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Analysis of thickness locking in classical, refined and mixed multilayered plate theories

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Abstract

This paper discusses the thickness locking (TL) mechanism, also known as Poisson locking, which is caused by the use of simplified kinematic assumptions in the plate analysis. Bending and vibration problems have been analyzed for isotropic, orthotropic and multilayered, composite plates. TL has been investigated for a large variety of plate theories: thin plate theory (TPT), First order shear deformation theory (FSDT), higher order theories (HOT), mixed theories and layer-wise (LW) theories. Transverse normal stress σ_{zz} and strain ϵ_{zz} zero conditions are discussed. Penalty numbers have been introduced to force $\epsilon_{zz}=0$ condition in the three-dimensional solution and refined plate theories. The unified formulation has been used to implement the whole considered plate modelings.

Analytical closed form solutions have been considered. A comprehensive numerical investigation has been performed. The following main conclusions have been acquired. (1) TL is strongly due to the coupling between transverse normal strain and in-plane strain in the constitutive law (Poisson effect). (2) TL appears if and only if transverse normal strains ϵ_{zz} are assumed constant in the thickness directions (that happens for TPT, FSDT and HOT with constant and linear transverse displacement expansion in the thickness direction). (3) TL can lead to large error (about 25% for deflections and 15% for circular frequency) in thin, isotropic plate analysis. (4) TL reduces significantly in orthotropic and laminated plates. (5) The use of LW models introduces benefits vs TL. (6) Mixed methods do not make any improvements with respect TL. (7) Penalties technique on elastic coefficients can be efficiently used to enforce $\epsilon_{zz}=0$ conditions in 3D solutions as well as in HOT, mixed and layer-wise plate theories.

Keywords: Thickness locking; Poisson locking; Plane-strain conditions; Plane-stress conditions; 3D constitutive equations; Higher order theories; Mixed theories

1. Introduction

Two-dimensional plate structures can be analyzed as particular case of three-dimensional (3D) continuum by eliminating, via ‘a priori’ integration, the thickness coordinate z. Such integration can be made by following two different methods: the method of asymptotic expansions, see [1,2]; the axiomatic methods, see [3–6]. The latter are the most popular ones: ‘intuitive’ assumptions are introduced for the behavior of some variables in the z-direction. Two well known examples of plate theories that have been derived on the basis of intuitions are the ‘Thin Plate Theory’ (TPT) by Cauchy [7], Poisson [8], or Kirchhoff [9] and the ‘First order Shear Deformation Theory’ (FSDT), by Reissner [10] and Mindlin [11].

The introduction of axiomatic and/or asymptotic approximations could introduce some not desired mechanisms which are not in the three-dimensional solution. One of these is the thickness locking which is related to the use of plane-strain/plane-stress hypothesis in thin plate theory. The analysis of thin plate problems is, in fact, often associated to plane-stress assumptions (thin surface problem) while plane-strain hypothesis is usually referred to beam theory (long cylinders) [3,12]. Discussion on plane strain, plane stress and/or plane elasto-static problems can be found also in [13]. However, in most of the applications,

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TPT assumptions are in a 'contradictory manner' made on strain fields. Plane strain assumptions (a thriorthogonal Cartesian reference surface is assumed, x and y are the coordinate on the reference plate surface Ω, normally the plate middle surface, and z is the thickness coordinate):

$$\text{plane strain: } \epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = 0, \tag{1}$$

are, in fact, used in-place of the more natural plane-stress conditions:

$$\text{plane stress: } \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0. \tag{2}$$

That contradiction introduces a 'locking mechanism' that make the plate/shell model not applicable in some cases. Thickness locking (TL, also known as Poisson locking) is the name assigned to that mechanism: TL does not permit to TPT analysis to lead to 3D solution in thin plate problems. A known technique to contrast TL consists of modifying the elastic stiffness coefficients by forcing the 'contradictory' conditions:

$$\text{transverse normal stress zero condition: } \sigma_{zz} = 0. \tag{3}$$

See the complete discussion reported in the books by Washizu [4], Librescu [5] and Reddy [6] as well as the discussion quoted in Section 3 of this paper.

The development of computational model, such as finite element method (FEM), can show additional locking mechanisms which are related to the further introduced computational approximations. An example is the 'Shear Locking' (SL) which is associated to the development of plate/shell elements in the framework of FSDT applications, see [14,15]. Locking mechanisms are not often easy to be understood and overcome. For instance, the scientific community has required about two decades to solve in a effective and definitive way SL phenomenon, see the work by Bishoff and Ramm [23] among the others.

TL is further exhibited by refined plate theories [17,16] and FEs [18,19] that account for a constant through-the-thickness deformation ϵ_{zz}; see the interesting discussion quoted by Bishoff and Ramm [21] and by Kulikov and Plotnikova [20]. On this respect, a recent paper by Vu-Quoc and Tan [22] on shell geometry, quotes *To incorporate 3D constitutive laws in shell formulations, the transverse normal strain must have at least a linear distribution over the shell thickness; otherwise the so-called Poisson thickness locking would occur*. That sentence clearly show a further manner to overcome TL.

Although there are many papers that discuss TL in FE applications, analytical analysis and results for classical and advanced plate theories, including laminated structures, are not available. The present work aims to contribute to that gap. It explores the TL mechanism in bending and vibration problems, in the case of closed form solutions, by considering various significant problems: (1) various plate lay-outs (one-layered plate, multilayered plates made by isotropic and orthotropic composite materials); (2) a large variety of plate theories (TPT, FSDT, higher order plate theories, mixed and layer-wise theories for laminated structures) including the 3D elasticity solutions. The quiet exhaustive plate theories treatment has been made by the use of unified formulation (UF) for plate/shell theories that was developed by the first author in earlier papers [23,24].

This paper has been organized as follows: Section 2 describes the addressed plate theories. A preliminary discussion on TL is given in Section 3. Numerical results and discussion are provided in Section 4.

2. Considered theories

In order to explore in which manner the various kinematic assumptions can affect the TL mechanisms, a large variety of plates theories for one-layer and multilayers plates have been considered in the present work. A short discussion is given in the following.

A plate of constant thickness h is considered. The geometry and reference system are shown in Fig. 1. x, y and z is the Cartesian reference system along whose direction the three displacement components u_x, u_y and u_z are measured. z denotes the thickness direction and Ω is the plate reference surface located with correspondence to the mid-plane of the plate.

2.1. Classical theories

2.1.1. Thin plate theories (TPT)

Classical thin plate thseories (TPT) that are based on Cauchy [7], Poisson [8], or Kirchhoff [9] type assumptions, discard transverse shear and trough-the-thickness deformation. The displacement model related to TPT can be written in the following form:

$$u_s(x,y,z;t) = u_{0s}(x,y;t) - z\frac{\partial u_{0z}(x,y;t)}{\partial s}, \quad s = x,y \tag{4}$$
$$u_z(x,y,z;t) = u_{0z}(x,y;t).$$

which states that the section remains plane and orthogonal to the plate reference surface Ω. t denotes time; u_0 denotes the displacement value with correspondence to the reference surface Ω. TPT is also known as classical lamination theory (CLT) in the case of application to laminated composites.

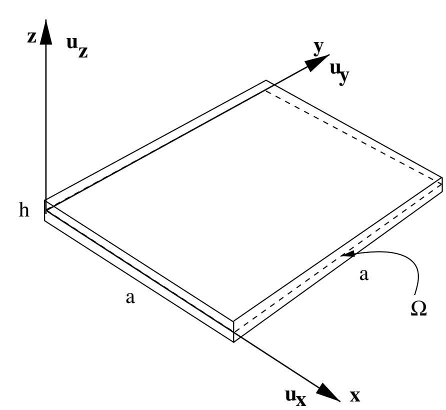

Fig. 1. Geometry and notations of the considered plate.

It should be noticed that TPT assumptions fulfill exactly the plane strain assumption at Eq. (1).

2.1.2. First order shear deformation theory (FSDT)

Transverse shear deformation can be introduced in TPT theories according to the following kinematic assumptions, known as Reissner–Mindlin [10,11], theory:

$$u_s(x,y,z;t) = u_{0s}(x,y;t) + z u_{1s}(x,y;t), \quad s = x,y \tag{5}$$
$$u_z(x,y,z;t) = u_{0z}(x,y;t).$$

which is also denoted as first order shear deformation theory (FSDT).

It should be noticed that FSDT assumptions fulfill the zero transverse strain condition:

$$\text{transverse normal strain zero condition: } \epsilon_{zz} = 0. \tag{6}$$

FSDT does not include any plane-stress/strain assumptions.

