Available online at www.sciencedirect.com
ScienceDirect
Composite Structures 77 (2007) 341–352
COMPOSITE STRUCTURES
www.elsevier.com/locate/compstruct

On the use of transverse shear stress homogeneous and non-homogeneous conditions in third-order orthotropic plate theory

Erasmo Carrera *

Aerospace Department, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Available online 12 September 2005

Abstract

This paper discusses the influence of non-homogeneous transverse shear stresses conditions on the accuracy of plate theories formulated in terms of displacement variables. The case of a third-order plate theories is considered and the attention has been focused on cylindrical bending problems. The following three models are developed and compared: (1) the 'original' third model with five displacement variables; (2) the 'reduced' third-order model with three displacement variables obtained by imposing 'homogeneous' stress conditions with correspondence to the plate top-surface; (3) the modification of case 2 which considers 'non-homogeneous' stress conditions. Variationally consistent governing equations have been derived by employing the principle of virtual displacement in the linear-elastic, static case. Closed form solutions results have been obtained for both stresses and displacements in the case of harmonic loadings and simply supported boundary conditions. The following main conclusions have been reached by the conducted numerical investigation: The use of non-homogeneous boundary conditions lead to a general improvement with respect to model 2 results. The 'original' model leads in general to better response evaluation than the other two models; an exception is made for the transverse shear stresses calculated by Hooke law for which case model 3 leads to the most accurate results.

Keywords: Plate theories; Stress conditions; Orthotropic plates; Refined models

1. Introduction

Early and recent developments of plate theories make use of assigned conditions on stresses to enhance the performance of formulations with only displacement variables. A relevant example of such applications is the so called third-order theories proposed first by Vlasov [1]. This was variationally substantiated and extended to laminated structures by Reddy [2]. This type of theory is often denoted as Vlasov–Reddy Theory (VRT). An historical review on related contributions has been provided in [3]. Improved theories which are based on similar approach have been considered by Jemielita [4] and over-viewed by the same author [5] and by Lewinsky [6]. An historical review of so called Zig-Zag theories that make use of stress conditions to build improved theories for laminated structures, have been recently presented by Carrera [7].

The interest is in this paper focuses on third-order Vlasov-type theories. The homogeneous conditions on top/bottom plate surfaces are used in VRT to reduce the number of the unknown displacement variables related to a third-order expansion in the thickness direction z of the in-plane displacement components. The resulting theory has the main advantage of preserving the same number of unknowns of the well-known Reissner–Mindlin Theory (RMT), which has a linear

* Tel.: +39 011 564 6836; fax: +39 011 564 6899.
E-mail address: erasmo.carrera@polito.it

0263-8223/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruct.2005.07.010

expansion z, see [3]. Non-homogeneous stress conditions have been also discussed in the already mentioned work by Jemielita and Lewinsky. However, no numerical results are available which are directly evaluate the effect of these non-homogeneous conditions on the plate response. Such an evaluation requires the use of a variational tool that treats in a 'consistent' manner these stress conditions in the governing equations of a given plate.

The aim of the present work is to give numerical evaluations of the effect of non-homogeneous conditions that has been discussed above. Attention has been restricted to cylindrical bending problems of an orthotropic plates. In order to make evident both advantages and disadvantages of the inclusion of non-homogeneous conditions on stresses, the following three plate models have been implemented and compared.

- Model 'O5'. It is the plate theory related to the original third-order in-plane displacement fields. This will be denoted by the acronym 'O5' (in which 'O' states for original and '5' denotes the total number of displacement variables involved in the model).
- Model 'V3-H'. The 5 displacement variables of Model O5 are reduced to 3 by imposing the homogeneous conditions on transverse shear stress. It is denoted by the acronym V3-H (which state Vlasov, 3-order, Homogeneous conditions).
- Model 'V3-NH'. 'O5' model is now reduced by enforcing non-homogeneous transverse stress conditions.

Preliminaries are quoted in Section 2. The governing equations related to the three models are given in Sections 3–5, respectively. These have been derived by direct application of the principle of virtual displacements (PVD) that permits to obtain governing equations in a form which is 'variationally consistent' with the made assumptions on displacements variables. Closed form solutions have been provided in the case of harmonic forms of applied loadings and simply supported boundary conditions. Numerical evaluations and conclusions are given the last section which compare the performances of the three considered models with respect to quasi-3D elasticity solutions.

2. Problem statement and preliminaries

Let us consider the cylindrical bending behavior of plate constituted by an orthotropic materials. L and h are the plate length and thickness, respectively. z and x are the thickness and in-plane coordinates, respectively. u and w are the displacement components in the in-plane and out-of-plane directions, respectively. Ω is a reference surface, that is chosen to coincide to the plate middle surface; Γ denoted the Ω boundary.

The plate is loaded on the top surface by transverse normal pressure $\bar{\sigma}_{zz}(x, h/2)$, and/or in-plane shear traction $\bar{\sigma}_{xz}(x, h/2)$. The bottom surface is considered free from loadings.

The constitutive equations which are consistent with the kinematic assumptions made in this paper are:

$$\sigma_{xx} = C_{11}\epsilon_{xx}, \quad \sigma_{xz} = C_{44}\epsilon_{xz} \tag{1}$$

where σ_{xx} and σ_{xz} are in-plane normal and transverse shear stresses while C_{11} and C_{44} are the stiffness coefficients of the considered orthotropic plates; ϵ_{xx} and ϵ_{xz} are the corresponding strains. In the case of small deflections, the strains can be expressed in terms of displacements by the following linear relations:

$$\epsilon_{xx} = u_{,x}, \quad \epsilon_{xz} = u_{,z} + w_{,x} \tag{2}$$

where comma denotes partial derivatives.

3. Model 'O5': the original third-order theory

3.1. Displacements model

The in-plane displacement u is assumed of third-order type in the z coordinate, while the transverse displacement w is considered as a constant. The resulting displacement model can be written:

$$\begin{aligned} u(x,z) &= u^0(x) + z\phi_1(x) + z^2\phi_2(x) + z^3\phi_3 \\ w(x,z) &= w^0(x) \end{aligned} \tag{3}$$

The apexes '0' denote values measured with correspondence to the reference surface Ω while ϕ_1, ϕ_2, ϕ_3 are additional variables that describe the trough-the-thickness variation of the in-plane displacements u. The displacement model related to RMT is obtained by simply neglecting ϕ_2 and ϕ_3 terms.

3.2. Governing equations

The governing differential equations are herein computed directly in terms of the unknown variables by employing the principle of virtual displacements (PVD) that in the static case states that the variation of internal work must be equilibrated by the variation of external work

$$\delta L_{\mathrm{i}} = \delta L_{\mathrm{e}} \tag{4}$$

δ denotes virtual variations. The external work related to the application of a transverse pressure and in-plane distribution of shear forces with correspondence to the top-plate surface is

$$\delta L_e = \int_\Omega \left[\bar{\sigma}_{zz}\left(x, \frac{h}{2}\right) \delta w \left(x, \frac{h}{2}\right) + \bar{\sigma}_{xz}\left(x, \frac{h}{2}\right) \delta u \left(x, \frac{h}{2}\right) \right] d\Omega \tag{5}$$

By introducing the variations

$$\begin{aligned} \delta w\left(x, \frac{h}{2}\right) &= \delta w^0, \\ \delta u\left(x, \frac{h}{2}\right) &= \delta u^0 + \frac{h}{2}\delta\phi_1 + \frac{h^2}{4}\delta\phi_2 + \frac{h^3}{8}\delta\phi_3 \end{aligned} \tag{6}$$

the external work becomes

$$\delta L_e = \int_\Omega \left[\bar{\sigma}_{zz}\delta w^0 + \bar{\sigma}_{xz}\left(\delta u^0 + \frac{h}{2}\delta\phi_1 + \frac{h^2}{4}\delta\phi_2 + \frac{h^3}{8}\delta\phi_3 \right) \right] d\Omega \tag{7}$$

