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# Analysis of thickness locking in classical, refined and mixed theories for layered shells

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**Abstract**

This paper is a sequel of the article: 'Analysis of Thickness Locking in Classical, Refined and Mixed Multilayered Plate Theories', Composite Structures, available online 9th February 2007. The analysis of thickness locking (TL) mechanism, (which is a plate/shell-theory mechanism, caused by the use of simplified kinematic assumptions) is herein extended to shell geometries. Bending problems have been analyzed for isotropic, one-layered and multilayered shells. TL has been investigated for a large variety of shell theories: thin shell theory, first order shear deformation theory, higher order theories, mixed theories and layer-wise theories. The unified formulation has been used to implement the whole considered shell modelings. Analytical closed form solutions have been considered. A comprehensive numerical investigation has been performed. The conclusions have been acquired: TL appears if and only if transverse normal strains $\epsilon_{zz}$ are assumed constant; the use of LW models introduces benefits vs TL; mixed methods do not make any 'relevant' improvements with respect to TL; TL does not depend on geometrical curvature parameters.


*Keywords:* Thickness locking; Poisson locking; Plane-strain conditions: Plane-stress conditions; Layered shells; Classical theories; Advanced theories

## 1. Introduction

Two-dimensional structures can be analyzed as particular case of three-dimensional (3D) continuum by eliminating, via 'a priori' integration, the thickness coordinate $z$. Such integration can be made by following two different methods: the method of asymptotic expansions, see [1,2]; the axiomatic methods, see [3–6]. The latter are the most popular ones: 'intuitive' assumptions are introduced for the behavior of some variables in the $z$-direction. Examples of axiomatic shell theories [7] are so called Love first approximation theories (LFAT) and Love second approximation theories (LSAT). Thickness strain and transverse shear deformation are discarded in LFAT and the shell sections remain plane and orthogonal to a reference surface $\Omega$ during deformation. LSAT are obtained whenever one of the above assumptions is discarded. Examples of LFAT and LSAT are the well known, thin shell theory (TST) and first order shear deformation theory (FSDT), respectively. Many review articles are available on that topic, see [8–11].

As discussed in the companion paper [12] the introduction of axiomatic and/or asymptotic approximations could introduce some unwanted mechanisms which do not appear in the three-dimensional solution. One of these is the thickness locking (TL) which is related to the use of plane-strain/plane-stress hypothesis in thin plate theory. The analysis of thin plate/shell problems is, in fact, often associated to plane-stress assumptions (thin surface problem) while plane-strain hypothesis is usually referred to beam theory (long cylinders) [3,13]. Discussion on plane strain, plane stress and/or plane elasto-static problems can be found also in [14]. However, in most of the applications, TST assumptions are in a 'contradictory manner' made on strain fields. Plane strain assumptions lead to (a thriorthogonal curvilinear reference surface is assumed, $\alpha$

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and $\beta$ are the coordinates on the reference shell surface $\Omega$, normally the shell middle surface, and $z$ is the thickness coordinate, see Fig. 1)

$$\text{plane strain}: \quad \epsilon_{\alpha z} = \epsilon_{\beta z} = \epsilon_{zz} = 0. \tag{1}$$

These last are, in fact, used in-place of the more natural plane-stress conditions

$$\text{plane stress}: \quad \sigma_{\alpha z} = \sigma_{\beta z} = \sigma_{zz} = 0. \tag{2}$$

That contradiction introduces a 'locking mechanism' that makes the plate/shell model not applicable in some cases. Thickness locking (TL, also known as Poisson locking) is the name assigned to that mechanism: TL does not permit to TST analysis to lead to 3D solution in thin shell problems. A known technique to contrast TL consists of modifying the elastic stiffness coefficients by forcing the 'contradictory' conditions

$$\text{transverse normal stress zero condition}: \quad \sigma_{zz} = 0. \tag{3}$$

See the complete discussion reported in the books by Washizu [4], Librescu [5] and Reddy [6] as well as the discussion quoted in Section 3 of [12].

TL is further exhibited by refined shell theories [15,16] that account for a constant through-the-thickness deformation $\epsilon_{zz}$; see the interesting discussion quoted by Bishoff and Ramm [17] and by Kulikov and Plotnikova [18,19]. On this respect, a recent paper by Vu-Quoc and Tan [20], quotes '*To incorporate 3D constitutive laws in shell formulations, the transverse normal strain must have at least a linear distribution over the shell thickness; otherwise the so-called Poisson thickness locking would occur*'. That sentence clearly shows a further manner to overcome TL. It is concluded that *TL has no relation with numerical aspects, it is a 'pure shell theory problem'*.