2.2. Higher order theories (HOT)

Refinement of TPT and FSDT analysis can be introduced by including higher order terms in the kinematic assumption for the displacements fields:

$$u_\tau(x,y,z;t) = u_{0\tau}(x,y;t) + z^i u_{i\tau}(x,y;t), \quad \tau = x,y,z, \ \ i = 1,N; \tag{7}$$

The summing convention for repeated indexes has been adopted; N is the order of expansion, which is taken as a free parameter. In this paper, the values $N = 1$ up to $N = 4$ are considered in the numerical investigation. According to Unified formulation, the related theories will be denoted as ED1–ED4. The letter E denotes that the kinematic is preserved for the whole layers of the plate, as in the so-called equivalent single layer (ESL). D denotes that only displacement unknowns are used and the last number denote the order of the expansion in z.

To be noticed that all stress and strain tensor components are different from zero in that case. ED2 case should be therefore free from thickness locking.

2.2.1. Use of Murakami zig-zag functions

The models at Eq. (7) are not able to describe zig-zag effect [25] in the case of laminates. The discontinuity of the first derivative with correspondence to the layer interfaces, can be accounted for by employing the Murakami zig-zag function (MZZF), that was proposed in the framework of RMVT applications [27,28]. The not dimensioned layer coordinate $\zeta_k = z_k/2h_k$ is further introduced (h_k is the thickness of the kth layer, z_k is the layer thickness coordinate). The Murakami zig-zag functions $M(z)$ was defined according to the following formula [27],

$$M(z) = (-1)^k \zeta_k \tag{8}$$

$M(z)$ has the following properties: it is a piece-wise linear function of the layer coordinates z_k; $M(z)$ has unit amplitude for the whole layers; the slope $M'(z) = \frac{dM}{dz}$ assumes opposite sign between two-adjacent layers (its amplitude is layer thickness dependent). The displacement including MZZF is written in the form:

$$u_\tau(x,y,z;t) = u_{0\tau}(x,y;t) + z^r u_{r\tau}(x,y;t) + (-1)^k \zeta_k u_{Z\tau}(x,y;t),$$
$$\tau = x,y,z \quad r = 1,2,\ldots,N-1 \tag{9}$$

Subscripts Z refers to the introduced Zig-Zag term. Higher order distributions in the z-direction are introduced by the r-polynomials.

Modification of EDN direct to include MZZF are herein denoted as EDZN analyses.

2.2.2. Theory with selective order of expansion for in-plane and out-of-plane displacements

It is of particular interest to consider plate theories which are based on a different order of expansion for in-plane and out-of-plane displacements:

$$u_s(x,y,z;t) = u_{0s}(x,y;t) + z^i u_{is}(x,y;t) \quad s = x,y \quad i = 1,N_p \tag{10}$$
$$u_Z(x,y,z;t) = u_{0Z}(x,y;t) + z^i u_{iZ}(x,y;t) \quad i = 1,N_w$$

N_p and N_w refer to the order of the expansion for in-plane and out-of-plane displacement components, respectively. $N_p = 1$ and $N_w = 0$ in both TPT and FSDT cases.

2.3. Layer-wise LW theories (LWT)

Multilayered plate can be analyzed by kinematic assumptions which are independent in each layer. According to [6] these approaches have herein stated as layer-wise theories.

Layer-wise description requires assuming independent displacement variables in each k-layer. The Taylor thickness expansion used for ESLM cases of the previous paragraphs is not convenient for layer-wise description. Interlaminar continuity for displacements can be more conveniently imposed by employing interface values as unknown variables. Therefore, layer-wise description is written according to the following expansion,

$$u^k_\tau = F_t u^k_{\tau t} + F_b u^k_{\tau b} + F_r u^k_{\tau r} \quad \tau = x,y,z;$$
$$r = 2,3,\ldots,N; \ \ k = 1,2,\ldots,N_l. \tag{11}$$

It is intended that the subscripts t and b denote values related to the layer top and bottom surface, respectively. The thickness functions $F_\tau(\zeta_k)$ have been defined by

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2},$$
$$r = 2,3,\ldots,N \tag{12}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the j-order defined in the ζ_k-domain : $-1 \leqslant \zeta_k \leqslant 1$. Fourth order case will be used in the numerical investigations; related polynomials are

$$P_0 = 1, \quad P_1 = \zeta_k, \quad P_2 = (3\zeta_k^2 - 1)/2,$$
$$P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2}, \quad P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8}.$$

The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1 & : \quad F_t = 1; \quad F_b = 0; \quad F_r = 0 \\ -1 & : \quad F_t = 0; \quad F_b = 1; \quad F_r = 0, \end{cases} \tag{13}$$

The top and bottom values have been used as unknown variables. The interlaminar compatibility of displacement can be therefore easily linked:

$$u^k_{\tau t} = u^{(k+1)}_{\tau b}, \quad k = 1, \ N_l - 1 \tag{14}$$

2.4. Mixed theories based on Reissner mixed variational theorem

It is well known fact, [25] that the kinematic above described, if applied to multilayered structures, are not able to furnish interlaminar continuous transverse shear and normal stresses at the interface between two adjacent layers. Reissner mixed variational theorem [26] offers a possibility to fulfill 'a priori' such an interlaminar continuity. Both displacements and transverse shear and normal stresses can be assumed in the RMVT framework.

2.4.1. Layerwise theories

In the layer wise case the displacement model at Eq. (11) is also used for the transverse stress variables:

$$\sigma^k_{\tau z} = F_t \sigma^k_{\tau zt} + F_b \sigma^k_{\tau zb} + F_r \sigma^k_{\tau zr} \tau = x, y, z; \quad r = 2, 3, \ldots, N; \quad k = 1, 2, \ldots, N_l. \tag{15}$$

The interlaminar transverse shear and normal stress continuity can be therefore easily linked:

$$\sigma^k_{\tau zt} = \sigma^{(k+1)}_{\tau zb}, \quad \tau = x, y, z, k = 1, \ N_l - 1 \tag{16}$$

These plate theories will be denoted as LM1–LM4.

2.4.2. Equivalent single layer theories

Mixed theories with Equivalent single layer description can be used by referring to the displacement model at Eq. (7) and layer-wise stress assumption at Eq. (15). These plate theories will be referred to as EMC1–EMC4. EMZC1–EMZC3 will instead denote those mixed theories that make use of the MZZF, that is Eq. (9) used for the displacement model.

2.5. Unified formulation

The previous models have been coded according to the unified formulation. Details can be found in previous first author's work [23,24].

3. A preliminary discussion on thickness locking

As stated in the introduction, the 'plane-strain' kinematic related to Eq. (4) originates TL in the case of thin plate analysis. To underline that fact, kinematic of Eq. (4) is denoted as $\text{TPT}_{\epsilon_{zz}}$. Most of the well known authors and books suggest the modification of material elastic coefficients by forcing the zero transverse stress conditions at Eq. (3). The related theory is herein denoted by the acronym $\text{TPT}_{\sigma_{zz}}$.

Hence transverse shear effect are neglected in a TPT, the plane-stress condition at Eq. (2) coincides to the pure zero transverse normal stress conditions Eq. (3). The notation 'plane-stress', is therefore inappropriate in the FSDT cases. It appears more correct to refer to the zero-transverse normal condition at Eq. (3) even though TPT is addressed.

In order to understand the TL mechanism, let us evaluate strains coming from geometrical and physical 'constitutive' relations. For sake of simplicity, reference is made to a plate made by isotropic materials.

3.1. Strains from kinematic

Let first consider a plate theory which is based on a different order of expansion of N_p and N_w in z-direction. The order of the linear strains coming from geometrical relations is:

$$\begin{aligned} \epsilon^G_{xx} &= u_{x,_x} \ N_p \\ \epsilon^G_{yy} &= u_{y,_y} \ N_p \\ \epsilon^G_{zz} &= u_{z,_z} \ N_w - 1 \end{aligned} \tag{17}$$

It can be remarked that:

1. The geometrical in-plane strains always show order N_p.
2. The order of transverse normal strain is always $N_w - 1$.
3. $\epsilon^G_{zz} = 0$ in the TPT and FSDT analyses.
4. The distribution of ϵ^G_{zz} would be constant through the thickness z if and only if $N_w = 1$ (that happens in the HOT with $N = 1$ cases, such as –ED1,EDZ1,EMZC1–).
5. To avoid constant transverse normal strains the following value must be used:

$$N_w \geqslant 2.$$

3.2. Strain from physical relation

A further 'physical' or 'constitutive' relation between the transverse strain and the in-plane ones (the x-direction is considered) is given by Poisson modulus that establish a proportional relation between these strains:

$$\epsilon^v_{zz} \propto v \epsilon^G_{xx} \tag{18}$$

It is clear that the transverse normal strain distribution due to Poisson modulus is of the same order of the in-plane ones.

