a/h=375$. Fundamental frequency f in Hz for different waves number. $\Delta\%=|f_{CLT}-f_i/f_{CLT}\cdot 100|$ is the difference between CLT and other theories.

m,n	CLT	FSDT	Δ%	ESL(N=4)	Δ%	LW(N=1)	Δ%
1,1	24.1486	24.1486	(0.00)	24.1486	(0.00)	24.1486	(0.00)
1,2	69.2606	69.1500	(0.16)	69.1500	(0.16)	69.1500	(0.16)
2,1	69.2606	69.1500	(0.16)	69.1500	(0.16)	69.1500	(0.16)
2,2	96.7100	96.5900	(0.12)	96.5900	(0.12)	96.5900	(0.12)
5,5	604.069	602.302	(0.29)	602.067	(0.33)	602.067	(0.33)
5,10	1730.22	1712.67	(1.01)	1709.25	(1.21)	1709.14	(1.22)
10,5	1730.22	1712.67	(1.01)	1709.25	(1.21)	1709.14	(1.22)
10,10	2415.33	2388.12	(1.13)	2382.82	(1.35)	2382.94	(1.34)
10,50	41,143.4	33,351.0	(18.9)	32,276.7	(21.6)	32,287.3	(21.5)
50,10	41,143.4	33,351.0	(18.9)	32,276.7	(21.6)	32,287.3	(21.5)
50,50	59,547.0	47,883.8	(19.6)	46,319.5	(22.2)	46,337.1	(22.2)
100,100	228,564	132,452	(42.1)	125,172	(45.2)	125,184	(45.2)
500,500	3,013,045	790,803	(73.7)	771,485	(74.4)	754,616	(74.9)

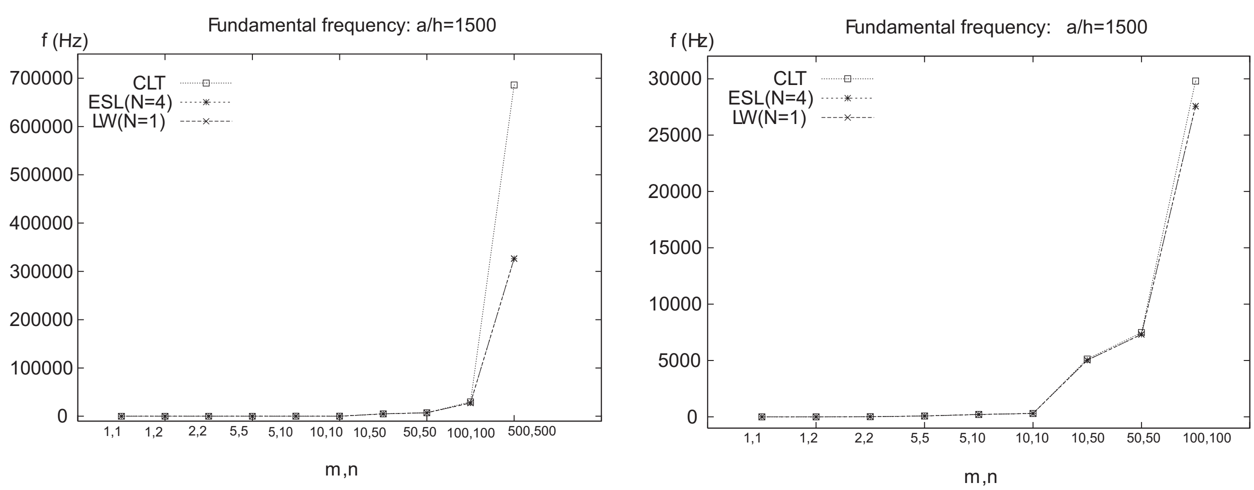

Fig. 14 Benchmark 1, fundamental frequencies in Hz for the ten-layered plate. Comparison between CLT, ESL(N=4), and LW(N=1) results. The figure on the right is a zoom of the figure on the left side.