Although there are many papers that discuss TL, most of them are made in the context finite element FE applications, analytical analysis in terms of differential equations and results for classical and advanced shell theories are not available. The problem has been already discussed in [12] for the case of plate geometries. In this work the analysis is extended to shell cases. Various shell lay-outs (one-layered and multilayered shells made by isotropic or carbon fiber reenforced materials) are analyzed. The TL effect has been investigated on a large variety of shell theories (TST, FSDT, higher order shell theories, mixed and layer-wise theories for laminated structures). The quite exhaustive shell theories treatment has been made by the use of unified formulation (UF) for plate/shell theories that was developed by the first author in earlier papers [21–30]. This paper has been organized as follows: Section 2 describes the addressed shell theories. Numerical results and discussion are provided in Section 3.

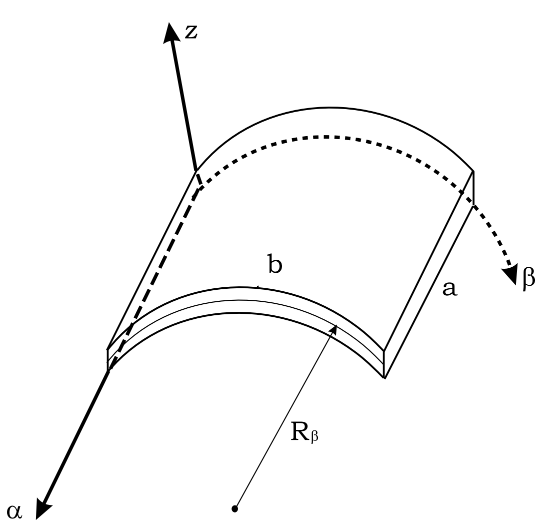


Fig. 1. Geometry and notations of the considered shell.

## 2. Brief outlines of the considered theories

In order to explore in which manner the various kinematic assumptions can affect TL mechanisms, a large variety of shell theories for one-layered and multilayered shells have been considered in the present work. A short discussion is given in the following. That description has been also given in [12] in the plate cases. However, for sake of completeness it is in this work re-made for the shell case. A shell of constant thickness $h$ is considered. The geometry and reference system are shown in Fig. 1. $\alpha$, $\beta$ and $z$ is the Cartesian curvilinear reference system along whose directions the three displacement components $u_\alpha$, $u_\beta$ and $u_z$ are measured. $z$ denotes the thickness direction and $\Omega$ is the shell reference surface located with correspondence to the mid-plane of the shell.

### 2.1. Thin shell theories, TST

Classical thin shell theories (TST) that are based on Cauchy [31], Poisson [32], or Kirchhoff [33] type assumptions, discard transverse shear and trough-the-thickness deformation. The displacement model related to TST can be written in the following form:

$$\begin{aligned} u_s(\alpha,\beta,z) &= u_{0s}(\alpha,\beta) - z\frac{\partial u_{0z}(\alpha,\beta)}{\partial s}, \quad s=\alpha,\beta, \\ u_z(\alpha,\beta,z) &= u_{0z}(\alpha,\beta), \end{aligned} \tag{4}$$

which states that the section remains plane and orthogonal to the shell reference surface $\Omega$. $u_0$ denotes the displacement value with correspondence to the reference surface $\Omega$. TST is also known as classical lamination theory (CLT) in the case of application to laminated composites.

It should be noticed that TST assumptions fulfill exactly the plane strain assumption at Eq. (1).

### 2.2. First order shear deformation theory, FSDT

Transverse shear deformation can be introduced in TST theories according to the following kinematic assumptions, known as Reissner–Mindlin [34,35] theory

$$u_s(\alpha,\beta,z) = u_{0s}(\alpha,\beta) + z u_{1s}(\alpha,\beta), \quad s = \alpha,\beta,$$
$$u_z(\alpha,\beta,z) = u_{0z}(\alpha,\beta), \tag{5}$$

which is also denoted as first order shear deformation theory (FSDT).

It should be noticed that FSDT assumptions fulfill the zero transverse strain condition

$$\text{transverse normal strain zero condition}: \quad \epsilon_{zz} = 0. \tag{6}$$

FSDT does not include any plane-stress/strain assumptions.

### 2.3. Higher order theories, HOT

Refinement of TST and FSDT analysis can be introduced by including higher order terms in the kinematic assumption for the displacements fields

$$u_\tau(\alpha,\beta,z) = u_{0\tau}(\alpha,\beta) + z^i u_{i\tau}(\alpha,\beta), \quad \tau = \alpha,\beta,z,$$
$$i = 1,\ldots,N. \tag{7}$$

The summing convention for repeated indexes has been adopted; $N$ is the order of expansion, which is taken as a free parameter. In this paper, the values $N=1$ up to $N=4$ are considered in the numerical investigation. According to unified formulation, the related theories will be denoted as $ED1$–$ED4$. The letter $E$ denotes that the kinematic is preserved for the whole layers of the shell, as in the so-called equivalent single layer (ESL). $D$ denotes that only displacement unknowns are used and the last number denote the order of the expansion in $z$.