If the TPT, FSDT and HOT with $N = 1$ cases are referred to, being ϵ^G_{xx} linear ($N_p = 1$), the related ϵ^v_{zz} will be also linear. That 'tragically' contradicts the in-plane strain kinematic of $\text{TPT}_{\epsilon_{zz}}$ (as well as of any other plate theories with $N_w \leqslant 1$). That contradiction originates the thickness locking or Poisson locking.

TL is is independent by that plate thickness; it is exhibited by both thick and thin plate geometries. As a consequence TL could make ineffective $TPT_{\epsilon_{zz}}$ analyses even though thin plate geometries are considered.

A question arise: how to contrast ϵ^{ν}_{zz}? That is how to contrast TL?

3.3. First remedy to TL: use of higher order kinematic

A natural remedy could be to using $N_w \geqslant 2$. That means to use a plate theory with at least quadratic expansion for the transverse displacement u_z:

$$N_w \geqslant 2 : u_z(x,y,z;t) = u_{0z}(x,y;t) + z u_{1z}(x,y;t) + z^2 u_{2z}(x,y;t) + \cdots \quad (19)$$

Unfortunately, that solution lead to governing plate equations which are more complicated that the classical ones. Simpler solutions were therefore tried.

In same cases, such as membrane deformation, the in-plane strain are constant. The order of the expansion N_w that is required to avoid TL is $N_w = 1$.

3.4. Second remedy to TL: modification of elastic coefficients

It is clear that ϵ^{ν}_{zz} is originates by constitutive equations of the given materials. It is due to the intrinsic coupling between out-of-plane and in-plane normal strain components. A proper solution could require the modification of such constitutive relations. That modification is discussed above.

The three-dimensional constitutive relation between stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$ in term of elastic coefficients:

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{C}]\{\boldsymbol{\epsilon}\} \quad (20)$$

In explicit form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \epsilon_{xz} \\ \epsilon_{yz} \\ \epsilon_{xy} \end{Bmatrix} \quad (21)$$

The elastic coefficients are related to the Engineering constants (Poisson ν, Young E and the shear G moduli) as follows:

$$\begin{aligned} C_{11} &= C_{22} = C_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \\ C_{12} &= C_{13} = C_{23} = \frac{\nu E}{(1+\nu)(1-2\nu)}, \\ C_{44} &= C_{55} = C_{66} = G \end{aligned} \quad (22)$$

That constitutive equations can be modified to contrast ϵ^{ν}_{zz}. A manner to do that, consists to impose a condition on transverse normal stress. Kulikov and Plotinka [20] have recently discussed several manners to introduce that. The most classical method introduce directly the $\sigma_{zz} = 0$ condition in Eq. (21). The modified elastic coefficients assume the following form:

$$\begin{aligned} \widetilde{C}_{11} &= \widetilde{C}_{22} = C_{11} - \frac{C_{13}^2}{C_{33}} = \frac{E}{1-\nu^2}, \\ \widetilde{C}_{12} &= C_{12} - \frac{C_{13}C_{23}}{C_{33}} = \frac{\nu E}{1-\nu^2}, \\ \widetilde{C}_{44} &= C_{44}, \quad \widetilde{C}_{55} = C_{55}, \ \widetilde{C}_{66} = C_{66} \end{aligned} \quad (23)$$

As its is well know that such a modification permits 'in a certain sense' to obtain the 3D solutions in thin plate analysis. The related theory is herein denoted by acronyms $TPT_{\sigma_{zz}}$. To be noticed that *only in-plane elastic coefficients are modified*. That could be confirmed for the orthotropic materials but not in the most general case of anisotropic materials.

The reduction of the magnitude of the elastic coefficients reduces the stiffness of the problem and it introduces intrinsic benefits in a displacement formulated theories.

The dual problems in terms of compliances would lead to same conclusions by starting on the inverse form of Hooke's law. A proper use of modified compliances due to plane-strain condition would require to refer to plate theories which are based on stress assumptions. Such a case has not been considered in the present paper. Some books show that by opportune definition of elastic coefficients the plane-strain and plane-stress governing equations are formally the same [13,15].

3.5. Use of penalty number to impose zero-transverse strain condition in HOT, mixed, LW and 3D solutions

Transverse plane-strain condition can be introduced in any higher order, mixed or layer-wise plate theories as well in any three-dimensional elasticity solutions by opportune manipulation of 3D constitutive equations, see also Kulikov and Plotinka [20]. The elastic constants of interest are:

$$C_{13}, \quad C_{23}, \quad C_{33}. \quad (24)$$

The first two terms,

$$C_{13}, \quad C_{23},$$

couple the in-plane stresses σ_{xx}, σ_{yy} to the thickness strain ϵ_{zz}: these lead to a lower plate stiffness for thin geometries.

The problem can also be seen as an overestimation of the contribution to σ_{zz} coming from transverse strain ϵ_{zz}, e.g. of the elastic constant

$$C_{33}.$$

Two penalty numbers γ and β can be introduced to modify the elastic constants according to the following formula:

$$\gamma : C^{\star}_{13} = \frac{C_{13}}{\gamma}, C^{\star}_{23} = \frac{C_{23}}{\gamma} \quad (25)$$

$$\beta : C^{\star}_{33} = C_{33} \times \beta \quad (26)$$

where the apex ★ denotes the penalized elastic coefficients.

4. Results and discussion

Navier-type, closed form solutions of simply supported isotropic and orthotropic, square plates are discussed. Readers are addressed to previous works [23,24] to find details of the implemented unified formulation and solution procedure.

A large numerical investigation has been conducted to compares the various theories that have been illustrated in the previous section. Selected results are discussed in the next paragraphs. These will give a quite exhaustive overviews on thickness locking mechanisms in classical and advanced theories as well as in one-layered and multi-layered plates. 3D solutions have been provided in same cases by the courtesy of Demasi [29].

4.1. Data of the considered problems

Various plate problems have been treated. Their data are described in the following.

4.1.1. Plate bending and vibration

Simply supported plates made by orthotropic materials have been considered. Plate bending loaded by bi-sinusoidal distribution of transverse pressure applied at the top plate-surface has been analyzed:

$$p_z(x,y) = \bar{p}_z \sin\left(\frac{mx}{a}\right)\sin\left(\frac{ny}{b}\right); \tag{27}$$

$\bar{p}_z$ is the applied load amplitude and m, n are the wave number in the two in-plane, plate directions. a and b are corresponding plate dimensions. Attention has been restricted to the case $m = n = 1$.

The companion cylindrical bending case has been also treated, in which the loading is:

$$p_z(x) = \bar{p}_z \sin\left(\frac{mx}{a}\right); \tag{28}$$

Attention has been restricted to the case $m = 1$.

The free vibration problem has been also investigated. Results are restricted to first fundamental bending circular frequency parameter $\bar{\omega}$.

4.1.2. Material and lay-outs data

Various multilayered configurations have been treated to investigate the effect of lay-outs on TL mechanisms. First the isotropic plate has been addressed. The mechanical and geometrical data are given in Table 1. E and ν are Young moduli and Poisson modulus, respectively. h represents the plate thickness. ρ is the mass density.

Table 1
Elastic and geometrical properties of isotropic plate in Al-2024

E (GPa)	73
ν	0.34
ρ (kg/m^3)	2800
h (m)	0.001

Table 2
Elastic and geometrical properties of orthotropic plate

E_L (GPa)	20,500
$E_T = E_z$ (GPa)	10
$\nu_{LT} = \nu_{Lz}$	0.25
$G_{LT} = G_{Lz}$ (GPa)	5
ρ (kg/m^3)	1600
h (m)	0.001

Table 3
Elastic and geometrical properties of multilayered plate in isotropic materials

Layer	1	2	3
E (GPa)	73	114	210
ν	0.34	0.3	0.3
ρ (kg/m^3)	2800	2768	7850
h (m)	$0.003\bar{3}$	$0.003\bar{3}$	$0.003\bar{3}$

Layer 1 is in Al 2024, layer 2 in titanium and layer 3 in steel.