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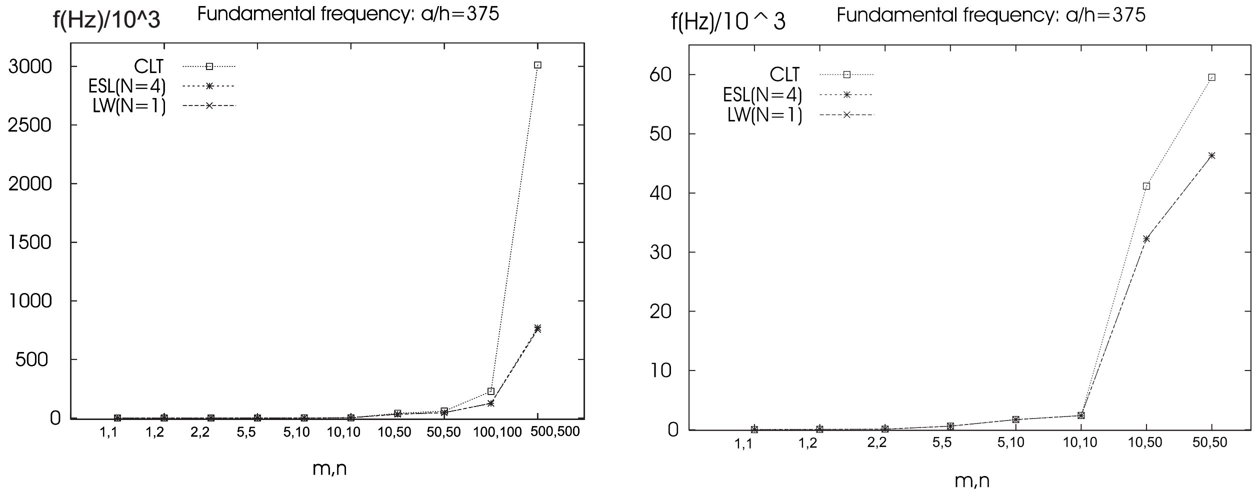

Fig. 15 Benchmark 2, fundamental frequencies in Hz for the 20-layered plate. Comparison between CLT, ESL(N=4), and LW(N=1) results. The figure on the right is a zoom of the figure on the left side.

wise with linear evaluation in z) has been addressed in this paper for the free vibration response of simply-supported multilayered composite plates. A closed-form solution has been discussed for a 10-layer and a 20-layered plate. A large range of frequencies (from low to high values of in-plane wave numbers) has been considered to show the importance of refined models in the numerical evaluation of higher modes. It has been concluded that the use of refined models is mandatory for higher mode evaluation even though thin plate geometries are considered. The presented results could be of some help to assess computational models, such as the finite element method and SEA, for the analysis of higher vibration modes in laminated plates.

