Extension of Reissner's mixed variational principle to thermopiezoelasticity

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presentata dal Socio corrispondente Ettore ANTONA
nell'adunanza del 22 novembre 2006

Abstract. *The Reissner's Mixed Variational Theorem as derived in [1] for a pure mechanical analysis is herein extended to thermopiezoelastic analysis starting from the thermodynamic principles and Maxwell's relations. The obtained variational principle and the related mixed constitutive equations are valid for the thermoelectroelastic analysis of multilayered structures embedding piezoelectric materials. The improvements of the proposed principle were demonstrated by referring to a numerical assessment of its application to finite element modeling of piezoelectric plates.*
Keywords: mixed variational principles, thermopiezoelasticity, layered plates.

Riassunto. *Il presente articolo propone l'estensione del teorema variazionale misto di Reissner, in origine pensato per analisi puramente meccanici [1], a problemi termo-elettro-meccanici basandosi sui principi termodinamici e le equazioni di Maxwell. Il principio variazionale così ottenuto e le corrispondenti relazioni costitutive trovano applicazione nell'analisi di piastre multistrato includenti materiali piezoelettrici sottoposti contemporaneamente a carichi di diversa natura: meccanici, elettrici e termici. I miglioramenti ottenibili in termini di accuratezza rispetto ai modelli classici sono comprovati da una serie di risultati numerici su casi di letteratura e riportati in forma di grafici comparativi.*
Parole chiave: principi variazionali misti, termopiezoelasticità, piastre multistrato.

1. Introduction

During the last decades, significant efforts have been devoted to the research and development of the so-called *smart* structures. The word *smart* is used here

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to redefine the concept of structures from a conventional passive elastic system to an active or adaptive system with its own self-sensing, diagnosis and control capabilities. These advanced structures can be designed to actively react to disturbance forces in the attempt to preserve structural integrity and, at the same time, maintaining, or even improving the level of performance. Smart structures are distinguished from conventional ones by the presence of integrated actuator, sensor and controller elements. In a typical application, the sensors are used to monitor the mechanical response of the structure through changes in the displacements, strains or accelerations. Once an undesirable structural response is detected by the sensors, a controller generates an ad-hoc input to the actuators. These respond to the input producing a corresponding change in the mechanical response of the structure in order to maintain it into an acceptable state.

Great potential can be found in advanced aerospace structural applications. In particular, the capability of *smart* structures to sense and adapt to the environment leads to a wide range of applications such as vibration suppression of aircraft structures, noise reduction of helicopter rotors, health monitoring, aeroelastic control of lifting surfaces, shape adaption of aerodynamic surfaces, shape control of optical and electromagnetic devices. Classical examples of mechanically adaptive structures are conventional aircraft wings with articulated leading and/or trailing edge surfaces. On the side of sensors particularly interesting are the possibilities of these structures to detect displacements, strains, temperature, heat flow or the presence and accumulation o damage.

A variety of different materials can be utilized in smart structures applications, depending on if they are controlled through electric, magnetic, thermal or light energy. Some of the most common active materials include: piezoelectric materials, shape memory alloys, fiber optics, electrostrictive materials and magnetostrictive materials. Of all these, only piezoelectric materials can to be used effectively both as actuator and sensor elements.

Another advantage of piezoelectric materials which can explain their widespread use in structural engineering applications is the simple integration with composite structures. Composite structures integrating piezoelectric materials offer the possibility to combine the low density, superior mechanical and thermal properties of composite materials along with the sensing, actuation and control. An adequate knowledge of the deflections and stresses in these structures is of prime interest for the structural analysts. The excessive stress levels at layer interfaces are, in fact, often the predominant cause of failure of laminated composite structures generating failure mechanisms such as the debonding of layers and the developing longitudinal cracks. Thus, an accurate description of local stress fields in each layer becomes mandatory also in the first stages of the design.

Although Mindlin derived the thermopiezoelectric equations for plates in 1974, limited research has been performed on this topic until the last decade of the twentieth century. A variety of analytical and numerical models has been developed to study the electroelastic response of laminated structures embedding piezoelectric materials. The recent advances in this field as been illustrated by Benjeddou in 2000 [2] basing on the review of more than 100 papers present in literature. A review of the theoretical developments in piezothermoelasticity has been given by Tauchert [3].

All the classical models based on the Kirchhoff assumptions result to be adequate only for thin laminates and even if the first-order shear deformation theory (FSDT) is appropriate for slightly thicker plates, all these models are not able to reproduce what is know as Zig-Zag (ZZ) effect for the displacement fields and to satisfy the Interlaminar Continuity (IC) for transverse stresses. In addition, for piezoelectric plates, lower order Equivalent Single Layer models such as CLT or FSDT, lead to a compromised electromechanical coupling since the electric field is coupled to an oversimplified displacement field. An enhanced kinematics can be achieved by introducing a layer-wise description of the unknowns as in [4]. It has been demonstrated how a higher order layer-wise model permits the obtained displacement field to be as accurate as that of a full 3-d model.