To be noticed that all stress and strain tensor components are different from zero in that case. $ED2$ case should be therefore free from thickness locking.

### 2.4. Theories with selective order of expansion for in-plane and out-of-plane displacements

It is of particular interest to consider shell theories which are based on a different order of expansion for in-plane and out-of-plane displacements

$$u_s(\alpha,\beta,z) = u_{0s}(\alpha,\beta) + z^i u_{is}(\alpha,\beta), \quad s = \alpha,\beta, \quad i = 1,\ldots,N_p,$$
$$u_z(\alpha,\beta,z) = u_{0z}(\alpha,\beta) + z^i u_{iz}(\alpha,\beta), \quad i = 1,\ldots,N_w. \tag{8}$$

$N_p$ and $N_w$ refer to the order of the expansion for in-plane and out-of-plane displacement components, respectively. $N_p = 1$ and $N_w = 0$ in both TST and FSDT cases.

### 2.5. Layer-wise theories, LWT

Multilayered shells can be analyzed by kinematic assumptions which are independent in each layer. According to [6] these approaches have herein stated as layer-wise theories. Layer-wise description requires assuming independent displacement variables in each $k$-layer. The Taylor thickness expansion used for ESL cases of the previous paragraphs is not convenient for layer-wise description. Interlaminar continuity for displacements can be more conveniently imposed by employing interface values as unknown variables. Therefore, layer-wise description is written according to the following expansion:

$$u_\tau^k = F_t u_{\tau t}^k + F_b u_{\tau b}^k + F_r u_{\tau r}^k, \quad \tau = \alpha,\beta,z,$$
$$r = 2,3,\ldots,N, \qquad k = 1,2,\ldots,N_l. \tag{9}$$

$N_l$ denotes the number of layers. It is intended that the subscripts t and b denote values related to the layer top and bottom surface, respectively. The thickness functions $F_\tau(\zeta_k)$ have been defined by

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2},$$
$$r = 2,3,\ldots,N, \tag{10}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the $j$th-order defined in the $\zeta_k$-domain: $-1 \leqslant \zeta_k \leqslant 1$. Fourth order case will be used in the numerical investigations; related polynomials are

$$P_0 = 1, \quad P_1 = \zeta_k, \quad P_2 = (3\zeta_k^2 - 1)/2,$$
$$P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2}, \quad P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8}.$$

The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1: & F_t = 1; \quad F_b = 0; \quad F_r = 0, \\ -1: & F_t = 0; \quad F_b = 1; \quad F_r = 0. \end{cases} \tag{11}$$

The top and bottom values have been used as unknown variables. The interlaminar compatibility of displacement can be therefore easily linked

$$u_{\tau t}^k = u_{\tau b}^{(k+1)}, \quad k = 1, \quad N_l - 1. \tag{12}$$

### 2.6. Layer-wise mixed theories based on Reissner's mixed variational theorem

It is well known fact, [11] that the kinematic above described, if applied to multilayered structures, are not able to furnish interlaminar continuous transverse shear and normal stresses at the interface between two adjacent layers. Reissner's mixed variational theorem [26] offers a possibility to fulfill 'a priori' such an interlaminar continuity. Both displacements and transverse shear and normal stresses can be assumed in the RMVT framework. In the layer-wise case the displacement model at Eq. (9) is also used for the transverse stress variables

$$\sigma_{\tau z}^k = F_t \sigma_{\tau z t}^k + F_b \sigma_{\tau z b}^k + F_r \sigma_{\tau z r}^k, \quad \tau = \alpha,\beta,z,$$
$$r = 2,3,\ldots,N, \quad k = 1,2,\ldots,N_l. \tag{13}$$

The interlaminar transverse shear and normal stress continuity can be therefore easily linked

$$\sigma_{\tau z t}^k = \sigma_{\tau z b}^{(k+1)}, \quad \tau = \alpha,\beta,z, \quad k = 1, \quad N_l - 1. \tag{14}$$

These shell theories will be denoted as $LM1$–$LM4$.

### 2.7. Unified formulation

The previous models have been coded according to the unified formulation. Details can be found in previous first author's works [21–30].

## 3. Results and discussion

As stated in the introduction, the 'plane-strain' kinematic related to Eq. (4) originates TL in the case of thin shell analysis. To underline that fact, kinematic of Eq. (4) is denoted as $TST_{\epsilon_{zz}}$. Most of the well known authors and books suggest the modification of material elastic coefficients by forcing the zero transverse stress conditions at Eq. (3). The related theory is herein denoted by the acronym $TST_{\sigma_{zz}}$. The same subscripts will be used for FSDT and $ED1$ theories.