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References

[1] Bogdanovich, A. E., and Sierakowski, R. L., 1999, "Composite Materials and Structures: Science, Technology and Applications—A Compendium of Books, Review Papers and Other Sources of Informations," Appl. Mech. Rev., **52**(12), pp. 351–366.
[2] Leissa, A. W., 1969, "Vibration of Plates," NASA Report No. SP-160.
[3] Leissa, A. W., 1973, "Vibration of Shells," NASA Report No. SP-288.
[4] Werner, S., 2004, *Vibrations of Shells and Plates*, 3rd ed., CRC, New York/Dekker, New York.
[5] De Rosa, S., Franco, F., and Ricci, F., 1999, *Introduzione alla Tecnica Statistico-Energetica (S.E.A.) per la Dinamica Strutturale e l'Acustica Interna*, Liguori Editore, Napoli.
[6] Noor, A. K., 1973, "Free Vibrations of Multilayered Composite Plates," AIAA J., **11**(7), pp. 1038–1039.
[7] Liew, K. M., Xiang, Y., and Kitipornchai, S., 1995, "Research on Thick Plate Vibration: A Literature Survey," J. Sound Vib., **180**(1), pp. 163–176.
[8] Shuyu, L., 2001, "Study on the Flexural Vibration of Rectangular Thin Plates With Free Boundary Conditions," J. Sound Vib., **239**(5), pp. 1063–1071.
[9] Leung, A. Y. T., and Zhu, B., 2004, "Transverse Vibration of Thick Polygonal Plates Using Analytically Integrated Trapezoidal Fourier p-Element," Comput. Struct., **82**(2–3), pp. 109–119.
[10] Batra, R. C., and Aimmanee, S., 2005, "Vibration of Thick Isotropic Plates With Higher Order Shear and Normal Deformable Plate Theories," Comput. Struct., **83**(12–13), pp. 934–955.
[11] Gorman, D. J., 2006, "Exact Solutions for the Free In-Plane Vibration of Rectangular Plates With Two Opposite Edges Simply Supported," J. Sound Vib., **294**(1–2), pp. 131–161.
[12] Gorman, D. J., 2004, "Accurate Analytical Type Solutions for the Free In-Plane Vibration of Clamped and Simply Supported Rectangular Plates," J. Sound Vib., **276**(1–2), pp. 311–333.
[13] Taher, H. R. D., Omidi, M., Zadpoor, A. A., and Nikooyan, A. A., 2006, "Free Vibration of Circular and Annular Plates With Variable Thickness and Different Combinations of Boundary Conditions," J. Sound Vib., **296**(4–5), pp. 1084–1092.
[14] Aimmanee, S., and Batra, R. C., 2007, "Analytical Solution for Vibration of an Incompressibile Isotropic Linear Elastic Rectangular Plate, and Frequencies Missed in Previous Solutions," J. Sound Vib., **302**(3), pp. 613–620.
[15] Batra, R. C., and Aimmanee, S., 2003, "Missing Frequencies in Previous Exact Solutions of Free Vibrations of Simply Supported Rectangular Plates," J. Sound Vib., **265**(4), pp. 887–896.
[16] Batra, R. C., Vidoli, S., and Vestroni, F., 2002, "Plane Wave Solutions and Modal Analysis in Higher Order Shear and Normal Deformable Plate Theories," J. Sound Vib., **257**(1), pp. 63–88.
[17] Bardell, N. S., Langley, R. S., and Dunsdon, J. M., 1996, "On the Free In-Plane Vibration of Isotropic Rectangular Plates," J. Sound Vib., **191**(3), pp. 459–467.
[18] Liew, K. M., and Yang, B., 1999, "Three-Dimensional Elasticity Solutions for Free Vibrations of Circular Plates: A Polynomial-Ritz Analysis," Comput. Methods Appl. Mech. Eng., **175**(1–2), pp. 189–201.
[19] Zhao, Y. B., Wei, G. W., and Xiang, Y., 2002, "Discrete Singular Convolution for the Prediction of High Frequency Vibration of Plates," Int. J. Solids Struct., **39**(1), pp. 65–88.
[20] Beslin, O., and Nicolas, J., 1997, "A Hierarchical Functions Set for Predicting Very High Order Plate Bending Modes With Any Boundary Conditions," J. Sound Vib., **202**(5), pp. 633–655.
[21] Oosterhout, G. M., van der Hoogt, P. J. M., and Spiering, R. M. E. J., 1995, "Accurate Calculation Methods for Natural Frequencies of Plates With Special Attention to the Higher Modes," J. Sound Vib., **183**(1), pp. 33–47.
[22] Sakiyama, T., and Huang, M., 1998, "Free Vibration Analysis of Rectangular Plates With Variable Thickness," J. Sound Vib., **216**(3), pp. 379–397.
[23] Mindlin, R. D., 2006, *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*, J. Yang, ed., World Scientific, Singapore/University of Nebraska-Lincoln.
[24] Bouthier, O. M., and Bernhard, R. J., 1995, "Simple Models of the Energetics of Transversely Vibrating Plates," J. Sound Vib., **182**(1), pp. 149–166.
[25] Wei, G. W., Zhao, Y. B., and Xiang, Y., 2002, "A Novel Approach for the Analysis of High-Frequency Vibrations," J. Sound Vib., **257**(2), pp. 207–246.
[26] Kim, H.-Y., and Hwang, W., 2001, "Estimation of Normal Mode and Other System Parameters of Composite Laminated Plates," Compos. Struct., **53**(3), pp. 345–354.
[27] Gorman D. J. and Ding, W., 2003, "Accurate Free Vibration Analysis of Completely Free Symmetric Cross-Ply Rectangular Laminated Plates," Compos. Struct., **60**(3), pp. 359–365.
[28] Gorman, D. J., 2000, "Free Vibration Analysis of Completely Free Rectangular Plates by the Superposition-Galerkin Method," J. Sound Vib., **237**(5), pp. 901–914.
[29] Ye, J. Q., 1997, "A Three-Dimensional Free Vibration Analysis of Cross-Ply Laminated Rectangular Plate With Clamped Edges," Comput. Methods Appl. Mech. Eng., **140**(3–4), pp. 383–392.
[30] Topal, U., Uzman, U. V., 2007, "Free Vibration Analysis of Laminated Plates Using Higher-Order Shear Deformation Theory," *Proceedings in Physics*, Vol. 111, Springer, Netherlands, pp. 493–498.
[31] Jung, S. N., Nagaraj, V. T., and Chopra, I., 2001, "Refined Structural Dynamics Model for Composite Rotor Blades," AIAA J., **39**(2), pp. 339–348.
[32] Cho, M., and Kim, J.-S., 2001, "Higher-Order Zig-Zag Theory for Laminated Composites With Multiple Delaminations," ASME J. Appl. Mech., **68**(6), pp. 869–877.
[33] Plagianakos, T. S., and Saravanos, D. A., 2004, "High-Order Layer Wise Mechanics and Finite Element for the Damped Dynamic of Sandwich Composite Beams," Int. J. Solids Struct., **41**(24–25), pp. 6853–6871.
[34] Rao, M. K., and Desai, Y. M., 2004, "Analytical Solutions for Vibrations of Laminated and Sandwich Plates Using Mixed Theory," Compos. Struct.,