In [5],[6] ZZ and IC were summarized by introducing the acronyms C_z^0-requirements. That is, both displacement and transverse stresses must be C^0–continuous functions in the thickness z direction in fact, the C^1 thickness continuity prevents the displacement field from presenting localized kinking at material interfaces where the mechanical properties change resulting in a loss of transverse stress equilibrium. The fulfillment of C_z^0–Requirements is a crucial point of two dimensional modeling of multilayered structures. Many amendments to classical theories have been proposed to fulfill partially the C_z^0-Requirements, see [6], [7].

A possible way to fulfill a-priori the C_z^0–requirements without any postprocessing procedure (such as those used in most of the available analyses) is to refer to mixed formulations [8]. As recently pointed out by Batra: *Mixed variational principles are important in theory and many applications in elasticity, for example, in the solution of problems by the finite element method, the mixed variational principles help determine accurately the stresses, displacements and the electric field* [9]. The Reissner's Mixed Variational Theorem as derived in [1] for a pure mechanical analysis is herein extended to thermopiezoelastic analysis and the related mixed constitutive equations are derived in analogy to what done by D'Ottavio [10] for the piezoelastic case and a brief numerical assessment of its application to finite element modeling of piezoelectric plates will be presented.

2. Generalized assumptions on the geometry and the materials

A multilayered plate is a laminate obtained stacking rectangular layers until the desired thickness and stiffness are reached. Generally each layer can be made of any kind of material, but in this work fiber reinforced composites are taken into account. In particular, our interest will be focused on laminates made by unidirectional fiber-reinforced laminae. The term unidirectional indicates that in each lamina all the fibers are lied along the same direction. In general in the building of a laminate, each lamina can be placed with a different orientation and this permits to design the right stiffness and strength to match the structural requirements. The sequence of the various lamina orientations is indicated as the *stacking sequence*.

If it's true that laminated structures give the designer a superior flexibility, on the other hand it's necessary, for their theoretical analysis, to introduce two different coordinate system: the material and the laminate one. The first one is given for each lamina and is commonly indicated by the axes *1,2,3*. The material coordinate 1-axis is taken to be parallel to the fiber direction, the 2-axis is transverse to the fiber direction in the plane of the lamina while the 3-axis is perpendicular to the lamina (see fig. 1a). The laminate coordinate system, indicated by the axes x,y,z is shown in fig. (1b) for a plate with 4 layers and dimensions a,b and h. All the material data such as Young moduli and piezoelectric coefficients, are given in the material reference system instead, the analysis of the structure is made in the laminate coordinate system. This implies that the governing equations are to be written first in the material coordinate system and then opportunely *rotated* to the laminate one.

In the present work the following notations are going to be used: standard tensor notation is used in a three-dimensional Euclidean space Ψ; at the same time, Einstein's summation convention is implied over repeated indices. A comma denotes partial differentiation with respect to the indicated space coordinate while a superposed dot stands for time differentiation. The symbol Ω denotes a regular, finite and bounded region contained in the space Ψ and Γ its boundary surface. Accordingly, $\bar{\Omega}$ indicates the closure of the thermopiezoelastic region. Latin indices run from 1 to 3.

The piezoelectric materials considered in this work are polarized polycrystalline ceramic materials with the crystal symmetry associated to the crystallographic class 2 mm of the hexagonal crystal system (see [11]) polarized along the thickness direction and as usual for composites, are assumed to be linear and orthotropic. This implies that in the laminate reference system *x,y,z* and using the single-subscript notation, the constitutive equations take the following compact form:

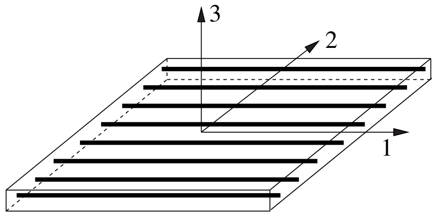
$$\sigma^{k} = C^{k}\varepsilon^{k} - e^{kT}E^{k} - \lambda^{k}\theta^{k} \tag{1}$$

$$D^{k} = e^{k}\varepsilon^{k} + \varepsilon^{k}E^{k} + p^{k}\theta^{k} \tag{2}$$