The internal work is

$$\delta L_i = \int_V [\sigma_{xx}\delta\epsilon_{xx} + \sigma_{xz}\delta\epsilon_{xz}] dV \tag{8}$$

where $dV = d\Omega\, dz$ and

$$\begin{aligned} \epsilon_{xx} &= u^0_{,x} + z\phi_{1,x} + z^2\phi_{2,x} + z^3\phi_{3,x}, \\ \epsilon_{xz} &= \phi_1 + 2z\phi_2 + 3z^2\phi_3 + w_{,x} \end{aligned} \tag{9}$$

The following stress resultants are introduced:

$$(\mathscr{R}^{q(z)}_{xx}, \mathscr{R}^{q(z)}_{xz}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} q(z)(\sigma_{xx}, \sigma_{xz}) dz \tag{10}$$

in which $q(z)$ is a generic function of the thickness coordinate. It can assume one of the following forms:

$$z,\ z^2,\ z^3,\ 1,\ 2z \text{ and } 3z^2 \tag{11}$$

The internal work can be therefore written in terms of the introduced stress resultants

$$\begin{aligned} \delta L_i = \int_\Omega (&\mathscr{R}^1_{xx}\delta u^0_{,x} + \mathscr{R}^z_{xx}\delta\phi_{1,x} + \mathscr{R}^{z^2}_{xx}\delta\phi_{2,x} + \mathscr{R}^{z^3}_{xx}\delta\phi_{3,x} \\ &+ \mathscr{R}^1_{xz}\delta\phi_1 + \mathscr{R}^{2z}_{xz}\delta\phi_2 + \mathscr{R}^{3z^2}_{xz}\delta\phi_3 + \mathscr{R}^1_{xz}\delta w_{,x}) d\Omega \end{aligned} \tag{12}$$

Integration by part is required in order to move the derivatives from the virtual variations

$$\begin{aligned} \delta L_i = \int_\Omega (&-\mathscr{R}^1_{xx_{,x}}\delta u^0 - \mathscr{R}^z_{xx_{,x}}\delta\phi_1 - \mathscr{R}^{z^2}_{xx_{,x}}\delta\phi_2 \\ &- \mathscr{R}^{z^3}_{xx_{,x}}\delta\phi_3 + \mathscr{R}^1_{xz}\delta\phi_1 + \mathscr{R}^{2z}_{xz}\delta\phi_2 + \mathscr{R}^{3z^2}_{xz}\delta\phi_3 \\ &- \mathscr{R}^1_{xz_{,x}}\delta w) d\Omega + \int_\Gamma (\mathscr{R}^1_{xx}\delta u^0 + \mathscr{R}^z_{xx}\delta\phi_1 \\ &+ \mathscr{R}^{z^2}_{xx}\delta\phi_2 + \mathscr{R}^{z^3}_{xx}\delta\phi_3 + \mathscr{R}^1_{xz}\delta w) d\Gamma \end{aligned} \tag{13}$$

The equilibrium equations in the plate domain Ω are

$$\begin{aligned} \delta u^0 &: \quad -\mathscr{R}^1_{xx_{,x}} = \bar{\sigma}_{xz} \\ \delta w^0 &: \quad \mathscr{R}^1_{xz_{,x}} = \bar{\sigma}_{zz} \\ \delta\phi_1 &: \quad -\mathscr{R}^z_{xx_{,x}} + \mathscr{R}^1_{xz} = \frac{h}{2}\bar{\sigma}_{xz} \\ \delta\phi_2 &: \quad -\mathscr{R}^{z^2}_{xx_{,x}} + \mathscr{R}^{z^2}_{xx} = \frac{h^2}{4}\bar{\sigma}_{xz} \\ \delta\phi_3 &: \quad -\mathscr{R}^{z^3}_{xx_{,x}} + \mathscr{R}^{z^3}_{xx} = \frac{h^3}{8}\bar{\sigma}_{xz} \end{aligned} \tag{14}$$

The geometric/mechanical associated boundary conditions on the boundary Γ are

$$\begin{aligned} \delta u^0 &: \quad u^0 = \bar{u}^0 \text{ or } \mathscr{R}^1_{xx} = \bar{\mathscr{R}}^1_{xx} \\ \delta w^0 &: \quad w^0 = \bar{w}^0 \text{ or } \mathscr{R}^1_{xz} = \bar{\mathscr{R}}^1_{xz} \\ \delta\phi_1 &: \quad \phi_1 = \bar{\phi}_1 \text{ or } \mathscr{R}^z_{xx} = \bar{\mathscr{R}}^z_{xx} \\ \delta\phi_2 &: \quad \phi_1 = \bar{\phi}_1 \text{ or } \mathscr{R}^{z^2}_{xx} = \bar{\mathscr{R}}^{z^2}_{xx} \\ \delta\phi_3 &: \quad \phi_1 = \bar{\phi}_1 \text{ or } \mathscr{R}^{z^3}_{xx} = \bar{\mathscr{R}}^{z^3}_{xx} \end{aligned} \tag{15}$$

in which over-bar denote imposed values.

The governing equations in Ω, can be expressed in terms of unknown displacements as follows:

$$\begin{aligned} \delta u^0 :\quad & -C_{11}\int_{1\times 1} u^0_{,xx} - C_{11}\int_{1\times z}\phi_{1,xx} - C_{11}\int_{1\times z^2}\phi_{2,xx} \\ & - C_{11}\int_{1\times z^3}\phi_{3,xx} = \bar{\sigma}_{xz} \\ \delta w^0 :\quad & -C_{44}\int_{1\times 1} w^0_{,xx} - C_{44}\int_{1\times 1}\phi_{1,x} - C_{44}2\int_{1\times z}\phi_{2,x} \\ & - C_{44}3\int_{1\times z^2}\phi_{3,x} = \bar{\sigma}_{zz} \\ \delta\phi_1 :\quad & -C_{11}\int_{z\times 1} u^0_{,xx} - C_{11}\int_{z\times z}\phi_{1,xx} - C_{11}\int_{z\times z^2}\phi_{2,xx} \\ & - C_{11}\int_{z\times z^3}\phi_{3,xx} + C_{44}\int_{1\times 1} w^0_{,x} + C_{44}\int_{1\times 1}\phi_1 \\ & + C_{44}2\int_{1\times z}\phi_2 + C_{44}3\int_{1\times z^2}\phi_3 = \frac{h}{2}\bar{\sigma}_{xz} \\ \delta\phi_2 :\quad & -C_{11}\int_{z^2\times 1} u^0_{,xx} - C_{11}\int_{z^2\times z}\phi_{1,xx} - C_{11}\int_{z^2\times z^2}\phi_{2,xx} \\ & - C_{11}\int_{z^2\times z^3}\phi_{3,xx} + C_{44}2\int_{z\times 1} w^0_{,x} + C_{44}2\int_{z\times 1}\phi_1 \\ & + C_{44}4\int_{z\times z}\phi_2 + C_{44}6\int_{z\times z^2}\phi_3 = \frac{h^2}{4}\bar{\sigma}_{xz} \\ \delta\phi_3 :\quad & -C_{11}\int_{z^3\times 1} u^0_{,xx} - C_{11}\int_{z^3\times z}\phi_{1,xx} - C_{11}\int_{z^3\times z^2}\phi_{2,xx} \\ & - C_{11}\int_{z^3\times z^3}\phi_{3,xx} + C_{44}3\int_{z^2\times 1} w^0_{,x} + C_{44}3\int_{z^2\times 1}\phi_1 \\ & + C_{44}6\int_{z^2\times z}\phi_2 + C_{44}9\int_{z^2\times z^2}\phi_3 = \frac{h^3}{8}\bar{\sigma}_{xz} \end{aligned} \tag{16}$$

The following notation has been introduced:

$$\int_{q\times t} = \int_{-h/2}^{h/2} q(z)t(z)\mathrm{d}z$$

where $q(z)$ and $t(z)$ can assume any of the forms in Eq. (11).

