Available online at www.sciencedirect.com
SCIENCE DIRECT®
Computers and Structures 82 (2004) 541–554
Computers & Structures
www.elsevier.com/locate/compstruc

On the use of the Murakami's zig-zag function in the modeling of layered plates and shells

E. Carrera *

Dipartimento di Ingegneria Aeronautica e Spaziale, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Received 12 June 2003; accepted 1 February 2004

Abstract

This paper discusses the use of the Murakami zig-zag function (MZZF) in the two-dimensional modeling of multilayered plates and shells. A literature overview of the available works is first presented. A 'simple use' of the MZZF is discussed: the MZZF is used to introduce the zig-zag effect in classical and higher order theories which are formulated with only displacement unknowns. An 'advanced use' of the MZZF is then considered to introduce the zig-zag effect in those theories which are formulated on the basis of both displacement and transverse stress assumptions. A number of new plate/shell theories has been considered. Numerical results encompassing, static, dynamic and thermally loaded orthotropic, simply supported plates and shells are presented to show both the effectiveness and limitations of the MZZF in the modeling of layered structures. Linear up to forth-order expansions for the in-plane and the out-of-plane displacements, in the thickness plate/shell direction, have been compared. It has been concluded that the MZZF is a valuable tool to enhance the performances of both classical and advanced theories. The conducted numerical evaluations have shown in particular that multilayered plate and shell theories can be greatly improved by the use of MZZF.

Keywords: Plates and shells; Multilayered structures; Composite materials; Zig-zag theories; Mixed methods; Vibration bending; Thermal stress

1. Introduction

Multilayered structures show a piece-wise continuous displacement field in the thickness plate/shell direction. This change in slope between two adjacent layers, that are considered to be perfectly bonded together, is known as the zig-zag (ZZ) effect. The different transverse (both shear and normal components) deformability of the layers is the source of the ZZ effect. Furthermore, these transverse strains come with transverse shear and normal stresses that, for equilibrium reasons, are continuous at the each layer interface. These equilibrium conditions are known as interlaminar continuity (IC) for transverse stresses.

Several possibilities are known to take ZZ and IC into account in the multilayered structures. Developments, see [1–6], have been made in both of layer-wise models [1] (LWM) and equivalent single layer models [2] framework. The resulting theories are often known as zig-zag theories (ZZT). Among the ZZTs that have been developed in the ESLM framework, three independent approaches are known. These were denoted in [7] as, the Lekhnitskii multilayered theory (LMT), the Ambartsumian multilayered theory (AMT) and the Reissner multilayered theory (RMT), respectively. The LMT and AWT describe the ZZ effect by enforcing IC via

[1] The number of the unknown variables is kept dependent on the number of the constitutive layers.

[2] The unknown variables are the same for the whole laminate.

* Tel.: +39-011-5646836; fax: +39-011-5646899.
E-mail address: erasmo.carrera@polito.it (E. Carrera).

0045-7949/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruc.2004.02.006

Nomenclature

1, 2, 3	subscripts denote displacement and stress components in the plate/shell reference system	AMT	Ambartsumian's multilayered theory
a, b, h	plate/shell geometrical parameters (length, width and thickness)	CLT	classical lamination theory
k	sub/super-script used to denote parameters related to the k-layer	ESLM	equivalent single layer models
$M(z)$	Murakami's zig-zag function	FSDT	first shear deformation theory
N	order of the expansions used for transverse stresses and displacements	HSDT	higher order shear deformation theory
N_l	number of constituent layers of multilayered plate/shell	IC	interlaminar continuity
R_β	radii of cylindrical shell	LMT	Lekhnitskii's multilayered theory
u	displacement component	LWM	layer-wise models
z	thickness coordinate	LM4	layer-wise mixed theory based on RMVT with fourth-order expansion for both displacements and transverse stresses
Ω	plate/shell reference surface	MZZF	Murakami's zig-zag function
σ	stress component	PVD	principle of virtual displacements
3D	three-dimensional analyses	RMVT	Reissner's mixed variational theorem
		RMT	Reissner's multilayered theory
		ZZ	zig-zag
		ZZT	zig-zag theory

constitutive equations of the layer along with strain–displacement relations. Independent assumptions for displacement and transverse stresses are instead made in the RMT applications.

In the framework of RMT applications, Murakami [8] introduced a function of the thickness coordinate that is able to emulate the ZZ effect. Such a function was denoted in [9] as the 'Murakami's zig-zag function' (MZZF). The MZZF was used in [8–13] to analyze the static response of layered plates and shells. Mixed finite elements were developed in [14–17] for plates and shells. The MZZF was also applied in the framework of plate/shell theories with only displacement variables [13,18], see also the review made in [9]. From an implementation point of view, the inclusion of the MZZF in existing plate/shell/beam models requires the same effort as those that are required to include an additional higher order term (by means of a higher order polynomial of the thickness coordinate). On the other hand, from a numerical point of view, as will be demonstrated in this paper, the MZZF leads to significant improvements of the existing plate/theories; these improvements are difficult to obtained when using other functions that differ from the MZZF.

This paper aims at contributing to a better understanding on the use of the MZZF as a tool to introduce ZZ effects in multilayered plate/shell structures. It has been organized as follows. The MZZF is described in Section 2. Section 3 discusses the use of the MZZF in the framework of plate/shell theories which are formulated on the basis of only displacement unknowns and the principle of virtual displacements (PVD) applications. This is the case of CLT, FSDT and HSDT, see [5]: it has been denoted as *simple use of the MZZF*. Section 4 presents the use of the MZZF in the case of the RMT which is formulated in terms of both displacement and transverse stress variables by referring to Reissner's mixed variational theorem (RMVT). This second possibility of using the MZZF has herein been denoted as *advanced use of the MZZF*. Numerical results are given in Sections 3 and 4. A comparison is made between the analyses, including and discarding the MZZF. Bending, vibration and thermal stress problems of plates and shells are treated. These results have been restricted to closed form solutions. Appendix A presents the extension of the MZZF to the Reddy–Vlasov theory (RVT) in the case of symmetrically laminated plates.

The full writing of the governing equations as well as their solution procedures have been omitted in the present work. Reference has been made to the unified, compact formulation for multilayered plate and shell theories (based on both PVD and RMVT applications) that have recently been proposed by the author. Those readers who are interested can refer to the exhaustive description given in [9,10].

2. The Murakami zig-zag function

Let us consider a multilayerd plate/shell composed by N_l layers, perfectly bonded together. z is the thickness coordinate of the whole multilayerd plate/shell with $z = 0$ at half o the plate thickness. Let z_k, the transverse coordinate of the layer k, be centered at half the individual layer thickness (h_k) with a nondimensional layer coordinate

$$\zeta_k = z_k/2h_k \quad -1 \leqslant \zeta_k \leqslant 1$$

Plate and shell geometry and notations have been depicted in Figs. 1 and 2. The Murakami zig-zag functions $M(z)$ was defined according to the following formula [8],

$$M(z) = (-1)^k \zeta_k \tag{1}$$

$M(z)$ has the following properties:

1. It is piece-wise linear function of the layer coordinates z_k;
2. $M(z)$ has unit magnitude for the whole layers;
3. The slope $M'(z) = \frac{dM}{dz}$ assumes opposite sign between two-adjacent layers. Its magnitude is layer thickness dependent.