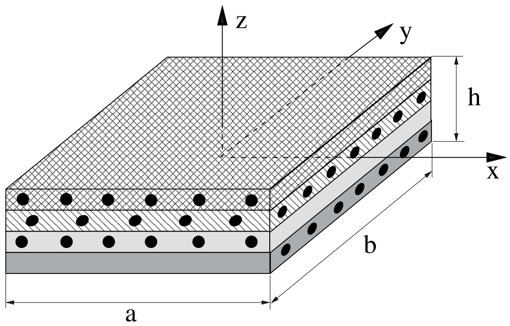
$$\eta^{k} = \lambda^{kT}\varepsilon^{k} + p^{kT}E^{k} + \alpha^{k}\theta^{k} \tag{3}$$

$$h^{k} = \kappa^{k}m^{k} \tag{4}$$

where:
σ^{k} is the symmetric Cauchy stress tensor
C^{k} is the matrix of the elastic moduli
ε^{k} is the symmetric Lagrange strain tensor
e^{k} is the matrix of the piezoelectric constants
E^{k} is the quasi-static electric field
λ^{k} is the vector of stress-temperature coefficients
θ^{k} is the temperature change from the reference one θ^{0}
D^{k} is the electric displacement vector
ε^{k} is the permittivity matrix
p^{k} is the vector of the pyroelectric coefficients
η is the entropy density
α^{k} is the vector of thermal expansion coefficients
h^{k} is the heat flux vector,
κ^{k} is the matrix of the thermal conductivities
m^{k} is the heat strain vector
and the superscripts k and T respectively are used to indicate that these equations are valid for the k-th lamina and a transposition of arrays.



(a) Material reference system



(b) Laminate reference system

Figure 1: Coordinate systems for multilayered plates.

From this point on stresses and strains are going to be separated in in-plane and normal components denoted respectively by the subscripts p and n. For an orthotropic lamina this separation gives:

$$\sigma_p^k = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}^k = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}^k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}^k + \begin{bmatrix} 0 & 0 & C_{13} \\ 0 & 0 & C_{23} \\ 0 & 0 & C_{36} \end{bmatrix}^k \begin{bmatrix} \varepsilon_5 \\ \varepsilon_4 \\ \varepsilon_3 \end{bmatrix}^k$$
$$= C_{pp}^k \, \varepsilon_p^k + C_{pn}^k \, \varepsilon_n^k \qquad (5)$$

$$\sigma_n^k = \begin{bmatrix} \sigma_5 \\ \sigma_4 \\ \sigma_3 \end{bmatrix}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13} & C_{23} & C_{36} \end{bmatrix}^k \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}^k + \begin{bmatrix} C_{55}^k & C_{45} & 0 \\ C_{45}^k & C_{44} & 0 \\ 0 & 0 & C_{33} \end{bmatrix}^k \begin{bmatrix} \varepsilon_5 \\ \varepsilon_4 \\ \varepsilon_3 \end{bmatrix}^k$$
$$= C_{pn}^{kT} \, \varepsilon_p^k + C_{nn}^k \, \varepsilon_n^k \qquad (6)$$

The same separation is made on the piezoelectric stiffness e of a PZT material obtaining:

$$e_p^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} & e_{32} & e_{36} \end{bmatrix}^k \quad \text{and} \quad e_n^k = \begin{bmatrix} e_{15} & e_{14} & 0 \\ e_{25} & e_{24} & 0 \\ 0 & 0 & e_{33} \end{bmatrix}^k \tag{7}$$

and on the stress-temperature coefficients λ_i:

$$\lambda_p^k = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_6 \end{bmatrix}^k \qquad \lambda_n^k = \begin{bmatrix} 0 \\ 0 \\ \lambda_3 \end{bmatrix}^k \tag{8}$$

As usual in literature pure-elastic laminae will be treated as piezoelectric ones with piezoelectric coefficients set to zero.

3. Extension of RMVT to thermopiezoelasticity

The extension of the Reissner's variational theorem to thermoelastic analysis can be achieved following the path illustrated in [1] and then to derive the new form of the constitutive equations. The coupling between the mechanical, electrical and thermal fields can be established using thermodynamic principles and Maxwell's relations and assuming the existence of an enthalpy function $H = H(\varepsilon, E, \theta)$

$$\begin{aligned} H(\varepsilon, E, \theta) = &\frac{1}{2}\varepsilon_p^T C_{pp}\varepsilon_p + \frac{1}{2}\varepsilon_p^T C_{pn}\varepsilon_n + \frac{1}{2}\varepsilon_n^T C_{np}\varepsilon_p + \frac{1}{2}\varepsilon_p^T C_{nn}\varepsilon_n \\ &- E^T e_p\varepsilon_p - E^T e_n\varepsilon_n - \frac{1}{2}E^T\varepsilon E \\ &- \varepsilon_p^T\lambda_p\theta - -\varepsilon_n^T\lambda_n\theta - E^T p\,\theta - \frac{\rho c_v}{2T_0}\theta^2 \end{aligned} \tag{9}$$