3.3. Closed form solutions

Closed form solutions of Eqs. (14) and (15) can be obtained by assuming the following harmonic forms for the unknown variables and applied loadings:

$$(u^0, \phi_1, \phi_2, \phi_3) = (U, \Phi_1, \Phi_2, \Phi_3)\cos(\alpha x), \quad w^0 = W\sin(\alpha x)$$
$$\bar{\sigma}_{xz} = \overline{S}_{xz}\cos(\alpha x), \quad \bar{\sigma}_{zz} = \overline{S}_{zz}\sin(\alpha x) \tag{17}$$

where capital letters denote amplitudes; $\alpha = \frac{m\pi}{L}$ and m denote the wave numbers.

The assumed forms Eqs. (17) coincide to so-called simply supported boundary conditions. These automatically fulfill the boundary conditions Eq. (15) while the equilibrium conditions Eq. (16) lead to the following linear system of algebraic equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{Bmatrix} U \\ W \\ \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{Bmatrix} \tag{18}$$

in which,

$$a_{11} = C_{11}\int_{1\times 1}\alpha^2, \quad a_{12} = 0, \quad a_{13} = C_{11}\int_{1\times z}\alpha^2,$$
$$a_{14} = C_{11}\int_{1\times z^2}\alpha^2, \quad a_{15} = C_{11}\int_{1\times z^3}\alpha^2$$
$$a_{22} = C_{44}\int_{1\times 1}\alpha^2, \quad a_{23} = C_{44}\int_{1\times 1}\alpha,$$
$$a_{24} = 2C_{44}\int_{1\times z}\alpha, \quad a_{25} = 3C_{44}\int_{1\times z^2}\alpha,$$
$$a_{33} = C_{11}\int_{z\times z}\alpha^2 + C_{44}\int_{1\times 1}, \quad a_{34} = C_{11}\int_{z\times z^2}\alpha^2 + 2C_{44}\int_{1\times z},$$
$$a_{35} = C_{11}\int_{z\times z^3}\alpha^2 + 3C_{44}\int_{1\times z^2},$$
$$a_{44} = C_{11}\int_{z^2\times z^2}\alpha^2 + 4C_{44}\int_{z\times z},$$
$$a_{45} = C_{11}\int_{z^2\times z^3}\alpha^2 + 6C_{44}\int_{z\times z^2},$$
$$a_{55} = C_{11}\int_{z^3\times z^3}\alpha^2 + 9C_{44}\int_{z^2\times z^2}$$
$$p_1 = \overline{S}_{xz}, \quad p_2 = \overline{S}_{zz}, \quad p_3 = \frac{h}{2}\overline{S}_{xz},$$
$$p_4 = \frac{h^2}{4}\overline{S}_{xz}, \quad p_5 = \frac{h^3}{8}\overline{S}_{xz} \tag{19}$$

The matrix is symmetric and taking into account the values assumed by the integrals one has

$$a_{12} = a_{13} = a_{15} = a_{24} = a_{34} = a_{45} = 0.$$

The uncoupled systems is therefore solved:

$$\begin{aligned}
U &= \frac{a_{44}p_1 - a_{14}p_4}{-(a_{14}a_{41}) + a_{11}a_{44}} \\
W &= \frac{-(a_{35}a_{53}p_2) + a_{33}a_{55}p_2 + a_{25}a_{53}p_3 - a_{23}a_{55}p_3 - a_{25}a_{33}p_5 + a_{23}a_{35}p_5}{-(a_{25}a_{33}a_{52}) + a_{23}a_{35}a_{52} + a_{25}a_{32}a_{53} - a_{22}a_{35}a_{53} - a_{23}a_{32}a_{55} + a_{22}a_{33}a_{55}}, \\
\Phi_1 &= \frac{a_{35}a_{52}p_2 - a_{32}a_{55}p_2 - a_{25}a_{52}p_3 + a_{22}a_{55}p_3 + a_{25}a_{32}p_5 - a_{22}a_{35}p_5}{-(a_{25}a_{33}a_{52}) + a_{23}a_{35}a_{52} + a_{25}a_{32}a_{53} - a_{22}a_{35}a_{53} - a_{23}a_{32}a_{55} + a_{22}a_{33}a_{55}} \\
\Phi_2 &= \frac{-(a_{41}p_1) + a_{11}p_4}{-(a_{14}a_{41}) + a_{11}a_{44}}, \\
\Phi_3 &= \frac{-(a_{33}a_{52}p_2) + a_{32}a_{53}p_2 + a_{23}a_{52}p_3 - a_{22}a_{53}p_3 - a_{23}a_{32}p_5 + a_{22}a_{33}p_5}{-(a_{25}a_{33}a_{52}) + a_{23}a_{35}a_{52} + a_{25}a_{32}a_{53} - a_{22}a_{35}a_{53} - a_{23}a_{32}a_{55} + a_{22}a_{33}a_{55}}
\end{aligned} \tag{20}$$

RMT results are easily obtained by considered the reduced problem related to the upper 3×3 part of the matrix in Eq. (18):

$$U = \frac{p_1}{a_{11}}, \quad W = \frac{p_2 a_{33} - p_3 a_{23}}{a_{22} a_{33} - a_{23} a_{32}},$$
$$\Phi_1 = \frac{p_3 a_{22} - p_2 a_{32}}{a_{22} a_{33} - a_{23} a_{32}} \tag{21}$$

The in-plane and transverse shear strain are

$$\begin{aligned} \epsilon_{xx} &= -(U + z\Phi_1 + z^2\Phi_2 + z^3\Phi_3)\alpha \sin \alpha x, \\ \epsilon_{xz} &= (\Phi_1 + 2z\Phi_2 + 3z^2\Phi_3 + \alpha W)\cos \alpha x \end{aligned} \tag{22}$$

Related stresses are computed via Hooke law

$$\begin{aligned} \sigma_{xx} &= -C_{11}(U + z\Phi_1 + z^2\Phi_2 + z^3\Phi_3)\alpha \sin \alpha x, \\ \sigma_{xz} &= C_{44}(\Phi_1 + 2z\Phi_2 + 3z^2\Phi_3 + \alpha W)\cos \alpha x \end{aligned} \tag{23}$$

Transverse shear stresses could be alternatively computed upon integration of the 3D indefinite equilibrium equations as it follows:

$$\begin{aligned} \sigma_{xz}(x,z) &= \sigma_{xz}\left(x, -\frac{h}{2}\right) - \int_{-\frac{h}{2}}^{z} \sigma_{xx,x} \mathrm{d}z \\ &= \sigma_{xz}\left(x, -\frac{h}{2}\right) - C_{11}\left[-\left(zU + \frac{z^2}{2}\Phi_1 + \frac{z^3}{3}\Phi_2 + \frac{z^4}{4}\Phi_3\right)\right. \\ &\quad \left. + \left(-\frac{h}{2}U + \frac{h^2}{8}\Phi_1 - \frac{h^3}{24}\Phi_2 + \frac{h^4}{64}\Phi_3\right)\right]\alpha^2 \cos(\alpha x) \end{aligned} \tag{24}$$

To be noticed that the order in z of transverse stress computed by Hooke's law (second order) can be significantly different from those computed by 3D elasticity equation (fourth order).

