Historical review of Zig-Zag theories for multilayered plates and shells

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This paper gives a historical review of the theories that have been developed for the analysis of multilayered structures. Attention has been restricted to the so-called Zig-Zag theories, which describe a piecewise continuous displacement field in the plate thickness direction and fulfill *interlaminar continuity* of transverse stresses at each layer interface. Basically, plate and shell geometries are addressed, even though beams are also considered in some cases. Models in which the number of displacement variables is kept independent of the number of constitutive layers are discussed to the greatest extent. Attention has been restricted to those plate and shell theories which are based on the so-called method of hypotheses or axiomatic approach in which assumptions are introduced for displacements and/or transverse stresses. Mostly, the work published in the English language is reviewed. However, an account of a few articles originally written in Russian is also given. The historical review conducted has led to the following main conclusions. 1) Lekhnitskii (1935) was the first to propose a Zig-Zag theory, which was obtained by solving an elasticity problem involving a layered beam. 2) Two other different and independent Zig-Zag theories have been singled out. One was developed by Ambartsumian (1958), who extended the well-known Reissner-Mindlin theory to layered, anisotropic plates and shells; the other approach was introduced by Reissner (1984), who proposed a variational theorem that permits both displacements and transverse stress assumptions. 3) On the basis of historical considerations, which are detailed in the paper, it is proposed to refer to these three theories by using the following three names: Lekhnitskii Multilayered Theory, (LMT), Ambartsumian Multilayered Theory (AMT), and Reissner Multilayered Theory (RMT). As far as subsequent contributions to these three theories are concerned, it can be remarked that: 4) LMT although very promising, has almost been ignored in the open literature. 5) Dozens of papers have instead been presented which consist of direct applications or particular cases of the original AMT. The contents of the original works have very often been ignored, not recognized, or not mentioned in the large number of articles that were published in journals written in the English language. Such historical unfairness is detailed in Section 3.2. 6) RMT seems to be the most natural and powerful method to analyze multilayered structures. Compared to other theories, the RMT approach has allowed from the beginning development of models which retain the fundamental effect related to transverse normal stresses and strains. This review article cites 138 references. [DOI: 10.1115/1.1557614]

1 INTRODUCTION

Two-dimensional (2D) modeling of multilayered plates and shells requires appropriate theories. The discontinuity of physical/mechanical properties in the thickness direction makes inadequate those theories which were originally developed for one-layered structures, eg, the Cauchy-Poisson-Kirchhoff-Love thin plate/shell theory [1–4], or the Reissner-Mindlin theory (Reissner [5] and Mindlin [6]), as well as higher order models such as the one by Hildebrand, Reissner, and Thomas [7]. These theories are, in fact, not able to reproduce piecewise continuous displacement and transverse stress fields in the thickness direction, which are experienced by multilayered structures [8]. In [9], these two effects have been summarized by the acronym C_z^0-requirements; that is, displacements and transverse stresses must be C^0-continuous functions in the z-thickness direction. A qualitative comparison of displacement and stress fields in a single-layered and a multi-layered structure is shown in Fig. 1. This picture clearly shows that theories designed for single-layered structures are not suitable to analyze multilay-

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ered ones. The piecewise form of transverse stress and displacement fields are often described in the open literature as *Zig-Zag* (ZZ) and *Interlaminar Continuity* (IC), respectively. The theories which describe these two effects are referred to as *Zig-Zag theories* in the present study.

A number of refinements of classical models, as well as theories developed for multilayered structures, have been proposed in the literature over the last four decades. Axiomatic and asymptotic approaches, along with other continuum-based ones, have been used to build 2D theories in both cases of equivalent single layer (ESL) and layer-wise (LW) variable description. Following Reddy [10], it is intended that the number of displacement variables is kept independent of the number of constitutive layers in the ESL models, while the same variables are independent in each layer for LW models. Koiter in [11] states, "A refinement of Love's first approximation theory is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time." Koiter's Recommendation (KR) should be taken into account in the development of Zig-Zag theories.

For a complete review of several approaches, computational techniques and numerical assessment, readers are referred to the many survey articles available on multilayered beams, plates, and shells. Among these, recommended reviews are the articles by Ambartsumian [12,13], Grigolyuk and Kogan [14], Librescu and Reddy [15], Leissa [16], Grigolyuk and Kulikov [17], Kapania and Raciti [18], Kapania [19], Noor and Burton [20,21], Jemlelita [22], Vasiliev and Lur'e [23], Reddy and Robbins [24], Noor, Burton, and Bert [25], Lur'e and Shumova [26], Grigorenko [27], Grigorenko and Vasilenko [28], Altenbach [29], Carrera [30], as well as the books by Lekhnitskii [31], Ambartsumian [32–34], Librescu [35], and Reddy [10].

Although these review works are excellent, in the author's opinion there still exists a need for a historical review with the aim of giving clear answers to the following questions:

1) Who first presented a zig-zag theory for a multilayered structure?

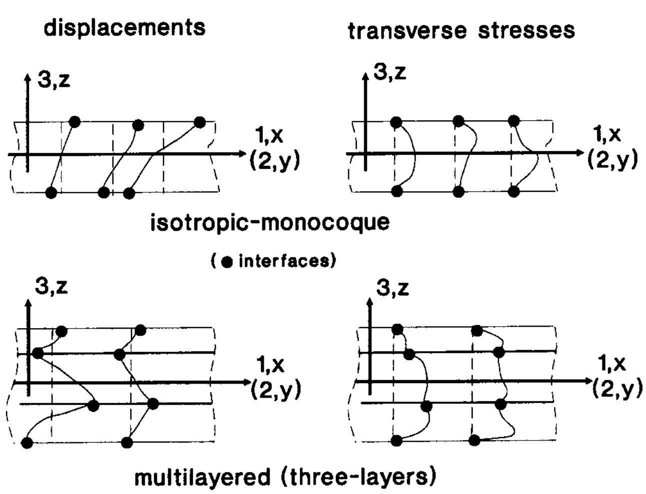

Fig. 1 C_z^0-requirements. Comparison of transverse stress field between a single-layered structure and a three-layered structure

2) How many different and independent ESL zig-zag theories have been proposed in the open literature?
3) Who first proposed the theories in question 2?
4) Are the original works recognized and mentioned correctly in the subsequent articles?
5) What are the main differences among the available zig-zag approaches to multilayered structures?

A response to these five points would be extremely useful to the analysts of layered structures. It will give an insight into the early and, at the same time, very interesting ideas and methods, such as those by Lekhnitskii [36], that could be extended and applied to further problems. Furthermore, as it is in any historical note, the author aims to establish a sort of historical justice, permitting to "give to Caesar what belongs to Caesar and to God what belongs to God."

The already mentioned works [12–35] show that a large number of different techniques, methods, and ideas have been applied to analyze multilayered structures. The attention of the present article has been focused on those zig-zag theories which fall into the two following categories:

I) The method of hypotheses is referred to and distributions of displacement and stress fields in the thickness direction of the plate/shell are introduced in an axiomatic sense.

II) With the exception of the approaches proposed by Reissner (Section 4) only those ZZ theories that have been developed in the framework of ESLM are discussed in this review.

Those analyses, such as the very interesting papers [37–46] which are based on an asymptotic approach have not been discussed in the present paper. A few examples of other approaches not discussed herein follows. Savin and Khoma [47] obtained multilayered shell equations by averaging 3D equations over the thickness coordinate. The concept of a uniform stress-strain state has been used by Khoroshun [48]. Full mixed methods were applied in [49–53]. Of certain relevance are the many developments made in the framework of LWM. Among these, the articles [54–61] are mentioned herein. For a complete and detailed review of the several topics and contributions, readers are referred to the overviews given in [10,12–35].

This paper, therefore, gives a historical review of available *Zig-Zag* theories in the context of questions 1–5, taking into account the limitations mentioned in points *I* and *II*.

The answer to question 1 is provided in Section 2, which shows that Lekhnitskii [36] was the first to propose a Zig-Zag theory that described the C_z^0-requirements.

As far as questions 2 and 3 are concerned, the author has recognized that apart from the method by Lekhnitskii [36], two other independent and different theories have been proposed in the literature in the second half of last century. The first of these was given by Ambartsumian in the articles [62,63] while the technique of building a second type of theory was traced later to Reissner [64]. The three approaches are discussed in Sections 2–4, along with a few details and literature. Based on historical reasons, which are detailed in the respective sections, it has been proposed to refer to these three different and independent contributions by the following names:

- Lekhnitskii Multilayered Theory (LMT)
- Ambartsumian Multilayered Theory (AMT)
- Reissner Multilayered Theory (RMT)

Concerning question 4, it will be pointed out that the original contribution by Lekhnitskii has been almost totally ignored in the subsequent literature (exception has to be made for the work by Ren [65,66]; see next section). The plate/shell theory developed by Ambartsumian has received differing attentions in the West and in the East. The Russian literature has recognized the Ambartsumian theory and improved it for its application to other problems, including the treatment of KR. On the other hand, the Western literature, with the exception made in a few works that appeared in the early 1970s, has not properly recognized Ambartsumian's works. This story has been detailed in Section 3.2. It appears that contributions made by Lekhnitskii and Ambartsumian have not received proper recognition. Probably because of World War II and the subsequent Cold War, these works have had little impact outside the Soviet Union.

As far as question 5 is concerned, discussion of the three approaches is given in Section 5. Among the three approaches discussed, the RMT theory has proved to be both the most versatile to describe completely the C_z^0-requirements and the most suitable for computational studies.

Concerning notation, the author has tried as much as possible to use those symbols that were quoted in the original articles. Such a choice would permit the readers to compare the formulas reported herein directly with those originally given. In any case, due to the large amounts of algebra along with the significant numbers of described theories, symbols used in a certain section refer only to that section. In those cases in which a given symbol is not defined in a given section, its definition has been provided on a earlier section. The present paper refers to beam, plate, and shell structures. Formulas related to the discussed theories do not consider both flat and curved geometries. For the sake of brevity, flat geometry is referred to in some cases, while doubly curved geometry has been considered in others.

2 LEKHNITSKII MULTILAYERED THEORY

To the best of the author's knowledge, Lekhnitskii should be considered the first contributor to the theory of multilayered structures. In [36], in fact, Lekhnitskii proposed a splendid method able to describe the Zig-Zag effect (for both in-plane and through the thickness displacements) and interlaminar continuous transverse stresses. To prove this point, Fig. 2, which is taken from the pioneering work of Lekhnitskii [36], shows an interlaminar continuous transverse shear stress field (τ^1 and τ^2 are shear stresses in layers 1 and 2, respectively) with discontinuous derivatives at the layer interface. In other words, C_z^0-requirements of Fig. 1 were entirely accounted for by Lekhnitskii [36].

The author believes it is relevant and of interest to quote the original derivations made by Lekhnitskii. It can, in fact, prove to be difficult to obtain the original article by Lekhnitskii which has not been translated into English. Furthermore, the theory proposed by Lekhnitskii is very interesting, and the method used could represent a starting point for future developments. The following detailed derivation is therefore taken directly from the original paper by Lekhnitskii, written in Russian. Few changes of notations were made. A more brief treatment can be found in the English translation of the book by Lekhnitskii [31] (Section 18 of Chapter III, p 74).

Let us consider a cantilever beam bent by a force P and a moment M and made up of N_l isotropic layers (see Fig. 3); x and z are the longitudinal and thickness directions, respectively. l and H are the length and thickness of the beam, respectively. c is the beam width and y is the related coordinate. Subscript k is used to denote variables and parameters related to the k-th layer. The layers are counted starting from the bottom. s_k is the layer thickness and h_k is the distance of a layer interface from the x-axis. σ_{xx}^k, σ_{zz}^k, σ_{xz}^k, u^k, and v^k are the stress and displacement components of the k-th layer. E^k, ν^k, and G^k are the Young's modulus, Poisson's ratio, and shear modulus of the k-th layer, respectively.

The problem considered is an $x-z$ plane stress problem. It can therefore be formulated in terms of a stress function φ^k defined in the k-th layer on the domain x,z. Stresses can be calculated using φ^k according to the following well-known relations:

$$\sigma_{xx}=\frac{\partial^2\varphi}{\partial z^2}, \quad \sigma_{zz}=\frac{\partial^2\varphi}{\partial x^2}, \quad \sigma_{xz}=-\frac{\partial^2\varphi}{\partial x\,\partial z} \tag{1}$$

Compatibility of strains can be written in terms the stress function according to the following compatibility equation:

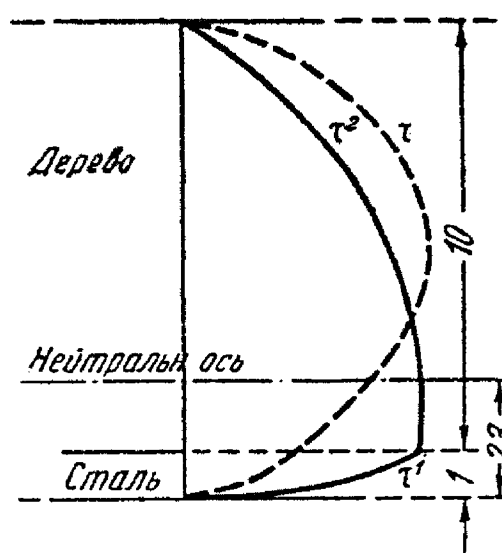

Fig. 2 C_z^0-form of a transverse shear stress in a two-layered structure. The interface and neutral axis are shown. The upper and lower layers are made of low and high stiffness materials, respectively. This graph was taken from the original work [36].

$$\frac{\partial^4 \varphi^k}{\partial x^4} + \frac{\partial^4 \varphi^k}{\partial x^2 \partial z^2} + \frac{\partial^4 \varphi^k}{\partial z^4} = 0 \qquad (2)$$

At this stage, the fundamental *axiomatic* assumption of the Lekhnitskii's theory is made, ie, a form for the stress function is assumed:

$$\varphi^k(x,z) = \frac{a_k}{6} z^3 + \frac{b_k}{2} z^2 + x\left(\frac{A_k}{3} z^3 + \frac{B_k}{2} z^2 + C_k z\right) \qquad (3)$$

A different choice for the stress function would lead to different results. According to Eq. (1) the chosen form is able to describe:

- Longitudinal stresses σ_{xx}, which are linear in the thickness z direction as well as in the longitudinal direction x,
  $$\sigma_{xx}^k = a_k z + b_k + x(2A_k z + B_k) \qquad (4)$$
- Thickness stresses σ_{zz}, which are zero in each x,z positions
  $$\sigma_{zz}^k = 0 \qquad (5)$$
- Transverse shear stress σ_{xz}, which is parabolic in z and independent of the longitudinal coordinate x
  $$\sigma_{xz}^k = -A_k z^2 - B_k z - C_k \qquad (6)$$

The $5 \times N_l$ unknown-constants a_k, b_k, A_k, B_k, and C_k must be determined according to the relations given in items $I-V$ which follow:

$I)$ Displacements are related to stresses by means of the strain-stress relations written for the constitutive equation of the k-th layer (eg, Hooke's Law written in terms of compliances):

$$\frac{\partial u^k}{\partial x} = \frac{1}{E^k}\sigma_{xx}^k - \frac{\nu^k}{E_2^k}\sigma_{zz}^k$$

$$\frac{\partial w^k}{\partial z} = -\frac{\nu^k}{E^k}\sigma_{xx}^k + \frac{1}{E^k}\sigma_{zz}^k \qquad (7)$$

$$\frac{\partial u^k}{\partial z} + \frac{\partial w^k}{\partial x} = \frac{1}{G^k}\sigma_{xy}^k$$