Hence transverse shear effects are neglected in a TST, the plane-stress condition at Eq. (2) coincides to the pure zero transverse normal stress conditions Eq. (3). The notation 'plane-stress', is therefore inappropriate in the FSDT cases. It appears more correct to refer to the zero-transverse normal condition at Eq. (3) even though TST is addressed.

Three remedies to TL were discussed in [12]: (1) the use of higher order kinematic; (2) the modification of elastic coefficients; and (3) the use of penalty number to impose zero-transverse strain condition. Only the first two remedies will be analyzed in this paper. Concerning method three the conclusions reached for plates [12] have been confirmed for shell geometries. These last, for sake of brevity have not been documented in the present work.

Table 1
TL effect in thin plate theories TPT

| $a/h$ | $TPT_{\epsilon_{zz}}$ | Err.% | $TPT_{\sigma_{zz}}$ | Err.% | 3D($LM4$) |
|---|---|---|---|---|---|
| $v = 0.34$ | | | | | |
| 2 | 2.0010 | (66.7) | 2.7238 | (54.7) | 6.0107 |
| 4 | 2.0009 | (44.3) | 2.7238 | (24.2) | 3.5958 |
| 10 | 2.0009 | (30.2) | 2.7238 | (4.94) | 2.8655 |
| 100 | 2.0009 | (26.6) | 2.7238 | (0.05) | 2.7252 |
| 1000 | 2.0009 | (26.5) | 2.7238 | (0.00) | 2.7238 |

Comparison of plane-strain and plane-stress results vs 3D solutions. Bending problem, $\overline{U}_z = U_z \frac{100Eh^3}{\bar{p}_z a^4}$ in $z = 0$, of a isotropic plate; $m, n = 1, 1$. This table is taken from [12].

Navier-type, closed form solutions of simply supported transversely isotropic cylindrical shell panels have been considered. Readers are addressed to previous works [22–30] to find details of the implemented unified formulation and solution procedure.

A large numerical investigation has been conducted to compare the various theories that have been illustrated in the previous section. Only selected results are discussed in this section.

Simply supported one-layered and multilayered cylindrical shell panels made by isotropic or carbon fiber reenforced materials have been considered. A bi-sinusoidal distribution of transverse pressure applied at the top shell-surface has been analyzed (cylindrical bending problem)

$$p_z(\beta) = \bar{p}_z \sin\left(\frac{n\pi\beta}{b}\right), \tag{15}$$

$\bar{p}_z$ is the applied load amplitude and $n$ is the wave number in the $\beta$ shell direction, $b$ is the corresponding panel dimensions. Attention has been restricted to the case $n = 1$. Transverse displacement amplitude $U_z$ will be compared.

Table 3
Isotropic one-layered shell

| $R_\beta/h$ | $TST_{\epsilon_{zz}}$ | Err.% | $TST_{\sigma_{zz}}$ | Err.% | 3D($LM4$) |
|---|---|---|---|---|---|
| $v = 0.15$ | | | | | |
| 2 | 2.1358 | (37.4) | 2.2045 | (35.4) | 3.4121 |
| 4 | 1.9789 | (15.4) | 2.0425 | (12.6) | 2.3383 |
| 10 | 1.8617 | (5.37) | 1.9216 | (2.33) | 1.9674 |
| 100 | 1.7846 | (3.14) | 1.8419 | (0.03) | 1.8424 |
| 1000 | 1.7767 | (3.11) | 1.8337 | (0.00) | 1.8337 |
| $v = 0.34$ | | | | | |
| 2 | 1.4652 | (52.4) | 1.9945 | (35.2) | 3.0776 |
| 4 | 1.3576 | (35.8) | 1.8480 | (12.6) | 2.1132 |
| 10 | 1.2772 | (28.2) | 1.7385 | (2.32) | 1.7798 |
| 100 | 1.2242 | (26.5) | 1.6665 | (0.02) | 1.6669 |
| 1000 | 1.2188 | (26.5) | 1.6591 | (0.00) | 1.6591 |
| $v = 0.45$ | | | | | |
| 2 | 0.5946 | (78.7) | 1.7985 | (35.7) | 2.7967 |
| 4 | 0.5509 | (71.2) | 1.6664 | (12.8) | 1.9104 |
| 10 | 0.5182 | (67.7) | 1.5677 | (2.36) | 1.6057 |
| 100 | 0.4968 | (66.9) | 1.5028 | (0.02) | 1.5031 |
| 1000 | 0.4946 | (66.9) | 1.4961 | (0.00) | 1.4961 |

Poisson's modulus effects on TL. Comparison between plane-strain and plane-stress assumption on transverse displacements $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 2
Isotropic one-layered shell