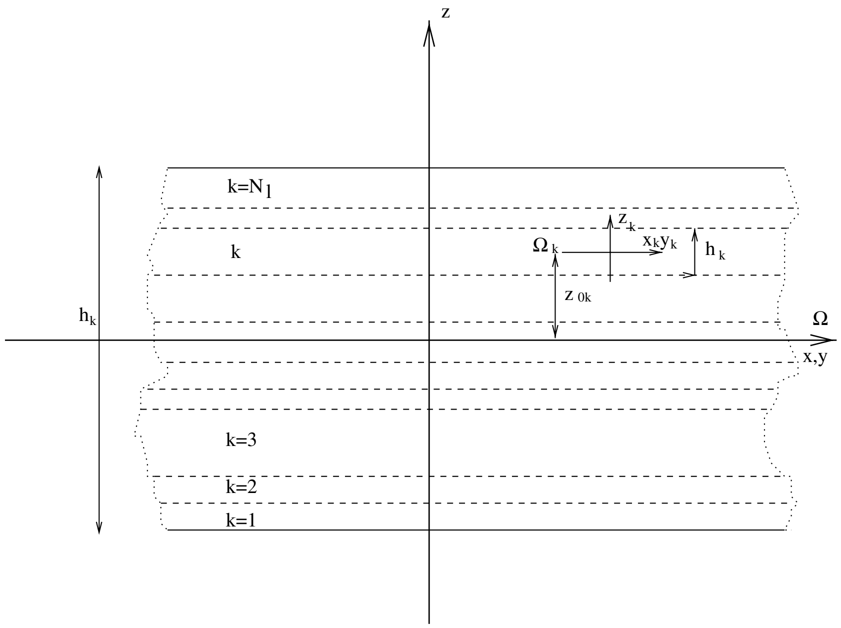

Fig. 1. Notations for a multilayered plate.

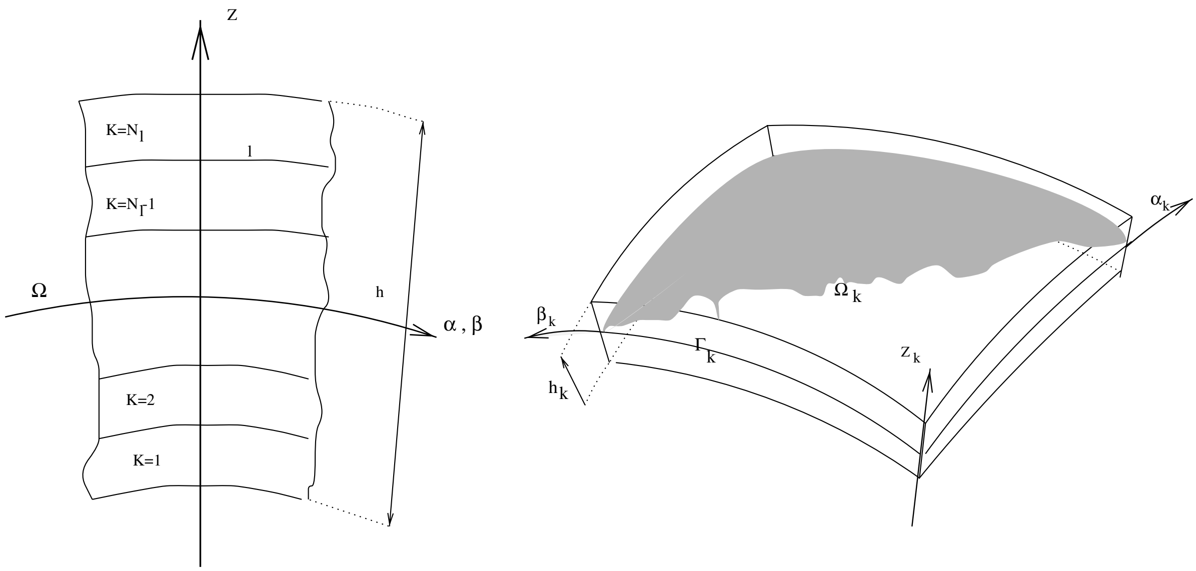

Fig. 2. Geometry and notations used for multilayered shells.

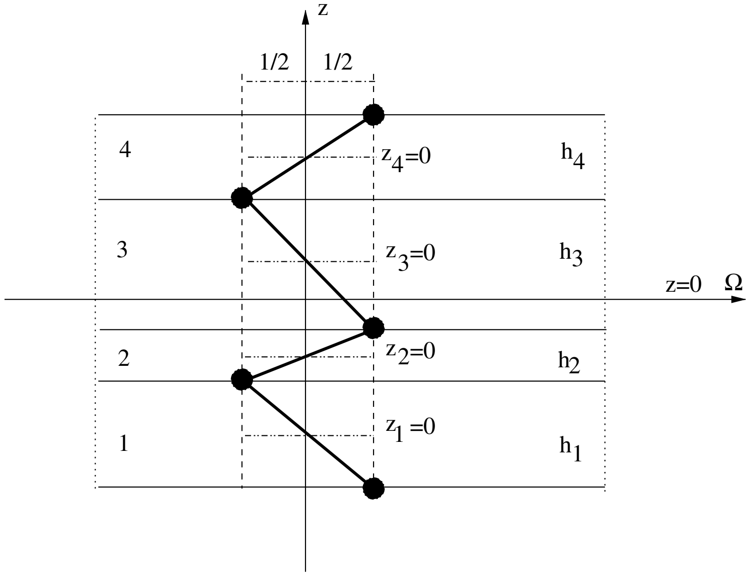

Fig. 3. Geometrical meaning of Murakami zig-zag function. A four layered structure is considered.

A plot of $M(z)$ is given in Fig. 3. The MZZF can be used to introduce discontinuous slopes with correspondence to the layer interfaces for any function $f(z)$. If a linear function is considered, one has

$$f_1(z) = c^0 + c^1 z \tag{2}$$

c^0, c^1 are the magnitudes of the constant and linear terms, respectively. By adding the MZZF one has,

$$f_{1M}(z) = c^0 + c^1 z + c^M M(z) \tag{3}$$

c^M is the effective magnitude of the ZZ effect. f_1 and f_{1M} are compared in Fig. 4 which makes evident how the $M(z)$ emulates the ZZ effects. $M(z)$ can also be used in conjunction to higher N-order expansion $f_N(z)$ (see Fig. 5),

$$f_{NM}(z) = c^0 + c^1 z + c^2 z^2 + \cdots + c^{N-1} z^{N-1} + c^N z^N + c^M M(z) \tag{4}$$