$II)$ Compatibility conditions for displacements at the interfaces are *Zig-Zag effects*,

$$u^{k-1} = u^k, \quad w^{k-1} = w^k, \quad k = 2, N_l \qquad (8)$$

$III)$ *Homogeneous conditions* at the bottom/top surface for the transverse stresses,

$$\sigma_{yy}^1 = \sigma_{yy}^{N_l} = 0, \quad \sigma_{xy}^1 = \sigma_{xy}^{N_l} = 0 \quad at \quad z = 0, H \qquad (9)$$

$IV)$ *Interlaminar equilibrium* for the transverse stresses

$$\sigma_{yy}^{k-1} = \sigma_{yy}^k, \quad \sigma_{xy}^{k-1} = \sigma_{xy}^k, \quad k = 2, N_l \qquad (10)$$

$V)$ Equivalence or equilibrium conditions between applied loads (M,P) and stresses

$$c\sum_1^{N_l} \int_{h_{k-1}}^{h_k} \sigma_{zz} dz = 0,$$

$$c\sum_1^{N_l} \int_{h_{k-1}}^{h_k} \sigma_{xx} z dz = \frac{M - Px}{H},$$

$$c\sum_1^{N_l} \int_{h_{k-1}}^{h_k} \sigma_{xz} dz = -\frac{P}{H} \qquad (11)$$

First Eqs. (7) are integrated in the x and z directions. The following expressions for the displacements u^k and w^k are obtained:

$$u^k = \frac{A_k}{E^k} x^2 z + \frac{B_k}{2E^k} x^2 + (a_k y + b_k)\frac{x}{E_k} + \left(\frac{\nu^k}{E^k} - \frac{1}{G^k}\right)\left(\frac{A_k}{E^k} z^3 + \frac{B_k}{2} z^2\right) - \alpha_k y + \beta_k \qquad (12)$$

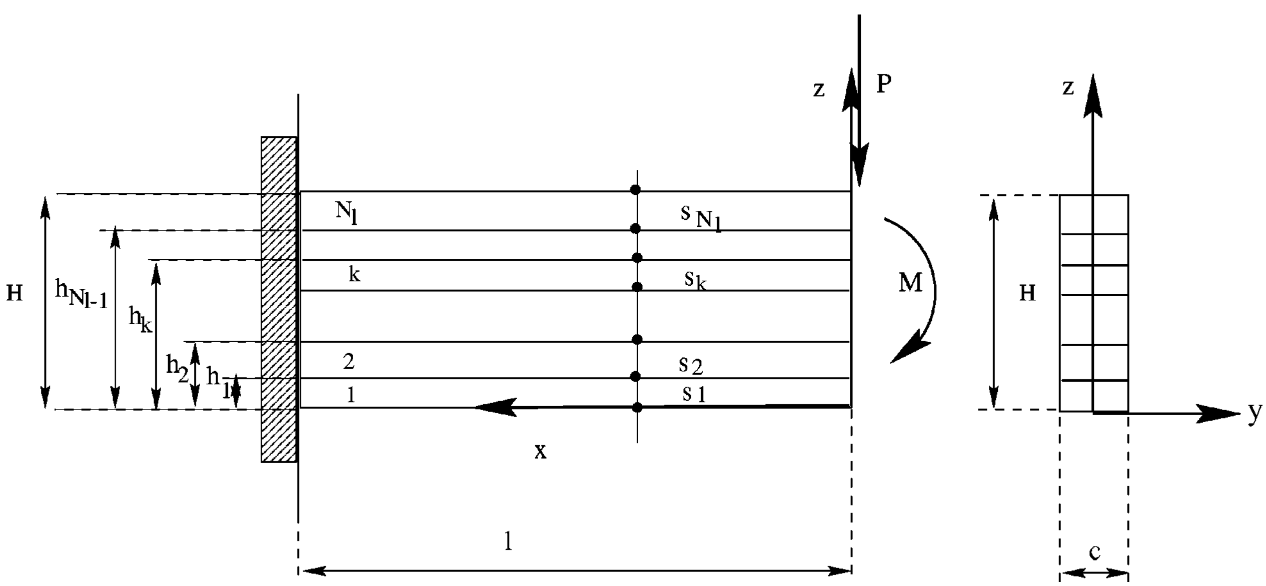

Fig. 3 Geometry and notations of Lekhnitskii's cantilever, multilayered beam

$$w^k = -\frac{A_k}{3E^k}x^3 - \frac{a_k}{2E^k}x^2 + \left[-\frac{\nu^k}{E^k}(A_k z^2 + B_k z) - \frac{C_k}{G^k} + \alpha_k \right] x - \frac{\nu^k}{E^k}\left(\frac{a^k}{2}z^2 + b_k z \right) + \gamma_k$$

α_k, β_k, and γ_k are the three integration constants. By using $II-IV$, and after some lengthy algebraic manipulations, the following relations are obtained,

$$\frac{a_k}{E^k} = \frac{a_{k-1}}{E^{k-1}}, \quad \frac{A_k}{E^k} = \frac{A_{k-1}}{E^{k-1}}$$

$$\frac{A_k}{E^k}h_{k-1} + \frac{B_k}{2E^k}h_{k-1} = \frac{A_{k-1}}{E^{k-1}}h_{k-1} + \frac{B_{k-1}}{2E^{k-1}}h_{k-1} \tag{13}$$

$$\frac{a_k h_{k-1} + b_k}{E^k} = \frac{a_{k-1}h_{k-1} + b_{k-1}}{E^{k-1}}$$

$$C_1 = 0, \; A_{N_l}h_{N_l}^2 + B_{N_l}h_{N_l} + C_{N_l} = 0$$

$$A_k h_{k-1}^2 + B_k h_{k-1} + C_k = A_{k-1}h_{k-1}^2 + B_{k-1}h_{k-1} + C_{k-1} \tag{14}$$

$$\left(\frac{\nu^k}{E^{k-1}} - \frac{1}{G^{k-1}} \right)\left(\frac{A_k h_{k-1}^3}{3} + \frac{B_k h_{k-1}^2}{2} \right) - \alpha_k h_{k-1} + \beta_k = \left(\frac{\nu^{k-1}}{E^k} - \frac{1}{G^k} \right)\left(\frac{A_{k-1}h_{k-1}^3}{3} + \frac{B_{k-1}h_{k-1}^2}{2} \right) - \alpha_{k-1}h_{k-1} + \beta_{k-1} \tag{15}$$

$$\frac{\nu^k}{E^k}(A_k h_{k-1}^2 + B_k h_{k-1}) - \frac{c_k}{G^k} + \alpha_k = \frac{\nu^{k-1}}{E^{k-1}}(A_{k-1}h_{k-1}^2 + B_{k-1}h_{k-1}) - \frac{c_{k-1}}{G^{k-1}} + \alpha_{k-1} - \frac{\nu^k}{E^k}\left(\frac{a_k}{2}h_{k-1}^2 + b_k h_{k-1} \right) + \gamma_k = -\frac{\nu^{k-1}}{E^{k-1}}\left(\frac{a_{k-1}}{2}h_{k-1}^2 + b_{k-1}h_{k-1} \right) + \gamma_{k-1} \tag{16}$$

The following recursive relations are obtained from Eqs. (13)

$$A_k = \frac{A_1 E^k}{E^1}, \quad B_k = \frac{B_1 E^k}{E^1}, \quad a_k = \frac{a_1 E^k}{E^1}, \quad b_k = \frac{b_1 E^k}{E^1} \tag{17}$$

while from Eqs. (15) it is found,

$$C_k = \frac{1}{E^1}\left\{ A_1 \left[\sum_{s=1}^{k-1} (h_s^2 - h_{s-1}^2)E^s - h_{k-1}E^k \right] + B_1 \left[\sum_{s=1}^{k-1} (h_s - h_{s-1})E^s - h_{k-1}E^k \right] \right\}$$

$$k = 2,3,N_l - 1 \tag{18}$$

Top and bottom layer values are:

$$C_1 = 0, \quad C_{N_l} = \frac{E^k}{E^1}(A_1 h_{N_l}^2 + B_1 h_{N_l}) \tag{19}$$

Eqs. (11) are used to determine the remaining bottom-layer constants,

$$A_1 = \frac{6PE_1}{Sc}, \quad B_1 = -\frac{6PE_1}{Sc}S_2,$$

$$a_1 = -\frac{12ME_1}{Sc}S_1, \quad b_1 = -\frac{6ME_1}{Sc}S_2 \tag{20}$$

where

$$S = 4\sum_{i=1}^{N_l} (h_i^3 - h_{i-1}^3)E^i \sum_{i=1}^{N_l} (h_i - h_{i-1})E^i - 3 \times \left[\sum_{i=1} N_l(h_i^2 - h_{i-1}^2)E^i \right]^2$$

$$S_1 = \sum_{i=1}^{N_l} (h_i^3 - h_{i-1}^3)E^i, \qquad S_1 = \sum_{i=1}^{N_l} (h_i^2 - h_{i-1}^2)E^i \tag{21}$$

It is noted that $h_0 = 0$ and $h_{N_l} = H$.

The final expressions for the stresses are:

$$\sigma_{xx}^k = \frac{6E^k}{Sc}(Px + M)(2S_1 z - S_2), \quad k = 1,2,..N_l$$

$$\sigma_{zz}^k = 0, \quad k = 1,2,..N_l$$

$$\sigma_{xz}^k = \frac{6P}{Sc}\left\{ S_1 \left[\sum_{s=1}^{k-1} (h_s^2 - h_{s-1}^2)E^s + (z^2 - h_{k-1}^2)E^k \right] \right\} - \left\{ S_2 \left[\sum_{s=1}^{k-1} (h_s - h_{s-1})E^s + (z^2 - h_{k-1})E^k \right] \right\},$$

$$k = 2,3,N_l - 1 \tag{22}$$

$$\sigma_{xz}^1 = -\frac{6PE^1}{Sc}z(S_1 z - S_2)$$

$$\sigma_{xz}^{N_l} = -\frac{6PE^{N_l}}{Sc}(h_{N_l} - z)[S_2 - (h_{N_l} + z)S_1]$$

The corresponding expression for the displacements could be obtained directly from Eqs. (12). It is noted that the amount of algebraic manipulations is quite impressive. At the present time, the use of software for symbolic calculation could be of some help to derive the above formulas or to apply the Lekhnitskii method to a different stress function assumption.

This section closes by making a few remarks on the theory proposed by Lekhnitskii.

1) Lekhnitskii's theory described Zig-Zag form of both longitudinal and through the thickness displacements; in particular:
   a) The longitudinal displacements u^k show a cubic order in the z-thickness direction
   b) The through thickness displacement w^k varies according to a parabolic equation in z
2) Lekhnitskii's theory furnishes interlaminar continuous transverse stresses σ_{zz} and σ_{xz} Eqs. (22)
3) Stresses obtained by Lekhnitskii fulfill the 3D indefinite equilibrium equations (this fundamental property is in

trinsic in the used stress function formulation)

4) Stresses and displacements have been obtained by employing:

   a) Compatibility conditions for stress functions
   b) Strain-displacement relations
   c) compatibility conditions for displacements at the interfaces

$$u^{k-1}=u^k, \quad w^{k-1}=w^k, \quad k=2,N_l \tag{23}$$

   d) Homogeneous conditions at the bottom and top surface for the transverse stresses

$$\sigma_{zz}^1=\sigma_{zz}^{N_l}=0, \quad \sigma_{xz}^1=\sigma_{xz}^{N_l}=0 \quad in z=0,h \tag{24}$$

   e) Interlaminar equilibrium for the transverse stresses

$$\sigma_{zz}^{k-1}=\sigma_{zz}^k, \quad \sigma_{xz}^{k-1}=\sigma_{xz}^k, \quad k=2,N_l \tag{25}$$

5) No post-processing was used to recover transverse stresses
6) Thickness normal stress σ_{zz} has been neglected. Nevertheless, the Poisson effects on thickness displacement w^k have been fully retained.
7) Full retainment of Koiter's recommendation would require a different assumption for the stress functions (the author is not aware of any work that was made in this direction)

2.1 Developments of LMT

Although Lekhnitskii's theory was published in the middle thirties of the last century and reported in a short paragraph of the English edition of his book [31], it has been systematically forgotten in the recent literature. An exception should be made for the work by Ren which is documented in this paragraph.

To the best of the author's knowledge, Ren is the only scientist who has used Lekhnitskii 's work. In the two papers [65,66] Ren has, in fact, extended Lekhnitskii 's theory to orthotropic and anisotropic plates. Further applications to vibration and buckling were made in a third paper written in collaboration with Owen [67]. These three papers are the only contributions known to the author that have been made in the framework of Lekhnitskii 's theory. As these three papers have been published in journals that are easily and worldwide available, the full description of the Ren extension to plates of LMT has, therefore, been omitted. Nevertheless, it is of interest to make a few remarks on Ren's works in order to quote explicitly the stress and displacement fields that were introduced by Ren to analyze the response of anisotropic plates. For the sake of simplicity, reference is made to the derivation made by Ren in [65], where cross-ply plates were considered. The extension to generally laminated plates can be found in [65,67].