As in [1], the enthalpy function is split into two parts: a first term including all the contributions coming from the transverse and normal mechanical strains and a second one for remaining ones.

$$H(\varepsilon, E, \theta) = H_0(\varepsilon_p, E, \theta) + H_1(\varepsilon_p, \varepsilon_n, E, \theta) \tag{10}$$

where the explicit expressions of H_0 and H_1 can be easily recovered from eq.(9) as:

$$\begin{aligned} H_0(\varepsilon_p, E, \theta) = & \frac{1}{2}\varepsilon_p^T C_{pp}\varepsilon_p \\ & - E^T e_p \varepsilon_p - \frac{1}{2} E^T \varepsilon E \\ & - \varepsilon_p^T \lambda_p \theta \\ & - E^T p\, \theta - \frac{\rho c_v}{2T_0}\theta^2 \\ H_1(\varepsilon_p, \varepsilon_n, E, \theta) = & \frac{1}{2}\varepsilon_p^T C_{pn}\varepsilon_n + \frac{1}{2}\varepsilon_n^T C_{np}\varepsilon_p + \frac{1}{2}\varepsilon_p^T C_{nn}\varepsilon_n \\ & - E^T e_n \varepsilon_n \\ & - \varepsilon_n^T \lambda_n \theta \end{aligned} \tag{11}$$

The mixed functional is now written introducing the Lagrange multiplier σ_{nM} and takes the following form:

$$\begin{aligned} \delta\Big\{ \int_{t_0}^{t_1} dt \int_\Omega & \left[H + \sigma_{nM}^T(\varepsilon_{nG} - \varepsilon_{nC}) - \frac{1}{2}\rho \dot{u}^T \dot{u} - b^T u \right] d\Omega \\ & - \int_\Gamma \left(\bar{t}^T u - \bar{Q}\Phi - \bar{h}\theta \right) d\Gamma \Big\} = 0 \end{aligned} \tag{12}$$

where the subscripts G and C indicate the strains defined through the geometric relations and those obtained by the constitutive equations, b are the internal body forces, $\bar{t}$ are the imposed surface tractions, $\bar{Q}$ is the imposed surface charge and $\bar{h}$ is the imposed heat flux.
The Euler equations associated to the functional are:

$$\sigma_{nM} = \frac{\delta H}{\delta \varepsilon_n} = \frac{\delta H_1}{\delta \varepsilon_n} = \frac{1}{2}C_{pn}^T \varepsilon_p + \frac{1}{2}C_{np}\varepsilon_p + C_{nn}\varepsilon_n - e_n^T E - \lambda_n \theta \tag{13}$$

since $C_{pn}^T = C_{np}$, eq.(13) can be rewritten as:

$$\sigma_{nM} = C_{np}\varepsilon_p + C_{nn}\varepsilon_n - e_n^T E - \lambda_n \theta \tag{14}$$

Now, inverting eq.(14) for ε_n, the transverse and normal strains can be obtained as functions of the transverse and normal stresses, the in-plane strains, the electric field and the temperature:

$$\varepsilon_{nC} = C_{nn}^{-1}\left(-C_{np}\varepsilon_{pG} + e_n^T E + \lambda_n \theta + \sigma_{nM} \right) \tag{15}$$

At this step, a complementary function G^* is defined through a partial Legendre transformation:

$$G^*(\varepsilon_p, \sigma_n M, E, \theta) = \sigma_{nM}{}^T \varepsilon_{nC} - H_1 \rightarrow H_1 = \sigma_{nM}^T \varepsilon_{nC} - G^* \qquad (16)$$

where:

$$\varepsilon_{nC} = \frac{\delta G^*}{\delta \sigma_{nM}} \qquad (17)$$

Substituting eq.(10) and eq.(16) into the mixed functional of eq.(12), the variational statement can be written as:

$$\delta\Big\{ \int_{t_0}^{t_1} dt \int_{\Omega} \Big[H_0 - G^* + \sigma_{nM}^T \varepsilon_{nG} - \frac{1}{2}\rho \dot{u}^T \dot{u} - b^T u \Big] d\Omega - \int_{\Gamma} \Big(\bar{t}^T u - \bar{Q}\Phi - \bar{h}\theta \Big) d\Gamma \Big\} = 0 \qquad (18)$$

