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Improved bending analysis of sandwich plates using a zig-zag function

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ABSTRACT

This paper shows the advantages of using the zig-zag function $(-1)^k\zeta_k$ (ZZF) in the bending analysis of sandwich flat panels. Higher order theories are developed by adding ZZF to displacement fields of known theories. The main advantage of ZZF lies in the fact that it introduces a discontinuity in the first derivative (so called zig-zag effect) of the displacements distribution with correspondence to the core–face interfaces. Results including and discarding ZZF are compared in the bending response of sandwich plates with soft core loaded by harmonic distribution of transverse pressure at the top surface. Different values of face-to-core-stiffness-ratio FCSR as well as length-to-thickness-ratio LTR have been analyzed. It is concluded that: (1) ZZF is highly recommended in the bending analysis of sandwich plates; (2) the use of ZZF makes the error almost independent by FCSR parameter; (3) ZZF is easy to implement and its use should be preferred with respect to other refined theories.

1. Introduction

Sandwich structures are nowadays used to build large portions of the advanced structural elements of aerospace, automotive and ship vehicles. These structures are characterized by a soft core between two stiffer faces. They consist of a three layered structure. The discontinuity of mechanical properties between faces and core introduces a discontinuity of the deformed core–faces planes at the interfaces, see Fig. 1. This is also known as zig-zag effect in laminated structures [1]. Such a discontinuities make difficult the use of classical theories such as Kirchhoff [2] or Reissner-Mindlin [3] type theories; to trace accurate responses of sandwich structures, see the books by Zenkert [4], Bitzer [5] and Vinson [6]. So called layer-wise models in which three layers are treated as three independent layers, can be used to capture the above ZZ form, see the overviews by Burton and Noor [7], Noor, Burton and Bert [8], Altenbach [9], Librescu and Hause [10], Vinson [11], Carrera and Brischetto [12], Demasi [13]. However, these models could result computational expensive.

In the framework of mixed multilayered plate theories, Murakami [14] proposed a zig-zag function ZZF ables to reproduce the described slope discontinuity. Equivalent single layer model (that is three layers are treated as one-layer equivalent plate) with only displacement unknowns can be developed on the basis of ZZF. The advantages of using the ZZF to analyze multilayered anisotropic plate and shells as well as the Finite Element implementation have been discussed by Carrera [15] and Demasi [16], respectively. In the present work the attention is restricted to the application of ZZF to bending analyses of sandwich plates. Various values of thickness ratios (thin and thick plates) are considered along with several face-to-core-stiffness-ratios to highlight the capability of ZZF. This paper consists of a first contribution of a work that aims to establish the limitations and advantages of ZZF in the analysis of sandwich structures. Future work could consider shell geometries, as well as dynamic problems and other lay-outs.

2. The zig-zag function

Let us consider a sandwich plate composed by three layers, perfectly bonded together. z is the thickness coordinate of the whole plate while z_k is the layer thickness coordinate. a and h are length and thickness of the square sandwich plate, respectively. The not dimensioned layer coordinate $\zeta_k = (2z_k)/h_k$ is further introduced (h_k is the thickness of the kth layer). The Murakami's zig-zag function $Z(z)$ was defined according to the following formula [14],

$$Z(z) = (-1)^k\zeta_k. \quad (1)$$

$Z(z)$ has the following properties:

1. It is piece-wise linear function of layer coordinates z_k,
2. $Z(z)$ has unit amplitude for the whole layers,
3. the slope $Z'(z) = \frac{dZ}{dz}$ assumes opposite sign between two-adjacent layers. Its amplitude is layer thickness independent.

A plot of $Z(z)$ is also given in Fig. 2. ZZF can be used to introduce discontinuous slopes with correspondence to layer interfaces for

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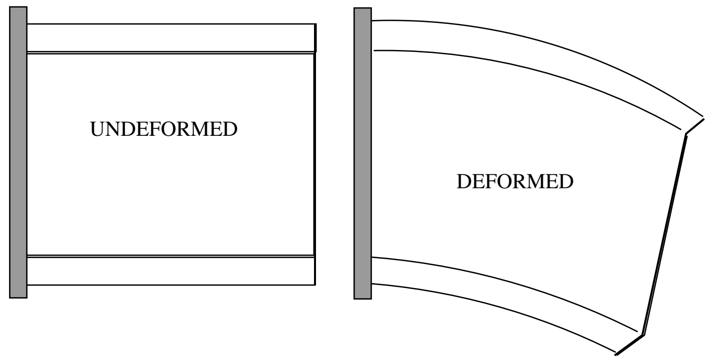

Fig. 1. Undeformed and deformed plane of a sandwich structure.