4. Model 'V3-H': The reduced model fulfilling homogeneous stress conditions

4.1. Displacement model

The homogeneous conditions for transverse shear stresses at the top and bottom plate

$$\begin{aligned} \bar{\sigma}_{xz}\left(x, \frac{h}{2}\right) &= C_{44}\epsilon_{xz}\left(x, \frac{h}{2}\right) = 0, \\ \bar{\sigma}_{xz}\left(x, -\frac{h}{2}\right) &= C_{44}\epsilon_{xz}\left(x, -\frac{h}{2}\right) = 0 \end{aligned} \tag{25}$$

are employed as in [1–3] to reduce the number of the displacement variables in Eq. (3). The two values of transverse shear strains Eq. (9) with correspondence to the top and bottom plate surfaces are

$$\begin{aligned} \epsilon_{xz}\left(x, \frac{h}{2}\right) &= \phi_1 + h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} \\ \epsilon_{xz}\left(x, -\frac{h}{2}\right) &= \phi_1 - h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} \end{aligned} \tag{26}$$

These strains must vanish according to Eq. (25):

$$\begin{aligned} &\phi_1 + h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} = 0, \\ &\phi_1 - h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} = 0 \end{aligned} \tag{27}$$

which lead to the following conditions:

$$\phi_2 = 0, \quad \phi_3 = -\frac{4}{3h^2}(\phi_1 + w^0_{,x}) \tag{28}$$

The resulting 'V3-H' displacement model is

$$\begin{aligned} u(x,z) &= u^0(x) + e(z)w^0_{,x} + g(z)\phi_1(x) \\ w(x,z) &= w^0(x) \end{aligned} \tag{29}$$

where

$$e(z) = -\frac{4}{3h^2}z^3, \quad g(z) = z + e(z)$$

It should be underlined that the use of a stress conditions has permitted to obtain a cubic displacement model which has the same number displacement variables of Reissner–Mindlin plate theory (u^0, ϕ_1, w^0).

4.2. Governing equations

The displacement variations with correspondence to the top surface are

$$\begin{aligned} \delta w\left(x, \frac{h}{2}\right) &= \delta w^0, \\ \delta u\left(x, \frac{h}{2}\right) &= \delta u^0 + e\left(\frac{h}{2}\right)\delta w^0_{,x} + g\left(\frac{h}{2}\right)\delta\phi_1 \end{aligned}$$

The external work assumes, therefore, the following form:

$$\begin{aligned} \delta L_{\mathrm{e}} = \int_{\Omega} &\left[\bar{\sigma}_{zz}\delta w^0 + \bar{\sigma}_{xz}\delta u^0 + e\left(\frac{h}{2}\right)\bar{\sigma}_{xz}\delta w^0_{,x}\right. \\ &\left. + g\left(\frac{h}{2}\right)\bar{\sigma}_{xz}\delta\phi_1^0\right]\mathrm{d}\Omega \end{aligned} \tag{30}$$

Upon integration by part the external work takes the following form:

$$\begin{aligned} \delta L_{\mathrm{e}} = \int_{\Omega} &\left[\left(\bar{\sigma}_{zz} - e\left(\frac{h}{2}\right)\bar{\sigma}_{xz_{,x}}\right)\delta w^0 + \bar{\sigma}_{xz}\delta u^0\right. \\ &\left. + g\left(\frac{h}{2}\right)\bar{\sigma}_{xz}\delta\phi_1^0\right]\mathrm{d}\Omega + \int_{\Gamma} e\left(\frac{h}{2}\right)\bar{\sigma}_{xz}\delta w^0 \mathrm{d}\Gamma \end{aligned} \tag{31}$$

The strains hold,

$$\epsilon_{xx} = u^0_{,x} + e(z)w_{,xx} + g(z)\phi_{1,x},$$
$$\epsilon_{xz} = g'(z)w_{,x} + g'(z)\phi_1 \tag{32}$$

where

$$e'(z) = -\frac{4}{h^2}z^2, \quad g'(z) = 1 + e'(z) = 1 - \frac{4}{h^2}z^2$$

The internal work is

$$\delta L_{\mathrm{i}} = \int_V \{\sigma_{xx}[\delta u^0_{,x} + e(z)\delta w_{,xx} + g(z)\delta\phi_{1,x}] + \sigma_{xz}g'(z)[\delta w_{,x} + \delta\phi_1]\}\mathrm{d}z\,\mathrm{d}\Omega \tag{33}$$

By using the already introduced stress results and the following values for $q(z)$:

$$1,\ e(z),\ g(z) \text{ and } g'(z) \tag{34}$$

the internal work can be expressed as

$$\delta L_{\mathrm{i}} = \int_V [\mathscr{R}^1_{xx}\delta u^0_{,x} + \mathscr{R}^{e(z)}_{xx}\delta w_{,xx} + \mathscr{R}^{g(z)}_{xx}\delta\phi_{1,x} + \mathscr{R}^{g'(z)}_{xz}\delta w_{,x} + \mathscr{R}^{g'(z)}_{xz}\delta\phi_1]\mathrm{d}\Omega \tag{35}$$

By integrating by part one has

$$\delta L_{\mathrm{i}} = \int_\Omega (-\mathscr{R}^1_{xx_{,x}}\delta u^0 + \mathscr{R}^e_{xx_{,xx}}\delta w^0 - \mathscr{R}^g_{xx_{,x}}\delta\phi_1 - \mathscr{R}^{g'}_{xz_{,x}}\delta w + \mathscr{R}^{g'}_{xz}\delta\phi_1)\mathrm{d}\Omega + \int_\Gamma (\mathscr{R}^1_{xx}\delta u^0 + \mathscr{R}^e_{xx}\delta w^0_{,x} - \mathscr{R}^e_{xx_{,x}}\delta w^0 + \mathscr{R}^g_{xx}\delta\phi_1 + \mathscr{R}^g_{xz}\delta w)\,\mathrm{d}\Gamma \tag{36}$$

The three equilibrium equations in terms of stress resultants on Ω are

$$\begin{aligned} \delta u^0:&\quad -\mathscr{R}^1_{xx_{,x}} = \bar{\sigma}_{xz} \\ \delta w^0:&\quad \mathscr{R}^e_{xx_{,xx}} - \mathscr{R}^{g'}_{xz_{,x}} = \bar{\sigma}_{zz} - e\left(\frac{h}{2}\right)\bar{\sigma}_{xz_{,x}} \\ \delta\phi_1:&\quad -\mathscr{R}^g_{xx_{,x}} + \mathscr{R}^{g'}_{xz} = g\left(\frac{h}{2}\right)\bar{\sigma}_{xz} \end{aligned} \tag{37}$$

The geometrical/mechanical associated conditions on the boundary Γ are

$$\begin{aligned} &\delta u^0:\quad u^0 = \bar{u}^0 \text{ or } \mathscr{R}^1_{xx} = \bar{\mathscr{R}}^1_{xx} \\ &\delta w^0:\quad w^0 = \bar{w}^0 \text{ or} \\ &e\left(\frac{h}{2}\right)\bar{\sigma}_{xz} - \mathscr{R}^e_{xx_{,x}} + \mathscr{R}^g_{xz} = \overline{e\left(\frac{h}{2}\right)\bar{\sigma}_{xz} - \mathscr{R}^e_{xx_{,x}} + \mathscr{R}^g_{xz}} \\ &\delta w^0_{,x}:\quad w^0_{,x} = \bar{w}^0_{,x} \text{ or } \mathscr{R}^e_{xx} = \bar{\mathscr{R}}^e_{xz} \\ &\delta\phi_1:\quad \phi_1 = \bar{\phi}_1 \text{ or } \mathscr{R}^g_{xx} = \bar{\mathscr{R}}^g_{xx} \end{aligned} \tag{38}$$

The corresponding governing equations on Ω in terms of displacement variables are

$$\begin{aligned} \delta u^0:&\quad -C_{11}\int_{1\times 1} u^0_{,xx} - C_{11}\int_{1\times e} w^0_{,xxx} - C_{11}\int_{1\times g}\phi^0_{1,xx} = \bar{\sigma}_{xz} \\ \delta w^0:&\quad C_{11}\int_{e\times 1} u^0_{,xxx} + C_{11}\int_{e\times e} w^0_{,xxxx} + C_{11}\int_{e\times g}\phi^0_{1,xxx} \\ &\quad - C_{44}\int_{g'\times g'} w^0_{,xx} - C_{44}\int_{g'\times g'}\phi^0_{1_{,x}} = \bar{\sigma}_{zz} - e(\frac{h}{2})\bar{\sigma}_{xz_{,x}} \\ \delta\phi^0_1:&\quad -C_{11}\int_{g\times 1} u^0_{,xx} - C_{11}\int_{g\times e} w^0_{,xxx} - C_{11}\int_{g\times g}\phi^0_{1,xx} \\ &\quad + C_{44}\int_{g'\times g'} w_{,x} + C_{44}\int_{g'\times g'}\phi_1 = g\left(\frac{h}{2}\right)\bar{\sigma}_{xz} \end{aligned} \tag{39}$$