On the basis of the form of τ_{xz}^k obtained by Lekhnitskii, see Eq. (22), it appeared reasonable to Ren, see [65], to assume *in an axiomatic sense* the following distribution of transverse shear stresses in a laminated plate, composed by N_l orthotropic layers, (x, y, and z are the coordinates of the reference system shown in Fig. 4):

$$\begin{aligned}\sigma_{xz}^k(x,y,z)&=\xi_x(x,y)a^k(z)+\eta_x(x,y)c^k(z)\\ \sigma_{yz}^k(x,y,z)&=\xi_y(x,y)b^k(z)+\eta_y(x,y)g^k(z)\end{aligned} \tag{26}$$

Four independent functions of x,y have been introduced to describe transverse shear stresses. The layer constants are parabolic functions of the thickness coordinate z according to Eqs. (22), and in view of this Ren adopted the following expressions

$$\begin{aligned}a^k(z)=S_{1x}&\left[\sum_{i=1}^{k-1}(h_i^2-h_{i-1}^2)Q_{11}^i(z^2-h_{k-1}^2)Q_{11}^k\right]\\ &-S_{2x}\left[\sum_{i=1}^{k-1}(h^i-h_{i-1})Q_{11}^i(z-h_{k-1})Q_{11}^k\right]\end{aligned}$$

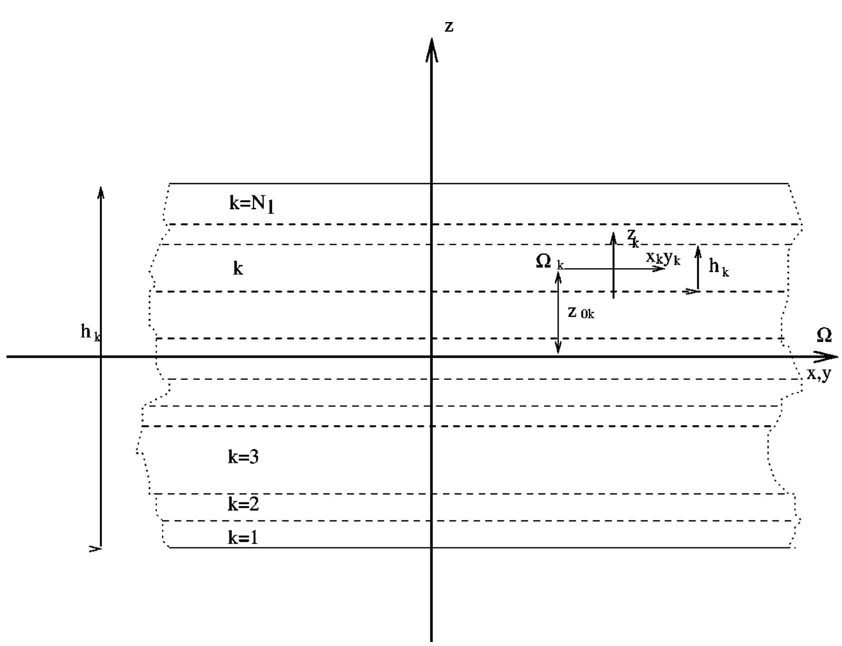

Fig. 4 Multilayered plate

$$c^k(z)=S_{1\nu}\left[\sum_{i=1}^{k-1}(h_i^2-h_{i-1}^2)\frac{Q_{11}^i}{\nu_{12}^i}(z^2-h_{k-1}^2)\frac{Q_{11}^k}{\nu_{12}^k}\right]-S_{2\nu}\left[\sum_{i=1}^{k-1}(h^i-h_{i-1})\frac{Q_{11}^i}{\nu_{12}^i}(z-h_{k-1})\frac{Q_{11}^k}{\nu_{12}^k}\right]$$

$$b^k(z)=S_{1y}\left[\sum_{i=1}^{k-1}(h_i^2-h_{i-1}^2)Q_{22}^i(z^2-h_{k-1}^2)Q_{22}^k\right]-S_{2y}\left[\sum_{i=1}^{k-1}(h^i-h_{i-1})Q_{22}^i(z-h_{k-1})Q_{22}^k\right] \quad (27)$$

$$g^k(z)=S_{1\nu}\left[\sum_{i=1}^{k-1}(h_i^2-h_{i-1}^2)\frac{Q_{22}^i}{\nu_{21}^i}(z^2-h_{k-1}^2)\frac{Q_{22}^k}{\nu_{21}^k}\right]-S_{2\nu}\left[\sum_{i=1}^{k-1}(h^i-h_{i-1})\frac{Q_{22}^i}{\nu_{21}^i}(z-h_{k-1})\frac{Q_{22}^k}{\nu_{21}^k}\right]$$

in which

$$S_{1x}=\sum_{K=1}^{N_l}(h_k-h_{k-1})Q_{11}^k, \quad S_{2x}=\sum_{K=1}^{N_l}(h_k-h_{k-1})Q_{11}^k$$

$$S_{1y}=\sum_{K=1}^{N_l}(h_k-h_{k-1})Q_{22}^k, \quad S_{2y}=\sum_{K=1}^{N_l}(h_k-h_{k-1})Q_{22}^k \quad (28)$$

$$S_{1\nu}=\sum_{K=1}^{N_l}(h_k-h_{k-1})\frac{Q_{22}^k}{\nu_{12}^k}, \quad S_{2\nu}=\sum_{K=1}^{N_l}(h_k-h_{k-1})\frac{Q_{22}^k}{\nu_{12}^k}$$

$$\frac{Q_{11}^k}{\nu_{12}^k}=\frac{Q_{22}^k}{\nu_{21}^k}$$

E_1^k, E_2^k, ν^k are Young's modulus of the k-th layer in the x and y directions and Poisson's ratio, respectively. h_k denotes the z-value of the k-th interface measured from the reference surface Ω. Q_{ij}^k layer-stiffnesses are those that appear in Hooke's law, hereby written in the reduced form [10] for orthotropic layers,

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\\ \sigma_{xz}\\ \sigma_{yz}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{21} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{66} & 0 & 0\\ 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & 0 & Q_{44}\end{bmatrix}\begin{Bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix} \quad (29)$$

Eqs. (26) represent an interlaminar continuous transverse shear stress field that is parabolic in each layer. As in Lekhnitskii [36], displacement fields are obtained by integrating strain-displacement relations after substituting into Hooke's law, Eq. (29). In contrast to the work by Lekhnitskii, it is emphasized that transverse strains ϵ_{zz} have been discarded by Ren. It is noted that such an assumption contrasts with the already mentioned Koiter's recommendation. Layer constants arising from the integration are determined by imposing compatibility conditions for the displacements at the interface. The displacement field assumes the following form,

$$u^k(x,y,z)=u_0(x,y)-w_{,x}(x,y)+\xi_x(x,y)A^k(z)+\eta_x(x,y)C^k(z)$$

$$v^k(x,y,z)=v_0(x,y)-w_{,y}(x,y)+\xi_y(x,y)B^k(z)+\eta_y(x,y)G^k(z) \quad (30)$$

$$w(x,y,z)=w_0(x,y)$$

where $A_z^k(z)$, $B_z^k(z)$, $B_z^k(z)$ and $G_z^k(z)$ are obtained by integrating the corresponding $a_z^k(z)$, $b_z^k(z)$, $c_z^k(z)$, and $g_z^k(z)$. Thus Eqs. (30) represent a piecewise Zig-Zag continuous displacement field in the thickness direction z which is cubic in each layer.

The displacement model given by Eqs. (30) can be used in the framework of known variational statements to formulate the governing equations of anisotropic plates in both strong end weak form. Strong forms and related closed form solutions have been discussed in the already mentioned works [65–67]. No weak form solutions, such as finite element applications, nor attempts to include KR as well as an extension to shell geometry of LMT are known to the author.

3 AMBARTSUMIAN MULTILAYERED THEORY

This section has been devoted to those Zig-Zag multilayered theories that have been mostly originated by attempts to extend the classical Reissner-Mindlin theory (Reissner [5], Mindlin [6]) for homogeneous, isotropic plates, to include the C_z^0-requirements. For convenience, these attempts are described here in the following points.

1) To describe plates/shells made of a single-layer of an anisotropic materials
2) To extend the single-layer case to the multilayered case by including Zig-Zag effects and satisfying interlaminar continuity for the transverse shear stresses
3) To include what was referred to as Koiter's recommendation in the introduction; that is, to include transverse normal stress/strain $\sigma_{zz}, \epsilon_{zz}$ effects, which are discarded by Reissner-Mindlin type theories

The classical Reissner-Mindlin theory in the case of a plate or shell assumes *in an axiomatic sense* the following displacement field,

$$u(\alpha,\beta,z)=u^0+z\phi_\alpha \quad \left(=u^0+z\left(\gamma_{\alpha z}-\frac{w_{,\alpha}}{A}-\frac{u^0}{R\alpha}\right)\right)$$

$$u(\alpha,\beta,z)=v^0+z\phi_\beta \quad \left(=v^0+z\left(\gamma_{\beta z}-\frac{w_{,\beta}}{B}-\frac{u^0}{R\beta}\right)\right) \quad (31)$$

$$w(\alpha,\beta,z)=w^0$$

Reissner-Mindlin theory includes transverse shear deformation, it has five degrees of freedom (three displacement $u^0(\alpha,\beta)$, $v^0(\alpha,\beta)$, $w^0(\alpha,\beta)$ on the reference surface plus two rotations $\phi_\alpha(\alpha,\beta), \phi_\beta(\alpha,\beta)$ or two transverse shear strains $\gamma_{\alpha z}(\alpha,\beta), \gamma_{\alpha z}(\alpha,\beta)$. Figure 5 shows the notation used for a doubly curved shell. A curvilinear reference system α, β, z has been considered; α and β are the curvilinear coordinates defined on a shell reference surface Ω. R_α, R_β

are the shell radii of principal curvatures. A and B are the Lamé shell parameters (see [68] for details). For convenience, the Reissner-Mindlin model has been written in two ways; the second one (the one in parentheses) expresses the rotation in terms of transverse displacement variations and shear strains. Usual approximations have been introduced as far as curvature terms of type $z/R_\alpha, z/R_\beta$, see [68,69]. The shear stress field related to an isotropic monocoque plate/shell with shear modulus G has a constant distribution in the thickness directions:

$$\begin{aligned}\sigma_{\alpha z}(\alpha,\beta,z)&=G(\phi_\alpha+w_{,\alpha})\\ \sigma_{\alpha z}(\alpha,\beta,z)&=G(\phi_\beta+w_{,\beta})\end{aligned} \tag{32}$$

Ambartsumian has been the first one to work on points 1 and 2. For that reason we refer to this type of Zig-Zag theories as Ambartsumian Multilayered Theory (AMT). Ambartsumian has, in fact, considered in a series of papers [62,63,70,71], and in the two books [32,33], refinements of Reissner-Mindlin theory directly to make it suitable for the application to anisotropic layered plates and shells. For this purpose, the classical hypotheses of Reissner-Mindlin theory were reformulated by Ambartsumian as follows,

a) The line elements of the plate/shell, normal to the middle surface Ω, do not change their length after deformation

b) The normal stresses σ_{zz} are small as compared to the in-plane ones $\sigma_{\alpha\alpha}$, $\sigma_{\beta\beta}$, $\sigma_{\alpha\beta}$

c) The transverse shear stresses $\sigma_{\alpha z}$, $\sigma_{\beta z}$ vary in the direction of the thickness (of the entire shell or layer) according to the law of the quadratic parabola

Single-layer case. Orthotropic plates and shells composed of a single layer were addressed in [62] and [63], respectively. The shell case is hereby considered. According to assumption c), the transverse shear field was assumed of the following type [63],

$$\sigma_{\alpha z}(\alpha,\beta,z)=\frac{1}{2}(\sigma_{\alpha z}^T-\sigma_{\alpha z}^B)+\frac{z}{h}(\sigma_{\alpha z}^T+\sigma_{\alpha z}^B)+\frac{1}{2}\left(z^2-\frac{h^2}{4}\right)\ \phi_\alpha(\alpha,\beta) \tag{33}$$

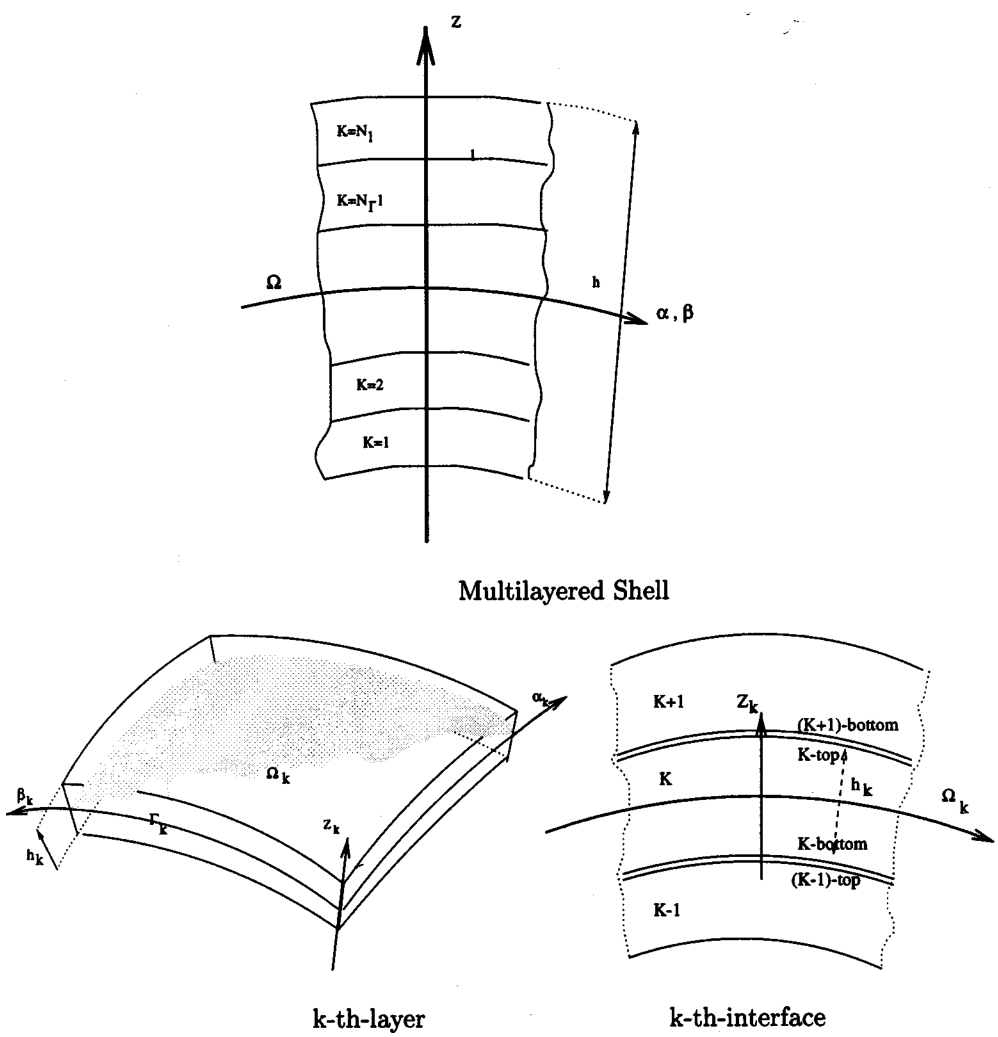

Fig. 5 Geometry and notations used for multilayered shells

$$\div \sigma_{\beta z}(\alpha,\beta,z)=\frac{1}{2}(\sigma_{\beta z}^{T}-\sigma_{\beta z}^{B})+\frac{z}{h}(\sigma_{\beta z}^{T}+\sigma_{\beta z}^{B}) + \frac{1}{2}\left(z^{2}-\frac{h^{2}}{4}\right)\phi_{\beta}(\alpha,\beta)$$

The applied values of transverse shear at the top T and bottom B shell surfaces are considered in the previous transverse shear model as:

$$\sigma_{\alpha z}^{T}=\sigma_{\alpha z}\left(\alpha,\beta,\frac{h}{2}\right),\quad \sigma_{\alpha z}^{B}=\sigma_{\alpha z}\left(\alpha,\beta,-\frac{h}{2}\right),$$
$$\sigma_{\beta z}^{T}=\sigma_{\beta z}\left(\alpha,\beta,\frac{h}{2}\right),\quad \sigma_{\beta z}^{B}=\sigma_{\beta z}\left(\alpha,\beta,-\frac{h}{2}\right)$$