The orthotropic plate made by one-layer with different orthotropic ratio E_L/E_T. It corresponds to an unidirectional layer of carbon fiber reinforced material. The data are those in Table 2. E_L and E_T are the Young Moduli in the longitudinal and transverse directions, respectively. The thickness Young modulus E_z coincides to E_T. G_{LT} is the shear modulus in LT directions; ν_{LT}, ν_{Lz} are the Poisson ratios.

The multilayer plate made by three layers of different metallic materials of Table 3 has been analyzed to investigate the effect of transverse anisotropy. It consists of a uni-directional laminated plates. The case of composite cross-ply multilayered plate made by unidirectional orthotropic lamina has been also addressed. Related data are given in Table 4.

4.2. Thin plate theory

Table 5 shows the values of elastic coefficients related to $\text{TPT}_{\sigma_{zz}}$ and $\text{TPT}_{\epsilon_{zz}}$ analysis for the isotropic plate. To be noticed the quite large differences among the coefficients which is closed to 25% and 50% for longitudinal and shear moduli, respectively. The amplitude of plate transverse deflection is compared in Table 6. That table makes clear the well known TL-phenomenon: TPT without any correc-

Table 4
Elastic and geometrical properties of cross-ply plate [0/90/0]

Properties	
E_L (GPa)	40
$E_T = E_z$ (GPa)	1
ν	0.25
G_{LT} (GPa)	0.5
G_{Lz} (GPa)	0.6
ρ (kg/m^3)	1
$h_1 = h_3$ (m)	0.0025
h_2 (m)	0.005

Table 5
Comparison of elastic coefficients for plane-stress and plane-strain cases

	C_{11}	C_{12}	C_{66}
$TPT_{\epsilon_{zz}}$ (3D)	1.5392	0.7929	0.3731
	$\widetilde{C}_{11}$	$\widetilde{C}_{12}$	$\widetilde{C}_{66}$
$TPT_{\sigma_{zz}}$	1.1307	0.3844	0.3731
Difference (%)	26.54	51.52	0.00

Case of isotropic plate Al-2024. Results are reported respect to $E = 73$ GPa.

Table 6
TL effect in TPT

a/h	$\nu = 0.34$				
	$TPT_{\epsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N = 4$)
2	2.0010	66.7	2.7238	54.7	6.0107
4	2.0009	44.3	2.7238	24.2	3.5958
10	2.0009	30.2	2.7238	4.94	2.8655
100	2.0009	26.6	2.7238	0.05	2.7252
1000	2.0009	26.5	2.7238	0.00	2.7238

Comparison of plane-strain and plane-stress results vs 3D solutions. Bending problem $\overline{w} = U_z \frac{100 E_T h^3}{\bar{p}_z a^4}$ of a isotropic plate; $m, n = 1, 1$.

tion on the elastic stiffness (that is $TPT_{\epsilon_{zz}}$) is not able to furnish thin plate results related to three-dimensional solutions. Furthermore, the modification introduced by imposing Eq. (3) condition, permits to overcome TL mechanisms. The error related to $TPT_{\epsilon_{zz}}$ almost coincides to the differences obtained in the evaluation of modified elastic coefficients of Table 5. Furthermore, the reduction of stiffness related to Eq. (23) makes $TPT_{\sigma_{zz}}$ results more effective with respect $TPT_{\epsilon_{zz}}$ in thick plate analyses.

The fundamental role of Poisson modulus is analyzed in Table 7. The results related to two values $\nu = 0.15$ and $\nu = 0.45$ are considered. It is confirmed the very strong relation between TL and Poisson effect, in fact: (1) a reduction of Poisson modulus leads to a direct reduction of TL and vice versa (the error of $TPT_{\epsilon_{zz}}$ reduces from 66.9% to 3.11% in thin plate case); (2) the use of zero-transverse-stress condition Eq. (3) is very suitable to contrast TL.

Table 7
Effect of Poisson modulus

a/h	$TPT_{\epsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N = 4$)
$\nu = 0.15$					
2	2.9168	51.7	3.0105	50.2	6.0451
4	2.9168	23.6	3.0105	21.2	3.8189
10	2.9168	7.17	3.0105	4.18	3.1420
100	2.9168	3.15	3.0105	0.04	3.0118
1000	2.9168	3.11	3.0105	0.00	3.0105
$\nu = 0.45$					
2	0.8120	86.1	2.4562	58.1	5.8569
4	0.8120	75.8	2.4562	26.8	3.3561
10	0.8119	68.8	2.4561	5.62	2.6023
100	0.8119	67.0	2.4561	0.06	2.4576
1000	0.8119	66.9	2.4562	0.00	2.4562

Comparison of plane stress and plane strain results for the thin plate theory vs 3D solutions. Bending problem $\bar{w} = U_z \frac{100 E_T h^3}{\bar{p}_z a^4}$ of a isotropic plate; $m, n = 1, 1$.

Table 8
As Table 6 for cylindrical bending

a/h	$TPT_{\epsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N = 4$)
2	8.0038	54.9	10.895	38.6	17.739
4	8.0037	36.8	10.895	13.9	12.656
10	8.0037	28.4	10.895	2.5	11.179
100	8.0037	26.5	10.895	0.02	10.897
1000	8.0037	26.5	10.895	0	10.895

$\nu = 0.34$; $m, n = 1, 0$.

Table 8 is related to cylindrical bending case. It is clearly shown that some benefits are obtained for thick plates cases; anyway the error that has been observed in the square plate cases is confirmed in thin plate results.

Vibration problem is examined in Table 9. The error appears lower than the bending cases of Tables 6–8. This is simply due to the fact that the circular frequency parameter is related to the square of the plate stiffness; if ω^2 is considered the same error of bending would be found.

An investigation on local parameters such displacement and stress distribution is made in Table 10 and Figs. 2–4. In-plane stress σ_{xx} and out-of-plane shear stress σ_{xz} are considered in Table 10. The transverse shear stress have been obtained upon integration of 3D indefinite equilibrium equations. The following should be remarked: *TL is*

Table 9
Comparison between plane stress and plane strain results of the thin plate theory vs 3D solutions

a/h	$TPT_{\epsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N = 4$)
2	5.4278	44.1	5.1005	35.4	3.7665
4	6.7318	32.8	5.7698	13.8	5.0688
10	7.0120	19.8	6.0100	2.7	5.8523
100	7.0688	16.7	6.0587	0.03	6.0570
1000	7.0694	16.6	6.0592	0	6.0592

Fundamental circular frequency parameter $\bar{\omega} = \omega \sqrt{\frac{a^4 \rho}{E_T h^2}}$ of isotropic plate; $m, n = 1, 1$.

Table 10
Comparison of in-plane ($\bar{\sigma}_{xx} = \frac{\sigma_{xx}}{\bar{p}_z (a/h)^2}$) and transverse shear stresses ($\bar{\sigma}_{xz} = \frac{\sigma_{xz}}{\bar{p}_z (a/h)}$)

a/h	$TPT_{\epsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N = 4$)
$\bar{\sigma}_{xx}$					
2	0.2303	27.23	0.2037	35.64	0.3165
4	0.2303	2.26	0.2037	9.55	0.2252
10	0.2303	11.36	0.2037	1.50	0.2068
100	0.2303	13.06	0.2037	0.00	0.2037
1000	0.2303	13.06	0.2037	0.00%	0.2037
$\bar{\sigma}_{xz}$					
2	0.2387	3.51	0.2387	3.51	0.2306
4	0.2387	0.76	0.2387	0.76	0.2369
10	0.2387	0.13	0.2387	0.13	0.2384
100	0.2387	0.00	0.2387	0.00	0.2387
1000	0.2387	0.00	0.2387	0.00	0.2387

Plate bending problem $m, n = 1, 1$ of Table 6. $\bar{\sigma}_{xx}$ is at the top of plate, $\bar{\sigma}_{xz}$ is at the middle surface.