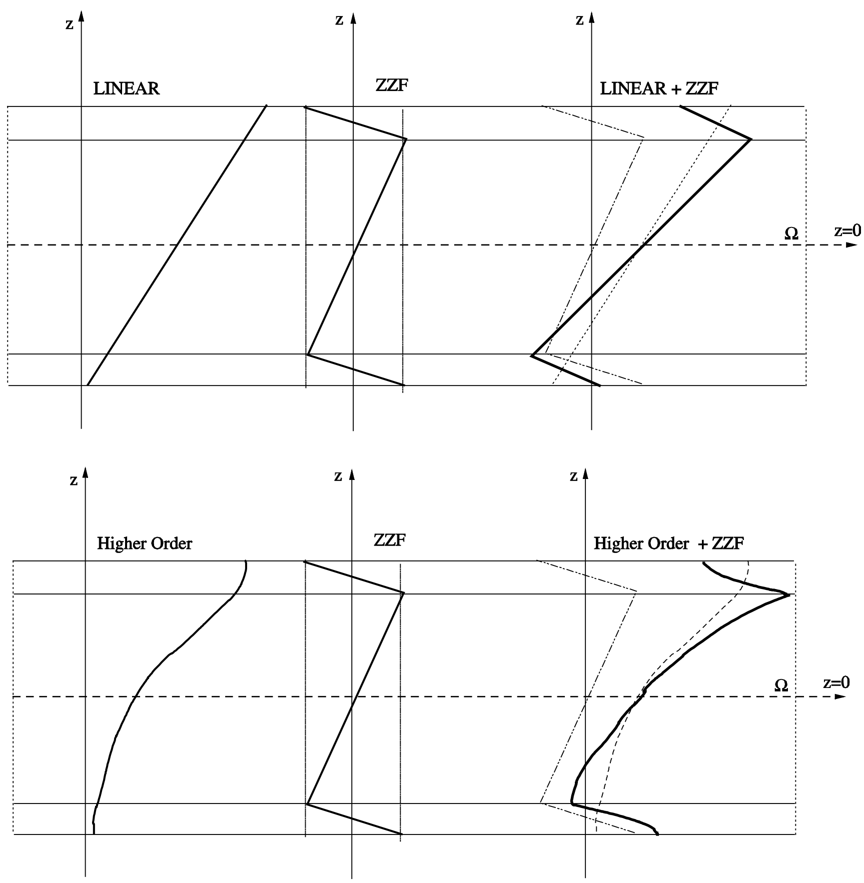

Fig. 2. Effect of adding the zig-zag function: linear case (top) and higher order case (bottom).

displacement $u(z)$. If a *linear function* is considered, to expand the displacement along the thickness one has:

$$u(z) = c^0 + c^1 z, \tag{2}$$

where c^0, c^1 are the amplitudes of the uniform and linear terms of $u(z)$, respectively. By adding ZZF one has:

$$u(z) = c^0 + c^1 z + c^Z Z(z), \tag{3}$$

c^Z is the amplitude of ZZF. Displacement with and without ZZF are compared in Fig. 2, which makes evident how $Z(z)$ emulates ZZ effects which is very relevant in sandwich structures. $Z(z)$ can be also used in conjunction to higher N-order expansion (see Fig. 2):

$$u(z) = c^0 + c^1 z + c^2 z^2 + \cdots + c^{N-1} z^{N-1} + c^N z^N + c^Z Z(z). \tag{4}$$

3. Considered theories

Classical theories for sandwich plates, such Kirchhoff (classical lamination theory, CLT), Reissner-Mindlin (first order shear defor-

Table 1
Number of degrees of freedom for the considered sandwich plate theories.

Theory	d.o.f.
CLT	3
FSDT	5
ED1	6
EDZ1	9
ED4	15
EDZ3	15

Table 2
Comparison of various theories to evaluate the transverse displacements amplitude (centre plate deflection) $\bar{w} = U_z \frac{100E_{skin}h^3}{p_z a^4}$ in $z = 0$.

LTR	4		10		100		1000	
FCSR = 10								
3D	13.294		6.3923		5.0649		5.0435	
CLT	5.0515	(62.3)	5.0515	(21.0)	5.0515	(0.26)	5.0515	(0.16)
FSDT	8.0821	(39.2)	5.5364	(13.4)	5.0563	(0.17)	5.0515	(0.16)
ED1	8.0821	(39.2)	5.5364	(13.4)	5.0563	(0.17)	5.0515	(0.16)
ED4	12.497	(6.00)	6.2671	(1.96)	5.0637	(0.02)	5.0516	(0.16)
EDZ1	13.550	(1.92)	6.4200	(0.12)	5.0652	(0.00)	5.0516	(0.16)
EDZ3	13.343	(0.37)	6.3935	(0.02)	5.0649	(0.02)	5.0516	(0.16)
FCSR = 10^5								
3D	1299.4		1230.6		126.70		6.8958	
CLT	5.5815	(99.6)	5.5814	(99.5)	5.5814	(95.6)	5.5814	(19.1)
FSDT	9.8240	(99.2)	6.2602	(99.5)	5.5852	(95.6)	5.5815	(19.1)
ED1	9.8240	(99.2)	6.2602	(99.5)	5.5852	(95.6)	5.5815	(19.1)
ED4	112.51	(91.3)	24.078	(98.0)	5.7692	(95.4)	5.5833	(19.0)
EDZ1	1344.2	(3.45)	1236.9	(0.51)	126.70	(0.00)	6.9101	(0.21)
EDZ3	994.51	(23.5)	933.11	(24.2)	122.92	(2.98)	6.9096	(0.20)