$\phi_{\alpha}(\alpha,\beta)$ and $\phi_{\beta}(\alpha,\beta)$ are the two functions of Reissner-Mindlin theory. It is noted that Eqs. (32) and (33) introduce a transverse shear stress field that is parabolic in the thickness direction; in addition Eqs. (33) include non-homogeneous conditions on top/bottom surfaces. The related transverse shear strains are,

$$\epsilon_{\alpha z}(\alpha,\beta,z)=\epsilon_{\alpha z}^{\star}+\frac{z}{h}\epsilon_{\alpha z}^{*}+\frac{1}{2}\left(z^{2}-\frac{h^{2}}{4}\right)\Phi_{\alpha}(\alpha,\beta) \tag{34}$$
$$\epsilon_{\beta z}(\alpha,\beta,z)=\epsilon_{\beta z}^{\star}+\frac{z}{h}\epsilon_{\beta z}^{*}+\frac{1}{2}\left(z^{2}-\frac{h^{2}}{4}\right)\Phi_{\beta}(\alpha,\beta)$$

where

$$\epsilon_{\alpha z}^{\star}=\frac{1}{2}[S_{55}(\sigma_{\alpha z}^{T}-\sigma_{\alpha z}^{B})+S_{45}(\sigma_{\beta z}^{T}-\sigma_{\beta z}^{B})],$$
$$\epsilon_{\beta z}^{\star}=\frac{1}{2}[S_{44}(\sigma_{\alpha z}^{T}-\sigma_{\alpha z}^{B})+S_{45}(\sigma_{\beta z}^{T}-\sigma_{\beta z}^{B})]$$
$$\epsilon_{\alpha z}^{*}=S_{55}(\sigma_{\alpha z}^{T}+\sigma_{\alpha z}^{B})+S_{45}(\sigma_{\beta z}^{T}+\sigma_{\beta z}^{B}),$$
$$\epsilon_{\beta z}^{*}=S_{44}(\sigma_{\alpha z}^{T}+\sigma_{\alpha z}^{B})+S_{45}(\sigma_{\beta z}^{T}+\sigma_{\beta z}^{B}) \tag{35}$$
$$\Phi_{\alpha}=S_{55}\phi_{\alpha}+S_{45}\phi_{\beta},\quad \Phi_{\beta}=S_{55}\phi_{\alpha}+S_{45}\phi_{\beta}$$

in which the following compliances have been introduced,

$$S_{44}^{k}=\frac{Q_{55}^{k}}{D^{k}},\quad S_{45}^{k}=-\frac{Q_{45}^{k}}{D^{k}},\quad S_{44}^{k}=\frac{Q_{44}^{k}}{D^{k}},$$
$$D^{k}=Q_{44}^{k}Q_{55}^{k}-(Q_{45}^{k})^{2}$$

Upon integration with respect to z of the strain/displacement relations by taking into account the assumptions and by introducing some of the usual approximations concerning the terms of the type $z/R_{\alpha},z/R_{\beta}$, Ambartsumian obtained the following displacement field,

$$u(\alpha,\beta,z)=\left(1+\frac{z}{R_{\alpha}}\right)u_{,\alpha}^{0}-\frac{z}{A_{1}}w_{,\alpha}-\frac{zh^{2}}{8}\left(1+\frac{z}{R_{\alpha}}\right)\Phi^{\alpha} + \frac{z^{3}}{6}\left(1+\frac{z}{4R_{\alpha}}\right)\Phi^{\alpha}+z\left(1+\frac{z}{2R_{\alpha}}\right)\epsilon_{\alpha z}^{\star} + \frac{z^{2}}{2h}\left(1+\frac{z}{3R_{\alpha}}\right)\epsilon_{\alpha z}^{\star}$$

$$v(\alpha,\beta,z)=\left(1+\frac{z}{R_{\beta}}\right)u_{,\beta}^{0}-\frac{z}{A_{1}}w_{,\beta}-\frac{zh^{2}}{8}\left(1+\frac{z}{R_{\beta}}\right)\Phi^{\beta} + \frac{z^{3}}{6}\left(1+\frac{z}{4R_{\beta}}\right)\Phi^{\beta}+z\left(1+\frac{z}{2R_{\beta}}\right)\epsilon_{\beta z}^{\star} + \frac{z^{2}}{2h}\left(1+\frac{z}{3R_{\beta}}\right)\epsilon_{\beta z}^{\star}$$
$$w(\alpha,\beta,z)=w_{0}(\alpha,\beta) \tag{36}$$

The important fact is noted that in view of Eqs. (31) the AMT gives in-plane displacement components which depend on both ϕ_{α} and ϕ_{β}.

Multilayer case. The case of two layers was addressed by Ambartsumian in the two papers [70,71] and further documented in [32,33]. It is noted that the extension to multilayer plates was given by Osternik and Barg [72] who first wrote transverse shear stresses in the form of Eqs. (37). Herein, it is preferred to present the Ambartsumian theory for the plate geometries as it has been given in [33], where due credit is given to the work done by Osternik and Barg [72]. Assumptions a) and b) are kept for the multilayered case while assumption c) for transverse shear stresses is now made for each layer. The transverse shear stresses are, in fact, written in the following layer-form,

$$\begin{aligned}\sigma_{xz}^{k}(x,y,z)&=[G_{13}^{k}\ f(z)+A_{k}]\phi_{x}(x,y)\\ \sigma_{yz}^{k}(x,y,z)&=[G_{23}^{k}\ f(z)+B_{k}]\phi_{y}(x,y)\end{aligned} \tag{37}$$

which consists of an interlaminar continuous transverse shear stress field. ϕ_{x} and ϕ_{y} are unknown functions that have the same meaning as ϕ_{α} and ϕ_{α} introduced in the shell case. These unknown functions are not affected by k-superscripts as is the case for any Equivalent Single Layer Theory. In [33] attention was restricted to the case in which $f(z)$ is a symmetric function of z ($f(z)$ is zero at top/bottom external plate surfaces) and to a symmetrically bent plate ($u^{0}=v^{0}=0$). G_{13}^{k} and G_{23}^{k} are the shear modulus k-th layer. A_{k} and B_{k} are layer constants to be determined by imposing IC and ZZ with respect to each layer interface. A complete list of boundary conditions on transverse stresses and displacements is rewritten,

$$\begin{aligned}&\sigma_{xz}^{N_{l}}(=\sigma_{xz}^{T})=0,\sigma_{xz}^{1}(=\sigma_{xz}^{B})=0,\\ &\sigma_{yz}^{N_{l}}(=\sigma_{yz}^{T})=0,\sigma_{yz}^{1}(=\sigma_{yz}^{B})=0\\ &\sigma_{xz}^{k-1}=\sigma_{xz}^{k},\sigma_{yz}^{k-1}=\sigma_{yz}^{k},\qquad k=2,N_{l}\\ &u^{k-1}=u^{k},\quad v^{k-1}=v^{k},\qquad k=2,N_{l}\end{aligned} \tag{38}$$

which lead to the following layer constants:

$$\begin{aligned}&A_{1}=A_{N_{l}}=0,\quad B_{1}=B_{N_{l}}=0\\ &A_{k}=\sum_{i=k}^{N_{l}}f(h_{i})(G_{13}^{i+1}-G_{13}^{i+1}),\\ &B_{k}=\sum_{i=k}^{N_{l}}f(h_{i})(G_{23}^{i+1}-G_{23}^{i+1}),\\ &k=(N_{l}+1)/2,\ldots,N_{l}\end{aligned} \tag{39}$$

$$A_k=A_{N_l+1-k}=0, \quad B_k=B_{N_l+1-k}=0,$$
$$k=1,\ldots(N_l+1)/2$$

Integration of strains gives the following Zig-Zag form of the displacement field,

$$\begin{aligned}
u^k(x,y,z)&=-zw_{,x}+J_0(z)\phi_x+z\frac{A_i}{G_{13}^k}\phi_x\\
&\quad+R_1^k\phi_x \quad sign(z)\\
v^k(x,y,z)&=-zw_{,x}+J_0(z)\phi_y+z\frac{A_i}{G_{23}^k}\phi_y\\
&\quad+R_2^k\phi_y \quad sign(z)\\
w(x,y,z)&=w_0
\end{aligned}\tag{40}$$

where

$$\begin{aligned}
&R_1=A_{(N_l+1)/2}=0, \quad R_2=A_{(N_l+1)/2}=0\\
&R_1^k=\sum_{i=(N_l+1)/2}^{k-1} h_i\left(\frac{A_i}{G_{13}^{i+1}}-\frac{A_{i+1}}{G_{13}^{i+1}}\right),\\
&R_2^k=\sum_{i=(N_l+1)/2}^{k-1} h_i\left(\frac{B_i}{G_{23}^{i+1}}-\frac{B_{i+1}}{G_{23}^{i+1}}\right),\ k=(N_l+3)/2,\ldots,N_l
\end{aligned}\tag{41}$$

$$R_1^k=R_1^{N_l+1-k}=0, \quad R_2^k=R_2^{N_l+1-k}=0,$$
$$k=1,\ldots(N_l+1)/2$$

Further details, along with governing equations and discussion, can be found in [33].

The displacements given in Eqs. (40) are affected by the k-superscripts. This can be formally avoided by using the Heaviside step function. These two ways of writing a Zig-Zag displacement field have been detailed in the Appendix for a simple 1D case.

Among the Zig-Zag theories discussed in the present article, the AMT is without doubt the theory that has mostly influenced the development of multilayered plate and shell theories. This is probably due to the fact that AMT uses the findings of a well-known theory such as the Reissner-Mindlin one. The manners in which Ambartsumian 's work has been used, referenced, and extended in the Western and Eastern scientific communities are discussed in the two following subsections.

3.1 Developments of Ambartsumian theory in the East

Many extensions of Ambartsumian's works have appeared in journals published in the Russian language. These works have been directed to extend AMT to generally laminated configurations, to geometrically nonlinear problems, as well as for the inclusion of KR. Refinements and applications of Ambartsumian Multilayered Theory to single-layer plates/shells were given in [73,74], while applications to multilayered plates and shells can be found in [75–78]. Applications to geometrically nonlinear problems have been given by Prusakov [79,80]. As an example, the developments presented by Andreev and Nemirovskii in [77] are briefly recounted here. These authors gave an extension of the AMT in Eqs. (37–40) to include nonsymmetrical multilayered plates and nonzero top and bottom surface transverse shear stress conditions. The displacement fields were written in the following form,

$$\begin{aligned}
u^k(x,y,z)&=\lambda_x^k+u_0-zw_{,x}+\mu^k\phi_x\\
v^k(x,y,z)&=\lambda_y^k+v_0-zw_{,y}+\mu^k\phi_y\\
w(x,y,z)&=w_0
\end{aligned}\tag{42}$$

where

$$\begin{aligned}
\lambda_x^k(x,y)&=\left[\sum_{j=1}^{k-1}(h_j-h_{j-1})S_{44}^j+(z-h_{k-1})S_{44}^k\right]\sigma_{xz}^B\\
&\quad+\left[\sum_{j=1}^{k-1}(h_j-h_{j-1})S_{45}^j+(z-h_{k-1})S_{45}^k\right]\sigma_{yz}^B\\
&\quad+\frac{1}{2h}\left[\sum_{j=1}^{k-1}(h_j^2-h_{j-1}^2)S_{44}^j\right.\\
&\quad\left.+(z^2-h_{k-1}^2)S_{44}^k\right](\sigma_{xz}^T-\sigma_{xz}^B)\\
&\quad+\frac{1}{2h}\left[\sum_{j=1}^{k-1}(h_j^2-h_{j-1}^2)S_{45}^j\right.\\
&\quad\left.+(z^2-h_{k-1}^2)S_{45}^k\right](\sigma_{yz}^T-\sigma_{yz}^B)\\
\lambda_y^k(x,y)&=\left[\sum_{j=1}^{k-1}(h_j-h_{j-1})S_{44}^j+(z-h_{k-1})S_{44}^k\right]\sigma_{xz}^B\\
&\quad+\left[\sum_{j=1}^{k-1}(h_j-h_{j-1})S_{45}^j+(z-h_{k-1})S_{45}^k\right]\sigma_{yz}^B\\
&\quad+\frac{1}{2h}\left[\sum_{j=1}^{k-1}(h_j^2-h_{j-1}^2)S_{44}^j\right.\\
&\quad\left.+(z^2-h_{k-1}^2)S_{44}^k\right](\sigma_{xz}^T-\sigma_{xz}^B)\\
&\quad+\frac{1}{2h}\left[\sum_{j=1}^{k-1}(h_j^2-h_{j-1}^2)S_{45}^j\right.\\
&\quad\left.+(z^2-h_{k-1}^2)S_{45}^k\right](\sigma_{yz}^T-\sigma_{yz}^B)
\end{aligned}\tag{43}$$

$$\begin{aligned}
\mu_x^k(x,y)&=\sum_{j=1}^{k-1}[(f(h_j)-f(h_{j-1})]S_{44}^j\\
&\quad+[f(z)-f(h_{k-1})]S_{44}^k]\sigma_{xz}^B\\
&\quad+\sum_{j=1}^{k-1}[(f(h_j)-f(h_{j-1})]S_{45}^j\\
&\quad+[f(z)-f(h_{k-1})]S_{45}^k]\sigma_{yz}^B
\end{aligned}$$

$$\mu_y^k(x,y)=\sum_{j=1}^{k-1}[(f(h_j)-f(h_{j-1})]S_{54}^j+[f(z)-f(h_{k-1})]S_{54}^k]\sigma_{xz}^B+\sum_{j=1}^{k-1}[(f(h_j)-f(h_{j-1})]S_{55}^j+[f(z)-f(h_{k-1})]S_{55}^k]\sigma_{yz}^B$$

The function $f(z)$ has the following properties

$$f_{,z}(0)=f_{,z}(h)=0 \tag{44}$$

The form of this function was taken as general in the theoretical developments while it was taken as a cubic function in the examples given in [77]. Related transverse shear stresses were give as,

$$\begin{aligned}\sigma_{xz}^k(x,y,z)&=\sigma_{xz}^B+\frac{z}{h}(\sigma_{xz}^T-\sigma_{xz}^B)+f_{,z}(z)\quad\phi_x\\ \sigma_{yz}^k(x,y,z)&=\sigma_{yz}^B+\frac{z}{h}(\sigma_{yz}^T-\sigma_{yz}^B)+f_{,z}(z)\quad\phi_y\end{aligned} \tag{45}$$

Inclusion of σ_{zz} and ϵ_{zz}. Many efforts have been made directly for the inclusion of Koiter's recommendation in AMT. Rybov [81,82] first made progress in this direction. He proposed a refined theory, taking the variation of transverse normal displacements according to the following form:

$$w^k(x,y,z)=w_0(x,y)+f_k^\star(z)\phi(x,y) \tag{46}$$

Only one unknown function $\phi(x,y)$ was used to express both transverse shear stresses,

$$\begin{aligned}\sigma_{xz}^k(x,y,z)&=G_{13}^k[f_k^\star(z)+f^*{}_{k,z}(z)]A_x^k\phi_{,x}(x,y)\\ \sigma_{yz}^k(x,y,z)&=G_{23}^k[f_k^\star(z)+f^*{}_{k,z}(z)]A_y^k\phi_{,y}(x,y)\end{aligned} \tag{47}$$

where $f_k^\star(z)$ and $f^*{}_k$ are two assigned functions of the thickness coordinate z in the k-th layer.