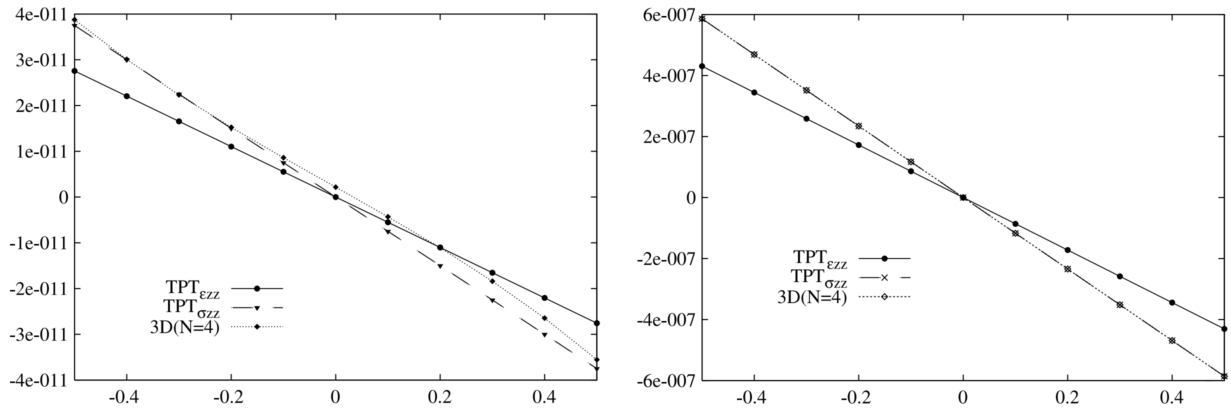

Fig. 2. Displacement u vs z. Isotropic plate. $a/h = 4$ (left) and $a/h = 100$ (right).

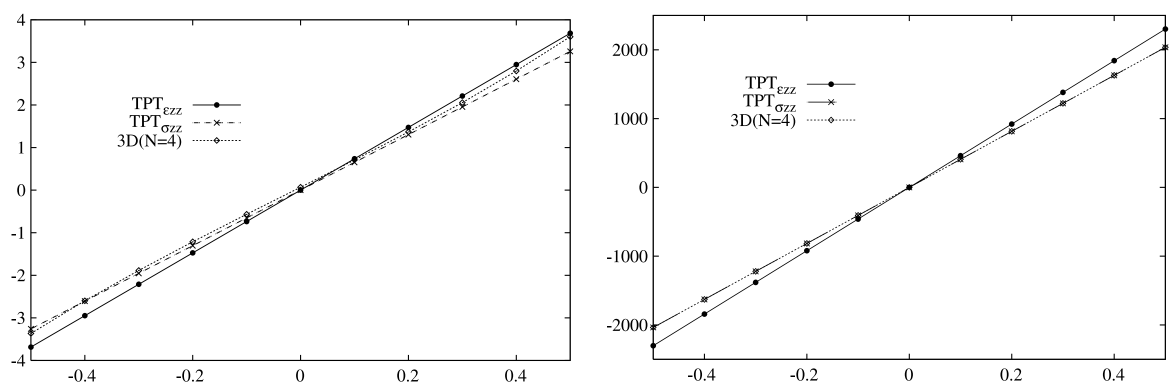

Fig. 3. In-plane stress σ_{xx} vs z. Isotropic plate. $a/h = 4$ (left) and $a/h = 100$ (right).

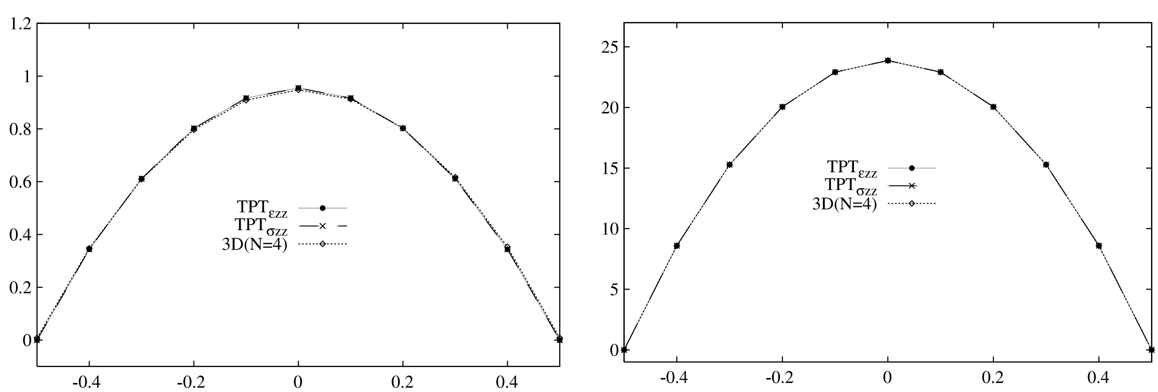

Fig. 4. Transverse shear stress σ_{xz} vs z. Isotropic plate. $a/h = 4$ (left) and $a/h = 100$ (right).

very much reduced with respect to transverse deflections; TL disappears in transverse shear stress evaluation.

In-plane displacement u_x and in-plane normal stress σ_{xx} distribution in the thickness directions have been plotted in Figs. 2 and 3, respectively. Thick and thin plates geometries are compared. It can be noticed that the use of plane-stress assumptions is not always the best choice. In some case, related to thick plate, the in-plane strain assumption could work well than TPT$_{\sigma_{zz}}$. That is, the reduction of the stiffness at Eq. (23), which is intrinsic in the plane-stress assumptions, could not be appropriate in the evaluation of local values of stresses and displacements.

The evaluation of transverse shear stress has been made in Fig. 4 to complete the pictures. Its is confirmed that plane-stress and plane-strain results do not differ.

4.3. Application of penalty techniques to 3D solution

It is of certain interest to investigated the effects of penalty technique discussed in Section 3 on plate theories and

Table 11
Application of penalties number to 3D exact solution and comparison to thin plate results

a/h	$3D_{\text{no penalties}}$	$3D_{\gamma=100}$	$3D_{\beta=100}$	$3D_{\beta=1000}$	$TPT_{\epsilon_{zz}}$	$TPT_{\sigma_{zz}}$
2	6.0107	5.8376	5.9880	5.9879	2.0010	2.7238
4	3.5958	3.0034	3.0175	3.0137	2.0009	2.7238
10	2.8655	2.1634	2.1689	2.1642	2.0009	2.7238
100	2.7252	2.0026	2.0079	2.0031	2.0009	2.7238
1000	2.7238	2.0010	2.0063	2.0015	2.0009	2.7238

Bending problem of Table 6. Results by the courtesy of Luciano Demasi from University of Washington, Seattle.

TL mechanisms. The effect on 3D elasticity solution is first considered. Results are given in Table 11 which quotes the 3D solution for the square plate bending problems. The various penalty techniques have been compared. To notice that $TPT_{\sigma_{zz}}$ coincides to 3D results without any penalties while $TPT_{\epsilon_{zz}}$ results are obtained by application of any of the penalty number to 3D elasticity solution.

Table 12
FSDT results related to plane-stress and plane-strain evaluation of elastic coefficients

a/h	$FSDT_{\epsilon_{zz}}$	Error%	$FSDT_{\sigma_{zz}}$	Error%	$3D(N=4)$
2	5.3952	10.2	6.1180	1.8	6.0107
4	2.8495	20.7	3.5723	0.6	3.5958
10	2.1367	25.4	2.8595	0.2	2.8655
100	2.0023	26.5	2.7251	0.004	2.7252
1000	2.0009	26.5	2.7238	0	2.7238

Plate bending deflection $\bar{w} = U_z \frac{100 E_T h^3}{\bar{p}_z a^4}$ of Table 6.

Table 13
Transverse shear stress evaluations $\bar{\sigma}_{xz} = \frac{\sigma_{xz}}{\bar{p}_z(a/h)}$ at the middle surface of the plate, Table 12 problem

a/h	From Hooke's law		From 3D indefinite equilibrium equations		
	$FSDT_{\epsilon_{zz}}$	$FSDT_{\sigma_{zz}}$	$FSDT_{\epsilon_{zz}}$	$FSDT_{\sigma_{zz}}$	$3D(N=4)$
$\bar{\sigma}_{xz}$					
2	0.1592	0.1592	0.2387	0.2387	0.2306
4	0.1592	0.1592	0.2387	0.2387	0.2369
10	0.1592	0.1592	0.2387	0.2387	0.2384
100	0.1592	0.1592	0.2387	0.2387	0.2387
1000	0.1592	0.1592	0.2387	0.2387	0.2387

The above assessed penalty technique could be useful to obtain plane-strain results in the case of refined and layer-wise theories that are considered in the next paragraphs.

4.4. First order shear deformation plate theory

Correction of in-plane stiffness coefficients due to the use of Eq. (3) are also used in conjunction with FSDT application. The improvements registered in TPT case are confirmed by FSDT analysis. The two cases $FSDT_{\epsilon_{zz}}$ and $FSDT_{\sigma_{zz}}$ are compared. As in TPT cases, the $FSDT_{\sigma_{zz}}$ or $FSDT_{\epsilon_{zz}}$ refer to $\sigma_{zz} = 0$ or $\epsilon_{zz} = 0$ conditions, respectively. FSDT results on bending are given in Table 12. FSDT improves TPT analyses in thick plate cases: but no reduction of TL with respect to TPT are obtained in thin plate geometries.