Applying the variational operator to the functional, the mixed variational theorem reads:

$$\int_{t_0}^{t_1} dt \Big\{ \int_{\Omega} \big[\delta H_0 - \delta G^* + \delta\sigma_{nM}{}^T \varepsilon_{nG} + \delta\varepsilon_{nG}{}^T \sigma_{nM} \big] d\Omega + \int_{\Omega} \delta u^T \rho \ddot{u}\, dV - \int_{\Omega} \delta u^T b\, dV \Big\} - \int_{\Gamma} \Big(\delta u^T \bar{t} - \bar{Q}\Phi - \bar{h}\theta \Big) d\Gamma \Big\} = 0 \qquad (19)$$

writing out each term explicitly:

$$\begin{aligned} \delta H_0 = &\delta H_0(\varepsilon, E, \theta) = \delta\varepsilon_{pG}^T (C_{pp}\varepsilon_p - e_p^T E - \lambda_p \theta) \\ &- \delta E^T (e_p \varepsilon_p - \varepsilon E - p\,\theta) + \delta\theta(-\lambda_p^T \varepsilon_p - p^T E - \frac{\rho c_v}{T_0}\theta) \end{aligned} \qquad (20)$$

$$\begin{aligned} \delta G^* = &- \delta\varepsilon_{pG}^T (\frac{\delta H_1}{\delta \varepsilon_p}) - \delta E^T (\frac{\delta H_1}{\delta E}) - \delta\theta(\frac{\delta H_1}{\delta\theta}) + \delta\sigma_{nM}^T (\frac{\delta G^*}{\delta\sigma_{nM}}) = \\ &- \delta\varepsilon_{pG}^T C_{pn}\varepsilon_n + \delta E^T e_n^T \varepsilon_n + \delta\theta\lambda_n^T \varepsilon_n + \delta\sigma_{nM}^T \varepsilon_{nC} \end{aligned} \qquad (21)$$

Finally, substituting eqs. (20), (21) into eq.(19) the mixed variational theorem takes the form:

$$\int_{t_0}^{t_1} dt \Big\{ \int_{\Omega} \Big[\delta\varepsilon_{pG}^{T}(C_{pp}\varepsilon_p - e_p^T E - \lambda_p\theta) - \delta E^T(e_p\varepsilon_p - \varepsilon E - p\,\theta)$$
$$+\delta\theta(-\lambda_p^T\varepsilon_p - p^T E - \frac{\rho c_v}{T_0}\theta) + \delta\varepsilon_{pG}^T C_{pn}\varepsilon_{nC} - \delta E^T e_n^T\varepsilon_n$$
$$-\delta\theta\lambda_n^T\varepsilon_{nC} - \delta\sigma_{nM}^T\varepsilon_{nC} + \delta\sigma_{nM}{}^T\varepsilon_{nG} + \delta\varepsilon_{nG}{}^T\sigma_{nM}\Big] d\Omega +$$
$$\int_{\Omega}\delta u^T\rho\ddot{u}\,dV - \int_{\Omega}\delta u^T b\,dV\Big\} - \int_{\Gamma}\Big(\delta u^T\bar{t} - \bar{Q}\Phi - \bar{h}\theta\Big)dA\Big\} = 0 \tag{22}$$

Substituting eq. (15) into eq.(22), the constitutive equations assume now the following expressions:

$$\varepsilon_{nC} = (C_{nn}^{-1}C_{np})\varepsilon_{pG} + (C_{nn}^{-1})\sigma_{nM} + (C_{nn}^{-1}e_n^T)E + (C_{nn}^{-1}\lambda_n)\theta \tag{23}$$

$$\begin{aligned}\sigma_{pC} =&(C_{pp} - C_{pn}C_{nn}^{-1}C_{np})\varepsilon_{pG} + (C_{pn}C_{nn}^{-1})\sigma_{nM}\\ &+(-e_p^T + C_{pn}C_{nn}^{-1}e_n^T)E + (-\lambda_p + C_{pn}C_{nn}^{-1}\lambda_n)\theta\end{aligned} \tag{24}$$

$$\begin{aligned}D_C =&(e_p - e_nC_{nn}^{-1}C_{np})\varepsilon_{pG} + (e_nC_{nn}^{-1})\sigma_{nM}\\ &+(\varepsilon + e_nC_{nn}^{-1}e_n^T)E + (e_nC_{nn}{}^{-1}\lambda_n + p)\theta\end{aligned} \tag{25}$$

$$\begin{aligned}\eta =&(\lambda_p^T - \lambda_n^TC_{nn}^{-1}C_{np})\varepsilon_{pG} + (\lambda_n^TC_{nn}^{-1})\sigma_{nM}\\ &+(p^T + \lambda_n^TC_{nn}^{-1}e_n^T)E + (\frac{\rho c_v}{T_0} + \lambda_n^TC_{nn}^{-1}\lambda_n)\theta\end{aligned} \tag{26}$$