4.3. Closed form solutions

By introducing the harmonic forms the previous section, the governing equations lead to the following system of algebraic equations:

$$\begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \end{bmatrix} \begin{Bmatrix} U \\ W \\ \Phi \end{Bmatrix} = \begin{Bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \end{Bmatrix} \tag{40}$$

The matrix is symmetric. Explicit form of stiffness coefficients and loadings follow:

$$\begin{aligned} &\tilde{a}_{11} = C_{11}\int_{1\times 1}\alpha^2, \quad \tilde{a}_{12} = C_{11}\int_{1\times e}\alpha^3, \\ &\tilde{a}_{13} = C_{11}\int_{1\times g}\alpha^2 \quad \tilde{a}_{22} = C_{11}\int_{e\times e}\alpha^4 + C_{44}\int_{g'\times g'}\alpha^2, \\ &\tilde{a}_{23} = C_{11}\int_{e\times g}\alpha^3 + C_{44}\int_{g'\times g'}\alpha, \\ &\tilde{a}_{33} = C_{11}\int_{g\times g}\alpha^2 + C_{44}\int_{g'\times g'} \quad \tilde{p}_1 = \overline{S}_{xz}, \\ &\tilde{p}_2 = \overline{S}_{zz} + e\left(\frac{h}{2}\right)\overline{S}_{xz}\alpha, \quad \tilde{p}_3 = g\left(\frac{h}{2}\right)\overline{S}_{xz} \end{aligned} \tag{41}$$

For the considered plate one has

$$\tilde{a}_{12} = \tilde{a}_{13} = 0$$

The uncoupled systems is therefore solved and the related displacement amplitudes are

$$U = \frac{\tilde{p}_1}{\tilde{a}_{11}}, \quad W = \frac{\tilde{p}_2\tilde{a}_{33} - \tilde{p}_3\tilde{a}_{23}}{\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32}}, \quad \Phi = \frac{\tilde{p}_3\tilde{a}_{22} - \tilde{p}_2\tilde{a}_{32}}{\tilde{a}_{22}\tilde{a}_{33} - \tilde{a}_{23}\tilde{a}_{32}},$$

The resulting displacements are

$$u = (U + \alpha e(z)W + g(z)\Phi_1)\cos\alpha x, \quad w = W\sin\alpha x \tag{42}$$

The strains are

$$\begin{aligned}\epsilon_{xx} &= u^0_{,x} + e(z)w_{,xx} + g(z)\phi_{1,x} \\ &= -[\alpha U + e(z)\alpha^2 W + \alpha g(z)\Phi_1]\sin(\alpha x) \\ \epsilon_{xz} &= g'(z)w_{,x} + g'(z)\phi_1 \\ &= g'(z)[\alpha W + \Phi_1]\cos(\alpha x)\end{aligned} \tag{43}$$

The stresses are

$$\begin{aligned}\sigma_{xx} &= C_{11}\epsilon_{xx} = -C_{11}(\alpha U + e(z)\alpha^2 W + \alpha g(z)\Phi_1)\sin(\alpha x) \\ \sigma_{xz} &= C_{44}\epsilon_{xz} = C_{44}g'(z)[\alpha W + \Phi_1]\cos(\alpha x)\end{aligned} \tag{44}$$

If the 3D indefinite equilibrium equations are integrated, the following transverse shear stress are obtained:

$$\begin{aligned}\sigma_{xz}(x,z) &= \sigma_{xz}\left(x,-\frac{h}{2}\right) - \int_{-\frac{h}{2}}^{z}\sigma_{xx,x}\mathrm{d}z \\ &= \sigma_{xz}\left(x,-\frac{h}{2}\right) - C_{11}\left[-(zU + E(z)\alpha W + G(z)\Phi_1)\right. \\ &\quad \left. + \left(-\frac{h}{2}U + E\left(-\frac{h}{2}\right)\alpha W + G\left(-\frac{h}{2}\right)\Phi_1\right)\right]\alpha^2\cos(\alpha x)\end{aligned} \tag{45}$$

where $E(z) = \int e(z)\mathrm{d}z = -\frac{z^4}{3h^2}$ and $G(z) = \int g(z)\mathrm{d}z = \frac{z^2}{2} - \frac{z^4}{3h^2}$.

5. Model 'V3-NH': The reduced model fulfilling non-homogeneous stress conditions

5.1. Displacement model

In this case, the top surface transverse shear condition is enforced. The transverse shear strains at the top/bottom plate surfaces are

$$\epsilon_{xz}\left(x,\frac{h}{2}\right) = \frac{\bar{\sigma}_{xz}\left(x,\frac{h}{2}\right)}{C_{44}}, \quad \epsilon_{xz}\left(x,-\frac{h}{2}\right) = 0 \tag{46}$$

Upon substitution of the displacement model Eq. (3), the last two conditions lead to

$$\begin{aligned}&\phi_1 + h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} = \frac{\bar{\sigma}_{xz}\left(x,\frac{h}{2}\right)}{C_{44}} \\ &\phi_1 - h\phi_2 + \frac{3}{4}h^2\phi_3 + w^0_{,x} = 0\end{aligned} \tag{47}$$

that is

$$\begin{aligned}\phi_2 &= \frac{\bar{\sigma}_{xz}\left(x,\frac{h}{2}\right)}{2hC_{44}}, \\ \phi_3 &= \frac{2}{3h^2}\left(-2\phi_1 - 2w^0_{,x} + \frac{\bar{\sigma}_{xz}\left(x,\frac{h}{2}\right)}{C_{44}}\right)\end{aligned} \tag{48}$$

The resulting 'V3-NH' model is therefore written

$$\begin{aligned}u(x,z) &= u^0(x) + e(z)w^0_{,x} + g(z)\phi_1(x) + f(z)/C_{44}\bar{\sigma}_{xz}\left(x,\frac{h}{2}\right) \\ w(x,z) &= w^0(x)\end{aligned} \tag{49}$$

with

$$f(z) = \left(\frac{z^2}{2h} + \frac{2z^3}{3h^2}\right)$$

To be noticed that the applied transverse shear stress influences directly the displacement model. This means that a contribution to the loadings will come from the plate internal work. Such a contribution could be find if and only if a variationally consistent technique is employed as it is the case of the present work.

5.2. Governing equations

The external work coincides to that of previous section. To compute the internal work it is convenient to split the strains according to the following formula:

$$\epsilon_{xx} = \epsilon^{\mathrm{H}}_{xx} + \epsilon^{\mathrm{NH}}_{xx}, \quad \epsilon_{xz} = \epsilon^{\mathrm{H}}_{xx} + \epsilon^{\mathrm{NH}}_{xx} \tag{50}$$

subscript H denote strains related to the V3-H model while subscript NH denotes strains coming from non-homogeneous boundary conditions on transverse shear stresses. That is

$$\epsilon^{\mathrm{NH}}_{xx} = \frac{f(z)}{C_{44}\bar{\sigma}_{xz,x}}, \quad \epsilon^{\mathrm{NH}}_{xz} = \frac{f'(z)}{C_{44}\bar{\sigma}_{xz}} \tag{51}$$

where $f'(z) = \frac{z}{h} + \frac{2z^2}{h^2}$. Since,

$$\delta\epsilon^{\mathrm{NH}}_{xx} = 0, \quad \delta\epsilon^{\mathrm{NH}}_{xz} = 0 \tag{52}$$

the strain variations are those of the homogeneous case and the governing equations are formally the same of those of the previous Model VR-H:

$$\begin{aligned}&\delta u^0: \quad -\mathscr{R}^{1_{\mathrm{NH}}}_{xx,x} = \bar{\sigma}_{xz} \\ &\delta w^0: \quad \mathscr{R}^{e_{\mathrm{NH}}}_{xx,xx} - \mathscr{R}^{g'_{\mathrm{NH}}}_{xz,x} = \bar{\sigma}_{zz} - e\left(\frac{h}{2}\right)\bar{\sigma}_{xz,x} \\ &\delta\phi_1: \quad -\mathscr{R}^{g_{\mathrm{NH}}}_{xx,x} + \mathscr{R}^{g'_{\mathrm{NH}}}_{xz} = g\left(\frac{h}{2}\right)\bar{\sigma}_{xz}\end{aligned} \tag{53}$$