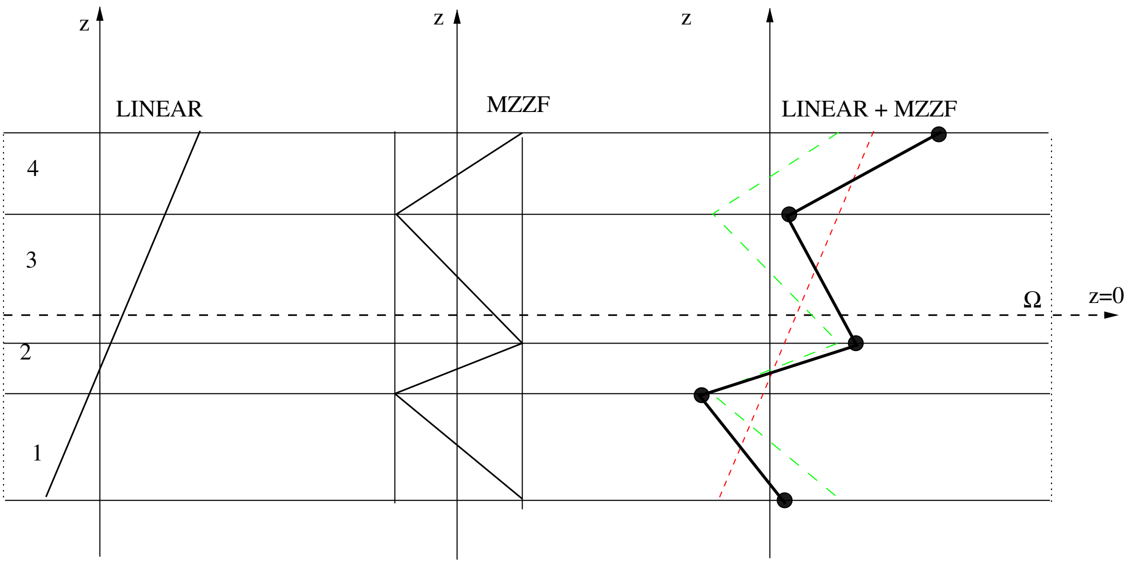

Fig. 4. Inclusion of the MZZF to a linear distribution.

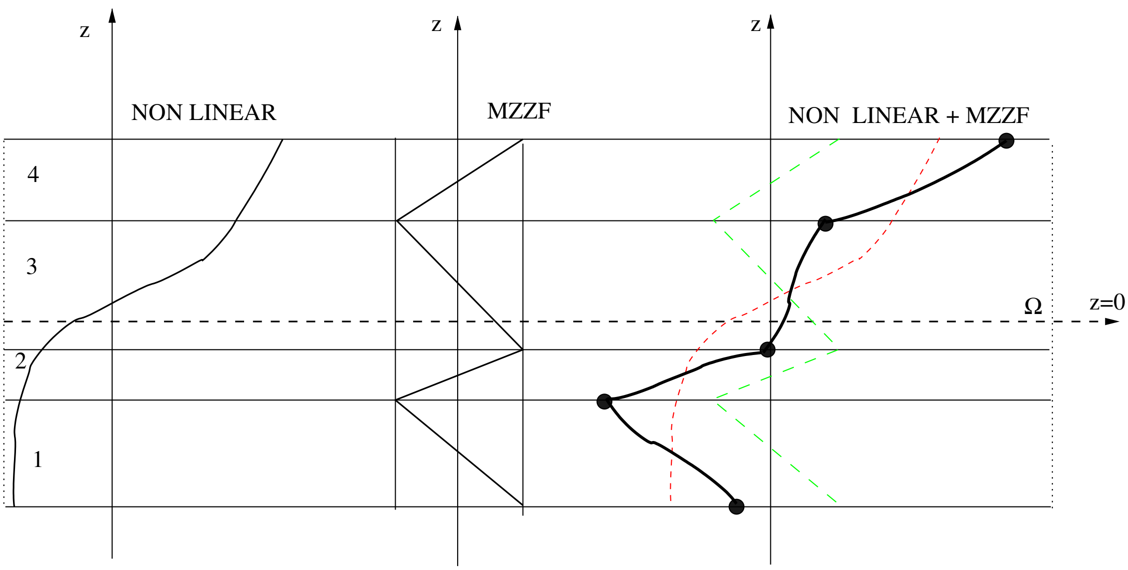

Fig. 5. Inclusion of the MZZF to a higher order distribution.

Power of z could be replaced by any other set of polynomials of z ($P_i,\ i=1,N$):

$$f_{NM}(z) = c^0P_0(z) + c^1P_1(z) + c^2P_2(z) + \cdots + c^{N-1}P^{N-1}(z) + c^NP^N(z) + c^MM(z) \quad (5)$$

3. Simple use of the MZZF: refinements of classical theories

Classical theories for multilayered plates and shells, such as CLT, FSDT and HSDT, see [5,6], do not account for the ZZ effect. A possible 'simple use' of the MZZF would consist to enhance classical models by 'simply' adding $M(z)$ in their displacement fields.

3.1. Formulation of displacement models

Let consider a linear distribution of a displacement component in the thickness direction z,

$$u = u^0 + zu^1 \quad (6)$$

in which: u is the displacement component along an assigned directions of the generic point P in a given reference system (Cartesian, for plate geometries, or curvilinear, for shell cases); u^0 is the value of u with correspondence to the plate/shell reference surface Ω to which correspondence one has $z=0$; u^1 is an additional variables (u^1 has the geometrical meaning of rotation of the normal to Ω in P); If CLT or FSDT applications are considered the Eq. (6) is retained only for the in-plane components (u_1 coincides to the derivative of the transverse displacement with respect to the in-plane coordinates in the CLT cases).

Table 1
Maximum transverse displacement $\overline{U}_z = U_z \times 100E_Th^3/(p^0a^4)$ ($z=0$) of thick plate ($a/h=4$) in cylindrical bending [20] $p_z = p^0 \sin \pi x/a$

	$N_l = 3$	$N_l = 4$
Exact [20]	2.887	4.181
HSDT ($N=3$) [22]	2.687	3.587
ZZT ($N=3$) [23]	–	4.083
ZZT ($N=1$) [23]	–	3.316
Present analysis Order-N	Discarding the MZZF	
1-CLT	0.508	1.115
1-FSDT	2.091	2.925
1	2.091	2.925
2	2.074	2.984
3	2.687	3.596
4	2.685	3.830
	Including the MZZF	
1-FSDT	2.798	3.178
1	2.798	3.171
2	2.781	3.378
3	2.876	4.089

Evaluations of benefits of simple use of the MZZF for cross-ply symmetrically (0/90/0) and unsymmetrically (0/90/90/0) laminated plates. Mechanical data of the lamina: $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$, $\nu_{LT} = \nu_{TT} = 0.25$.

Table 2
Transverse displacements $\overline{U}_z = U_z \times 100E_T h^3/(p^0 a^4)(z=0)$ and transverse shear stresses $\overline{S}_{xz} = S_{xz}/(p_0 a/h)$, ($z=0$ unless denoted) for a rectangular ($b=3a$) three layered plates (0/90/0) loaded by $p_z = p^0 \sin \pi x/a \sin \pi y/b$ in [20]

a/h	$\overline{U}_z$		$\overline{S}_{xz}$		
	4	20	4	z	20
Exact [20]	2.820	0.610	0.387	–	0.434
ZZT [27]	2.729	0.609	0.378	–	0.451
ZZT [21]	2.80	–	0.317	–	–
Present analyses					
Order-*N*	Discarding the MZZF				
1-CLT	0.5010	0.5010	0.439	0.0	0.440
1-FSDT	2.051	0.5633	0.436	0.0	0.439
1	2.051	0.5633	0.437	0.1	0.439
2	2.035	0.5668	0.437	0.060	0.439
3	2.627	0.5955	0.378	0.166	0.436
4	2.625	0.5955	0.378	0.166	0.436
	Including the MZZF				
1-FSDT	2.736	0.6020	0.394	±0.233	0.435
1	2.736	0.6020	0.389	0.233	0.435
2	2.719	0.6043	0.394	0.267	0.435
3	2.811	0.6095	0.388	0.267	0.434

Evaluations of benefits of simple use of the MZZF. Mechanical data of the lamina are those of Table 1.