The percentage error respect to the 3D solution is given in brackets.

mation theory, FSDT) and higher order shear deformation theory, HSDT (see the review papers by Librescu and Hause [10] and Vinson [11]) do not account for ZZ effect. A possible use of ZZF would consist to enhance classical models by 'simply' adding $Z(z)$ in their displacement fields. Let consider a linear distribution of a displacement component in the thickness direction z, as it is in CLT and FSDT:

$$u = u^0 + zu^1, \tag{5}$$

in which: u is the displacement component along an assigned direction of the generic point P in a given Cartesian reference system; u^0 is the value of u with correspondence to the reference surface Ω to which correspondence one has $z = 0$; u^1 is an additional variable (u^1 has the geometrical meaning of rotation of the normal to Ω in P).

If FSDT applications are considered, Eq. (5) is retained only for the in-plane components (u_1 coincides to the derivative of the transverse displacement with respect to the in-plane coordinates in the CLT cases). ZZF offers a possibility to introduce ZZ effect in Eq. (5), in fact:

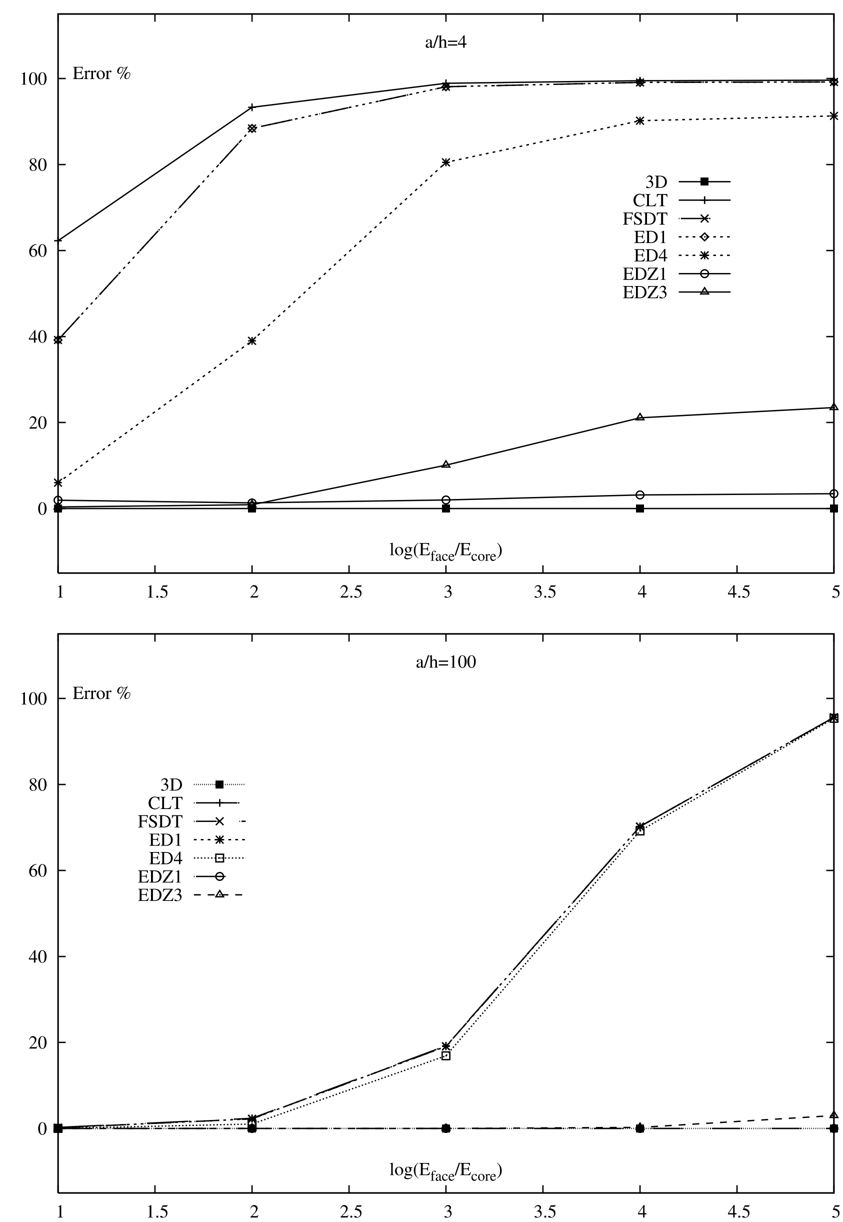

Fig. 3. Error of various theories vs. FCSR for thick (top) and thin (bottom) sandwich panels.