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D($LM4$) | 2.1132 | | 1.7798 | | 1.6763 | | 1.6669 | |
| $TST_{\epsilon_{zz}}$ | 1.3576 | (35.8) | 1.2772 | (28.2) | 1.2303 | (26.6) | 1.2242 | (26.5) |
| $TST_{\sigma_{zz}}$ | 1.8480 | (12.5) | 1.7385 | (2.32) | 1.6747 | (0.09) | 1.6665 | (0.02) |
| $FSDT_{\epsilon_{zz}}$ | 1.6212 | (23.3) | 1.3167 | (26.0) | 1.2318 | (26.5) | 1.2246 | (26.5) |
| $FSDT_{\sigma_{zz}}$ | 2.1116 | (0.07) | 1.7781 | (0.09) | 1.6762 | (0.00) | 1.6669 | (0.00) |

TL Effect in shell theories TST and FSDT. Comparison of plane-strain and plane-stress results vs 3D solutions. Bending problem, $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 4
Isotropic one-layered shell

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D($LM4$) | 2.1132 | | 1.7798 | | 1.6763 | | 1.6669 | |
| $ED1_{\epsilon_{zz}}$ | 1.6196 | (23.4) | 1.3175 | (26.0) | 1.2316 | (26.5) | 1.2246 | (26.5) |
| $ED1_{\sigma_{zz}}$ | 2.1098 | (0.16) | 1.7790 | (0.04) | 1.6763 | (0.00) | 1.6669 | (0.00) |
| $ED2$ | 2.0682 | (2.13) | 1.7727 | (0.40) | 1.6761 | (0.12) | 1.6668 | (0.00) |
| $ED3$ | 2.1128 | (0.02) | 1.7797 | (0.00) | 1.6763 | (0.00) | 1.6669 | (0.00) |
| $ED4$ | 2.1132 | (0.00) | 1.7798 | (0.00) | 1.6763 | (0.00) | 1.6669 | (0.00) |

Effect of TL on HOT (EDN). Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 5
Isotropic one-layered shell

| $R_\beta/h$ | 4 | 10 | 50 | 100 |
|---|---|---|---|---|
| 3D($LM4$) | 2.1132 | 1.7798 | 1.6763 | 1.6669 |
| $\text{TST}_{\epsilon_{zz}}$ | 1.3576 | 1.2772 | 1.2303 | 1.2242 |
| $\text{TST}_{\sigma_{zz}}$ | 1.8480 | 1.7385 | 1.6747 | 1.6665 |
| $N_p = 4,\ N_w = 4$ | 2.1132 | 1.7798 | 1.6763 | 1.6669 |
| $N_p = 4,\ N_w = 3$ | 2.1132 | 1.7798 | 1.6763 | 1.6669 |
| $N_p = 4,\ N_w = 2$ | 2.1043 | 1.7782 | 1.6763 | 1.6669 |
| $N_p = 4,\ N_w = 1$ | 1.6713 | 1.3254 | 1.2322 | 1.2247 |
| $N_p = 4,\ N_w = 0$ | 1.6730 | 1.3246 | 1.2321 | 1.2247 |

Selective order of expansion $N_p$ and $N_w$ influence on TL. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Three multilayered configurations have been treated to investigate the effect of lay-outs on TL mechanisms. First the isotropic one-layered shell has been addressed. The material consists of 2024-T6 aluminum alloys ($E = 73{,}000$ MPa, $\nu = .34$). The length has been fixed to the value $b = 10.471963$, $R_\alpha = \infty$, $R_\beta = 10$ and $b/R_\beta = \pi/3$ as in [25].

Table 1, taken from [12] compares the plane-strain and plane-stress results in the case of plate. The transverse displacement not-dimensioned amplitude $\overline{U}_z$ has been compared in the whole analyses. TL phenomena is very much evident for thin geometries: plane strain results $\text{TPT}_{\epsilon_{zz}}$ leads to about 26% error with respect to 3D solution. Table 2 shows correspondent shell results. FSDT analysis has been also considered. 3D solutions have been recovered by using $LM4$ shell models as in [12]. Thin/thick shells are obtained by varying the shell panel thickness $h$. A first conclusion is obtained: by shell thickness decreasing TL leads to the