Apexes NH denote stress resultants related to whole stresses calculated according to the strains of Eq. (50)

$$\begin{aligned}(\mathscr{R}^{q(z)_{\mathrm{NH}}}_{xx}, \mathscr{R}^{q(z)}_{xz}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} q(z)(\sigma_{xx},\sigma_{xz})\mathrm{d}z \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} q(z)[C_{11}(\epsilon^{\mathrm{H}}_{xx} + \epsilon^{\mathrm{NH}}_{xx}), C_{44}(\epsilon^{\mathrm{H}}_{xz} + \epsilon^{\mathrm{NH}}_{xz})]\mathrm{d}z\end{aligned} \tag{54}$$

The geometrical/mechanical associated boundary conditions on the boundary Γ are

$$\begin{aligned}
&\delta u^0: \quad u^0=\bar{u}^0 \text{ or } \mathscr{R}^{1_{\rm NH}}_{xx}=\bar{\mathscr{R}}^{1_{\rm NH}}_{xx}\\
&\delta w^0: \quad w^0=\bar{w}^0 \text{ or}\\
&e\left(\frac{h}{2}\right)\bar{\sigma}_{xz}-\mathscr{R}^{e{\rm NH}}_{xx_{,x}}+\mathscr{R}^{g{\rm NH}}_{xz}=\overline{e\left(\frac{h}{2}\right)\bar{\sigma}_{xz}-\mathscr{R}^{e{\rm NH}}_{xx_{,x}}+\mathscr{R}^{g{\rm NH}}_{xz}}\\
&\delta w^0_x: \quad w^0_{,x}=\bar{w}^0_{,x} \text{ or } \mathscr{R}^{e{\rm NH}}_{xx}=\bar{\mathscr{R}}^{e{\rm NH}}_{xz}\\
&\delta\phi_1: \quad \phi_1=\bar{\phi}_1 \text{ or } \mathscr{R}^{g{\rm NH}}_{xx}=\bar{\mathscr{R}}^{g{\rm NH}}_{xx}
\end{aligned} \tag{55}$$

The governing equations on the domain Ω are written in terms of unknown variables as it follows:

$$\begin{aligned}
\delta u^0:\ & -C_{11}\int_{1\times 1}u^0_{,xx}-C_{11}\int_{1\times e}w^0_{,xxx}-C_{11}\int_{1\times g}\phi^0_{1,xx}\\
&-C_{11}/C_{44}\int_{1\times f}\bar{\sigma}_{xz,xx}=\bar{\sigma}_{xz}\\
\delta w^0:\ & C_{11}\int_{e\times 1}u^0_{,xxx}+C_{11}\int_{e\times e}w^0_{,xxxx}+C_{11}\int_{e\times g}\phi^0_{1,xxx}\\
&-C_{44}\int_{g'\times g'}w^0_{,xx}-C_{44}\int_{g'\times g'}\phi^0_{1_{,x}}+C_{11}/C_{44}\int_{f\times e}(z)\bar{\sigma}_{xz,xxx}\\
&-\int_{f'\times g'}\bar{\sigma}_{xz_{,x}}=\bar{\sigma}_{zz}-e\left(\frac{h}{2}\right)\bar{\sigma}_{xz_{,x}}\\
\delta\phi^0_1:\ & -C_{11}\int_{g\times 1}u^0_{,xx}-C_{11}\int_{g\times e}w^0_{,xxx}-C_{11}\int_{g\times g}\phi^0_{1,xx}\\
&+C_{44}\int_{g'\times g'}w_{,x}+C_{44}\int_{g'\times g'}\phi_1-C_{11}/C_{44}\int_{f\times g}\bar{\sigma}_{xz_{,xx}}\\
&+\int_{f'\times g'}\bar{\sigma}_{xz}=g\left(\frac{h}{2}\right)\bar{\sigma}_{xz}
\end{aligned} \tag{56}$$

In which, we have a sort of internal loadings coming from the imposed non-homogeneous boundary conditions on the top surface of the plate.

5.3. Closed form solution

The governing system of differential equations takes the same form of previous section. Related algebraic equations are

$$\begin{bmatrix}\hat{a}_{11}&\hat{a}_{12}&\hat{a}_{13}\\\hat{a}_{21}&\hat{a}_{22}&\hat{a}_{23}\\\hat{a}_{31}&\hat{a}_{32}&\hat{a}_{33}\end{bmatrix}\begin{Bmatrix}\hat{U}\\\hat{W}\\\hat{\Phi}\end{Bmatrix}=\begin{Bmatrix}\hat{p}_1\\\hat{p}_2\\\hat{p}_3\end{Bmatrix} \tag{57}$$

whose coefficients coincide to that of Eq. (41)

$$\hat{a}_{ij}=\tilde{a}_{ij}$$

while the loading terms are

$$\hat{p}_1=p_1^{\rm H}+p_1^{\rm NH},\quad \hat{p}_2=p_2^{\rm H}+p_2^{\rm NH},\\ \hat{p}_3=p_3^{\rm H}+p_3^{\rm NH} \tag{58}$$

where

$$p_1^{\rm H},=\tilde{p}_1,\ p_2^{\rm H}=\tilde{p}_2,\ p_3^{\rm H}=\tilde{p}_3$$

and

$$\begin{aligned}
p_1^{\rm NH}&=-C_{11}I_{1f}\bar{\sigma}_{xz_{,xx}},\\
p_2^{\rm NH}&=C_{11}I_{fe}\bar{\sigma}_{xz_{,xxx}}-C_{44}I_{f'g'}\bar{\sigma}_{xz_{,x}},\\
p_3^{\rm NH}&=-C_{11}I_{fg}\bar{\sigma}_{xz_{,xx}}-C_{44}I_{f'g'}\bar{\sigma}_{xz}
\end{aligned} \tag{59}$$

As far as the closed form solutions is concerned the following new loading terms are obtained:

$$\begin{aligned}
p_1^{\rm NH}&=\overline{S}_{xz}\left(1-C_{11}/C_{44}\int_{1\times f}\alpha^2\right)\\
p_2^{\rm NH}&=\overline{S}_{zz}+\left(e\left(\frac{h}{2}\right)\alpha-C_{11}/C_{44}\int_{f\times e}\alpha^3-\int_{f'\times g'}\alpha\right)\overline{S}_{xz}\\
p_3^{\rm NH}&=\left(g\left(\frac{h}{2}\right)-C_{11}/C_{44}\int_{f\times g}\alpha^2-\int_{f'\times g'}\right)\overline{S}_{xz}
\end{aligned} \tag{60}$$

The solution is

$$\widehat{U}=\frac{\hat{p}_1}{\tilde{a}_{11}},\quad \widehat{W}=\frac{\hat{p}_2\tilde{a}_{33}-\hat{p}_3\tilde{a}_{23}}{\tilde{a}_{22}\tilde{a}_{33}-\tilde{a}_{23}\tilde{a}_{32}},\quad \hat{\Phi}=\frac{\hat{p}_3\tilde{a}_{22}-\hat{p}_2\tilde{a}_{32}}{\tilde{a}_{22}\tilde{a}_{33}-\tilde{a}_{23}\tilde{a}_{32}}$$

The strains and stresses are

$$\begin{aligned}
\epsilon_{xx}&=u^0_{,x}+e(z)w_{,xx}+g(z)\phi_{1,x}+f(z)/C_{44}\bar{\sigma}_{xz,x}\\
&=-\left[\alpha U+e(z)\alpha^2W+\alpha g(z)\Phi_1+\frac{f(z)}{C_{44}}\alpha S_{xz}\right]\sin(\alpha x)\\
\epsilon_{xz}&=g'(z)w_{,x}+g'(z)\phi_1+f'(z)/C_{44}\bar{\sigma}_{xz}\\
&=\left[g'(z)(\alpha W+\Phi_1)+\frac{f'(z)}{C_{44}}S_{xz}\right]\cos(\alpha x)
\end{aligned} \tag{61}$$

Table 1
Response to transverse loadings

a/h	RMT	V3-H,V3-NH	O5	Q-3D
2	63.773	70.478	71.288	70.145
4	24.933	27.285	27.301	27.222
10	14.069	14.477	14.477	14.476
100	12.021	12.025	12.025	12.028
1000	12.000	12.000	12.000	12.004

Transverse displacement amplitude $\widetilde{W}C_{11}\alpha^4\times/\overline{S}_{zz}$.