Concerning transverse shear stresses, FSDT permits their evaluation in two ways: by the use of Hooke's law and by integration of 3D indefinite equilibrium equations. These evaluations are compared in Table 13. It is confirmed the superiority of transverse shear stresses evaluated by 3D indefinite equilibrium equations. It is further concluded that TL does not affect transverse shear stresses even though these are evaluated directly by Hooke's Law. Trough the thickness distributions of transverse shear stress is given in Fig. 5. The same comments made for TPT cases are confirmed.

4.5. Higher order theory

Results related to bending problems of isotropic plates are considered in Tables 14 and 15. Various, selective N_p

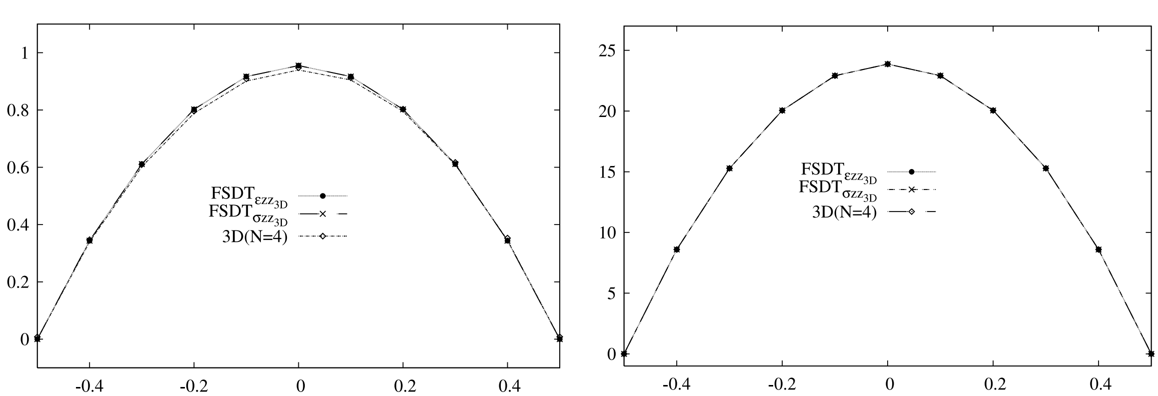

Fig. 5. FSDT analysis of transverse shear stress σ_{xz} vs z. Isotropic plate. $a/h = 4$ (left) and $a/h = 100$ (right).

Table 14
Higher order theories analysis

a/h	3D($N=4$)	$\mathrm{TPT}_{\varepsilon_{zz}}$	$\mathrm{TPT}_{\sigma_{zz}}$	$N_p=4$ $N_w=0$	$N_p=3$ $N_w=0$	$N_p=2$ $N_w=0$	$N_p=1$ $N_w=0$
2	6.0107	2.0010	2.7238	5.9878	5.9878	5.3952	5.3952
4	3.5958	2.0009	2.7238	3.0132	3.0132	2.8495	2.8495
10	2.8655	2.0009	2.7238	2.1637	2.1637	2.1367	2.1367
100	2.7252	2.0009	2.7238	2.0026	2.0026	2.0023	2.0023
1000	2.7238	2.0009	2.7238	2.0009	2.0009	2.0009	2.0009

Results related to selective order of expansion for N_p and N_w. Isotropic plate, bending problem $\overline{w}=U_z\frac{100E_Th^3}{\bar{p}_za^4}$. $m, n=1,1$.

Table 15
HOT analysis: Table 14 continue

a/h	$N_p=4$ $N_w=1$	$N_p=3$ $N_w=1$	$N_p=2$ $N_w=1$	$N_p=4$ $N_w=2$	$N_p=3$ $N_w=2$	$N_p=4$ $N_w=3$
2	5.9878	5.9878	5.3952	6.0444	6.0444	6.0444
4	3.0132	3.0132	2.8495	3.5976	3.5976	3.5976
10	2.1637	2.1637	2.1367	2.8655	2.8655	2.8655
100	2.0026	2.0026	2.0023	2.7252	2.7252	2.7252
1000	2.0009	2.0009	2.0009	2.7238	2.7238	2.7238

Table 16
HOT analysis

a/h	$\mathrm{TPT}_{\varepsilon_{zz}}$	$\mathrm{TPT}_{\sigma_{zz}}$	$(N=1)_{\varepsilon_{zz}}$	$(N=1)_{\sigma_{zz}}$
2	2.0010	2.7238	5.3952	6.1180
4	2.0009	2.7238	2.8495	3.5723
10	2.0009	2.7238	2.1367	2.8595
100	2.0009	2.7238	2.0023	2.7251
1000	2.0009	2.7238	2.0009	2.7238

Use of plane-stress condition in the case $N=1$ for bending problems of Table 14.

and N_w values have been compared. The following comments can be made:

- HOT with constant transverse displacement ($N_w=0$) distribution shows TL which is independent by the order of the expansion for the in-plane components (N_p values).
- TL is confirmed for in the $N_w=1$ cases.
- TL disappears in bending problems if and only if $N_w \geqslant 2$, that is, if and only if the transverse shear strain ϵ_{zz} is not constant in the thickness direction.

It is of some interest to notice that TL still exists in the $N_w=1$ case which implementation require the use of a full 6×6 constitute equations Eq. (21). To imposition of the plane stress condition could appear questionable in this case. What above confirms the conclusion reached in Table 6 concerning the transverse normal strain field generated by Poisson effects.

$N=1$ case has been analyzed in Table 16 which shows that benefits are still obtained by the modification of elastic coefficients, as in the TPT and FSDT cases. At the same time Table 17 shows that if the plane-stress conditions is imposed in the case of $N_w \geqslant 2$ the obtained results are completely wrong. In conclusion the conditions $\sigma_{zz}=0$ makes sense if and only if a constant transverse normal strain distribution is a allowed by the considered plate model. TL would disappears in the $N=1$ type theories if membrane deformations are considered (constant transverse normal strain fields are, in fact, associated to that case).

Table 18
Comparison of elastic coefficients for plane-stress and plane strain cases

	$E_L/E_T=2$			
	C_{11}	C_{12}	C_{22}	C_{66}
$\mathrm{TPT}_{\varepsilon_{zz}}$ (3D)	2.1820	0.3636	1.1270	0.5000
	$\widetilde{C}_{11}$	$\widetilde{C}_{12}$	$\widetilde{C}_{22}$	$\widetilde{C}_{66}$
$\mathrm{TPT}_{\sigma_{zz}}$	2.0650	0.2581	1.0320	0.5000
Difference (%)	5.36	29.01	8.43	0.00
	$E_L/E_T=50$			
	C_{11}	C_{12}	C_{22}	C_{66}
$\mathrm{TPT}_{\varepsilon_{zz}}$ (3D)	50.170	0.3344	1.0690	0.5000
	$\widetilde{C}_{11}$	$\widetilde{C}_{12}$	$\widetilde{C}_{22}$	$\widetilde{C}_{66}$
$\mathrm{TPT}_{\sigma_{zz}}$	50.060	0.2503	1.0010	0.5000
Difference (%)	0.22	25.14	6.36	0.00

Case of orthotropic plate. Results are reported respect to $E_T=10$ GPa.

Table 17
Application of plane-stress condition to various HOTs

a/h	$(N=2)_{\varepsilon_{zz}}$	$(N=2)_{\sigma_{zz}}$	$(N=3)_{\varepsilon_{zz}}$	$(N=3)_{\sigma_{zz}}$	$(N=4)_{\varepsilon_{zz}}$	$(N=4)_{\sigma_{zz}}$
2	5.5938	6.9169	6.0444	7.2922	6.0107	7.2570
4	3.4815	4.9724	3.5976	5.0648	3.5958	5.0630
10	2.8469	4.3796	2.8655	4.3942	2.8655	4.3942
100	2.7250	4.2653	2.7252	4.2655	2.7252	4.2655
1000	2.7238	4.2642	2.7238	4.2642	2.7238	4.2642

Plate bending problem $\bar{w}=U_z\frac{100E_Th^3}{\bar{p}_za^4}$.