Table 6
Isotropic one-layered shell

| | $\frac{b}{h} = 4.19$ | Err.% | $\frac{b}{h} = 10.47$ | Err.% | $\frac{b}{h} = 104.72$ | Err.% |
|---|---|---|---|---|---|---|
| | $b/R_\beta = \pi$ | | | | | |
| 3D($LM4$) | 171.17 | | 144.16 | | 135.02 | |
| $\text{TST}_{\epsilon_{zz}}$ | 109.96 | (35.8) | 103.45 | (28.2) | 99.165 | (26.5) |
| $\text{TST}_{\sigma_{zz}}$ | 149.69 | (12.5) | 140.82 | (2.32) | 134.99 | (0.02) |
| $\text{FSDT}_{\epsilon_{zz}}$ | 131.32 | (23.3) | 106.65 | (26.0) | 99.105 | (26.6) |
| $\text{FSDT}_{\sigma_{zz}}$ | 171.04 | (0.07) | 144.03 | (0.09) | 135.02 | (0.00) |
| $ED3$ | 171.14 | (0.02) | 144.16 | (0.00) | 135.02 | (0.00) |
| | $b/R_\beta = \pi/6$ | | | | | |
| 3D($LM4$) | 0.1321 | | 0.1112 | | 0.1042 | |
| $\text{TST}_{\epsilon_{zz}}$ | 0.0848 | (35.8) | 0.0798 | (28.2) | 0.0765 | (26.6) |
| $\text{TST}_{\sigma_{zz}}$ | 0.1155 | (12.6) | 0.1087 | (2.25) | 0.1042 | (0.00) |
| $\text{FSDT}_{\epsilon_{zz}}$ | 0.1013 | (23.3) | 0.0823 | (26.0) | 0.0765 | (26.6) |
| $\text{FSDT}_{\sigma_{zz}}$ | 0.1319 | (0.15) | 0.1113 | (0.09) | 0.1042 | (0.00) |
| $ED3$ | 0.1320 | (0.07) | 0.1112 | (0.00) | 0.1042 | (0.00) |
| | $b/R_\beta = \pi/3$ | | | | | |
| 3D($LM4$) | 2.1132 | | 1.7798 | | 1.6669 | |
| $\text{TST}_{\epsilon_{zz}}$ | 1.3576 | (35.8) | 1.2772 | (28.2) | 1.2242 | (26.5) |
| $\text{TST}_{\sigma_{zz}}$ | 1.8480 | (12.5) | 1.7385 | (2.32) | 1.6665 | (0.02) |
| $\text{FSDT}_{\epsilon_{zz}}$ | 1.6212 | (23.3) | 1.3167 | (26.0) | 1.2246 | (26.5) |
| $\text{FSDT}_{\sigma_{zz}}$ | 2.1116 | (0.07) | 1.7781 | (0.09) | 1.6669 | (0.00) |
| $ED3$ | 2.1128 | (0.02) | 1.7797 | (0.00) | 1.6669 | (0.00) |
| | $b/R_\beta = \pi/2$ | | | | | |
| 3D($LM4$) | 10.698 | | 9.0100 | | 8.4388 | |
| $\text{TST}_{\epsilon_{zz}}$ | 6.8728 | (35.7) | 6.4657 | (28.2) | 6.1978 | (26.5) |
| $\text{TST}_{\sigma_{zz}}$ | 9.3556 | (12.5) | 8.8014 | (2.31) | 8.4367 | (0.02) |
| $\text{FSDT}_{\epsilon_{zz}}$ | 8.2072 | (23.3) | 6.6658 | (26.0) | 6.1997 | (26.5) |
| $\text{FSDT}_{\sigma_{zz}}$ | 10.690 | (0.07) | 9.0016 | (0.09) | 8.4386 | (0.00) |
| $ED3$ | 10.696 | (0.02) | 9.0100 | (0.00) | 8.4388 | (0.00) |

Effect of curvature parameter $b/R_\beta$ on TL for transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 7
Isotropic three-layered shell

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D($LM4$) | 2.1285 | | 1.7436 | | 1.6255 | | 1.6145 | |
| $TST_{\epsilon_{zz}}$ | 1.4274 | (32.9) | 1.3171 | (24.5) | 1.2555 | (22.8) | 1.2478 | (22.7) |
| $TST_{\sigma_{zz}}$ | 1.8554 | (12.8) | 1.7073 | (2.08) | 1.6252 | (0.02) | 1.6148 | (0.02) |
| $FSDT_{\epsilon_{zz}}$ | 1.6627 | (21.9) | 1.3511 | (22.5) | 1.2568 | (22.7) | 1.2481 | (22.7) |
| $FSDT_{\sigma_{zz}}$ | 2.0906 | (1.78) | 1.7413 | (0.13) | 1.6264 | (0.05) | 1.6151 | (0.04) |