$$u = u^0 + zu^1 + (-1)^k \zeta_k u^Z. \quad (6)$$

The following remarks can be made:

1. The additional degree of freedom u^Z has a meaning of displacement,
2. the amplitude u^Z is layer independent: u^Z has, in fact, an intrinsic equivalent single layer description. At a first glance this fact could appear as a strong limitation of ZZF. In reality ZZF does not differ from other zig-zag theories, such as Ambartsumian Multilayered Theories as well as Lekhnitskii Multilayered Theories, as it has been detailed in reference [1],
3. ZZF can be used for both in-plane and out-of-plane displacement components.

Table 3
Transverse shear stress σ_{xz} evaluation in $z = 0$.

LTR	4		410		100	
FCSR = 10^5						
3D	0.0111		0.1627		16.039	
CLT	0.7045	(>100)	1.7612	(>100)	17.612	(9.81)
FSDT	0.7045	(>100)	1.7612	(>100)	17.612	(9.81)
ED1	0.7045	(>100)	1.7612	(>100)	17.612	(9.81)
ED4	0.6502	(>100)	1.7379	(>100)	17.609	(9.79)
EDZ1	0.0112	(0.90)	0.1627	(0.00)	16.038	(0.01)
EDZ3	0.0082	(26.1)	0.1225	(24.7)	15.537	(3.13)

The percentage error respect to the 3D solution is given in brackets.

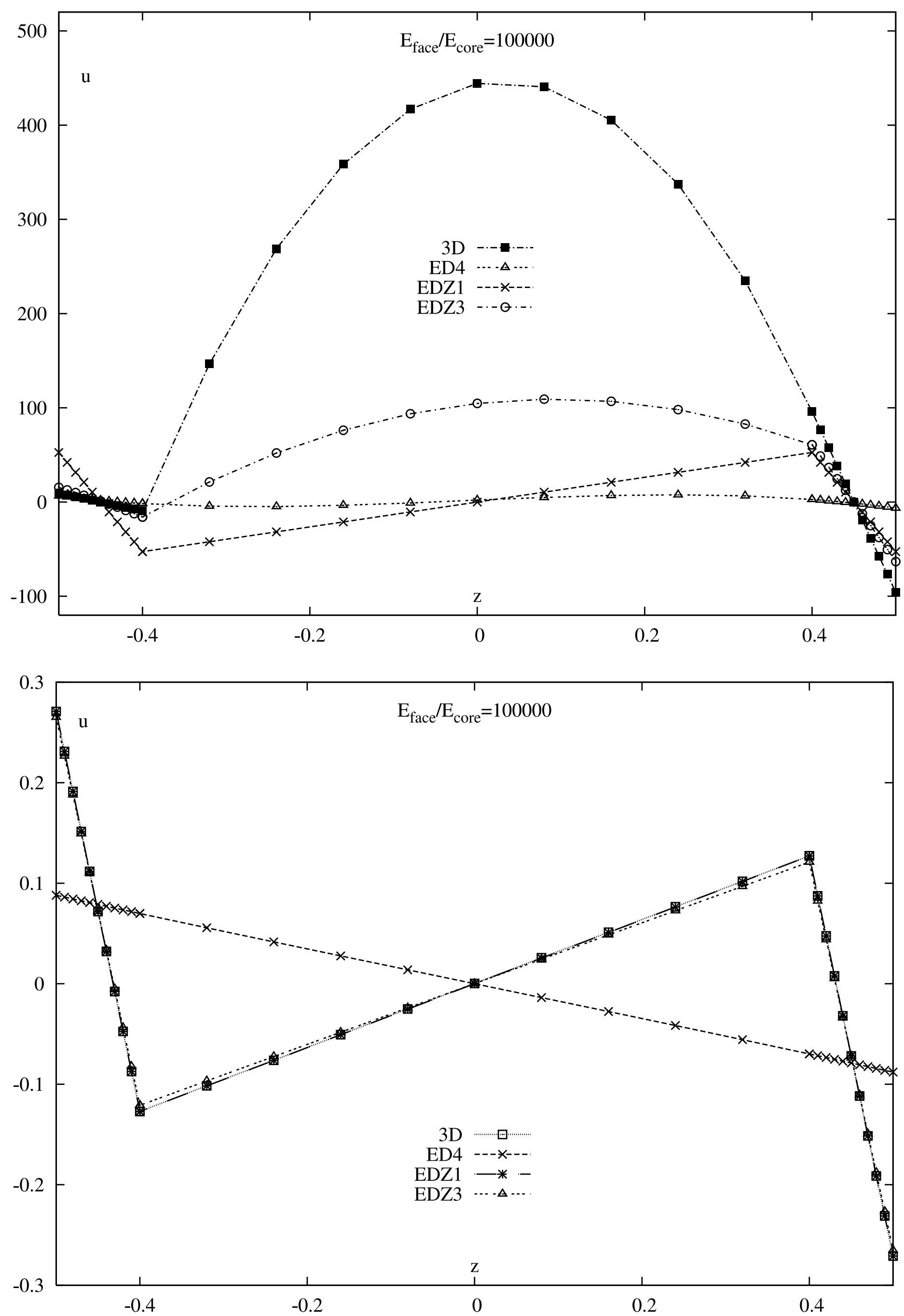

Fig. 4. In-plane displacement distribution of $\bar{u}$ vs. z for thick (top) and thin (bottom) sandwich plates.