Table 2
Shear loading case

a/h	FSDT	V3-H	V3-NH	O5	Q-3D
2	6.0000	10.873	9.3726	9.7775	9.7214
4	6.0000	7.2737	7.1757	7.1840	7.1827
10	6.0000	6.2064	6.2039	6.2039	6.2058
100	6.0000	6.0021	6.0021	6.0021	6.0039
1000	6.0000	6.0000	6.0000	6.0000	6.0019

Transverse displacement amplitude $\widehat{W}C_{11}\alpha^3/\overline{S}_{xz}$.

Table 3
Comparison of the loadings amplitudes for the V3-H and V3-NH analyzes considered in Table 2

a/h	V3-H			V3-NH		
	$p_1/\overline{S}_{xz}$	$p_2/\overline{S}_{xz}$	$p_3/\overline{S}_{xz}$	$\hat{p}_1/\overline{S}_{xz}$	$\hat{p}_2/\overline{S}_{xz}$	$\hat{p}_3/\overline{S}_{xz}$
2	1.0	−2.6180E−1	3.3333	−1.1555	−2.0528E−1	−0.61798
4	1.0	−1.3090E−1	3.3333	0.46111	−1.6311E−1	0.18455
10	1.0	−5.2360E-2	3.3333	0.91378	−7.2014E−2	0.25353
100	1.0	−5.2360E−3	3.3333	0.99914	−7.3291E−3	0.26654
1000	1.0	−5.2360E−4	3.3333	0.99999	−7.3304E−4	0.26667

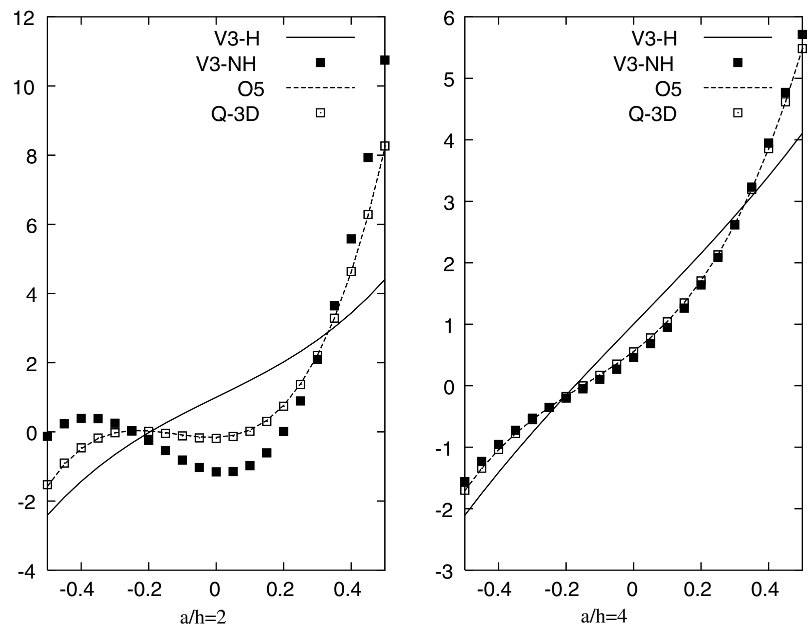

Fig. 1. Distribution of in-plane displacement amplitude in the thickness directions $U \times C_{11}h\alpha^2/\overline{S}_{zz}$.

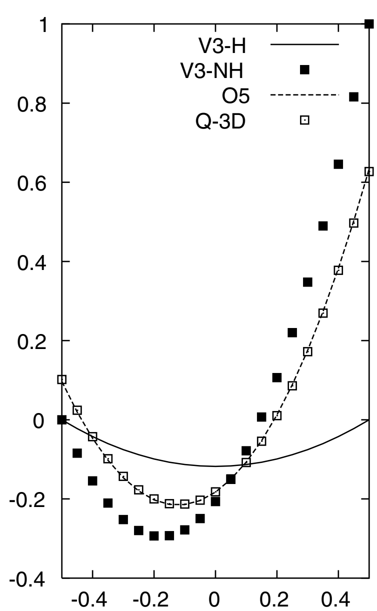

Fig. 2. Distribution of transverse shear stresses from Hooke law $S_{xz}/\bar{S}_{xz}$ of very thick plate $a/h = 2$.

$$\sigma_{xx} = -C_{11}\left[\alpha U + e(z)\alpha^2 W + \alpha g(z)\Phi_1 + \frac{f(z)}{C_{44}}\alpha S_{xz}\right]\sin(\alpha x)$$
$$\sigma_{xz} = C_{44}\left[g'(z)(\alpha W + \Phi_1) + \frac{f'(z)}{C_{44}}S_{xz}\right]\cos(\alpha x) \tag{62}$$

If the 3D indefinite equilibrium equations are integrated the following transverse shear stress are obtained:

$$\begin{aligned}\sigma_{xz}(x,z) &= \sigma_{xz}(x,0) + \int_{-\frac{h}{2}}^{z}\sigma_{xx,x}\,dz \\ &= \sigma_{xz}(x,0) - C_{11}(zU + E(z)\alpha W \\ &\quad + (z)\Phi_1 + \frac{f(z)}{C_{44}}\overline{S}_{xz}\Big)\alpha^2\cos(\alpha x)\end{aligned} \tag{63}$$

where $F(z) = \int f(z)dz = \frac{z^3}{6h} + \frac{z^4}{6h^2}$.

6. Numerical evaluations and discussion

Numerical evaluation and comparison of the three discussed models are given in this section. An orthotropic