Table 19
Bending of orthotropic plate $\bar{w} = U_z \frac{100E_T h^3}{\bar{p}_z a^4}$ ($a/b = 1$, $m = n = 1$)

a/h	$TPT_{\varepsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	$3D(N=4)$
$\frac{E_L}{E_T} = 2$					
2	2.0409	57.3	2.1948	54.1	4.7810
4	2.0408	29.6	2.1948	24.3	2.8981
10	2.0408	11.6	2.1948	5.0	2.3100
100	2.0408	7.1	2.1948	0.05	2.1960
1000	2.0408	7.0	2.1948	0	2.1948
$\frac{E_L}{E_T} = 50$					
2	0.2286	93.8	0.2300	93.7	3.6764
4	0.2285	83.4	0.2300	83.3	1.3782
10	0.2285	48.5	0.2300	48.2	0.4442
100	0.2285	1.6	0.2300	0.9	0.2322
1000	0.2285	0.6	0.2300	0	0.2300

Thin plate theories analysis vs 3D solutions. Comparison between low ($E_L/E_T = 2$) and high ($E_L/E_T = 50$) orthotropic ratio.

Table 20
Vibration results $\bar{\omega} = \omega\sqrt{\frac{a^4\rho}{E_T h^2}}$ of Table 19 problem

a/h	$TPT_{\varepsilon_{zz}}$	Error%	$TPT_{\sigma_{zz}}$	Error%	3D($N=4$)
$\frac{E_L}{E_T} = 2$					
2	5.8924	39.8	5.6820	34.8	4.2154
4	6.6657	17.8	6.4276	13.6	5.6592
10	6.9431	6.4	6.6951	2.6	6.5229
100	6.9994	3.7	6.7494	0.03	6.7476
1000	7.0000	3.7	6.7500	0	6.7500
$\frac{E_L}{E_T} = 50$					
2	7.8344	62.7	7.6690	59.2	4.8154
4	15.669	89.2	15.338	85.2	8.2833
10	20.748	39.1	20.682	38.7	14.912
100	20.916	0.8	20.850	0.5	20.750
1000	20.918	0.3	20.852	0.005	20.851

4.6. Orthotropic plates

The above mode analyses have been restricted to isotropic plates. Orthotropic plates are considered in this section with two significant values of orthotropic ratio $\frac{E_L}{E_T} = 2$ and $\frac{E_L}{E_T} = 50$. The related values of 3D and modified elastic coefficients are given in Table 18. Results for bending and vibration are given in Tables 19 and 20, respectively. The attention has been restricted to TPT analyses. TL is very much reduced by orthotropic ratio increasing. Orthotropic ratio, in fact, plays exactly the same role of Poisson modulus: it reduces the coupling between transverse normal strain distribution and in-plane strain distribution related to the elastic coefficients. The error for the bending case is reduced by 26%–7% by increasing the orthotropic ratio from 1 to 2 (isotropic case in Table 6); for the higher orthotropic ratio ($E_L/E_T = 50$) the error due to TL is reduced to 0.6%. TL almost disappears for $E_L/E_T = 50$. Such a results could make somehow difficult to recognize TL in orthotropic plate analyses, such as those plate made by advanced high modulus composite materials.

Correspondent FSDT and HSDT analyses are given in Tables 21 and 22. Higher order terms play the same role observed in the case of plate made by isotropic materials. TL, even very much reduced, has still some influence on the results related to FSDT and HSDT related to $N_w = 0$ and $N_w = 1$ cases. For sake of completeness Table 22 shows the results related to plate theories with selective order of expansion for in-plane and out-of-plane displacement components. Attention has been restricted to static analysis.

Table 21
Higher order theories results for orthotropic plates

a/h	2	4	10	100	1000
$\frac{E_L}{E_T} = 2$					
$TPT_{\varepsilon_{zz}}$	2.0409	2.0408	2.0408	2.0408	2.0408
$TPT_{\sigma_{zz}}$	2.1948	2.1948	2.1948	2.1948	2.1948
$FSDT_{\varepsilon_{zz}}$	4.6256	2.6913	2.1452	2.0419	2.0408
$FSDT_{\sigma_{zz}}$	4.7853	2.8471	2.2995	2.1958	2.1948
$N = 1$	4.6256	2.6913	2.1452	2.0419	2.0408
$N = 2$	4.4044	2.7958	2.2933	2.1958	2.1948
$N = 3$	4.8187	2.9003	2.3101	2.1960	2.1948
$N = 4$	4.7810	2.8981	2.3100	2.1960	2.1948
$\frac{E_L}{E_T} = 50$					
$TPT_{\varepsilon_{zz}}$	0.2286	0.2285	0.2285	0.2285	0.2285
$TPT_{\sigma_{zz}}$	0.2300	0.2300	0.2300	0.2300	0.2300
$FSDT_{\varepsilon_{zz}}$	3.7158	1.2657	0.4101	0.2304	0.2285
$FSDT_{\sigma_{zz}}$	3.7482	1.2753	0.4125	0.2318	0.2300
$N = 1$	3.7158	1.2657	0.4101	0.2304	0.2285
$N = 2$	3.4449	1.2514	0.4115	0.2318	0.2300
$N = 3$	3.7016	1.3802	0.4443	0.2322	0.2300
$N = 4$	3.6764	1.3782	0.4442	0.2322	0.2300

Bending problem $\bar{w} = U_z \frac{100E_T h^3}{\bar{p}_z a^4}$. $a/b = 1$; m, $n = 1$, 1.

Table 22
Use of selective order of expansion in orthotropic plate for the bending problems: $\bar{w} = U_z \frac{100E_T h^3}{\bar{p}_z a^4}$

a/h	2	4	10	100	1000
$\frac{E_L}{E_T} = 2$					
$TPT_{\varepsilon_{zz}}$	2.0409	2.0408	2.0408	2.0408	2.0408
$TPT_{\sigma_{zz}}$	2.1948	2.1948	2.1948	2.1948	2.1948
$N_p = 4$ $N_w = 4$	4.7810	2.8981	2.3100	2.2237	2.1948
$N_p = 4$ $N_w = 3$	4.8187	2.9003	2.3101	2.2237	2.1948
$N_p = 4$ $N_w = 2$	4.8187	2.9003	2.3101	2.2237	2.1948
$N_p = 4$ $N_w = 1$	5.0881	2.8175	2.1659	2.0721	2.0408
$N_p = 4$ $N_w = 0$	5.0881	2.8175	2.1659	2.0721	2.0408
$\frac{E_L}{E_T} = 50$					
$TPT_{\varepsilon_{zz}}$	0.2286	0.2285	0.2285	0.2285	0.2285
$TPT_{\sigma_{zz}}$	0.2300	0.2300	0.2300	0.2300	0.2300
$N_p = 4$ $N_w = 4$	3.6764	1.3782	0.4442	0.2851	0.2300
$N_p = 4$ $N_w = 3$	3.7016	1.3802	0.4443	0.2851	0.2300
$N_p = 4$ $N_w = 2$	3.7016	1.3802	0.4443	0.2851	0.2300
$N_p = 4$ $N_w = 1$	3.9759	1.3965	0.4431	0.2836	0.2286
$N_p = 4$ $N_w = 0$	3.9759	1.3965	0.4431	0.2836	0.2286

4.7. Multilayered plates: layer-wise and mixed theories

It is of certain interest to explore TL mechanism in multilayered plates and to related dedicated theories. Tables 23–26 consider the two multilayered plates described in Section 4.1. Results related to mixed theories as well as

Table 23
Multilayered plate made by three isotropic layers of different materials

a/h	2	4	10	100	1000
$TPT_{\varepsilon_{zz}}$	1.3042	1.3042	1.3042	1.3042	1.3042
$TPT_{\sigma_{zz}}$	1.6875	1.6875	1.6875	1.6875	1.6875
$FSDT_{\varepsilon_{zz}}$	3.1307	1.7608	1.3773	1.3049	1.3042
$FSDT_{\sigma_{zz}}$	3.5140	2.1441	1.7606	1.6882	1.6875
Higher order theories					
ED1	3.0221	1.7575	1.3789	1.3062	1.3054
ED2	3.5316	2.1812	1.7659	1.6858	1.6850
ED3	3.7454	2.2385	1.7767	1.6878	1.6869
ED4	3.7428	2.2425	1.7776	1.6878	1.6869
EDZ1	3.3105	1.8468	1.4010	1.3150	1.3141
EDZ2	3.6243	2.2139	1.7729	1.6877	1.6868
EDZ3	3.7165	2.2373	1.7771	1.6881	1.6873
ED4($N_p = 4$, $N_w = 0$)	3.5873	1.8904	1.3989	1.3051	1.3042
ED4($\gamma = 100$)	3.4870	1.8848	1.3987	1.3052	1.3042
Mixed theories					
EMC1	3.0595	1.7677	1.3810	1.3067	1.3060
EMC2	3.5595	2.1889	1.7674	1.6861	1.6853
EMC3	3.7635	2.2434	1.7776	1.6879	1.6870
EMC4	3.7564	2.2463	1.7783	1.6879	1.6870
EMZC1	3.3536	1.8608	1.4039	1.3156	1.3147
EMZC2	3.6421	2.2197	1.7740	1.6878	1.6870
EMZC3	3.7286	2.2407	1.7777	1.6882	1.6873
EMZC3($N_p = 4$, $N_w = 0$)	3.5919	1.8899	1.3987	1.3052	1.3043
EMC4($\gamma = 100$)	3.5003	1.8884	1.3993	1.3052	1.3042

Comparison of classical, HOT and Mixed Equivalent Single Layer plate theories. Bending problem $\bar{w} = U_z \frac{100 E_T h^3}{\bar{p}_z a^4}$, $\frac{a}{b} = 1$.