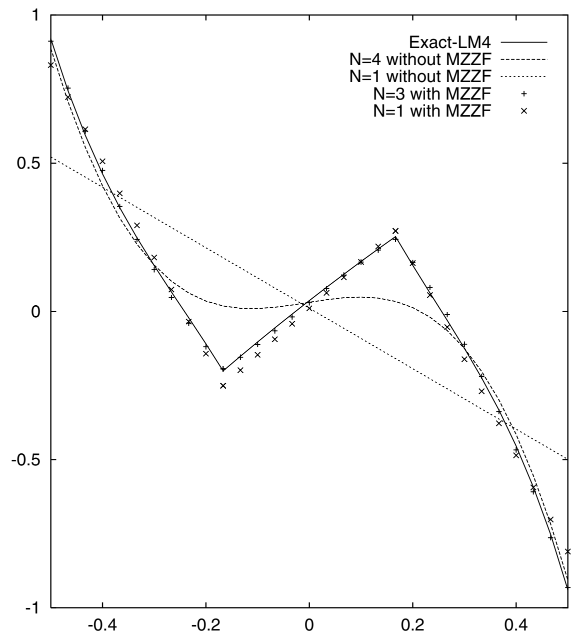

Fig. 6. Through the thickness distribution of the amplitude u_1 vs z. Simple use of the MZZF. Table 1 plate problem.

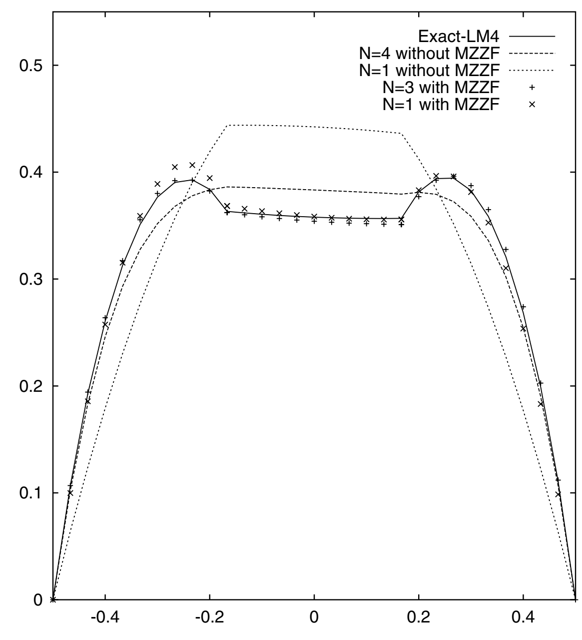

Fig. 7. Through the thickness distribution of the magnitude $\sigma_{13}/(P_0 a/h))$ vs z. Simple use of the MZZF. Table 1 plate problem.

The MZZF offers the possibility to introduce in a simple manners the ZZ effect in Eq. (6),

$$u = u^0 + zu^1 + (-1)^k \zeta_k u^M \quad (7)$$

The following remarks can be made:

1. The additional degree of freedom u^M has been introduced with respect to FSDT. It has a meaning of displacement.

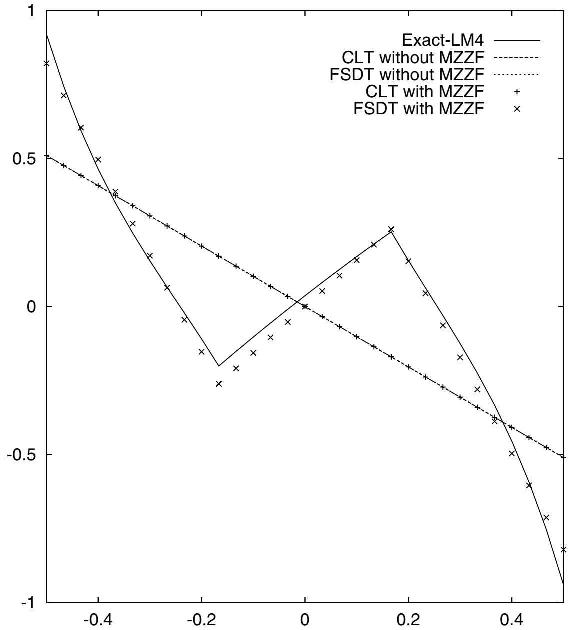

Fig. 8. Through the thickness distribution of the amplitude u_1 vs z. Simple use of the MZZF in the case of CLT and FSDT analyses. Table 1 plate problem.

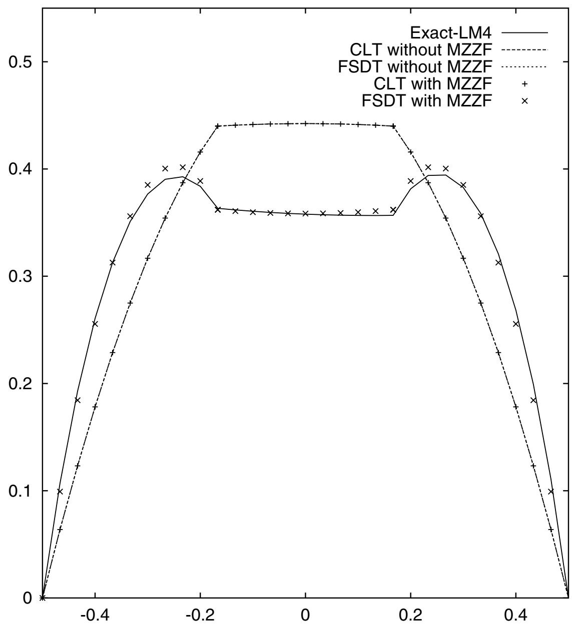

Fig. 9. Through the thickness distribution of the amplitude $\sigma_{13}/(P_0a/h)$ vs z. Simple use of the MZZF in the case of CLT and FSDT analyses. Table 1 plate problem.

2. The magnitude u^M is layer independent: u^M has, in fact, an intrinsic ESLM description. At a first glance this fact could appear as a strong limitation of the MZZF. In reality ZZ effect has ESLM description also in others ZZ theories, such as AWT and LMT, see [7].
3. The MZZF can be used for both the in-plane and the out-of-plane displacement components. This is a valuable advantage with respect to LMT and AMT. These two type of theories could, in fact, become difficult to be used if the ZZ effect is required for the transverse displacement component u_3, see [7].

Table 3
Ren's shells problem

R_β/h	2	4	10	50
Exact [24]	1.436	0.457	0.144	0.0808
CLT [24]	0.0799	0.0781	0.0777	0.0776
FSDT [25]	–	0.342	0.120	0.0793
HSDT-($N=3$) [25]	1.141	0.382	0.128	0.0796
Present analysis				
Order-N	Discarding the MZZF			
1	1.105	0.3292	0.1187	0.0795
2	1.112	0.3303	0.1190	0.0798
3	1.364	0.4225	0.1363	0.0805
4	1.355	0.4213	0.1363	0.0805
	Including the MZZF			
1	1.382	0.4398	0.1406	0.0804
2	1.376	0.4402	0.1408	0.0807
3	1.397	0.4527	0.1440	0.0809

Symmetric three layers (90/0/90) loaded by $p_z = p^0 \sin \pi\beta/b$. Transverse displacement magnitude. $\overline{U}_z = U_z \times \frac{10E_T h^3}{p^0 R_\beta^4}$, $z=0$. Exact solution by Ren [24]. Mechanical data of the lamina are those of Table 1.