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63(3–4), pp. 361–373.
[35] Plagianakos, T. S., and Saravanos, D. A., 2003, "Mechanics and Finite Elements for the Damped Dynamic Characteristics of Curvilinear Laminates and Composite Shell Structures," J. Sound Vib., **263**(2), pp. 399–414.
[36] Sun, C. T., and Whitney, J. M., 1972, "On Theories for the Dynamic Response of Laminated Plates," 13th ASME and SAE Structures, Structural Dynamics and Materials Conference, San Antonio, TX, April 10–12.
[37] Wang, C. M., 1996, "Vibration Frequencies of Simply Supported Polygonal Sandwich Plates Via Kirchhoff Solutions," J. Sound Vib., **190**(2), pp. 255–260.
[38] Nayak, A. K., Moy, S. S. J., and Shenoi, R. A., 2002, "Free Vibration Analysis of Composite Sandwich Plates Based on Reddy's Higher-Order Theory," Composites, Part B, **33**(7), pp. 505–519.
[39] Messina, A., and Soldatos, K. P., 1999, "Vibration of Completely Free Composite Plates and Cylindrical Shell Panels by a Higher-Order Theory," Int. J. Mech. Sci., **41**(8), pp. 891–918.
[40] Tessler, A., Saether, E., and Tsui, T., 1995, "Vibration of Thick Laminated Plates," J. Sound Vib., **179**(3), pp. 475–498.
[41] Han, W., and Petyt, M., 1997, "Geometrically Nonlinear Vibration Analysis of Thin, Rectangular Plates Using the Hierarchical Finite Element Method—II: 1st Mode of Laminated Plates and Higher Modes of Isotropic and Laminated Plates," Comput. Struct., **63**(2), pp. 309–318.
[42] Zhang, Q. J., and Sainsbury, M. G., 2000, "The Galerkin Element Method Applied to the Vibration of Rectangular Damped Sandwich Plates," Comput. Struct., **74**(6), pp. 717–730.
[43] Carrera, E., 1999, "A Study of Transverse Normal Stress Effect on Vibration of Multilayered Plates and Shells," J. Sound Vib., **225**(5), pp. 803–829.
[44] Carrera, E., 1999, "A Reissner's Mixed Variational Theorem Applied to Vibrational Analysis of Multilayered Shells," ASME J. Appl. Mech., **66**(1), pp. 69–78.
[45] Carrera, E., 1998, "Layer-Wise Mixed Models for Accurate Vibration Analysis of Multilayered Plates," ASME J. Appl. Mech., **65**(4), pp. 820–828.
[46] Carrera, E., 1995, "A Class of Two Dimensional Theories for Multilayered Plates Analysis," Atti Accad. Sci. Torino, Cl. Sci. Fis., Mat. Nat., **19–20**, pp. 49–87.
[47] Carrera, E., 2002, "Theories and Finite Elements for Multilayered, Anisotropic, Composite Plates and Shells," Arch. Comput. Methods Eng., **9**(2), pp. 87–140.
[48] Reddy, J. N., 2004, *Mechanics of Laminated Composite Plates and Shells, Theory and Analysis*, 2nd ed., CRC, New York.
[49] Carrera, E., and Brischetto, S., 2008, "Analysis of Thickness Locking in Classical, Refined and Mixed Multilayered Plate Theories," Compos. Struct., **82**(4), pp. 549–562.
[50] Noor, A. K., and Burton, W. S., 1989, "Stress and Free Vibration Analysis of Multilayered Composite Plates," Compos. Struct., **11**(3), pp. 183–204.
[51] Reddy, J. N., and Phan, N. D., 1985, "Stability and Vibration of Isotropic, Orthotropic and Laminated Plates According to a Higher Order Shear Deformation Theory," J. Sound Vib., **98**(2), pp. 157–170.

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