Now introducing these new definitions:

$$\hat{C}_{pp} = C_{pp} - C_{pn}C_{nn}{}^{-1}C_{np} \tag{27}$$
$$\hat{C}_{pn} = C_{pn}C_{nn}{}^{-1} \tag{28}$$
$$\hat{C}_{se} = C_{pn}C_{nn}{}^{-1}e_n^T - e_p^{kT} = \hat{C}_{pn}e_n^T - e_p^T \tag{29}$$
$$\hat{C}_{s\theta} = C_{pn}C_{nn}{}^{-1}\lambda_n - \lambda_p = \hat{C}_{pn}\lambda_n - \lambda_p \tag{30}$$
$$\hat{C}_{np} = -C_{nn}{}^{-1}C_{np} \tag{31}$$
$$\hat{C}_{nn} = C_{nn}{}^{-1} \tag{32}$$

$$\hat{C}_{de} = C_{nn}^{-1} e_n^T = \hat{C}_{nn} e_n^T \tag{33}$$
$$\hat{C}_{d\theta} = C_{nn}^{-1} \lambda_n = \hat{C}_{nn} \lambda_n \tag{34}$$
$$\hat{C}_{ed} = e_p - e_n C_{nn}^{-1} C_{np} = e_p + e_n \hat{C}_{np} \tag{35}$$
$$\hat{C}_{es} = e_n C_{nn}^{-1} = e_n \hat{C}_{nn} \tag{36}$$
$$\hat{C}_{ee} = \varepsilon + e_n C_{nn}^{-1} e_n^{kT} = \varepsilon + e_n^k \hat{C}_{nn} e_n^{kT} \tag{37}$$
$$\hat{C}_{e\theta} = e_n C_{nn}^{-1} \lambda_n + p = \hat{C}_{es} \lambda_n + p \tag{38}$$
$$\hat{C}_{\theta d} = \lambda_p^T - \lambda_n^T C_{nn}^{-1} C_{np} \tag{39}$$
$$\hat{C}_{\theta s} = \lambda_n^T C_{nn}^{-1} \tag{40}$$
$$\hat{C}_{\theta e} = p^T + \lambda_n^T C_{nn}^{-1} e_n^T \tag{41}$$
$$\hat{C}_{\theta\theta} = \frac{\rho c_v}{T_0} + \lambda_n^T C_{nn}^{-1} \lambda_n \tag{42}$$

and pointing out that:

$$\hat{C}_{nn} = \hat{C}_{nn}^T \qquad \hat{C}_{pp} = \hat{C}_{pp}^T \qquad \hat{C}_{ee} = \hat{C}_{ee}^T \tag{43}$$
$$\hat{C}_{pn} = -\hat{C}_{np}^T \qquad \hat{C}_{de} = \hat{C}_{es}^T \qquad \hat{C}_{se} = -\hat{C}_{ed}^T \tag{44}$$
$$\hat{C}_{\theta d} = -\hat{C}_{s\theta}^T \qquad \hat{C}_{\theta s} = -\hat{C}_{d\theta}^T \qquad \hat{C}_{\theta e} = -\hat{C}_{e\theta}^T \tag{45}$$

because:

$$C_{pn} = C_{np}^{kT} \qquad \text{and} \qquad C_{nn} = C_{nn}^T \tag{46}$$

The mixed constitutive equations take the following compact form:

$$\sigma_{pC} = \hat{C}_{pp} \varepsilon_{pG} + \hat{C}_{pn} \sigma_{nM} + \hat{C}_{se} E_G + \hat{C}_{s\theta} \theta_M \tag{47}$$
$$\varepsilon_{nC} = \hat{C}_{np} \varepsilon_{pG} + \hat{C}_{nn} \sigma_{nM} + \hat{C}_{de} E_G + \hat{C}_{d\theta} \theta_M \tag{48}$$
$$D_C = \hat{C}_{ed} \varepsilon_{pG} + \hat{C}_{es} \sigma_{nM} + \hat{C}_{ee} E_G + \hat{C}_{e\theta} \theta_M \tag{49}$$
$$\eta = \hat{C}_{\theta d} \varepsilon_{pG} + \hat{C}_{\theta s} \sigma_{nM} + \hat{C}_{\theta e} E + \hat{C}_{\theta\theta} \theta \tag{50}$$

Using eqs.(23)-(26) the variational theorem can be rewritten in a more compact way:

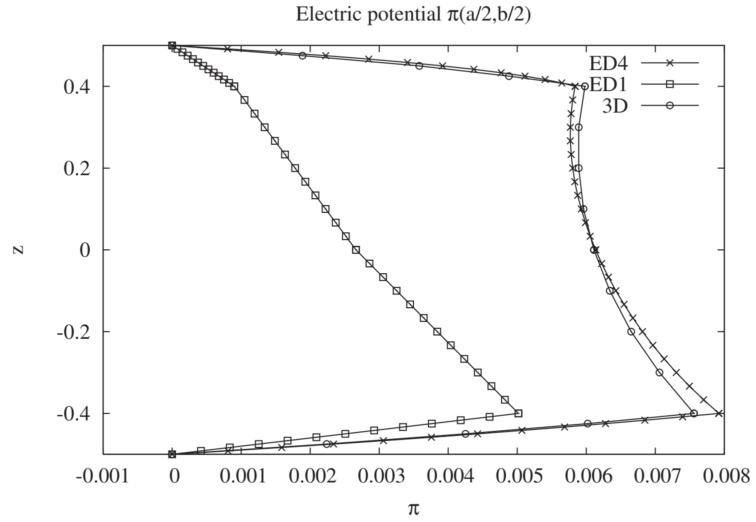
$$\int_{t_0}^{t_1} dt \left\{ \int_{\Omega} \left[\delta\varepsilon_{pG}^{T} \sigma_{pC} - \delta E_{G}^{T} D_{C} + \delta\varepsilon_{nG}{}^{T} \sigma_{nM} + \delta\sigma_{nM}{}^{T} (\varepsilon_{nG} - \varepsilon_{nC}) - \delta\theta\eta_{C} \right] d\Omega + \int_{\Omega} \delta u^{T} \rho \ddot{u} \, d\Omega - \int_{\Omega} \delta u^{T} b \, d\Omega - \int_{\Gamma} \left(\delta u^{T} \bar{t} - \delta\Phi \bar{Q} - \delta\theta \bar{h} \right) d\Gamma \right\} = 0 \tag{51}$$

If a thermal uncoupled problem is addressed and thus the temperature field is not considered as an unknown of the problem but it is treated the same way of an external load and its values along the thickness of the plate are imposed, the contribution given by the entropy as well as the fourth constitutive equation can be neglected since the variation of the temperature is null.

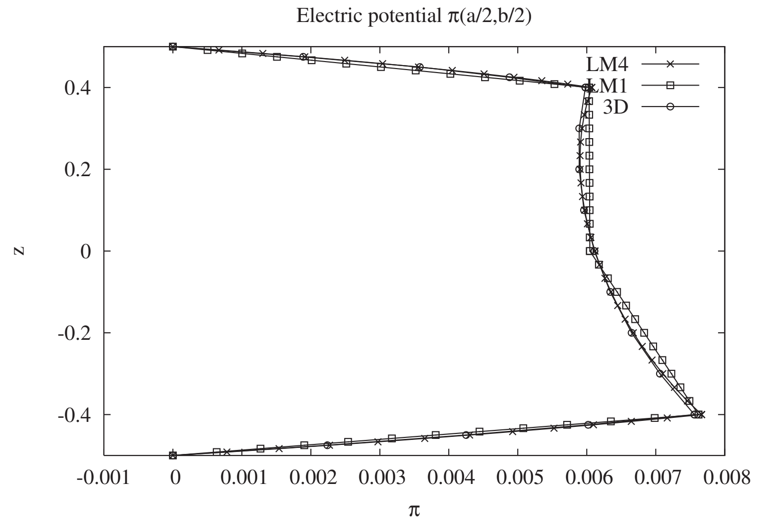
The expressions of the *mixed* constitutive equations result in a manipulation of the original ones and could have been obtained by solving eqs.(1)-(3) for the new independent variables.

4. Numerical Assessment and conclusions

The improvements given by the use of the present variational principles have been verified developing a set of plate finite elements and applying them in solving several benchmarking cases present in literature. In [12], new finite elements for the coupled analysis of multilayered plates embedding piezoelectric layers have been developed extending the work by Carrera and Demasi [13]. In that work the performance of several finite elements based on the Reissner's mixed variational theorem has been compared to that of finite elements derived from the classical principle of virtual displacement. The various elements differ from one another by the laminate kinematic assumptions upon which each of them was based. In particular, basing on a unified formulation, see [8], both the effects of the order of the expansion as well as the description along the thickness of the plate of the problem unknowns in the response of composite plates has been investigated. In so doing, combining the possibilities given by global (Equivalent Single Layer models ESL) or layerwise (LW) kinematic assumptions and varying the order of the expansion of the unknowns up to the fourth one as well as the variational statement, more than 24 different plate elements have been tested.