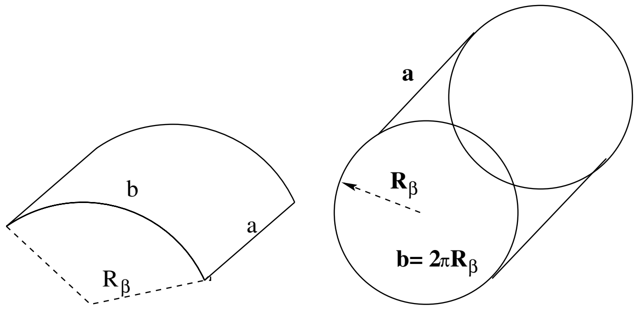

Fig. 10. Geometrical notations used for the investigated cylindrical panels and shells.

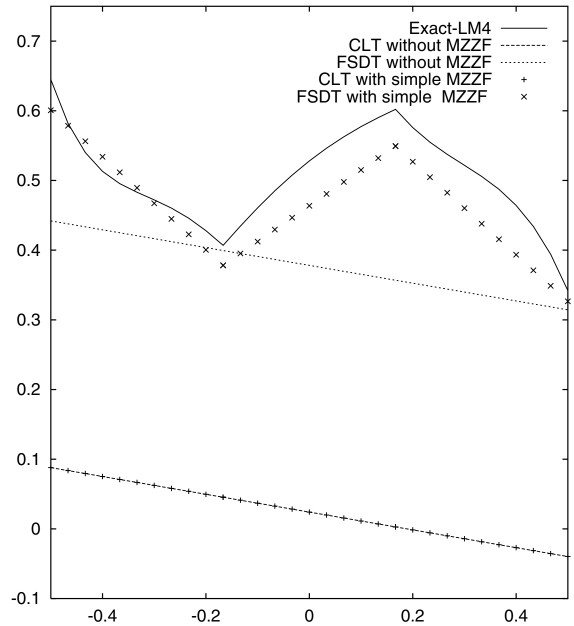

Fig. 11. Through the thickness distribution of the amplitude u_1 vs z. Simple use of the MZZF in the case of CLT and FSDT analyses. Table 3 shell problem.

3.1.1. Refinements of FSDT

The application the MZZF to FSDT leads to the following displacement model:

$$
\begin{aligned}
u_1 &= u_1^0 + zu_1^1 + (-1)^k \zeta_k u_1^M \\
u_2 &= u_2^0 + zu_2^1 + (-1)^k \zeta_k u_2^M \\
u_3 &= u_3^0
\end{aligned} \tag{8}
$$

Subscripts 1, 2 and 3 denote displacement components in three orthogonal directions of a given plate/shell reference systems. The third one refers to the thickness, transverse z-direction. The enhanced FSDT model has seven degrees of freedom, two more than the classical FSDT.

3.1.2. Refinement of FSDT by inclusion of ZZ effects and transverse normal strains ϵ_{zz}

The displacement model which include transverse normal strains as well as ZZ effect in the FSDT is

$$
\begin{aligned}
u_1 &= u_1^0 + zu_1^1 + (-1)^k \zeta_k u_1^M \\
u_2 &= u_2^0 + zu_2^1 + (-1)^k \zeta_k u_2^M \\
u_3 &= u_3^0 + zu_3^1 + (-1)^k \zeta_k u_3^M
\end{aligned} \tag{9}
$$

A discussion on the implication of the extension of the MZZF to transverse displacement u_3 was provided in [13].

3.1.3. Use of the MZZF in conjunction to higher order expansion

The MZZF can be used to introduce the ZZ effect in any HSDT type expansion. The expansion considered in this work makes use of power of z polynomials,

$$
u_i = u_i^0 + zu_i^1 + z^2 u_i^2 + \cdots z^N u_i^N + (-1)^k \zeta_k u_i^M, \quad i = 1, 3 \tag{10}
$$

N is the order of the expansion. The cases $N = 1$, 2 , 3 ,4 will be considered in the numerical discussion.

The MZZF can also be used in the framework of plate/shell theories which refer to displacement models which are different by those in Eq. (10). An example developed in the framework of the HSDT which is known as Reddy–Vlasov theory, as been detailed in Appendix A.

3.2. Numerical evaluations

In order to show the effectiveness of the MZZF a few numerical results are presented in this section. Comparison is made between theories which include the MZZF and correspondent theories in which the MZZF is discarded. Theories which show linear ($N = 1$) up to forth-order ($N = 4$) expansions are compared. The expansion of Eq. (10) is referred to the three displacements components. The acronim 1-FSDT implements Eq. (8); the acronim 1-CLT is a sub-case of 1-FSDT,

Table 4
Circular frequency parameter $\omega\sqrt{\frac{a^4\rho}{E_T h^2}}$ of simply supported square plates cross-ply skew-symmetric and symmetric laminates (layers of equal thickness); $\frac{E_L}{E_T}=40$, $\frac{G_{LT}}{E_T}=\frac{G_{Lz}}{E_T}=0.50$, $\frac{G_{TT}}{E_T}=0.60$, $\nu_{LT}=\nu_{Lz}=\nu_{TT}=0.25$

a/h	2	4	10	20	100
0/90					
LM4-Exact	4.703	7.345	10.088	10.859	11.151
Order-N	Discarding the MZZF				
1-CLT	8.576	10.388	11.115	11.230	11.267
1-FSDT	5.544	8.314	10.544	11.072	11.261
1	5.544	8.314	10.545	11.072	11.261
2	4.968	7.701	10.254	10.911	11.154
3	4.883	7.647	10.235	10.906	11.154
4	4.745	7.425	10.132	10.874	11.152
	Including the MZZF				
1-FSDT	4.874	7.605	10.279	10.992	11.257
1	4.848	7.562	10.215	10.921	11.184
2	4.838	7.545	10.189	11.891	11.153
3	4.780	7.490	10.165	10.884	11.152
0/90/90/0					
LM4-Exact	5.260	9.224	15.148	17.626	18.753
Order-N	Discarding the MZZF				
1-CLT	15.892	17.977	18.725	18.840	18.877
1-FSDT	5.927	9.960	15.573	17.829	18.833
1	5.927	9.960	15.573	17.829	18.833
2	5.920	9.938	15.522	17.763	18.759
3	5.392	9.389	15.232	17.655	18.754
4	5.380	9.384	15.232	17.655	18.754
	Including the MZZF				
1-FSDT	5.927	9.595	15.572	17.829	18.833
1	5.926	9.956	15.563	17.817	18.819
2	5.920	9.938	15.522	17.763	18.760
3	5.390	9.388	15.232	17.655	18.754

Table 5
Thermal stress transverse shear stress $\sigma_{13}/(E_T\alpha_L T_0)$ at $z=\mp h/6$

a/h	2	4	10	20	50	100
Exact [29]	−3.508	2.830	2.580	1.441	0.5948	0.2987
Order-N	Discarding the MZZF					
1	19.80	9.900	3.960	1.980	0.7920	0.3960
2	15.03	7.489	2.993	1.496	0.5984	0.2992
3	−20.92	0.2403	2.434	1.424	0.5938	0.2987
4	−20.91	0.2463	2.436	1.425	0.5938	0.2987
	Including the MZZF					
1	33.44	13.09	4.230	2.015	0.7943	0.3963
2	1.544	4.271	2.719	1.460	0.5961	0.2989
3	−4.814	2.687	2.580	1.442	0.5949	0.2988

Cylindrical bending problem considered by Bhaskar et al. [29]; temperature distribution in the thickness: $T=T_0\frac{2z}{h}\sin\frac{\pi x}{a}$. Three layered plate 0°/90°/0°. Thermo-mechanical data of the lamina: $E_L/E_T=25$, $G_{LT}/E_T=0.5$, $G_{TT}/E_T=0.2$, $\nu_{LT}=\nu_{TT}=0.25$, $\alpha_T/\alpha_L=1125$.

which is obtained by neglecting transverse shear strains (this is done by implementing a penalty technique on the shear correction factor).