3.1. Refinements of FSDT

The application of ZZF to FSDT leads to the following displacement model:

$$u_1 = u_1^0 + zu_1^1 + (-1)^k \zeta_k u_1^Z$$
$$u_2 = u_2^0 + zu_2^1 + (-1)^k \zeta_k u_2^Z \quad (7)$$
$$u_3 = u_3^0.$$

Subscripts 1, 2 and 3 denote displacement components in the three orthogonal directions of a given plate/shell reference system. The third one refers to the thickness, transverse z-direction. The enhanced FSDT model has seven degrees of freedom, two more than classical FSDT.

3.2. Refinement of FSDT by inclusion of ZZ effects and transverse normal strains ε_{zz}

The displacement model which include transverse normal strains as well as ZZ effect in FSDT is,

$$u_1 = u_1^0 + zu_1^1 + (-1)^k \zeta_k u_1^Z$$
$$u_2 = u_2^0 + zu_2^1 + (-1)^k \zeta_k u_2^Z \quad (8)$$
$$u_3 = u_3^0 + zu_3^1 + (-1)^k \zeta_k u_3^Z.$$

The related theories are denoted as ED1 and EDZ1 depending on the inclusion or not of the ZZF, respectively.

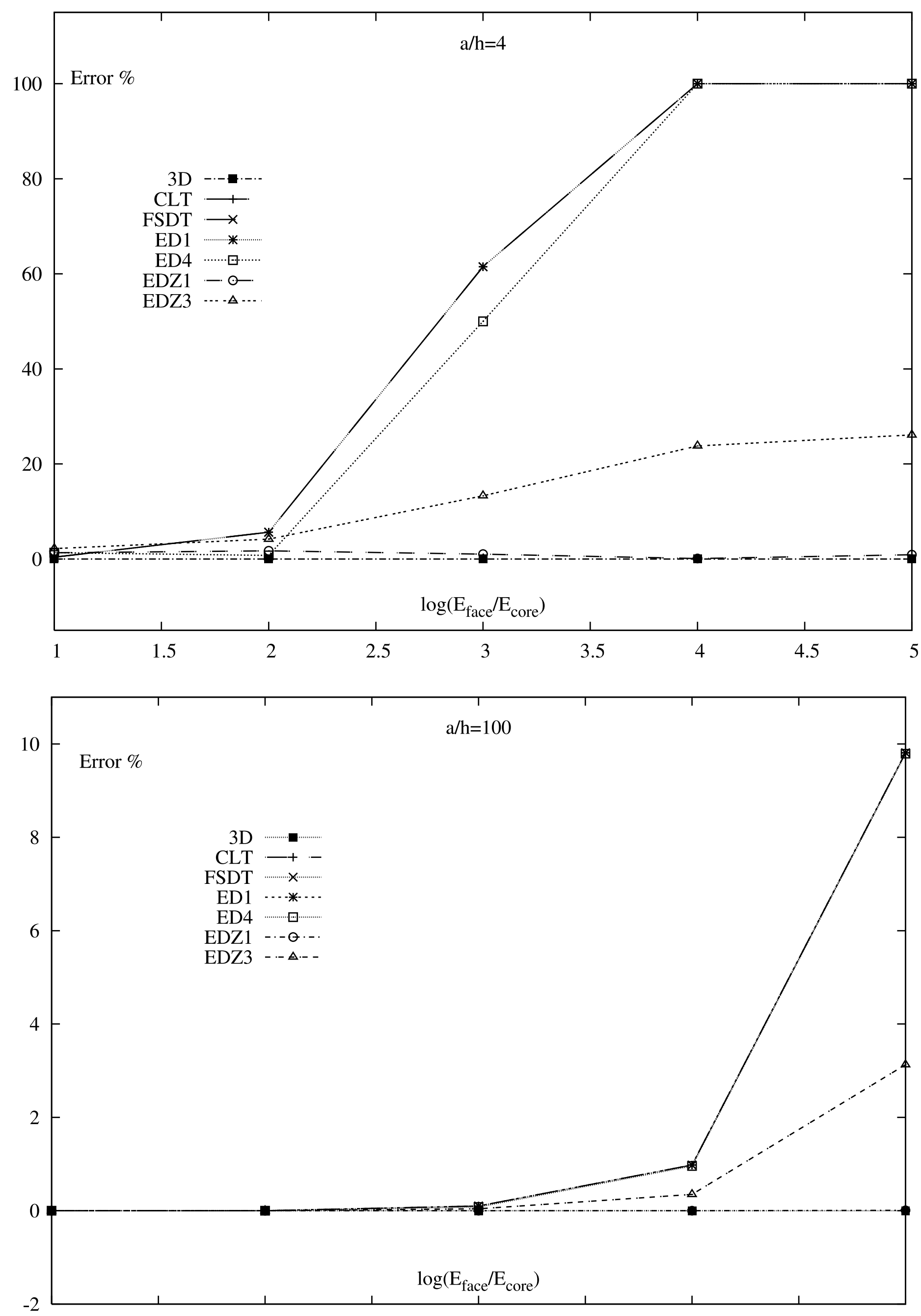

Fig. 5. Errors in transverse shear stress σ_{xz} evaluation vs. FCSR ($\frac{E_{skin}}{E_{core}}$). Thick (top) and thin (bottom) sandwich plates.

3.3. Higher order theories with ZZ function

ZZF can be used to introduce ZZ effect in any HSDT type expansions. The expansions considered in this work make use of power of z polynomials:

$$u_i = u_i^0 + zu_i^1 + z^2u_i^2 + \cdots + z^N u_i^N + (-1)^k \zeta_k u_i^Z, \qquad i = 1, 2, 3. \qquad (9)$$

N is the order of the expansion. The cases N = 1, 2, 3, 4 will be considered in the numerical discussion. The application will refer to two cases, ED4 and EDZ3, that correspond to cases N = 4 and N = 3 (the latter includes ZZF). A summary of the degrees of freedom of the considered theories is given in Table 1. It appears clear that the computational cost of EDZ1 is more less than ED4. The EDZ3 has the same number of d.o.f. of the ED4.

4. Numerical results and discussion

The various models described above have been adopted in the framework of the Carrera's unified formulation CUF [17] for plates and shells which has been detailed in [18]. Closed form solutions are herein discussed for the case of simply supported flat sandwich plates made by isotropic layers and loaded by a transverse distribution of in-plane bisinusoidal pressure. Details of related governing equations and solution procedures are herein omitted. Thickness locking problem has been approached according to the findings in [19]. The attention has been focused to evaluate the effectiveness of ZZ function to improve the bending response of sandwich plates with respect to classical and refined theories that do not make use of ZZ function. CLT, FSDT, and higher order theo-