Table 24
As Table 23

a/h	2	4	10	100	1000
$TPT_{\varepsilon_{zz}}$	1.3042	1.3042	1.3042	1.3042	1.3042
$TPT_{\sigma_{zz}}$	1.6875	1.6875	1.6875	1.6875	1.6875
$FSDT_{\varepsilon_{zz}}$	3.1307	1.7608	1.3773	1.3049	1.3042
$FSDT_{\sigma_{zz}}$	3.5140	2.1441	1.7606	1.6882	1.6875
Higher order theories					
LD1	3.6304	2.1728	1.7247	1.6383	1.6376
LD2	3.7637	2.2542	1.7803	1.6884	1.6876
LD3	3.7745	2.2556	1.7804	1.6884	1.6876
LD4	3.7741	2.2556	1.7804	1.6884	1.6876
LD4($\gamma = 100$)	3.5203	1.8976	1.4010	1.3052	1.3044
Mixed theories					
LM1	3.7169	2.2375	1.7722	1.6819	1.6811
LM2	3.7767	2.2551	1.7803	1.6884	1.6875
LM3	3.7741	2.2556	1.7804	1.6884	1.6874
LM4	3.7741	2.2556	1.7804	1.6884	1.6873
LM4($\gamma = 100$)	3.5203	1.8976	1.4010	1.3052	1.3040

Layer wise plate theories.

Table 25
Cross-ply [0/90/0] plate bending problem $\bar{w} = U_z \frac{100 E_T h^3}{\bar{p}_z a^4}$, $\frac{a}{b} = 1$

a/h	2	4	10	100	1000
$TPT_{\varepsilon_{zz}}$	0.28061	0.2806	0.28058	0.28058	0.28058
$TPT_{\sigma_{zz}}$	0.28281	0.2828	0.28278	0.28278	0.28278
$FSDT_{\varepsilon_{zz}}$	2.7920	0.9859	0.40820	0.28191	0.28059
$FSDT_{\sigma_{zz}}$	2.7981	0.9903	0.41092	0.28411	0.28279
Higher order theories					
ED1	2.7920	0.9859	0.40820	0.28191	0.28059
ED2	2.5361	0.9692	0.40974	0.28411	0.28279
ED3	3.1140	1.0943	0.42590	0.28426	0.28279
ED4	3.0813	1.0920	0.42584	0.28426	0.28279
EDZ1	3.2981	1.0897	0.41910	0.28200	0.28059
EDZ2	3.0547	1.0725	0.42033	0.28420	0.28279
EDZ3	3.1849	1.1125	0.42823	0.28428	0.28279
ED4($N_p = 4$, $N_w = 0$)	3.3458	1.1114	0.42479	0.28207	0.28059
ED4($\gamma = 100$)	3.1395	1.0987	0.42445	0.28207	0.28059
Mixed theories					
EMC1	2.7999	0.9881	0.40863	0.28191	0.28059
EMC2	2.5387	0.9699	0.40988	0.28411	0.28279
EMC3	3.1162	1.0949	0.42601	0.28426	0.28279
EMC4	3.0826	1.0924	0.42590	0.28426	0.28279
EMZC1	3.3418	1.1035	0.42131	0.28202	0.28059
EMZC2	3.0845	1.0815	0.42175	0.28421	0.28279
EMZC3	3.1901	1.1141	0.42843	0.28428	0.28279
EM4($N_p = 4$, $N_w = 0$)	3.0826	1.0924	0.42590	0.28426	0.28279
EM4($\gamma = 100$)	3.1408	1.0990	0.42451	0.28207	0.28059

Equivalent single layer theories.

Table 26
Layer-wise theories for Table 25 problem

a/h	2	4	10	100	1000
$TPT_{\varepsilon_{zz}}$	0.28061	0.28059	0.28058	0.28058	0.28058
$TPT_{\sigma_{zz}}$	0.28281	0.28279	0.28278	0.28278	0.28278
$FSDT_{\varepsilon_{zz}}$	2.7920	0.98592	0.40820	0.28191	0.28059
$FSDT_{\sigma_{zz}}$	2.7981	0.99028	0.41092	0.28411	0.28279
Higher order theories					
LD1	3.1536	1.0783	0.42009	0.28385	0.28245
LD2	3.1424	1.1102	0.42878	0.28429	0.28279
LD3	3.1970	1.1282	0.43056	0.28430	0.28279
LD4	3.1950	1.1280	0.43056	0.28430	0.28279
LD4($\gamma = 100$)	3.2602	1.1347	0.42912	0.28211	0.28059
Mixed theories					
LM1	3.2955	1.1324	0.42983	0.28426	0.28276
LM2	3.1730	1.1190	0.42958	0.28430	0.28279
LM3	3.1987	1.1283	0.43056	0.28430	0.28279
LM4	3.1960	1.1280	0.43056	0.28430	0.28279
LM4($\gamma = 100$)	3.2612	1.1347	0.42912	0.28211	0.28060

to layer-wise modelings are given. The following main remarks can be made.

- TL is still present in multilayered cases, but it appears in very different manners for the two considered plates. With respect to one-layer isotropic plate, the TL is only barely reduced by the multilayered plate made with isotropic layer. TL is almost removed by the multilayered plate made by orthotropic unidirectional layers.
- TL disappears if LW analyses are introduced even thought the linear expansion in each constitutive layer is employed. Linear LW model, in fact, contrasts TL by introducing piece-wise constant distribution of transverse normal strain.
- Mixed plate theories, which make use of a different form of the constitutive Hooke's law, do not introduce any improvements as far as TL is concerned.

It is further shown that plane-strain results can be recovered by application of penalties numbers to layer-wise results as well as to mixed theories.

5. Conclusions

This paper has presented a numerical investigation on thickness locking in classical, refined and advanced mixed theories for one-layered and multilayered isotropic and orthotropic plates. Closed form solutions have been given. The following already known conclusions have been confirmed.

- The use $\sigma_{zz} = 0$ condition appears a suitable technique to contrast TL in thin-plate analysis. It is very effective to contrast TL in the case of TPT and FSDT applications for the evaluation of global parameters such as displacement amplitudes and/or circular frequencies.
- $\sigma_{zz} = 0$ preserves its advantages if applied to plate theories (HOT) with linear distribution of transverse displacement field u_z.
- TL appears if and only if a plate theory shows a constant distribution of transverse normal strain ϵ_{zz}; that is to avoid TL the plate theories would require at least a parabolic distribution of transverse displacement component u_z.

Furthermore, to the best of the author's knowledge it appears that the following noire remarks could represent an original contribution to the TL mechanism analysis:

- There is not available theoretical study which justify the modification of the elastic constant by forcing $\sigma_{zz} = 0$ conditions. On this respect, the present work has shown that the modification of elastic stiffness coefficients has the unique aim to contrast the 'parasitic' strain coming from Poisson effect. On the other hand such a modification appear questionable if the employed plate theories make use of full 3D Hooke's law, such as the HOT with linear distribution of transverse displacements fields.
- The use of $\sigma_{zz} = 0$ condition could not introduce effective improvements if local distribution of stresses and displacements are considered. In some case the plane-strain conditions could be more effective.
- The use of enhanced transverse displacement fields is more effective of imposing $\sigma_{zz} = 0$.
- TL does not appear in the evaluation of transverse shear stresses of TPT and FSDT.
- The use $\sigma_{zz} = 0$ condition leads to large error if applied to plate theories accounting parabolic, cubic and higher transverse displacements fields.
- TL is very much reduced in orthotropic plate analysis.
- TL can be contrasted by the use of Layer-Wise plate theories.
- Mixed theories for multilayered structures does not introduce any improvement as far as TL is concerned.
- Plane strain results can be obtained as particular case of 3D (as well as of LW) analyses by appropriate penalization of elastic coefficients.

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