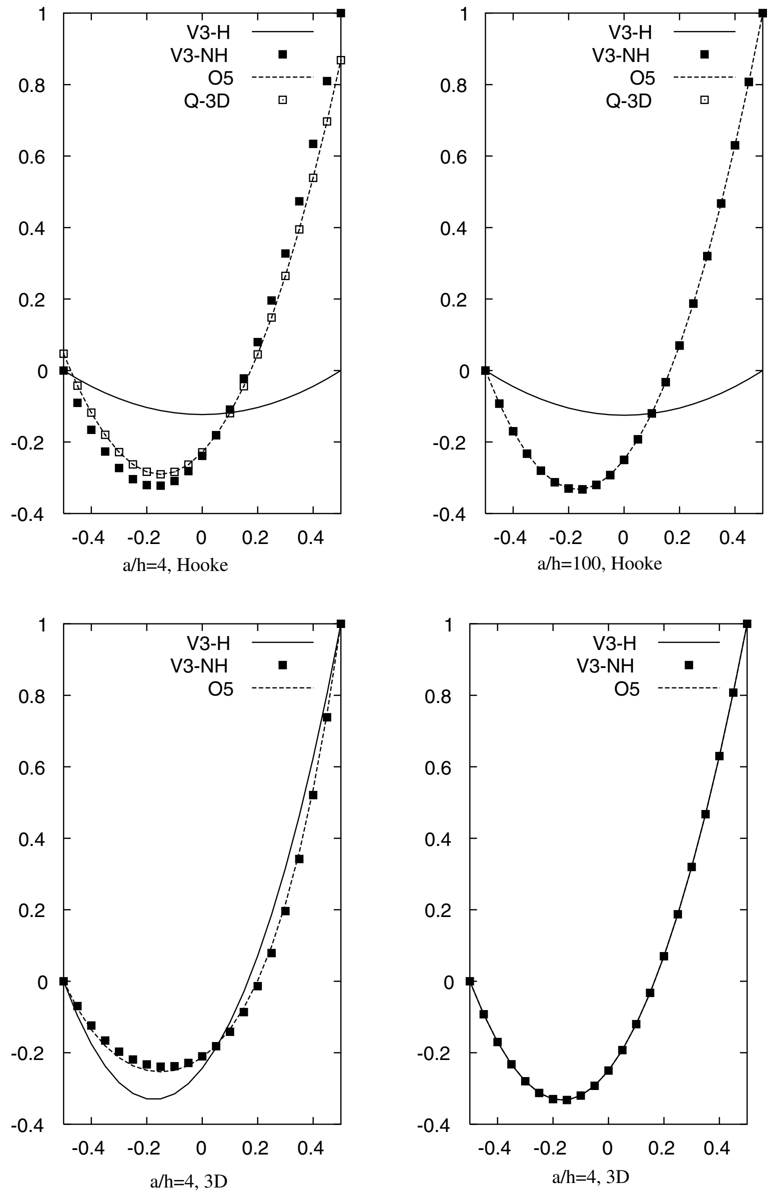

Fig. 3. Distribution of transverse shear stresses amplitude $S_{xz}/\overline{S}_{xz}$ of moderately thick and thin plates.

plate with stiffness coefficients $\frac{C_{11}}{C_{44}} = \frac{25.16}{1.2}$, has been considered. The following two loading cases have been considered:

1. only transverse normal pressure distribution $\bar{\sigma}_{zz}$;
2. only transverse shear distribution $\bar{\sigma}_{xz}$.

Both have been applied to the top plate surface. First loading case has been considered in Table 1. The applied transverse shear stress being zero, it follows that the theories denoted as V3-H and V3-NH coincide. The amplitudes of transverse displacement with correspondence to the reference surface Ω is quoted in Table 1. The following four theories are compared: RMT, V3-H, O5 and Q-3D. The latter result consists of quasi-3D solutions; these refer to third-order expansion for both the in-plane and out-of-plane displacements, as described in [8]. Q-3D results are herein taken as the reference solutions.

The following comments can be made:

- Differences among different theories arise in both very thick ($a/h = 2$) and moderately thick ($a/h = 4$) plate cases. Results merge in thin plate geometries ($a/h = 100$).
- V3-H gives almost the same accuracy of O5 analysis. However, V3-H analysis requires two displacement variables less than O5 one.

- V3-H result improves very much the classical RMT description, that happens without increasing the number of the independent variables.

The last comment is the historical reason why V3-H type analysis is one of the most diffused refined theory for isotropic and laminated beams, plates and shells.

The more interesting case of applied transverse shear loading has been addressed in Table 2. The models V3-H and V3-NH can be compared in this case. Values of transverse displacement amplitudes have been compared. The following remarks can be made.

- Considered analyses merge in thin plate cases (the quoted value of the displacement is independent by the thickness ratio in the RMT analysis).
- V3-H and V3-NH analyzes can improve very much RMT results.
- V3-H and V3-NH results are not better than those related to the original (but more expensive) O5 model.
- V3-NH analyzes are always better than corresponding V3-H ones. That means that the presented variationally consistent manner of including not homogeneous stress conditions is very effective.

The values of the amplitude of the loading vector related to V3-H and V3-NH analyzes of Table 2 are given in Table 3. It is of interest to notice that the loadings are quite different in the two cases; these values do not merge even though thin plates are considered. At the same time the variation of the loadings do not influence the results on displacements. This is a further proof of the consistency of the variational technique herein employed to derive governing equations which are consistent with the made hypothesis on displacements.

A better understanding of the behavior of the various theories could be achieved by considering the distribution of the unknown variables in the plate thickness direction. This is done in Figs. 1–3. The distribution of the in-plane displacements in the thickness direction z is plotted in Fig. 1. The following comments can be made.

- The differences among the various theories is very much subordinate to the value of the thickness coordinate z.
- V3-NH results are much better than corresponding V3-H. That is the square term $f(z)$ of Eq. (49) has a very strong influence.
- V3-NH analysis is still poorer than O5 ones which coincide to quasi-3D results.
- The $a/h = 4$ geometry shows that V3-NH tends to merge with O5 and Q-3D results while V3-H results are still quite different.

Previous remarks are confirmed by Figs. 2 and 3 which consider transverse shear stress. Very thick ($a/h = 2$), thick ($a/h = 4$) and thin ($a/h = 100$) plates are considered. Transverse stresses from Hooke law are compared. Stresses obtained upon integration of the 3D indefinite equilibrium equations are also considered for the thick and thin geometries (these stresses were not available for the Q-3D cases). The following further comments can be made.

- Transverse shear stress evaluations can be quite different for the different theories.
- V3-H stresses from Hooke law are very ineffective even though thin plates are considered.
- V3-NH theory seems instead very effective to compute transverse shear stress from Hooke's law. In particular V3-H fulfill 'as definition' the top-surface condition $\sigma_{xz} = \bar{\sigma}_{xz}$.
- O5 and Q-3D analyzes from Hooke law show some difficulties to fulfill non-homogeneous top-surface conditions in the case of thick geometries.
- The integration of 3D equations leads to benefits for the considered V3-H analyzes. Anyway V3-NH results very closed to O5 ones.

7. Conclusions

The paper has investigated the effects of non-homogeneous conditions on transverse shear stresses in third-order plate theories formulated on the basis of displacement assumptions. These conditions have been used to reduce the number of the unknown variables. Attention has been restricted to orthotropic plates in cylindrical bending. A comparison has been made among three models: Model O5 is the 'original' third model with five displacement variables; Model V3-H is the reduced third-order model with three displacement variables obtained by imposing 'homogeneous' stress conditions with correspondence to the plate top-surface; Model V3-NH is the modification of previous case obtained by considering 'non-homogeneous' stress conditions. The principle of virtual displacements has been used to derive variationally consistent governing equations. Closed form solutions, related to simply supported plates loaded by harmonic distribution of applied loadings have led to the following main conclusions.

1. It is confirmed that VR-H models leads to significant improvements with respect to classical Reissner–Mindlin plate theory.
2. VR-NH evaluations are more accurate than VR-H ones. This results prove the convenience of taking

into account the non-homogeneous boundary conditions whenever plate theories are formulated.

3. However, the original model ‘O5’ has led to better description of displacement and stresses than VR-NH analyzes. An exception is made for transverse shear stresses computed from Hooke law, in which case VR-NH analyzes could be more accurate than O5 ones.

Future investigation should be addressed to theories that consider transverse normal strains effects. This problem would permit to extend the analysis herein given to those problems in which non-homogeneous boundary conditions on transverse normal stresses can be imposed. Furthermore, the differences among the various theories could be more significant in the case of laminated, composite plates.

References

[1] Vlasov BF. On the equations of bending of plates. Dokla Ak Nauk Azerbeijanskoi SSR 1957;3:955–9 [in Russian].

[2] Reddy JN. A simple higher order theory for laminated plates. J Appl Mech 1984;51:745–52.

[3] Reddy JN. A general non-linear third-order theory of plates with moderate thickness. Int J Nonlinear Mech 1990;25:677–86.

[4] Jemielita. Technical theory of plates with moderate thickness. Rozprawy Inz 1975;23:483–99 [in Polish].

[5] Jemielita. On kinematical assumptions of refined theories of plates. J Appl Mech 1990;57:1088–91.

[6] Lewinsky T. On refined plate models based on kinematical assumptions. Ing Archiv 1987;57:133–46.

[7] Carrera E. Historical review of Zig-Zag theories for multilayered plates and shells. Appl Mech Rev 2003;56:287–308.

[8] Carrera E. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Arch Comput Methods Eng, State-of-the-art Rev 2003;10:215–96.