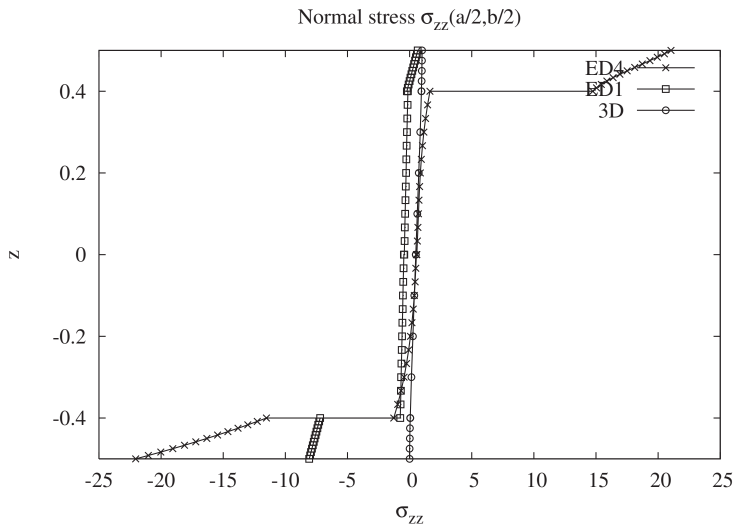


(a) $\phi(\frac{a}{2},\frac{b}{2})$ Comparison of the PVD ESL models (first and fourth order) and 3D solution.

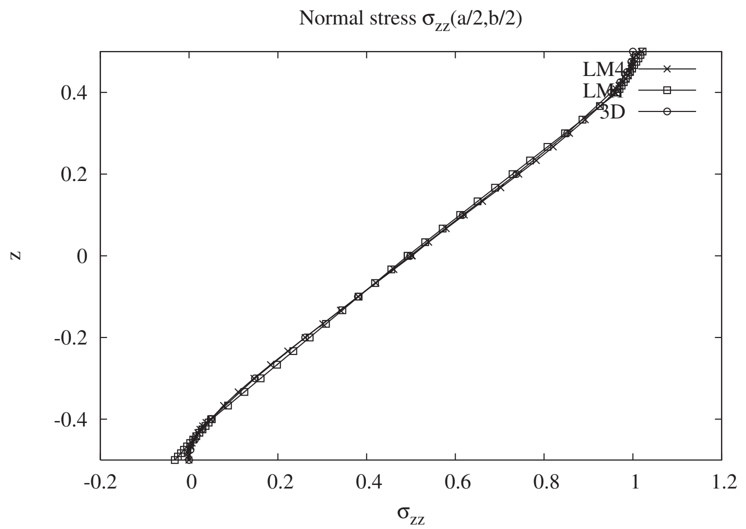


(b) $\phi(\frac{a}{2},\frac{b}{2})$ Comparison of the RMVT LW models (first and fourth order) and 3D solution.

Figure 2: Electric potential $\phi(\frac{a}{2},\frac{b}{2})$ for a plate with thick-to-side ratio S=4.



(a) $\sigma_{zz}(\frac{a}{2},\frac{b}{2})$ Comparison of the PVD ESL models (first and fourth order) and 3D solution.



(b) $\sigma_{zz}(\frac{a}{2},\frac{b}{2})$ Comparison of the RMVT: LW models (first and fourth order) and 3D solution.

Figure 3: Normal stress $\sigma_{zz}(\frac{a}{2},\frac{b}{2})$ for a plate with thick-to-side ratio S=4.

The assessment has been conducted referring to the analysis case from [14] and results are given for a simply supported cross-ply laminate [0/90] composed of an elastic material with piezoelectric layers bonded to the upper and lower surfaces. The plate has side lengths $a = b$ and a total thickness oh h. The elastic layers have thickness of $0.4h$ and the thickness of the piezoelectric layers is $0.1h$. From the results reported in [12], the following conclusions can be pointed out for thermopiezoelastic analysis of multilayered composite plates:

- Independently of the variational principle, the ESL first order models lead to an inaccurate description of the displacement and electric fields especially for thick plates: compare figs. (2a) and (2b).
- Slight differences are appreciable in the changing of the variational principle for displacements and electric fields but the use of the RMVT is fundamental in first order models for the modelization of normal stress: compare figs. (3a) and (3b).
- The use of the mixed variational principle improves the results of the electric potential especially for lower order models.
- Normal and transverse stresses results are satisfactory only for mixed higher order LW models.

The numerical assessment confirm that the mixed variational principle of Reissner can be a powerful tool for the design of multilayered composite plate embedding piezoelectric materials and result particularly suitable for delamination problems.

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