The most relevant contribution for the inclusion of σ_{zz} and ϵ_{zz} in the AMT has been made by Rasskazov and coauthors. In fact, Rasskazov and coauthors proposed the inclusion of both σ_{zz} and ϵ_{zz} in a series of papers [83–88] for both plate and shell geometries, linear and nonlinear problems, and for analytical solution methods as well as computational techniques such as the finite element method. Some details of the Rasskazov refinement of AMT theory follows; the complete derivations can be found in the above mentioned papers by Rasskazov which are published in journals, English translations of which are available worldwide. The transverse shear stress field was taken similar to that by Ambartsumian,

$$\begin{aligned}\sigma_{xz}^k(x,y,z)&=G_{13}^k(z)f_{1,z}(z)\phi_x(x,y)\\ \sigma_{yz}^k(x,y,z)&=G_{23}^k(z)f_{2,z}(z)\phi_y(x,y)\end{aligned} \tag{48}$$

An interlaminar piecewise continuous transverse normal stress was introduced in the following form,

$$\sigma_{zz}=\sigma_{zz}^T\frac{h_1+z}{h}+\sigma_{zz}^B\frac{h_2-z}{h}+f^\star(z)\varphi^\star(x,y) \tag{49}$$

Transverse normal strains were taken as

$$\epsilon_{zz}=f_{3,zz}(z)\psi^\star(x,y) \tag{50}$$

The method used to compute the function $\varphi^\star(x,y)$ can be found in [83]. The displacements field was given in the following form

$$\begin{aligned}u^k(x,y,z)&=u_0-zw_{,x}+f_x(z)\phi_x+f_3(z)\quad\psi^\star_{,x}\\ v^k(x,y,z)&=w_0-zw_{,y}+f_y(z)\phi_y+f_3(z)\quad\psi^\star_{,y}\\ w(x,y,z)&=w_0+f_3(z)_{,z}\psi^\star\end{aligned} \tag{51}$$

The mechanical properties of the layers were considered as piecewise functions of z. The introduced functions are given as

$$\begin{aligned}f_x(z)&=\int_0^z\left[G_{13}^{-1}\int_{-h_1}^z A_{11}(z)(z-\delta_x)dz\right]dz\\ f_x(z)&=\int_0^z\left[G_{23}^{-1}\int_{-h_1}^z A_{22}(z)(z-\delta_y)dz\right]dz\\ f_3(z)&=\int_0^z\left(\int_0^z z\,\eta(z)dz\right)dz\\ f^\star(z)&=(z+h_1)(z+h_2)\end{aligned} \tag{52}$$

where

$$\begin{aligned}A_{11}&=\frac{1}{2}Q_{11}[1-\nu_{12}(z)]+G_{12}(z),\\ A_{22}&=\frac{1}{2}Q_{22}[1-\nu_{21}(z)]+G_{12}(z)\\ \eta(z)&=\frac{1}{2}\,\frac{\nu_{13}(z)[1+\nu_{21}(z)]+\nu_{23}(z)[1+\nu_{12}(z)]}{1-\nu_{12}(z)\nu_{21}(z)}\\ \delta_x&=\frac{\int A_{11}(z)zdz}{\int A_{11}(z)dz},\qquad \delta_y=\frac{\int A_{22}(z)zdz}{\int A_{22}(z)dz}\end{aligned} \tag{53}$$

Further developments on the inclusion of transverse compression have been made by Grikoliuk and Vasilenko [89,90] and Vasilenko [91].

3.2 Developments of Ambartsumian theory in the West

A few articles have appeared in journals written in English in the early 1970's, in which developments of AMT have been considered, and which made direct reference to the original works by Ambartsumian. These are discussed below.

Whitney's contribution. Whitney [92] first applied and extended AMT to generally anisotropic and symmetrical and nonsymmetrical plates. It was clearly stated by Whitney that his own work was based on that by Ambartsumian [33]. For the sake of simplicity, the case of symmetrically laminated plates is hereby outlined. Details can be found in [92]. Interlaminar continuous transverse shear stresses were assumed as follows,

$$\sigma_{xz}^k(x,y,z)=[Q_{55}^k f(z)+a_{55}^k]\phi_x(x,y)+[Q_{45}^k f(z)+a_{45}^k]\phi_y(x,y) \tag{54}$$

$$\sigma_{yz}^k(x,y,z)=[Q_{45}^k f(z)+a_{55}^k]\phi_x(x,y) + [Q_{44}^k f(z)+a_{44}^k]\phi_y(x,y)$$

Equations (37) can be obtained by setting $Q_{45}^k = a_{45}^k = 0$. $f(z)$ is a function of the thickness coordinate, the form of which should be assumed differently as far as symmetrical and unsymmetrical laminated cases. A parabolic form for $f(z)$ has mostly been considered (explicit formulas for unsymmetrical cases were also given by Whitney). The layer constants $a_{44}^k, a_{45}^k, a_{55}^k$ are determined by imposing the continuity conditions of transverse shear stresses at the interfaces, while top-bottom stress-free conditions are used to determine the form of $f(z)$. Explicit forms of layer constants were omitted by Whitney. Transverse shear strains related to the assumed transverse shear stress fields are

$$\begin{aligned}\gamma_{xz}^k(x,y,z)&=[f(z)+S_{55}^k a_{55}^k+S_{45}^k a_{45}^k]\phi_x(x,y)\\&\quad+[S_{55}^k a_{45}^k+S_{45}^k a_{44}^k]\phi_y(x,y)\\ \gamma_{yz}^k(x,y,z)&=[S_{44}^k a_{44}^k+S_{45}^k a_{45}^k]\phi_x(x,y)+[f(z)\\&\quad+S_{44}^k a_{44}^k+S_{45}^k a_{55}^k]\phi_y(x,y)\end{aligned} \tag{55}$$

By assuming the transverse displacement constant in the thickness direction, ie, $\epsilon_{zz}=0$, and integrating the shear strains, the following Zig-Zag displacement fields were obtained:

$$\begin{aligned}u^k(x,y,z)&=-zw_{,x}+[J(z)+g_1^k(z)]\phi_x(x,y)\\&\quad+g_2^k(z)\phi_y(x,y)\\ v^k(x,y,z)&=-zw_{,y}+[J(z)+g_3^k(z)]\phi_y(x,y)\\&\quad+g_4^k(z)\phi_x(x,y)\\ w(x,y,x)&=w_0(x,y,z)\end{aligned} \tag{56}$$

where

$$J(z)=\int f(z)dz$$

$$\begin{aligned}g_1^k(z)&=[S_{55}^k a_{55}^k+S_{45}^k a_{45}^k]z+d_1^k\\ g_2^k(z)&=[S_{55}^k a_{55}^k+S_{45}^k a_{45}^k]z+d_2^k\\ g_3^k(z)&=[S_{55}^k a_{55}^k+S_{45}^k a_{45}^k]z+d_3^k\\ g_4^k(z)&=[S_{55}^k a_{55}^k+S_{45}^k a_{45}^k]z+d_4^k\end{aligned} \tag{57}$$

where $d_1^k, d_2^k, d_3^k, d_4^k$ are calculated by imposing compatibility of in-plane displacement at each interface.

Rath and Das's contribution. A second relevant work on the application of AMT was given by Rath and Das [93] who extended the work done by Whitney to doubly curved shells and dynamic problems. The transverse shear stress fields in each layer was taken the same as those by Whitney. Integration in z of related shear strains led to the following displacement fields [93],

$$\begin{aligned}u^k(\alpha,\beta,z)&=\left(1+\frac{z}{R_\alpha}\right)u_{,\alpha}^0-\frac{z}{A_1}w_{,\alpha}+\left[z+\frac{z^2}{R_\alpha}-4\frac{z^3}{3h^2}\right.\\&\quad\left.-\frac{z^4}{3h^2R_\alpha}\right)+G_1^k\left(z+\frac{z^2}{R_\alpha}\right)+d_1^k\left(1+\frac{z}{R_\alpha}\right)\phi_\alpha\\ v^k(\alpha,\beta,z)&=\left(1+\frac{z}{R_\beta}\right)v_{,\alpha}^0-\frac{z}{A_2}w_{,\beta}+\left[z+\frac{z^2}{R_\beta}-4\frac{z^3}{3h^2}\right.\\&\quad\left.-\frac{z^4}{3h^2R_\beta}\right)+G_3^k\left(z+\frac{z^2}{R_\beta}\right)+d_3^k\left(1+\frac{z}{R_\beta}\right)\phi_\beta\end{aligned} \tag{58}$$

$$w(\alpha,\beta,z)=w_0(\alpha,\beta)$$

where G_1^k, d_1^k, G_3^k, d_3^k are layer constants directly derived from those which were assumed for the transverse shear stress Eqs. (54).

Other contributions which make direct reference to Ambartsumian 's works. Further to the two papers by Whitney [92] and Rath and Das [93], a third article coauthored by Sun and Whitney [94] compared Whitney's Zig-Zag theory to simplified ones which discard interlaminar continuity and/or Zig-Zag effects. Further contributions by Hsu and Wang [95,96] employed the original AMT at layer level as it was given in [63]. A layer-wise theory was thereby developed for the cylindrical shell geometry.

Other contributions which do not make direct reference to Ambartsumian 's works. With the exception of the five papers [92–96], the author is not aware of any further *direct* applications of the AMT. Dozens of papers have instead been presented over the last decades that deal with Zig-Zag effects and interlaminar continuous transverse shear stresses, and which mostly ignore the original work by Ambartsumian, as well as those by Whitney, and Rath and Das. In the author's opinion, most of these articles should be considered as simplified cases of the Ambartsumian Multilayered Theory as well as of the developments in [92,93]. Unfortunately, the original works and authors are not mentioned (or barely cited) in the reference lists of this large amount of articles. In order to try to explain such historical unfairness, a reconstruction of what happened has been attempted here.

A pioneering article by Yu [97] should first be mentioned in which a Zig-Zag theory was presented. Because of the short time between Ambartsumian's works and Yu's article, together with the Cold War, it could be surely assumed that the works of Ambartsumian were unknown to Yu. A 1D sandwich plate made of an isotropic core and faces was considered in [97]. The three slopes of the displacement field in the three layers were derived by imposing transverse shear continuity at the two interfaces. The in-plane displacement fields were assumed linear in each layer. Yu did not start his derivation by a direct assumption of transverse stress field as was done by Ambartsumian; Yu, in fact, preferred to start from a Reissner-Mindlin type displacement model and compute the faces and core slopes by imposing transverse shear stress continuity at the interfaces. It was also mentioned in

[97] that the method could easily be extended to displacement fields which are taken as cubic functions of z. Details of this last development were not given by Yu.

Almost 15 years later, Chou and Carleone [98] presented a Zig-Zag theory of anisotropic plates. As in Yu (even though the original work by Yu was not mentioned) a piecewise linear displacement field in each layer was considered and a *Zig-Zag* theory was proposed. It seems that Chou and Carleone were not aware of the works by Yu and Ambartsumian, as well as those by Whitney. Chou and Carleone as well as Yu's analyses, in fact, consist of a particular case of the AMT theory in which only those terms in $a_{44}^k, a_{45}^k, a_{55}^k$ of Eqs. (54) which are independent of z are retained.

Disciuva and coauthors [99–101] employed Yu's and Chou and Carleone's Zig-Zag displacement field, which was linear in each layer, by employing the Heaviside step function, and gave finite element applications. It should be pointed out, once again, that Yu, Chou and Carleone, and Disciuva and co-authors, namely the YCCD analyses, deal with a particular case of what is herein called the AMT theory. As previously mentioned, the linear piecewise continuous displacement field used in the YCCD cases can in fact be obtained from the AMT by simply neglecting the higher order terms (which multiply z^2 and z^3) in Eqs. (58). By doing this, the resulting YCCD models:

i) are not able to fulfill homogeneous conditions for transverse shear stresses at the top/bottom plate/shell surfaces;

ii) are not suitable for unsymmetrically laminated structures.

Most the the subsequent works on Zig-Zag theories did not refer to the work by Ambartsumian, nor to those by Whitney and Rath and Das. In the same manner, the developments which appeared in journals in the Russian language, as well as those reviewed in the previous subsection were mostly ignored. Subsequent works were instead very much influenced by Yu, Chou and Carleone, and Disciuva's articles; that is, *reference has been made to simplified analyses while the most complete and exhaustive AMT has not been recognized, nor mentioned!* As a matter of fact, most of the subsequent 15 years of literature were devoted to introducing the improvements i) and ii) in the YCCD studies. The result of this *unuseful work* was that, almost ten years later, the original AMT was re-obtained by Cho and Parmerter [102] who gave *the best* refinements of YCCD studies. A few details of the story behind this are given below.

First Bhaskar and Varadan [103] and then Savithri and Varadan [104] and Lee *et al* [105] introduced the top-bottom zero stress conditions mentioned in point i. This was done by extending, in a YCCD type theory, the Vlasov [106] third order in-plane displacement fields (which were frequently applied to laminated structures by Reddy [107]). The resulting model, which did not have the quadratic terms in the expressions for displacement fields, was still not suitable for unsymmetrical laminated plates. A final *best version*, which includes what is mentioned in points i and ii, was proposed by Cho and Parmerter [102], and then (among others) by Soldatos and Timarci [108], Timarci and Soldatos [109] Lee and Waas [110], Lee, Waas and Karnopp [111], Iblidi, Karama, and Touratier [112], Aitharaju and Averill [113], Cho and Averill [114], and Polit and Touratier [115]. Some (but not extremely relevant) differences can be found in these various articles. The Heaviside step function was used by Cho and Parmerter [102], whose displacement fields for a generally laminated plate were written in the form shown below,

$$
\begin{aligned}
u(x,y,z)=u_0&+\sum_{k=0}^{N_u-1} S_x(z-z_k)H(z-z_k)\\
&+\sum_{k=0}^{N_s-1} T_x(z-\zeta_k)H(-z+\zeta_k)\\
&-\frac{z^2}{2h}\left(\sum_{k=0}^{N_u-1} S_x^k+\sum_{k=0}^{N_s-1} T_x^k\right)\\
&-\frac{z^3}{3h^2}\left\{w_{,x}+\frac{1}{2}\left(\sum_{k=0}^{N_u-1} S_x^k+\sum_{k=0}^{N_s-1} T_x^k\right)\right\}\\
v(x,y,z)=v_0&+\sum_{k=0}^{N_u-1} S_y(z-z_k)H(z-z_k)\\
&+\sum_{k=0}^{N_s-1} T_y(z-\zeta_k)H(-z+\zeta_k)\\
&-\frac{z^2}{2h}\left(\sum_{k=0}^{N_u-1} S_x^k+\sum_{k=0}^{N_s-1} T_y^k\right)\\
&-\frac{z^3}{3h^2}\left\{w_{,x}+\frac{1}{2}\left(\sum_{k=0}^{N_u-1} S_y^k+\sum_{k=0}^{N_s-1} T_y^k\right)\right\} \qquad (59)\\
w(x,y,x)=w_0&(x,y)
\end{aligned}
$$

N_u and N_s are the number of layers in the upper and lower half, respectively. z and ζ are the two thickness coordinates for the upper and lower half, respectively; z_k and ζ_k are the corresponding interface values. The mid-plane rotations ϕ_x, ϕ_y at z^+ (z^+ and z^- are the top and bottom layer interfaces, respectively) are

$$
\phi_x=\frac{\partial u}{\partial z}_{|z=0^+}=S_x^0, \quad \phi_y=\frac{\partial v}{\partial z}_{|z=0^+}=S_y^0
$$

and

$$
\begin{aligned}
&S_x^k=a_{x\tau}^k(w_{,\tau}^0+\phi_{,\tau})+b_{x\tau}^k w_{,\tau}^0,\\
&S_y^k=a_{y\tau}^k(w_{,\tau}^0+\phi_{,\tau})+b_{y\tau}^k w_{,\tau}^0,\\
&\qquad k=0,1,..,N_u-1, \quad \tau=x,y\\
&T_x^k=c_{x\tau}^k(w_{,\tau}^0+\phi_{,\tau})+d_{x\tau}^k w_{,\tau}^0,\\
&T_y^k=c_{y\tau}^k(w_{,\tau}^0+\phi_{,\tau})+d_{y\tau}^k w_{,\tau}^0, \quad k=0,1,..,N_s-1, \quad \tau=x,y
\end{aligned}
$$

where $a_{x\tau}^k, b_{x\tau}^k, c_{x\tau}^k, d_{x\tau}^k$ $a_{y\tau}^k, b_{y\tau}^k, c_{y\tau}^k, d_{y\tau}^k$ are layer stiffnesses (see [102] for further details).