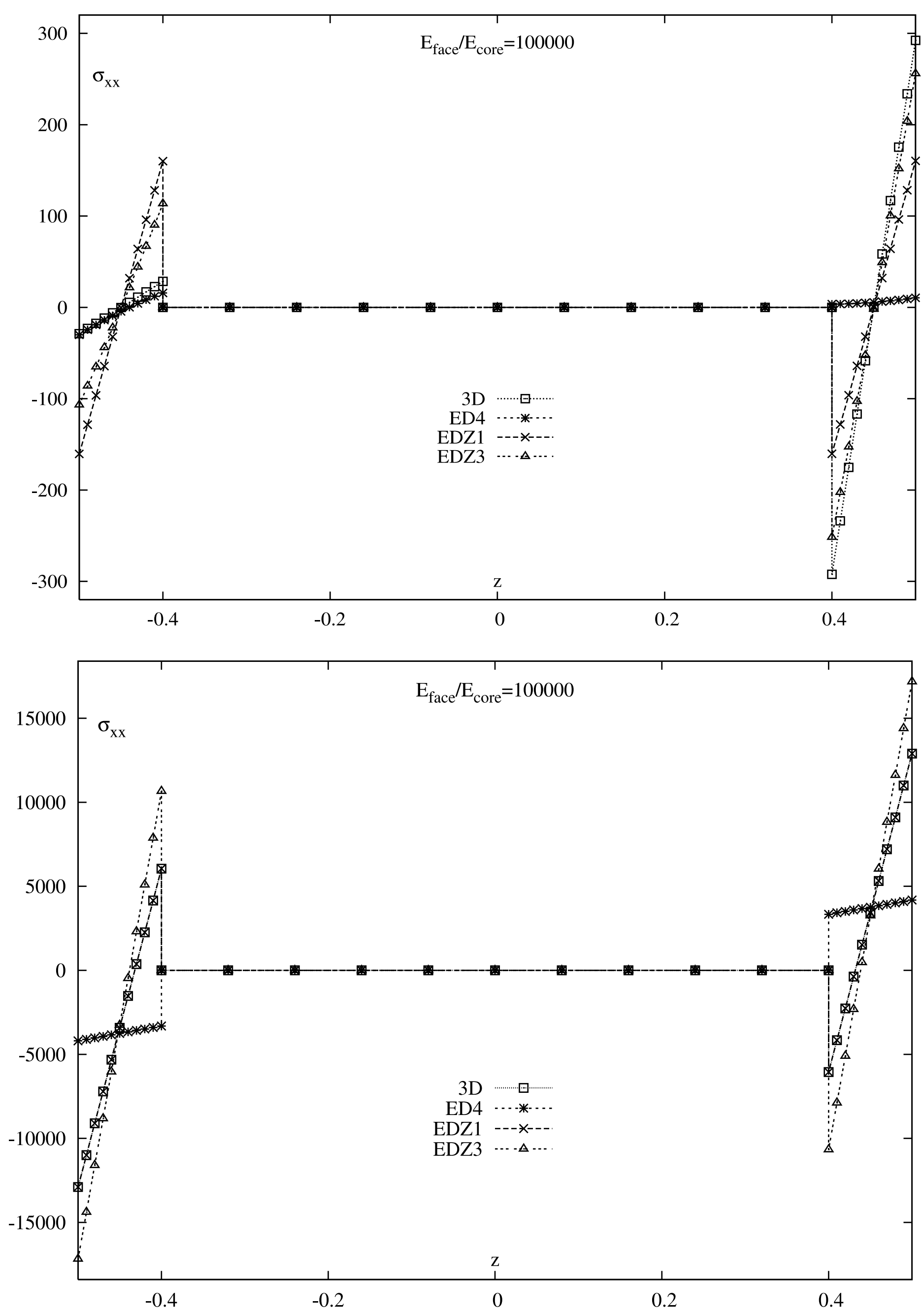

Fig. 6. In-plane stress σ_{xx} evaluation vs. z for thick (top) and thin (bottom) plates.

ries (ED1, ED4) are therefore compared to theories that make use of ZZ functions (EDZ1, EDZ3). Results related to intermediate cases such as ED2, ED3 either EDZ2 have been omitted for sake of brevity. 3D solution has been also provided via layer-wise mixed theories, see [18].

Various values of length-to-thickness ratio (a/h) LTR have been considered, from thick ($a/h = 4$) to thin ($a/h = 1000$) sandwich square plates. The physical reason of zig-zag form for displacements field in the thickness direction is strongly due to the variation of mechanical properties between core and faces. A mechanical parameter FCSR (face-to-core-stiffness-ratio) has been therefore introduced to make evident such a discontinuity. Faces made by aluminum alloy (Al2024, $E = 73{,}000$ [MPa], $\nu = .34$) with thickness $h_f = 1$ [mm] have been considered. A foam core with thickness $h_c = 8$ [mm] and 'isotropic' materials have been addressed; the mechanical properties of the core have been varied from 10^{-1} to 10^{-5}, that is the values FCSR = 10^1, 10^2, 10^3, 10^4, 10^5, have been implemented. Results in tables are given only for the two extreme cases which emulate the two cases of hard core and very soft core, respectively.

Table 2 compares the transverse displacement of the considered theories according to various LTR and FCSR values. The error with respect to quasi 3D solutions has been given. The following interesting conclusions can be made: – CLT, FSDT and ED1–ED4 analyses are quite accurate for the analyses of sandwich plate if and only if low values of FCSR are considered, the error can become quite