Bending, vibration and thermal stress problems of plates and shells have been considered. PVD applications along with related governing equations of these problems as well as solution procedures have been omitted in this paper. Interested readers are addressed to the two papers [9,10].

3.2.1. Bending of plates and shells

Results on the MZZF influence on the response of laminated plates are discussed in Tables 1 and 2 and Figs. 6–9. Symmetrical and unsymmetrically laminated cross-ply plates are considered in Table 1. Harmonic distributions of transverse pressure have been considered. Data are given in the captions of tables and figures. Young moduli in the longitudinal and transverse direction of the fiber are denoted by E_L and E_T, respectively; G_{LT} and G_{TT} are the shear moduli, and ν_{LT} is the Poisson ratio. However, the notation that have been used for the thermo-mechanical properties of the lamina are those quoted in [5].

Present analyses have been compared to 3D exact solutions and to others HSDT and ZZT results. The following comments can be made:

1. The simple use of the MZZF shows improvement in the static response of the considered thick laminated plates. Such an improvement is very much subordinate to the order N.
2. To notice that the inclusion of the MZZF in FSDT analysis ($N = 1$) leads to improvements that cannot be obtained by HSDT which discard the MZZF (as it is the case of $N = 4$ or of the well known HSDT

Table 6
As Table 1, advanced use of the MZZF

	$N_l = 3$	$N_l = 4$
Exact	2.887	4.181
	Discarding the MZZF	
1-FSDT	2.196	3.142
1	2.196	3.142
2	2.109	3.029
3	2.744	3.660
4	2.717	3.869
	Including the MZZF	
1-FSDT	2.801	3.181
1	2.801	3.174
2	2.783	3.378
3	2.879	4.094

Table 8
As Table 3, advanced use of the MZZF

R_β/h	4	10	50
Exact [24]	0.457	0.144	0.0808
Order-N	Discarding the MZZF		
1	0.3453	0.1212	0.0796
2	0.3357	0.1198	0.0798
3	0.4315	0.1379	0.0806
4	0.4263	0.1372	0.0806
	Including the MZZF		
1	0.4403	0.1410	0.0804
2	0.4405	0.1411	0.0807
3	0.4531	0.1440	0.0809

Table 7
As Table 2, advanced use of the MZZF

a/h	0/90				
	2	4	10	20	100
LM4-Exact	4.703	7.345	10.088	10.859	11.151
Order-N	Discarding the MZZF				
1	11.259	7.822	10.367	11.018	11.259
2	4.711	7.352	10.092	10.861	11.152
3	4.799	7.527	10.181	10.889	11.153
4	4.731	7.408	10.124	10.871	11.152
	Including the MZZF				
1	4.672	7.340	10.106	10.878	11.170
2	4.727	7.395	10.119	10.896	11.152
3	4.685	7.444	10.144	10.877	11.152

in [22]). That is the introduction of the MZZF is very powerful in the case $N = 1$.

3. The simple use of the MZZF which includes transverse strain ϵ_{zz} effect can lead to better description with respect to those ZZT in which IC is included but ϵ_{zz} is discarded (as it is the case of the ZZT in [23], see also the discussion of Koiter's recommendation provided in [13]).
4. The order N influences in a different manner symmetrically and unsymmetrically laminated plates. The latter requires higher order terms.

The through the thickness distributions of the stress and the displacement related to the Table 1 problems of three layered plates have been traced in Figs. 6–9. CLT and FSDT results including and discarding the MZZF have been compared in Figs. 6 and 7. The plotted exact solutions correspond to the LWM mixed analyses with four-order expansion in each layer (these results have been denoted as Exact-LM4 in the made Figures and Tables; full description of LM4 implementations was provided in [9]). Even though only displacement variables have been considered, the inclusion of the MZZF function improves very much the performance of FSDT analyses for both displacement and transverse stress fields. It is confirmed that the inclusion of the MZZF in a CLT analyses is meaningless. hence transverse shear deformations are neglected by CLT: as expected the MZZF is ineffective in CLT cases. Figs. 6–9 confirm the comments already made for Table 1 discussion. Thick and moderately thin plates have been considered in Table 2 in which displacement and transverse stress evaluations are given. The effectiveness of the MZZF is confirmed for both linear and higher order expansions.

Bending of thick and thin cylindrical shells has been considered in Table 3 and Figs. 10 and 11. Fig. 10 shows geometry and notations of the investigated shells. The effectiveness of the simple use of the MZZF as a tools to improve the performance of classical theories has been confirmed for shell geometries. To notice as the MZZF becomes ineffective for thin shell cases in which transverse strains can be neglected with respect to in-plane ones.

Table 9
Simple and advanced use of MMZF

R_β/a	10	100
Exact [28]	10.027	9.815
PAR_{ds} [26]	10.223	10.013
HYP_{ds} [26]	10.226	10.018
UNI_{cs} [26]	10.187	9.977
PAR_{cs} [26]	10.051	9.840
HYP_{cs} [26]	10.050	9.839
Present analysis		
'Simple Use of the MZZF'		
Order-N	Discarding the MZZF	
1	11.065	10.862
2	11.040	10.842
3	10.179	9.969
4	10.178	9.968
	Including the MZZF	
1	10.104	9.889
2	10.090	9.879
3	10.030	9.818
'Advanced Use of the MZZF'		
	Including the MZZF	
1	10.090	9.876
2	10.082	9.872
3	10.029	9.818

Effect of radii to length ratio R_β/a on $\omega \times a^2\sqrt{\frac{\rho}{h^2E_T}}$. Comparison to exact solution by Ye and Soldatos [28] and to other refined analyses. $a/h = 10$, $m = 1$, $n = 2$ unless given in brackets. Three layered ringed shell 0/90/0, $h_1 = h_3 = h_2/2$. $E_L/E_T = 25$, $G_{LT}/E_T = 0.5$, $G_{TT}/E_T = 0.2$, $\nu_{LT} = \nu_{TT} = 0.25$.

3.2.2. Free vibration of plates and shells

The fundamental frequency parameter of vibrational response of symmetrically and unsymmetrically laminated plates have been compared in Table 4. From very thick ($a/h = 2$) to thin geometries ($a/h = 100$) have been compared. The benefits of inclusion of the MZZF for thick plates analyses are clearly shown by Table 4 results. Further results on shell vibration will be given in the subsequent Table 9.

3.2.3. Thermal stress in plates

The MZZF can be also used in thermoelastic analyses of plates. An example has been considered in Table 5. A plate loaded by a linear trough the thickness temperature field has been considered. The applied temperature has in-plane bi-sinusoidal distribution. Exact 3D solution to this problem was provided by Bhaskar et al. [29]. Very thick to thin geometries have been considered. Table 5 shows that the MZZF can be very effective to improve classical HSDT. The limitation of the presented ESLM analyses for very thick plates are herein underlined.