The previous displacement field, even though written in a more tedious form, coincides exactly with that of Eqs. (31). The use of the Heaviside step function is not essential. Its use is, in fact, preferred by some authors and omitted by others. The author's opinion is that the use of the Heaviside function, although it has some formal advantages, is not useful as far as calculations and/or computer implementations are concerned.

More exhaustive discussions on the developments of Zig-Zag theory in the West can be found in most of the previously mentioned review articles [12–34] and in the papers [108–115]. Those developments that were directed to include transverse normal strain effects in the AMT are more of an interest. Among these, the contributions by Savithri and Varadan [104] and Cho and Averill [114] should be mentioned.

4 MULTILAYERED THEORY BASED ON REISSNER MIXED VARIATIONAL THEOREM

A third approach to laminated structures was originated by two papers by Reissner [64,116] in which a mixed variational equation, namely Reissner's Mixed Variational Theorem (RMVT) was proposed. For this reason we denote such an approach as Reissner Multilayered Theory. This third approach is the only one developed in the West.

Reissner Mixed Variational Theorem. RMVT permits one to satisfy, completely and *a priori*, the C_z^0-requirements by assuming two independent fields for displacements $\boldsymbol{u}=\{u,v,w\}$, and transverse stresses $\boldsymbol{\sigma}_n=\{\sigma_{xz},\sigma_{yz},\sigma_{zz}\}$, (bold letters denote arrays). Briefly, RMVT expresses 3D indefinite equilibrium equations (and related equilibrium conditions at the boundary surfaces which are, for the sake of brevity, not written here) and compatibility equations for transverse strains in a variational form. The 3D equilibrium equations in the dynamic case are,

$$\sigma_{ij,i}-\rho\ \ddot{u}_i=p_i \quad i,j=1,2,3 \tag{60}$$

ρ is the mass density and double dots denote accelerations, while $(p_1,p_2,p_3)=\boldsymbol{p}$ are body forces. The compatibility conditions for transverse stresses can be written by evaluating transverse strains in two ways: by Hooke's law $\boldsymbol{\epsilon}_{nH}=\{\epsilon_{xz_H},\epsilon_{yz_H},\epsilon_{zz_H}\}$ and by geometrical relations $\boldsymbol{\epsilon}_{nG}=\{\epsilon_{xz_G},\epsilon_{yz_G},\epsilon_{zz_G}\}$; the subscript n denotes transverse/normal components. They satisfy the equation

$$\boldsymbol{\epsilon}_{nH}-\boldsymbol{\epsilon}_{nG}=0 \tag{61}$$

RMVT therefore states.

$$\int_V(\delta\boldsymbol{\epsilon}_{pG}^T\boldsymbol{\sigma}_{pH}+\delta\boldsymbol{\epsilon}_{nG}^T\boldsymbol{\sigma}_{nM}+\delta\boldsymbol{\sigma}_{nM}^T(\boldsymbol{\epsilon}_{nG}-\boldsymbol{\epsilon}_{nH}))dV = \int_V\rho\,\delta\boldsymbol{u}\ddot{\boldsymbol{u}}dV+\delta L_e \tag{62}$$

The superscript T signifies an array transpose and V denotes the 3D multilayered body volume, while the subscript p denotes in-plane components, respectively. Therefore, $\boldsymbol{\sigma}_p=\{\sigma_{xx},\sigma_{yy},\sigma_{xy}\}$ and $\boldsymbol{\epsilon}_p=\{\epsilon_{xx},\epsilon_{yy},\epsilon_{xy}\}$. The subscript H indicates that stresses are computed via Hooke's law. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometrical relations (subscript G). δL_e is the virtual variation of the work done by the external layer-force $\boldsymbol{p}$. Subscript M indicates that transverse stresses are those of the assumed model.

RMVT leads to a set of both equilibrium and constitutive equations which are variationally consistent with the assumptions made on displacements and transverse stresses.

Murakami's contribution to RMT. The first application of RMVT was made by Murakami [117,118], who developed a refinement of Reissner-Mindlin type theories. First, a Zig-Zag form of displacement field was introduced by means of two *Zig-Zag* functions (the Murakami's Zig-Zag functions $\xi_k(-1)^kD_x,\xi_k(-1)^kD_y$),

$$\begin{aligned} u^k(x,y,z)&=u_0(x,y)+z\,\phi_x(x,y)+\xi_k(-1)^kD_x(x,y)\\ v^k(x,y,z)&=v_0(x,y)+z\,\phi_y(x,y)+\xi_k(-1)^kD_y(x,y)\\ w(x,y,z)&=w_0(x,y)\end{aligned} \tag{63}$$

$\xi_k=2z_k/h_k$ is a nondimensioned layer coordinate (z_k is the physical coordinate of the k-layer whose thickness is h_k). The exponent k changes the sign of the Zig-Zag term in each layer. Such a trick permits one to reproduce the discontinuity of the first derivative of the displacement variables in the z-direction. The geometrical meaning of the Zig-Zag function is explained in Figs. 6 and 7.

Transverse shear stress fields were assumed parabolic by Murakami [117] in each layer, and interlaminar continuous according to the following formula,

$$\begin{aligned} \sigma_{xz}^k(x,y,z)&=\sigma_{xz}^{kt}(x,y)F_0(z_k)+F_1(z_k)R_x^k(x,y)\\ &\quad+\sigma_{xz}^{kb}(x,y)F_2(z_k)\\ \sigma_{yz}^k(x,y,z)&=\sigma_{yz}^{kt}(x,y)F_0(z_k)+F_1(z_k)R_y^k(x,y)\\ &\quad+\sigma_{yz}^{kb}(x,y)F_2(z_k)\end{aligned} \tag{64}$$

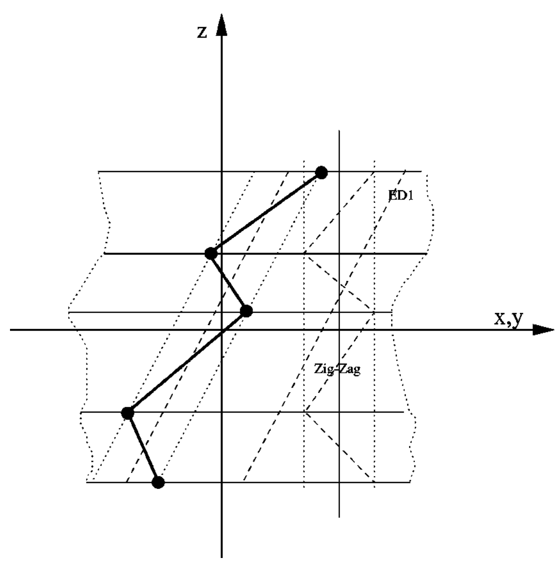

Fig. 6 Geometrical meaning of Murakami's Zig-Zag function—linear case

where $\sigma_{xz}^{kt}(x,y)$, $\sigma_{yz}^{kt}(x,y)$, $\sigma_{xz}^{kb}(x,y)$, $\sigma_{yz}^{kb}(x,y)$ are the top and bottom values of transverse shear stresses, while $R_x^k(x,y)$, $R_y^k(x,y)$ are layer stress resultants. The introduced layer thickness coordinate polynomials are given by

$$F_0=-1/4+\xi_k+3\xi_k^2,$$

$$F_1=\frac{3-12\xi_k^2}{2h_k},$$

$$F_2=-1/4-\xi_k+3\xi_k^2$$

Homogeneous and nonhomogeneous boundary conditions at the top/bottom plate surfaces can be linked to the introduced stress field.

Toledano and Murakami [119] introduced transverse normal strain and stress effects by using a third-order displacement field for both in-plane and out-of-plane components and a fourth-order transverse stress field for both shear and normals components. Koiter's recommendation is retained in [119].

Carrera's contribution to RMT. A generalization of RMVT to develop ESL and LW plate/shell theories, as well as finite element applications, has been give by Carrera [9,120–130]. Displacement and transverse stress components were assumed as follows

$$\boldsymbol{u}^k=F_t\boldsymbol{u}_t^k+F_b\boldsymbol{u}_b^k+F_r\boldsymbol{u}_r^k=F_\tau\boldsymbol{u}_\tau^k,\quad \tau=t,b,r$$

$$r=2,3,..,N \tag{65}$$

$$\boldsymbol{\sigma}_{nM}^k=F_t\boldsymbol{\sigma}_{nt}^k+F_b\boldsymbol{\sigma}_{nb}^k+F_r\boldsymbol{\sigma}_{nr}^k=F_\tau\boldsymbol{\sigma}_{n\tau}^k,\quad k=1,2,..,N_l$$

The subscripts t and b denote values related to the top and bottom layer surface, respectively. These two terms constitute the linear part of the expansion. The thickness functions $F_\tau(\xi_k)$ have now been defined at the k-th layer level,

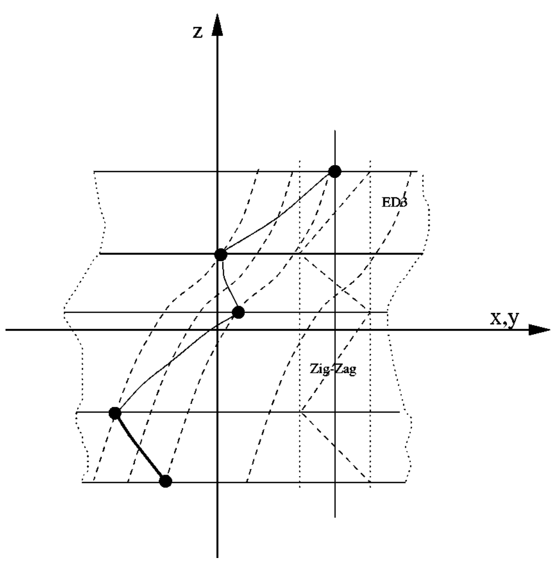

Fig. 7 Geometrical meaning of Murakami's Zig-Zag function—higher degree case

$$F_t=\frac{P_0+P_1}{2},\quad F_b=\frac{P_0-P_1}{2},$$

$$F_r=P_r-P_{r-2},\quad r=2,3,..,N \tag{66}$$

in which $P_j=P_j(\xi_k)$ is the Legendre polynomial of the j-order defined in the ξ_k-domain: $-1\leqslant\xi_k\leqslant 1$. For instance, the first five Legendre Polynomials are

$$P_0=1,\quad P_1=\xi_k,\quad P_2=(3\xi_k^2-1)/2,$$

$$P_3=\frac{5\xi_k^3}{2}-\frac{3\xi_k}{2},\quad P_4=\frac{35\xi_k^4}{8}-\frac{15\xi_k^2}{4}+\frac{3}{8}$$

The chosen functions have the following values:

$$\xi_k=\begin{cases} 1: & F_t=1;\quad F_b=0;\quad F_r=0\\ -1: & F_t=0;\quad F_b=1;\quad F_r=0,\end{cases} \tag{67}$$

The top and bottom values have been used as unknown variables. Such a choice makes the model particularly suitable in view of the fulfillment of the C_z^0-requirements. The interlaminar transverse shear and normal stress continuity can therefore be linked by simply writing:

$$\boldsymbol{\sigma}_{nt}^k=\boldsymbol{\sigma}_{nb}^{(k+1)},\quad k=1,N_l-1 \tag{68}$$

In those cases in which the top/bottom plate/shell stress values are prescribed (zero or imposed values), the following additional equilibrium conditions must be satisfied for:

$$\boldsymbol{\sigma}_{nb}^1=\overline{\boldsymbol{\sigma}}_{nb},\quad \boldsymbol{\sigma}_{nt}^{N_l}=\overline{\boldsymbol{\sigma}}_{nt} \tag{69}$$

where the over-bar denotes the imposed values on the plate boundary surfaces.

The Weak Form of Hooke's Law. Full use of Reissner's theorem requires solving a problem in terms of both displacement and transverse stress variables. This can be computationally expensive. In order to preserve the advantages of a classical displacement formulation, a *Weak Form of Hooke's Law* (WFHL) was proposed in [9]. The WFHL, which was completely inspired by RMVT, permits one to express, in a weak sense, transverse stress variables in terms of the displacement ones.

As shown in [9], the truncated Legendre expansion for displacement and transverse stress variables can be expressed in a weighted residual form in the thickness direction,

$$\int_{A_k}F_s(\boldsymbol{\epsilon}_{nG}^k-\boldsymbol{\epsilon}_{nH}^k)dz=0,\quad s=t,b,2,3,..,N \tag{70}$$

As in the RMVT, Eqs. (70) impose compatibility of transverse strains. The difference is that the integral is now introduced only in the z-direction.

On substitution of given displacement and the transverse stress models, as well as a given Hooke's law and strain displacement relation, and by integrating along z, the set of Eqs. (70) leads to a relation between stress and displacement variables that can be formally written in the following array form:

$$\boldsymbol{H}_u^k\ \boldsymbol{u}^k-\boldsymbol{H}_\sigma^k\ \boldsymbol{\sigma}^k=0 \tag{71}$$

where $\boldsymbol{H}_\sigma^k$ is a square symmetric nonsingular matrix, while $\boldsymbol{H}_u^k$ is rectangular, singular and nonsymmetric. Examples of these matrices are given in [9,120] Under certain conditions, see [9], Eq. (71) can be explicitly written as,

$$\boldsymbol{\sigma}^k = -\boldsymbol{H}_\sigma^{k-1} \ \boldsymbol{H}_u^k \ \boldsymbol{u}^k \tag{72}$$

which gives the sought relation between stresses and displacement variables.

Other available works based on the RMVT. Examples of application of RMVT to laminated plates in the frame of an Equivalent Single-Layers Model were presented in the previously mentioned articles of Murakami [117,118] and Toledano and Murakami [119]. The results obtained for cylindrical bending of cross-ply, symmetrically laminated plates showed improvement in the description of the in-plane response with respect to first order shear deformation theory [118] and applications to unsymmetrically laminated plates were presented in [119]. Shell applications of the results in [118] were developed by Bhaskar and Varadan [131] and Jing and Tzeng [132]. Bhaskar and Varadan [131] underlined the severe limitation of transverse shear stresses which are evaluated *a priori* by the assumed model. Finite element applications of this model have also been developed. Linear analysis of thick plates were discussed by Rao and Meyer [133]. Linear and geometrically nonlinear static and dynamic analyses were considered by Carrera and coauthors [120–122]. Systematic application of RMVT to develop plate elements has been provided in a recent work by Carrera and Demasi [134]. Partial implementation to shell elements was proposed by Bhaskar and Varadan [135]. Full shell implementation has been given by Brank and Carrera [136].