Evaluation of TL mechanism in TST and FSDT cases. The Young's modulus of Titanium has been used to compute $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 8
Isotropic three-layered shell

| $R_\beta/h$ | 4 | 10 | 50 | 100 |
|---|---|---|---|---|
| 3D($LM4$) | 2.1285 | 1.7436 | 1.6255 | 1.6145 |
| $TST_{\epsilon_{zz}}$ | 1.4274 | 1.3171 | 1.2555 | 1.2478 |
| $TST_{\sigma_{zz}}$ | 1.8554 | 1.7073 | 1.6252 | 1.6148 |
| $N_p = 4,\ N_w = 4$ | 2.1049 | 1.7361 | 1.6237 | 1.6134 |
| $N_p = 4,\ N_w = 3$ | 2.1044 | 1.7360 | 1.6236 | 1.6133 |
| $N_p = 4,\ N_w = 2$ | 2.0907 | 1.7300 | 1.6212 | 1.6112 |
| $N_p = 4,\ N_w = 1$ | 1.7268 | 1.3611 | 1.2580 | 1.2491 |
| $N_p = 4,\ N_w = 0$ | 1.7296 | 1.3611 | 1.2572 | 1.2482 |

Analysis of selective order of expansion $N_p$, $N_w$ on TL. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

same error of plate cases. That is TL of thin shells behaves as in thin plates.

The strong dependency of TL by Poisson's ratio is clearly shown in Table 3. A reduction of TL is, in fact, obtained in the $\nu = .15$ case; on the contrary, TL increases very much for $\nu = .45$.

Higher order shell theories are compared in Table 4. TL disappear if and only if at least quadratic distribution in $z$ is assumed for the displacements. A comparison of different assumptions for the expansion used for in-plane $N_p$ and out-of-plane $N_w$ displacements is made in Table 5. The obtained results confirm that quadratic distribution of transverse displacement $u_z$ is enough to remove TL from shell theories.

To evaluate the influence of curvature parameter on shell theories, Table 6 considers panels with the same length ($b = 10.471963$) with three values of thickness ratio ($b/h = 4.19, 10.47, 104.72$) and four values of the curvature parameter ($b/R = \pi/6, \pi/3, \pi/2, \pi$). Panels with equal thickness ratio and different curvatures can be compared. It appears clear that TL is not influenced by the curvature parameter but only by thickness parameter.

Multilayered shells made by three isotropic layers have been considered in the subsequent tables. The mechanical properties of the three materials are: (1) first layer (Al 2024) $E = 73{,}000$ MPa, $\nu = .34$; (2) second layer (Titanium) $E = 114{,}000$ MPa, $\nu = .3$; and (3) third layer (Iron) $E = 210{,}000$ MPa, $\nu = .3$. Table 7 shows that a reduction of TL is obtained by considering layers with different Poisson's ratio. Table 8 confirms the effect of varying $N_w$ order of expansion as in Table 5 case. Higher order theories are examined in Table 9. It is confirmed that HOT with higher than linear transverse displacement fields does not exhibit any TL mechanism. Layer-wise results, both classical and mixed ones, are given in Table 10. It is clearly shown that layer-wise description, both classical and mixed ones, introduces benefits vs TL mechanisms: the piece-wise continuous displacement fields strongly reduces TL.

Table 9
Isotropic three-layered shell

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D($LM4$) | 2.1285 | | 1.7436 | | 1.6255 | | 1.6145 | |
| $ED1_{\epsilon_{zz}}$ | 1.6603 | (22.0) | 1.3510 | (22.5) | 1.2576 | (22.6) | 1.2491 | (22.6) |
| $ED1_{\sigma_{zz}}$ | 2.0880 | (1.90) | 1.7429 | (0.04) | 1.6305 | (0.31) | 1.6196 | (0.31) |
| $ED4$ | 2.1049 | (1.11) | 1.7361 | (0.43) | 1.6237 | (0.11) | 1.6134 | (0.07) |

Analysis of HOT vs TL. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 10
Isotropic three-layered shell

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D($LM4$) | 2.1285 | | 1.7436 | | 1.6255 | | 1.6145 | |
| $LD1$ | 2.0531 | (3.54) | 1.6899 | (3.08) | 1.5772 | (2.97) | 1.5666 | (2.97) |
| $LD4$ | 2.1285 | (0.00) | 1.7436 | (0.00) | 1.6255 | (0.00) | 1.6145 | (0.00) |
| $LM1$ | 2.1136 | (0.70) | 1.7358 | (0.45) | 1.6191 | (0.39) | 1.6082 | (0.39) |
| $LM4$ | 2.1285 | (0.00) | 1.7436 | (0.00) | 1.6255 | (0.00) | 1.6145 | (0.00) |

Analysis of TL on classical ($LD1$, $LD4$) and mixed ($LM1$, $LM4$) layer-wise shell theories. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 11
Cross ply multilayered shell [90/0/90]