4. Advanced use of the MZZF in the framework of mixed variational methods

As stated in the Introduction, the MZZF was originally introduced in the framework of RMT applications. According to RMVT [19] statement two independent fields are assumed for the displacements and the

Table 10
As Table 5, advanced use of the MZZF

a/h	2	4	10	20	50	100
Exact	−3.508	2.830	2.580	1.441	0.5948	0.2987
Order-N	Discarding the MZZF					
1	19.80	9.901	3.960	1.980	0.7920	0.3960
2	14.99	7.484	2.992	1.496	0.5984	0.2992
3	−15.10	0.9014	2.461	1.427	0.5940	0.2987
4	−16.80	0.6969	2.453	1.426	0.5940	0.2987
	Including the MZZF					
1	34.07	13.37	4.263	2.020	0.7946	0.3963
2	3.050	4.535	2.735	1.462	0.5962	0.2990
3	−3.814	2.840	2.589	1.443	0.5950	0.2987

transverse stresses. These last are assumed independent in each layer and interlaminar continuous. AMT and LMT analyses introduced ZZ effect by imposing IC conditions for transverse shear and normal stresses. That cannot be done by mixed approach. In this context the MZZF plays a quite fundamental role. With respect to 'simple use of the MZZF' discussed in the previous section, the 'advanced use of the MZZF' permits the a priori fulfillment of both ZZ and IC in a multilayered plate/shell theories. As disadvantage an increase of the number of unknown variables is obtained. This fact can be somehow avoid by referring to the weak form of Hooke's law, presented in [9], which permits to express the transverse stress variables in terms of the displacement unknowns.

4.1. Stress and displacements models

The displacement models including or discarding the MZZF are the same of those used in the previous section. Transverse stress fields (both shear and normal components) in each layer are conveniently written [9] in terms of Legendre polynomials according to the following expansion,

$$\boldsymbol{\sigma}_\tau^k = F_{\mathrm{T}}(z_k)\boldsymbol{\sigma}_{\tau\mathrm{T}}^k + F_{\mathrm{B}}(z_k)\boldsymbol{\sigma}_{\tau\mathrm{B}}^k + F_2(z_k)\boldsymbol{\sigma}_{\tau 2}^k + \cdots + \cdots + F_{N-1}(z_k)\boldsymbol{\sigma}_{\tau(N-1)}^k + F_N(z_k)\boldsymbol{\sigma}_{\tau N}^k \quad (11)$$

where $\boldsymbol{\sigma}_{\tau\mathrm{T}}^k$, $\boldsymbol{\sigma}_{\tau\mathrm{B}}^k$ are the top and bottom layer values of the transverse stresses, while $\boldsymbol{\sigma}_{n\tau}^k$ $(\tau = 2, N)$ are the higher order terms; $F_{\mathrm{T}}(z_k)$, $F_{\mathrm{T}}(z_k)$ $F_r(z_k)$ are appropriate combinations of Legendre polynomials.

Bold letter denotes arrays. Subscript τ denotes the three out-of-plane stress components $\tau = 13$, 23, 33, for instance $\boldsymbol{\sigma}_\tau^k = (\sigma_{13}^k, \sigma_{23}^k, \sigma_{33}^k)$.

The IC requires the fulfillment of the following relation:

$$\boldsymbol{\sigma}_{\tau\mathrm{B}}^k = \boldsymbol{\sigma}_{\tau\mathrm{T}}^{k-1} \quad (12)$$

4.2. Numerical evaluations

Most of the investigations that were conducted in Section 3 have been re-analyzed in the framework of the mixed assumptions and RMVT applications. These are presented in Tables 6–10. The following main conclusions can be made on these analyses.

1. It is confirmed that it is convenient to refer to the mixed RMVT approach with respect to the PVD one, that is RMT leads to better results than those known as LMT and AWT applications.
2. The effectiveness of MZZF has been confirmed for mixed multilayered theories.

5. Concluding remarks

This work has evaluated the numerical performance of the inclusion of the Murakami zig-zag functions in multilayered plate/shell theories. Classical theories based on PVD applications as well as advanced ones which are based on RMVT applications, were considered. Bending, vibration and thermal stress response were treated. The following main conclusions were reached:

1. It has been shown that the MZZF offers a very simple method to include the zig-zag effect in existing plate/shell theories that were originally formulated for traditional isotropic, one-layered structures.
2. The inclusion of the MZZF in a given displacement model has resulted to be more effective than the introduction of higher order polynomials.
3. The MZZF can be used for both in-plane and transverse displacements, that is, the extension of the ZZ effect to transverse displacements does not require any additional effort. Additional efforts are instead required if AMT or LMT are employed.

4. However, the use of the MZZF appears constrained by its intrinsic ESLM description. On the other hand, such a limitation is common to any other zig-zag theory which is developed in the ESLM framework.

This work was restricted the to closed form solutions. The obtained results should encourage the use of the MZZF in the framework of approximated methods, such as the finite element method. In particular, the MZZF would permit the inclusion of the ZZ effect in existing plate/shell finite elements by preserving the C^0-continuity of the nodal variables. The evaluation of the MZZF in the framework of FE applications could be subject of future investigation.

Appendix A. Introduction of the MZZF in a Reddy–Vlasov theory

A.1. The original RVT

So called Reddy–Vlasov HSDT [5,6] introduces the fulfillment of homogeneous conditions for transverse shear stresses in a plate/shell theories. For the sake of simplicity the plate geometry has been herein addressed.

According to RVT, the original FSDT displacement model is enhanced to third-order for the in-plane components,

$$\begin{aligned} u_1 &= u_1^0 + zu_1^1 + z^2u_1^2 + z^3u_1^3 \\ u_2 &= u_2^0 + zu_2^1 + z^2u_2^2 + z^3u_2^3 \\ u_3 &= u_3^0 \end{aligned} \tag{A.1}$$

By imposing the homogenoues conditions of transverse shear stress

$$\begin{aligned} &\sigma_{xz}(x,y,+h/2) = 0, \quad \sigma_{xz}(x,y,-h/2) = 0, \\ &\sigma_{yz}(x,y,+h/2) = 0, \quad \sigma_{yz}(x,y,-h/2) = 0, \end{aligned} \tag{A.2}$$

and upon substitution of displacement field in the transverse shear strains one has

$$\begin{aligned} &u_1^1 + hu_1^2 + 3\frac{h^2}{4}u_1^3 + u_{3,x}^1 = 0, \\ &u_1^1 - hu_1^2 + 3\frac{h^2}{4}u_1^3 + u_{3,x}^1 = 0, \\ &u_2^1 + hu_2^2 + 3\frac{h^2}{4}u_2^3 + u_{3,y}^1 = 0, \\ &u_2^1 - hu_2^2 + 3\frac{h^2}{4}u_2^3 + u_{3,y}^1 = 0 \end{aligned}$$

It follows that the additional variables can be expressed in terms of those related to FSDT,

$$\begin{aligned} &u_1^2 = u_2^2 = 0, \quad u_1^3 = -\frac{4}{3h^2}(u_1^1 + u_{3,x}^1), \\ &u_2^3 = -\frac{4}{3h^2}(u_2^1 + u_{3,y}^1) \end{aligned} \tag{A.3}$$

The resulting displacement fields of the RVT is

$$\begin{aligned} u_1 &= u_1^0 - \frac{4z^3}{3h^2}u_{3,x}^1 + \left(z - \frac{4z^3}{3h^2}\right)u_1^1 \\ u_2 &= u_2^0 - \frac{4z^3}{3h^2}u_{3,y}^1 + \left(z - \frac{4z^3}{3h^2}\right)u_2^1 \\ u_3 &= u_3^0 \end{aligned} \tag{A.4}$$

To notice that VRT has the same number of degrees of freedom of FSDT. In addition VRT include a parabolic distribution of transverse shear strains.