The limitations of Equivalent Single-Layer analyses were known to Toledano and Murakami [137], who applied the RMVT to Layer-Wise models. A linear in-plane displacement expansion was expressed in terms of the interface values in each layer, while transverse shear stresses were assumed parabolic. It was shown that the accuracy of the resulting theories was layout independent. Transverse normal stress and related effects were discarded and the analysis showed severe limitations to analyze thick plates. A more comprehensive evaluation of Layer-Wise theories for the case of linear and parabolic expansions was considered by the author in [123] where applications to the static analysis of plates were presented. Subsequent works extended the analysis to dynamics and shell geometry [124–130]. A more comprehensive review on works based on Reissner's Theorem has been recently provided in [30].

5 REMARKS ON LEKHNITSKII, AMBARTSUMIAN, AND REISSNER MULTILAYERED THEORIES

From a historical point of view, Lekhnitskii [36] has probably made the first remarkable contribution to multilayered structure modeling: his work was the first to show the importance of satisfying the C_z^0-requirements.
A list of the main features of LMT follows.

- LMT-0. Lekhnitskii gave the *first* theory which accounts for the C_z^0-requirements.
- LMT-1. Such a Zig-Zag theory was obtained by solving an elasticity problem related to a layered beam.
- LMT-2. Even though Lekhnitskii restricted his analysis to a cantilever, multilayer beam, he presented explicit formulas for transverse stresses and displacement fields (Eqs. (12–22)), which are valid at any point of the considered beam. This could result in being extremely useful to assess newly developed analytical/numerical models.
- LMT-3. The work by Lekhnitskii shows a manner in which multilayered structure problems can be handled. For instance, the inclusion of transverse normal stress would require a different choice of stress function in Eq. (3).
- LMT-4. The use of a stress function formulation leads to in-plane and transverse stress fields which satisfy *by definition* the 3D equilibrium equations. Stresses have been calculated by Lekhnitskii by solving a boundary value problem related to the compatibility equations which were written in terms of a stress function. In particular, the evaluation of transverse stresses does not require any post-processing procedure, such as the use of Hooke's law or integration of 3D equilibrium equations.
- LMT-5. Although transverse normal stresses are neglected, the transverse displacement varies in the thickness direction according to a piecewise parabolic distribution given in Eq. (12). An attempt for the inclusion of the transverse normal stress effect would require an appropriate choice of the stress function in Eq. (3).

As far as the extensions made by Ren to the plate geometry of LMT are concerned, the following remarks can be made:

- LMT-6. Transverse shear stresses are continuous at the interfaces, and parabolic in each layer. Furthermore, stress-free conditions are fulfilled at top and bottom plate surfaces.
- LMT-7. Four independent functions defined on a reference surface are used to express transverse shear stresses which are assumed parabolic in each layer.
- LMT-8. The form of transverse shear stresses has been given *a priori* by Ren in terms of the above mentioned four independent functions and layer constants which are parabolic in z. The relation between the layer constants and the mechanical and geometrical properties of the layers was explicitly written by Ren. In other words, their calculation does not require any imposition of transverse shear stress continuity at each interface, as will be the case for AMT and RMT.
- LMT-9. In-plane displacements are continuous at each interface and are cubic in each layer.
- LMT-10. Seven independent variables, which are defined in the region Ω, have been used to describe displacement and stress fields in the laminated plates. Four were used for the transverse shear stresses, plus three corresponding to the three values of displacements were given corresponding to the reference surface.
- LMT-11. In agreement with Lekhnitskii, Ren neglects transverse normal stresses σ_{zz}. In contrast to Lekhnitskii, transverse normal strains ϵ_{zz} are discarded by Ren.

- LMT-12. Transverse shear stresses are calculated by Ren directly by Eqs. (26). That is, Hooke's Law is not used, nor is integration of the 3D equilibrium equations required.

As far as the AMT theory is concerned, it could be remarked as follows:

- AMT-0. It is a natural extension of Reissner-Mindlin theory which was originally developed for isotropic, monocoque structures to multilayered, anisotropic plates and shells.
- AMT-1. As LMT-6, nonzero conditions on top/bottom surfaces could also be fulfilled.
- AMT-2. Two independent functions defined on Ω are used to express transverse shear stresses, as in Eqs. (54), two less than those used in LMT.
- AMT-3. Layer constants, parabolic in each layer, must be computed by imposing transverse shear stress continuity at each interface, while the form of the $f(z)$ function could be found by imposing top-bottom stress-free conditions.
- AMT-4. As LMT-9.
- AMT-5. Five independent variables, which are defined on Ω, are used to describe displacement and stress fields in the laminated plate/shells, two less than the LMT case.
- AMT-6. As LMT-11.
- AMT-7. Literature has shown that much better evaluations for transverse shear stresses can be obtained via integration of the 3D equilibrium equations, as compared to the use of Eqs. (54).
- AMT-8. Extension to a shell requires a reformulation of displacement models and related layer-constants.
- AMT-9. Additional unknown functions are required to include transverse normal stress/strain effects related to KR.

As far as the RMT is concerned, the following remarks can be made:

- RMT-0. It is the only Zig-Zag approach entirely developed in the West. It is based on a variational theorem that permits both displacements and transverse stress assumptions.
- RMT-1. As LMT-6. In this case, zero as well as nonzero conditions for transverse shear and normal stresses can be included at the top and/or bottom of the plate.
- RMT-2. At least $2N_l+1$ independent variables must be used for each transverse shear and normal stress component. Additional constitutive equations are therefore obtained by applying RMVT. However, these variables can be expressed in terms of the displacement ones by using a weak form of Hooke's Law Eqs. (70).
- RMT-3. In-plane displacements are continuous at each interface and can be chosen linear or of higher order in each layer. ESL applications require the introduction of *Murakami Zig-Zag functions*.
- RMT-4 Layer constants do not appear in the expressions of displacements and transverse stresses. In practice, IC are *variationally* imposed by writing the constitutive equations between transverse stresses and displacement variables.
- RMT-5. The number of independent variables can be arbitrarily chosen as explained in RMT-3.
- RMT-6. Interlaminar continuous transverse normal stresses/strains can be easily described by the RMT theory. KR was, in fact, already included in the early developments of RMT.
- RMT-7. Much better evaluations of transverse stresses are obtained via integration of the 3D equilibrium equations, as compared to the use of assumed forms, eg, Eqs. (64).
- RMT-8. The extension to shell does not require any changes in displacement and stress fields.
- RMT-9. Among the Zig-Zag theories examined, RMT is probably the most suitable from a computational point of view.

Which of the three theories is the best one? Certainly an answer to such a question would be of great use to every scientist who works on 2D modelings of layered structures. The present historical note, as stated in the Introduction, was not aimed at giving an answer to such a question. The author's opinion is that an unequivocal answer to such a point does not exist. None of the introduced developments provide an exact solution. These are all *axiomatic theories* and, as a consequence, they will all violate, in same way or another, the fundamental equations of 3D elasticity (for instance, if one takes the 3D equilibrium equations as valid and neglects transverse normal stress/strain, it follows that a z-parabolic transverse shear stress field requires z-linearity for the in-plane stresses, eg, z-linear displacement field; this is not the case for LMT, AMT, or RMT which use z-cubic displacement fields). In this respect, the RMT has the advantage that transverse stresses and displacements are assumed independent of each other and that these assumptions are made according to a desired accuracy.

Exhaustive benchmark problems that compare the three approaches are not available. Some attempts have been made in [127–129,134]. Available results have shown that, in the framework of ESL theory, the LMT analysis leads to a better description than the AMT one, and that the RMT could lead a to better description than LMT and AMT. In any case, it must be taken into account that a general conclusion related to any ESL theory is the following: available numerical evaluations show that although ESL theories can describe transverse shear and normal strains, including transverse warping of cross section, the approach is *kinematically homogeneous* in the sense that the kinematics are insensitive to individual layers. If detailed response of individual layers is required and if significant variations in displacement gradients between layers exist, as it is the case for the description of the local phenomena, this approach will necessitate the use of especially higher order theories in each of the constitutive layers along with a correspondent increase in the number of unknowns in the solution process, as well as in the complexity of the analyses. That is, LW analysis is required in such cases.

Finally, we mention that Gurtovoi and Piskunov [138] have recently proposed a method to compare the accuracy of two different plate theories. The hypotheses employed in [138] make the method of Gurtovoi and Piskunov not applicable for comparing the three approaches considered in this work. However, attempts in this direction should be tried in the future.

6 CONCLUSIONS

The historical review documented in this paper has shown that, within axiomatic framework, three independent ways of introducing *Zig-Zag* theories have been proposed for the analysis of multilayered plates and shells. In particular, it has been established that:

- Lekhnitskii [36] was the first to propose a theory for multilayered structures which describe the Zig-Zag behavior of a displacement field in the thickness direction and interlaminar equilibrium of transverse stresses. Lekhnitskii's work was originally presented as an elasticity solution for layered, cantilever beams.
- Apart from the method by Lekhnitskii, which was extended to plate structures by Ren, the other two approaches were proposed by Ambartsumian (who extended the well-known Reissner-Mindlin theory to anisotropic layered plates and shells); and by Reissner who proposed a variational theorem that permits both displacement and transverse stress assumptions.
- Based on the author's historical considerations, which are documented in this paper, it has been suggested to refer to these three approaches as:
  1) Lekhnitskii Multilayered Theory
  2) Ambartsumian Multilayered Theory
  3) Reissner Multilayered Theory
- A point-by-point comparison of the three approaches has been discussed extensively in Section 5.

Future developments could be directed to extend LMT theory to make it suitable for the inclusion of KR. A more extensive comparison of the three approaches and of their numerical performances vs elasticity solutions would be welcomed. Benchmark problems could be set out for this purpose. Other theories which are not based on axiomatic methods (such as asymptotic theories, full mixed formulations, etc) should be conveniently included in such a comparison.

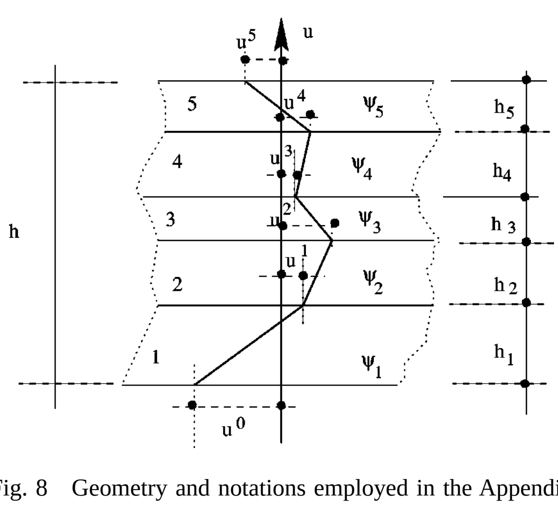

Fig. 8 Geometry and notations employed in the Appendix

As a final remark, it is clearly stated that the author is aware that this historical review could not be complete. The author is aware that other, significant articles and papers could exist on this subject that might have escaped the present work. In particular, there are many articles published in the Russian language whose English translations are not available, and they might report relevant information not considered here. However, what has been quoted in this article could at least be of some help for assigning the right credentials as far as contributions and contributors to multilayered theory are concerned.

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APPENDIX: TWO DIFFERENT WAYS OF WRITING THE AMT

This appendix shows two different manners in which the displacement fields related to the AMT type theories can be derived. The same analysis could also be extended to LRT. For the sake of simplicity, a 1D flat case is considered. Extension to the 2D case and shell geometry should be obvious. The origin of the thickness coordinates, for the sake of simplicity, is placed in the bottom layer; attention has been restricted to a piecewise continuous, linear displacement field.

The displacement field u in each layer can be first written by using displacement values at the interfaces (see Fig. 8):

$$
\begin{aligned}
&u^1(z)=u^0+z\psi_1, \quad \leqslant z\leqslant h_1\\
&u^2(z)=u^1(h_1)+(z-h_1)\psi_2, \quad h_1\leqslant z\leqslant h_2\\
&\ \ \cdots \qquad\qquad \cdots\cdot\\
&u^k(z)=u^{k-1}(h_{k-1})+(z-h_{k-1})\psi_k, \quad h_{k-1}\leqslant z\leqslant h_k\\
&\ \ \cdots \qquad\qquad \cdots\cdot\\
&u^{N_l}(z)=u^{N_l-1}(h_{N_l-1})+(z-h_{N_l-1})\psi_{N_l},\\
&h_{N_l-1}\leqslant z\leqslant h_{N_l}
\end{aligned}
\tag{73}
$$

where

u^0 is the value of the displacement u with corresponding to the bottom surface.

$u^k(h_k)$, $k=1,N_l$ are the interface values of u.

ψ_k, $k=1,N_l$ are the values which identify the rotations in the layers.
Moreover

$$
\begin{aligned}
&u^1(h_1)=u^0+h_1\psi_1\\
&u^2(h_2)=u^1(h_1)+(h_2-h_1)\psi_2\\
&\cdot\cdot=\cdot\cdot\cdot\\
&\cdot\cdot\cdot\cdot=\cdot\cdot\cdot \qquad (74)\\
&u^k(h_k)=u^{k-1}(h_{k-1})+(h_k-h_{k-1})\psi_k\\
&\cdot\cdot\cdot\cdot\cdot=\cdot\cdot\\
&u^{N_l-1}(z)=u^{N_l-2}(h_{N_l-2})+(h_{N_l-1}-h_{N_l-2})\psi_{N_l-1}
\end{aligned}
$$

The generic interface values is then re-written,

$$u^k(h_k)=u^0+\sum_{k=1}^{N_l-1}(h_k-h_{k-1})\psi_k, \quad k=1,N_l \qquad (75)$$

It is noted that $h_0=0$. It follows that the displacement field in each layer can be written in the following unified form,

$$u^k(z)=u^0+\sum_{k=1}^{N_l-2}(h_{k-1}-h_{k-2})\psi_{k-1}+(z-h_{k-1})\psi_k, \quad k=1,N_l \qquad (76)$$

The N_l rotations ψ_k can be expressed in terms of one of them (for instance the rotation in the bottom layer) by imposing the N_l-1 interlaminar continuity conditions for transverse shear stresses,

$$\psi_k=a^k\psi_1, \quad k=2,N_l-1$$

where a^k are layer constants defined by interlaminar transverse stresses. To be more precise, ψ_1 appears as a combination of an in-plane derivative of transverse displacement w_x, and a transverse shear strain γ_{xz}. The previous relation could be written as

$$\psi_k=-w_x+a^k\gamma_{xz}$$

as it is in Eqs. (33,36). For the sake of simplicity we take the first of these as valid: as a consequence the displacement u is

$$u^k(z)=u^0+\sum_{k=1}^{N_l-2}(h_{k-1}-h_{k-2})a^{k-1}\psi_1+(z-h_{k-1})a^k\psi_1, \quad k=1,N_l \qquad (77)$$

Equations (76) give the displacement field in each layer. Such a displacement field can be written in a form which is applicable to the whole multilayer by using the Heaviside step function. Such a function is defined as follows

$$H(z-z_k)=\begin{cases}0 & z\leqslant z_k\\ 1 & z\geqslant z_k\end{cases} \qquad (78)$$

By means of H, the displacement u can be written in a form which is formally not affected by k,

$$u(z)=u^0+\sum_{k=1}^{N_l-1}(z-z_{k-1})\psi_k H(z-z_k) \qquad (79)$$

or

$$u(z)=u^0+\sum_{k=1}^{N_l-1}(z-z_{k-1})a_k\psi_1 H(z-z_k) \qquad (80)$$

Judging from the available literature, the use of the Heaviside function could be considered questionable. It has been used in early works [99,102], and abandoned in some of the more recent papers [109,112]. The Heaviside function has the formal advantage of permitting one to write u in a form which is not affected by k index. Nevertheless, such an advantage could be ineffective in the calculation and related computer implementations.