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D([36]) | 0.4570 | | 0.1440 | | 0.0808 | | 0.0787 | |
| $\mathrm{TST}_{\epsilon_{zz}}$ | 0.0862 | (81.1) | 0.0810 | (43.7) | 0.0780 | (3.46) | 0.0776 | (1.40) |
| $\mathrm{TST}_{\sigma_{zz}}$ | 0.0865 | (81.0) | 0.0813 | (43.5) | 0.0783 | (3.09) | 0.0780 | (0.89) |
| $\mathrm{FSDT}_{\epsilon_{zz}}$ | 0.3314 | (27.5) | 0.1179 | (18.1) | 0.0794 | (1.73) | 0.0780 | (0.89) |
| $\mathrm{FSDT}_{\sigma_{zz}}$ | 0.3318 | (27.4) | 0.1182 | (17.9) | 0.0798 | (1.24) | 0.0783 | (0.51) |

Evaluation of TL mechanism in TST and FSDT cases. Transverse displacement $\overline{U}_z = U_z \frac{10E_T h^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 12
Cross ply multilayered shell [90/0/90]

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D([36]) | 0.4570 | | 0.1440 | | 0.0808 | | 0.0787 | |
| $ED1_{\epsilon_{zz}}$ | 0.3292 | (28.0) | 0.1187 | (17.6) | 0.0795 | (1.61) | 0.0780 | (0.89) |
| $ED1_{\sigma_{zz}}$ | 0.3296 | (27.9) | 0.1191 | (17.3) | 0.0798 | (1.24) | 0.0783 | (0.51) |
| $ED4$ | 0.4232 | (7.40) | 0.1363 | (5.35) | 0.0805 | (0.37) | 0.0785 | (0.25) |

Analysis of HOT vs TL. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Table 13
Cross ply multilayered shell [90/0/90]

| $R_\beta/h$ | 4 | Err.% | 10 | Err.% | 50 | Err.% | 100 | Err.% |
|---|---|---|---|---|---|---|---|---|
| 3D([36]) | 0.4570 | | 0.1440 | | 0.0808 | | 0.0787 | |
| $LD1$ | 0.4407 | (3.57) | 0.1408 | (2.22) | 0.0807 | (0.12) | 0.0785 | (0.25) |
| $LD4$ | 0.4581 | (0.24) | 0.1440 | (0.00) | 0.0808 | (0.00) | 0.0787 | (0.00) |
| $LM1$ | 0.4424 | (3.19) | 0.1425 | (1.04) | 0.0808 | (0.00) | 0.0786 | (0.13) |
| $LM4$ | 0.4581 | (0.24) | 0.1440 | (0.00) | 0.0808 | (0.00) | 0.0787 | (0.00) |

Analysis of TL on classical ($LD1$, $LD4$) and mixed ($LM1$, $LM4$) layer-wise shell theories for a three-layered shell. Transverse displacement $\overline{U}_z = U_z \frac{10Eh^3}{\bar{p}_z R_\beta^4}$ in $z = 0$.

Finally, a multilayered cross-ply shell panel made by composite othotropic layers has been analyzed in Tables 11–13. 3D solution to that problem has been provided by Ren [36]. The three layers with fiber orientation [90/0/90] have the following properties: $E_L = 25$ Psi, $E_T = E_z = 1$ Psi, $G_{LT} = 0.5$ Psi, $G_{Lz} = 0.2$ Psi and $\nu_{LT} = \nu_{Lz} = 0.25$. The results in Tables 11 and 13 confirm those already obtained for multilayered shells made by isotropic layers: classical, refined and layer-wise theories are compared to the 3D solution. Furthermore, as it was found in the plate cases [12], the high orthotropic ratio of used composite materials, makes TL mechanism less dangerous with respect to the case of shells made by isotropic layers.

Further analyses, which are not documented herein, have been conducted to the dynamic response. Conclusions already reached for plates [12] have been confirmed for the present shell geometries.

## 4. Conclusions

This paper has presented a numerical investigation on Thickness Locking in classical, refined and advanced mixed theories for one-layered and multilayered shells. The following conclusions have been confirmed, see also [12]:

- It is confirmed that TL consists of a shell theory problem; it has no relation with numerical methods, such as FE approximations.
- The use of $\sigma_{zz} = 0$ condition appears a suitable technique to contrast TL in thin-shell analysis. It is very effective to contrast TL in the case of TST and FSDT applications.
- $\sigma_{zz} = 0$ preserves its advantages if applied to shell theories (HOT) with linear distribution of transverse displacement field $u_z$.
- TL appears if and only if a shell theory shows a constant distribution of transverse normal strain $\epsilon_{zz}$; that is to avoid TL the shell theories would require at least a parabolic distribution of transverse displacement component $u_z$.
- TL can be contrasted by the use of layer-wise shell theories.
- Mixed theories for multilayered structures do not introduce significant improvement as far as TL is concerned.
- Curvature parameters have no influence on TL which is purely dominated by shell thickness parameter.

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