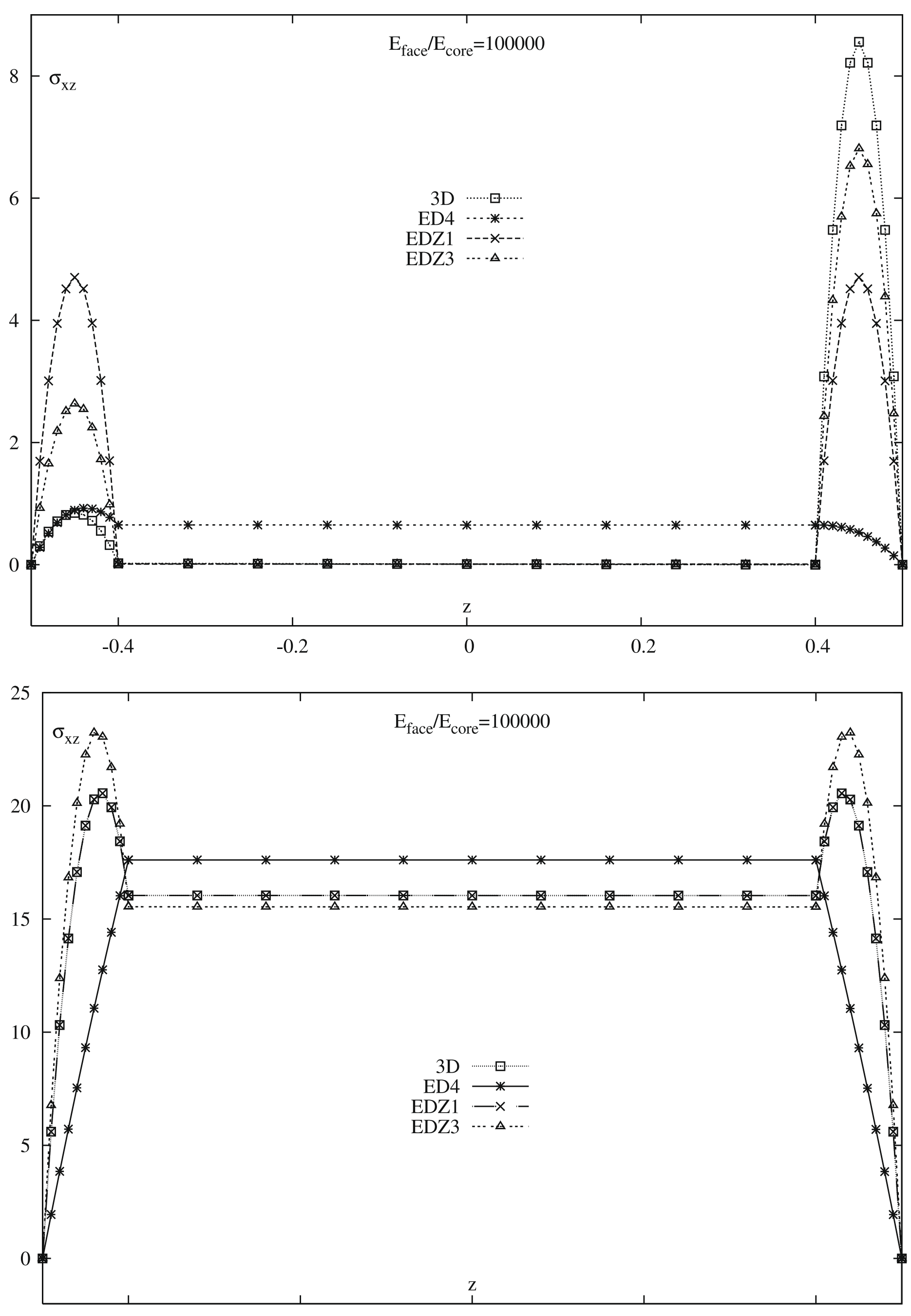

Fig. 7. Transverse shear stress σ_{xz} evaluation vs. z for thick (top) and thin (bottom) plates.

significant in the case of soft core plate; – the use of zig-zag function leads to significant benefits in both thin and thick geometries as well as in the cases of soft and hard core; – the error is reduced from 19.1% of CLT, FSDT, ED1, ED4 theories to .21% and .20% of EDZ1 and EDZ3 analyses, respectively; – significant improvements are also obtained in the case of thick plate geometries. The latter comments clearly show the benefits of using ZZ function in the modelling of sandwich plates. The computational cost of EDZ1 is less than that of ED4 analysis. In other words it appears not convenient to introduce higher orders z^2, z^3 and z^4 in the displacement field since a simple ZZ function leads to much better description. To notice that EDZ1 can lead in some cases to better results than EDZ3 analysis; that is due to the fact that a local value of displacement is compared, which can appear, by chance, better in a specific position such a $z = 0$. That will be made clear below.

Fig. 3 shows the error of various theories for two values of thickness ratio and five values of FCSR. It is once again evident the effectiveness of ZZ function to trace the bending response of sandwich plates. The obtained results have been substantiated by evaluating stress values as well as displacement and stress distributions in the thickness direction. Fig. 4 traces the distribution of in-plane displacement in the thickness sandwich direction in both cases of thick and thin geometries. The capability of zig-zag function to describe the change in slope, as in the 3D solution, with correspondence to face-to-core interfaces is made clearly evident by these two plots. Other higher order theories have no chance to emulated that zig-zag effect. To notice that EDZ3 leads to better description than EDZ1 analysis even though in some point EDZ1 could be, by chance, more accurate. The benefits of using zig-zag function are confirmed by the transverse shear stress evaluation given in Table 3. The error has been traced in Fig. 5. The capability of ZZ function to improve the description of in-plane and transverse shear stress fields in both core and faces is confirmed in Figs. 6 and 7: the high shear stress level in the faces can be only captured by introducing a good description of ZZ phenomena.

5. Conclusions

The present paper has demonstrated the convenience of using the zig-zag function to build higher order sandwich plate theories. The conducted numerical investigation has shown that very significant improvements are obtained by ZZF for both displacement and stress evaluations. The use of EDZ1 theory is high recommended for thin sandwich structures with soft core. Such an improvements are almost independent by the face-to-core-stiffness-ratio; this is not if ZZF is discarded by the sandwich theory. Scientists working in the field of sandwich structures modelling are encouraged to use the ZZF. It is in authors' opinion that they will be surprised by its effectiveness and benefits as well as by the simplicity of its implementation. Future work could be directed to investigates ZZF in dynamic cases, shell geometries as well as unsymmetrically sandwich lay-outs.

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