A.2. Enhancement of RVT via MMZF

The MZZF can be also introduced in the framework of the RVT,

$$\begin{aligned} u_1 &= u_1^0 + zu_1^1 + z^2u_1^2 + z^3u_1^3 + \zeta_k(-1)^k u_1^M \\ u_2 &= u_2^0 + zu_2^1 + z^2u_2^2 + z^3u_2^3 + \zeta_k(-1)^k u_2^M \\ u_3 &= u_3^0 \end{aligned} \tag{A.5}$$

Taking into account that the derivative of MZZF is

$$\frac{2}{h_k}(-1)^k$$

which is constant in each layer but its magnitude is different in those layers that have different thicknesses. The zero transverse shear stress conditions lead to

$$\begin{aligned} &u_1^1 + hu_1^2 + 3\frac{h^2}{4}u_1^3 + u_{3,x}^1 + \frac{2}{h_k}(-1)^k u_1^M = 0, \\ &u_1^1 - hu_1^2 + 3\frac{h^2}{4}u_1^3 + u_{3,x}^1 + \frac{2}{h_k}(-1)^k u_1^M = 0, \\ &u_2^1 + hu_2^2 + 3\frac{h^2}{4}u_2^3 + u_{3,y}^1 + \frac{2}{h_k}(-1)^k u_2^M = 0, \\ &u_2^1 - hu_2^2 + 3\frac{h^2}{4}u_2^3 + u_{3,y}^1 + \frac{2}{h_k}(-1)^k u_2^M = 0 \end{aligned} \tag{A.6}$$

Also in this case the higher order variables are expressed to those related to the enhanced FSDT case,

$$\begin{aligned} &u_1^2 = u_2^2 = 0, \\ &u_1^3 = -\frac{4}{3h^2}\left(u_1^1 + u_3^1,x + \frac{2}{h_k}(-1)^k u_1^M\right), \\ &u_2^3 = -\frac{4}{3h^2}\left(u_2^1 + u_3^1,y + \frac{2}{h_k}(-1)^k u_2^M\right) \end{aligned}$$

The resulting displacement field of RVT is,

$$\begin{aligned} u_1 &= u_1^0 - \frac{4z^3}{3h^2}u^1_{3_{,x}} + \left(z - \frac{4z^3}{3h^2}\right)u_1^1 \\ &\quad + \left(\zeta_k - \frac{8z^3}{3h_k h^2}\right)(-1)^k u_1^M \\ u_2 &= u_2^0 - \frac{4z^3}{3h^2}u^1_{3_{,y}} + \left(z - \frac{4z^3}{3h^2}\right)u_2^1 + \left(\zeta_k - \frac{8z^3}{3h_k h^2}\right) \\ &\quad (-1)^k u_2^M \\ u_3 &= u_3^0 \end{aligned} \tag{A.7}$$

Eq. (A.7) represents the displacement field of a modified RVT which includes ZZ effects according to the MZZF statement. Numerical evaluations of this theory have not been provided in this work.

References

[1] Lekhnitskii SG. Strength calculation of composite beams. Vestnik inzhen tekhnikov 1935;(9).
[2] Lekhnitskii SG. Anisotropic plates. 2nd ed. Translated from the 2nd Russian Ed by Tsai SW, Cheron. Gordon and Breach; 1968.
[3] Ambartsumian SA. Theory of anisotropic shells. Fizmatzig, Moskwa, 1961. Translated from Russian, NASA TTF–118, 1964.
[4] Ambartsumian SA. Theory of anisotropic plates. Translated from Russian by Cheron T, Edited by JE Ashton Tech Pub Co, 1969.
[5] Reddy JN. Mechanics of laminated composite plates, theory and analysis. CRC Press; 1997.
[6] Carrera E. Theories and finite elements for multilayered anisotropic, composite plates and shells. Arch Comput Meth Eng State Art Rev 2002;9:87–140.
[7] Carrera E. Historical review of zig-zag theories for multilayered plates and shells. Appl Mech Rev 2003;56: 287–308.
[8] Murakami H. Laminated composite plate theory with improved in-plane response. J Appl Mech 1986;53:661–6.
[9] Carrera E. Developments, ideas and evaluations based upon the Reissner's mixed theorem in the modeling of multilayered plates and shells. Appl Mech Rev 2001;54:301–29.
[10] Carrera E. Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking. Arch Comput Meth Eng State Art Rev 2003;10:215–97.
[11] Toledano A, Murakami H. A high-order laminated plate theory with improved in-plane responses. Int J Solids Struct 1987;23:111–31.
[12] Jing H, Tzeng KG. Refined shear deformation theory of laminated shells. Am Inst Aeronaut Astronaut J 1993;31:765–73.
[13] Carrera E. A study of transverse normal stress effects on vibration of multilayered plates and shells. J Sound Vib 1999;225:803–29.
[14] Rao KM, Meyer-Piening HR. Analysis of thick laminated anisotropic composites plates by the finite element method. Compos Struct 1990;15:185–213.
[15] Brank B, Carrera E. Multilayered shell finite element with interlaminar continuous shear stresses: a refinement of the Reissner–Mindlin formulation. Int J Numer Meth Eng 2000;48:843–74.
[16] Carrera E, Parisch H. Evaluation of geometrical nonlinear effects of thin and moderately thick multilayered composite shells. Compos Struct 1998;40:11–24.
[17] Carrera E, Demasi L. Multilayered finite plate element based on Reissner mixed variational theorem. Part I: theory and Part II: numerical analysis. Int J Numer Meth Eng 2002;55:191–231, 253–91.
[18] Bhaskar K, Varadan TK. A higher-order theory for bending analysis of laminated shells of revolution. Comput Struct 1991;40:815–9.
[19] Reissner E. On a certain mixed variational theory and a proposed applications. Int J Numer Meth Eng 1984;20: 1366–8.
[20] Pagano NJ. Exact solutions for composite laminates in cylindrical bending. J Compos Mater 1969;3:398–411.
[21] Ren JG. A new theory for laminated plates. Compos Sci Technol 1986;26:225–39.
[22] Lo KH, Christensen RM, Wu EM. A higher-order theory of plate deformation. Part 2: laminated plates. J ApplMech 1977;44:669–76.
[23] Cho M, Parmerter RR. Efficient higher order composite plate theory for general lamination configurations. Am Inst Aeronaut Astronaut J 1993;31:1299–305.
[24] Ren JG. Exact solutions for laminated cylindrical shells in cylindrical bending. Compos Sci Technol 1987;29:169–87.
[25] Dennis ST, Palazotto AN. Laminated shell in cylindrical bending, two-dimensional approach vs exact. Am Inst Aeronaut Astronaut J 1991;29:647–50.
[26] Timarci T, Soldatos KP. Comparative dynamic studies for symmetric cross-ply circular cylindrical shells on the basis a unified shear deformable shell theories. J Sound Vib 1995;187:609–24.
[27] Idlbi A, Karama M, Touratier M. Comparison of various laminated plate theories. Compos Struct 1997;37:173–84.
[28] Ye JQ, Soldatos KP. Three-dimensional vibration of laminated cylinders and cylindrical panels with symmetric or antisymmetric cross-ply lay-up. Compos Eng 1994;4:429–44.
[29] Bhaskar K, Varadan TK, Ali JSM. Thermoelastic solution for orthotropic and anisotropic composites laminates. Composites Part B 1996;27B:415–20.