It appears clear that a linear, piecewise form for u leads to layer continuity stiffnesses a^k which are independent of z. In fact, top-bottom homogeneous transverse shear stress conditions cannot be imposed in this case. This was known to Ambartsumian [33] and Whitney [92] who assumed a^k as a cubic function of z. In fact, the two additional functions related to quadratic and cubic terms of z are determined by the two homogeneous conditions of transverse shear stresses with corresponding to top and bottom plate/shell surfaces.

LIST OF ACRONYMS

Acronyms that are used frequently in this article are listed below.

AMT	-	Ambartsumian Multilayered Theory
ESLM	-	Equivalent Single Layer Models
IC	-	Interlaminar Continuity
KR	-	Koiter's Recommendation
LMT	-	Lekhnitskii Multilayered Theory
LWM	-	Layer-Wise Models
RM	-	Reissner-Mindlin
RMT	-	Reissner Multilayered Theory
RMVT	-	Reissner's Mixed Variational Theorem
WFHL	-	Weak Form of Hooke's Law
ZZ	-	Zig-Zag

REFERENCES

[1] Cauchy AL (1828), Sur l'équilibre et le mouvement d'une plaque solide, *Exercises de Matematique* **3**, 328–355.
[2] Poisson SD (1829), Memoire sur l'équilibre et le mouvement des corps elastique, *Mem. Acad. Sci.* **8**, 357.
[3] Kirchhoff G (1850), Über das Gleichgewicht und die Bewegung einer elastischen Scheibe, *J Angew Math* **40**, 51–88.
[4] Love AEH (1927), *The Mathematical Theory of Elasticity, 4th Edition*, Cambridge Univ Press, Cambridge.
[5] Reissner E (1945), The effect of transverse shear deformation on the bending of elastic plates, *ASME J. Appl. Mech.* **12**, 69–76.
[6] Mindlin RD (1951), Influence of rotatory inertia and shear in flexural motions of isotropic elastic plates, *ASME J. Appl. Mech.* **18**, 1031–1036.
[7] Hildebrand FB, Reissner E, and Thomas GB (1938), Notes on the foundations of the theory of small displacements of orthotropic shells, *NACA TN-1833*, Washington DC.

[8] Pagano NJ (1969), Exact solutions for composite laminates in cylindrical bending, *J. Compos. Mater.* **3**, 398–411.
[9] Carrera E (1995), A class of two dimensional theories for multilayered plates analysis, *Atti Acc Sci Torino, Mem Sci Fis* **(19–20)**, 49–87.
[10] Reddy JN (1997), *Mechanics of Laminated Composite Plates, Theory and Analysis*, CRC Press.
[11] Koiter WT (1959), A consistent first approximations in the general theory of thin elastic shells, *Proc of Symp. on the Theory of Thin Elastic Shells, Aug 1959*, North-Holland, Amsterdam, 12–23.
[12] Ambartsumian SA (1962), Contributions to the theory of anisotropic layered shells, *Appl. Mech. Rev.* **15**, 245–249.
[13] Ambartsumian SA (2002), Nontraditional theories of shells and plates, *Appl. Mech. Rev.* **55**(5), R35–R44.
[14] Grigolyuk EI and Kogan FA (1972), The present status of the theory of multilayered shells, *Prikl. Mekh.* **8**, 3–17.
[15] Librescu L and Reddy JN (1986), A critical review and generalization of transverse shear deformable anisotropic plates, Euromech Colloquium 219, Kassel, Sept (1986), *Refined Dynamical Theories of Beams, Plates and Shells and Their Applications*, I Elishakoff and Irretier (eds), Springer Verlag, Berlin (1987), 32–43.
[16] Leissa AW (1987), A review of laminated composite plate buckling, *Appl. Mech. Rev.* **40**(5), 575–591.
[17] Grigolyuk EI and Kulikov GM (1988), General directions of the development of theory of shells, *Mekhanica Kompozitnykh Materialov* **24**, 287–298.
[18] Kapania RK and Raciti S (1989), Recent advances in analysis of laminated beams and plates, *AIAA J.* **27**(7), 923–946.
[19] Kapania RK (1989), A review on the analysis of laminated shells, *ASME J. Pressure Vessel Technol.* **111**(2), 88–96.
[20] Noor AK and Burton WS (1989), Assessment of shear deformation theories for multilayered composite plates, *Appl. Mech. Rev.* **42**(1), 1–18.
[21] Noor AK and Burton WS (1990), Assessment of computational models for multilayered composite shells, *Appl. Mech. Rev.* **43**(4), 67–97.
[22] Jemlelita G (1990), On kinematical assumptions of refined theories of plates: A survey, *ASME J. Appl. Mech.* **57**, 1080–1091.
[23] Vasiliev VV and Lur'e SA (1992), On refined theories of beams, plates and shells, *J. Compos. Mater.* **26**, 422–430.
[24] Reddy JN and Robbins DH (1994), Theories and computational models for composite laminates, *Appl. Mech. Rev.* **47**(6), 147–165.
[25] Noor AK, Burton S, and Bert CW (1996), Computational model for sandwich panels and shells, *Appl. Mech. Rev.* **49**(3), 155–199.
[26] Lur'e SA and Shumova NP (1996), Kinematic models of refined theories concerning composite beams, plates and shells, *Int., Appl. Mech.* **32**, 422–430.
[27] Grigorenko YaM (1996), Approaches to numerical solution of linear and nonlinear problems in shell theory in classical and refined formulations, *Int., Appl. Mech.* **32**, 409–442.
[28] Grigorenko YaM and Vasilenko AT (1997), Solution of problems and analysis of the stress strain state of non-uniform anisotropic shells (survey), *Int., Appl. Mech.* **33**, 851–880.
[29] Altenbach H (1998), Theories for laminated and sandwich plates. A review, *Int., Appl. Mech.* **34**, 243–252.
[30] Carrera E (2001), Developments, ideas and evaluations based upon the Reissner's mixed theorem in the modeling of multilayered plates and shells, *Appl. Mech. Rev.* **54**(4), 301–329.
[31] Lekhnitskii SG (1968), *Anisotropic Plates, 2nd Edition*, Translated from the 2nd Russian Edited by SW Tsai and Cheron, Bordon and Breach.
[32] Ambartsumian SA (1961), *Theory of Anisotropic Shells*, Fizmatzig, Moskwa; Translated from Russian, NASA TTF-118 (1964).
[33] Ambartsumian SA (1969), *Theory of Anisotropic Plates*, Translated from Russian by T Cheron and Edited by JE Ashton Tech Pub Co.
[34] Ambartsumian SA (1991), *Fragments of the Theory of Anisotropic Shells*, World Scientific Publ Co, Singapore.
[35] Librescu L (1975), *Elasto-statics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures*, Noordhoff Int, Leyden, Netherland.
[36] Lekhnitskii SG (1935), Strength calculation of composite beams, *Vestnik inzhen i tekhnikov, No 9*.
[37] Ponyatovskii VV (1962), On the theory of plates of medium thickness, *Prikl. Mat. Mekh.* **26**, 335–441.
[38] Agolovyan LA (1966), Use of the method of asymptotic integration to construct an approximate theory of anisotropic shells, *Prikl. Mat. Mekh.* **30**, 388–398.
[39] Gold'enveizer LP (1968), Method of veryfing the theory of shells, *Prikl. Mat. Mekh.* **32**, 684–695.
[40] Fettahlioglu OA and Steele CR (1974), Asymptotic solutions for orthotropic non-homogeneous shells of revolution, *ASME J. Appl. Mech.* **44**, 753–758.
[41] Prusakov AP (1975), Construction of a theory of flexure plates of moderate thickness by an ergonomic asymptotic method, *Prikl. Mekh.* **11**, 44–51.
[42] Prusakov AP and Plekhanov AV (1977), Constructing an improved theory of two-layered plates, *Prikl. Mekh.* **13**, 59–75.
[43] Widera GEO and Fan H (1988), On the derivation of a refined theory for non-homogeneous anisotropic shells of revolution, *ASME J. Appl. Mech.* **110**, 102–105.
[44] Hodges DH, Lee BW, and Atilgan AR (1993), Application of the variational-asymptotic method to laminated composite plates, *AIAA J.* **31**, 1674–1983.
[45] Satyrin VG and Hodges DH (1996), On asymptotically correct linear laminated plate theory, *Int. J. Solids Struct.* **33**, 3649–3671.
[46] Prusakov AP (1996), Analysis of the theory of bending of transversally isotropic plates, *Int., Appl. Mech.* **32**, 554–559.
[47] Savin GN and Khoma IYu (1971), On the theory of anisotropic shells which are free from the kinematic hypothesis of a normal element, *Prikl. Mekh.* **7**, 9–15.
[48] Khoroshun LP (1978), Derivation of equations for laminate plates and shells, *Prikl. Mekh.* **14**, 3–21.
[49] Prusakov AP (1971), The non linear bending of gently sloping multilayer shells, *Prikl. Mekh.* **7**, 3–8.
[50] Pagano NJ (1978), Stress fields in composite laminates, *Int. J. Solids Struct.* **14**, 385–400.
[51] Putcha NS and Reddy JN (1986), A refined mixed shear flexible finite element for the nonlinear analysis of laminated plates, *Compos. Struct.* **22**, 529–538.
[52] Prusakov AP (1997), Toward a theory of bending of laminated plates, *Int., Appl. Mech.* **33**, 227–232.
[53] Fares ME and Zenkour AM (1998), Mixed variational formula for the thermal bending of laminated plates, *J. Therm. Stresses* **22**, 347–365.
[54] Grigolyuk EI and Chulkov PP (1964), Theory of viscolesatic multilayered shells with hard filler, *Prikl. Mekh. Tekh. Fiz.* **5**, 109–117.
[55] Grigolyuk EI and Chulkov PP (1972), On the theory of multilayered shells, *Contribution to the Theory of Aircraft Structures*, Delft, 171–183.
[56] Cheung YK and Wu CI (1972), Free vibrations of thick, layered cylinders having finite length with various boundary conditions, *J. Sound Vib.* **24**, 189–200.
[57] Srinivas S (1973), A refined analysis of composite laminates, *J. Sound Vib.* **30**, 495–507.
[58] Chepiga S (1976), Refined theory of laminated plates, *Prikl. Mekh.* **12**, 45–49.
[59] Grigolyuk EI and Kulikov PP (1986), Modern method of designing pneumatic tires, *Summary of Documents of the II All-Union Scientific-Technical Conf*, Moscow.
[60] Cho KN, Bert CW, and Striz AG (1991), Free vibrations of laminated rectangular plates analyzed by higher order individual-layer theory, *J. Sound Vib.* **145**, 429–442.
[61] Robbins Jr. DH and Reddy JN (1993), Modeling of thick composites using a layer-wise theory, *Int. J. Numer. Methods Eng.* **36**, 655–677.
[62] Ambartsumian SA (1958), On a theory of bending of anisotropic plates, Investiia Akad Nauk SSSR, Ot Tekh Nauk, No. 4.
[63] Ambartsumian SA (1958), On a general theory of anisotropic shells, *Prikl. Mat. Mekh.* **22**(2), 226–237.
[64] Reissner E (1984), On a certain mixed variational theory and a proposed application, *Int. J. Numer. Methods Eng.* **20**, 1366–1368.
[65] Ren JG (1986), A new theory of laminated plates, *Compos. Sci. Technol.* **26**, 225–239.
[66] Ren JG (1986), Bending theory of laminated plates, *Compos. Sci. Technol.* **27**, 225–248.
[67] Ren JG and Owen DRJ (1989), Vibration and buckling of laminated plates, *Int. J. Solids Struct.* **25**, 95–106.
[68] Kraus H (1967), *Thin Elastic Shells*, John Wiley, NY.
[69] Carrera E (1991), The Effects of shear deformation and curvature on buckling and vibrations of cross-ply laminated composite shells, *J. Sound Vib.* **151**, 405–433.
[70] Ambartsumian SA (1957), Analysis of two-layer orthotropic shells, *Investiia Akad Nauk SSSR, Ot Tekh Nauk, No. 7.*
[71] Ambartsumian SA (1957), Two analysis method for two-layer orthotropic shells, *Izv An Arm SSR Seiya Fiz-Matem nauk* **X**(2).
[72] Ostermik ES and Barg YaA (1964), Industrial computing method for multi-layers anisotropic plates, Trudy konf. Izd. AN. Arm. SSR.
[73] Piskunov VG, Gurtovoi AG, and Sipetov VS (1984), Improvements

of theory of anisotropic plates, *Prikl. Mekh.* **20**, 82–87.
[74] Piskunov VG and Rasskazov AA (1998), Investigation of stress-strained state of tapered orthotropic shells and plates on the basis of second order shear theory, *Prikl. Mekh.* **20**, 82–87.
[75] Teregulov AG (1970), On the theory of multilayered anisotropic shells, *Inv. Theory of Plates and Shells, No 6–7, Izd. Kaz., Un., Kazan*.
[76] Grigorenko YaM and Vasilenko AT (1977), Taking account of non-uniformity of transverse displacements deformation in the thickness of layered shells, *Prikl. Mekh.* **13**, 36–42.
[77] Andreev AN and Nemirovoskii YuV (1978), A variation of the theory of elastic multilayer anisotropic plates, *Prikl. Mekh.* **14**, 55–62.
[78] Grikolyuk EI and Vasilenko AT (1992), Problems of the statics of anisotropic shells with finite shear stiffness, Nauk Dumka, Moskow.
[79] Prusakov AP (1971), Finite deflections of multilayered shallow shells, Izv Aka Nauk SSSR, Mech Tverd Tela, No 3.
[80] Prusakov AP (1971), Nonlinear bending equations of shallow multilayered shells, *Prikl. Mekh.* **7**(3), 3–8.
[81] Rybov AF (1964), Fundamental equations of the theory of multilayered shells taking account of transverse shear strain, *Strength of Mat and Th of Struct, No 3*, Budivelnik, Kiev.
[82] Rybov AF (1967), On the theory multilayered shell shells with filler, Air Force Construction and Air Force Eng No 11, Izd Khakosvsk Univ, Kharkov.
[83] Rasskazov AO (1976), Theory of multilayer orthotropic shallow shells, *Prikl. Mekh.* **12**, 50–76.
[84] Rasskazov AO (1977), Vibration theory of multilayer orthotropic shells, *Prikl. Mekh.* **13**, 23–29.
[85] Rasskazov AO (1978), Calculation of multilayer orthotropic shallow shells by the method of finite element, *Prikl. Mekh.* **14**, 51–56.
[86] Rasskazov AO (1978), Stability of multilayer orthotropic shells, *Prikl. Mekh.* **14**, 62–66.
[87] Rasskazov AO, Karpilovskii VS, Iosenko VI, and Kharchenko NG (1986), Analysis of multilayered orthotropic shells of revolution with finite shear stiffness, *Prikl. Mekh.* **24**, 24–28.
[88] Rasskazov AO and Burygina NG (1986), Refinement of shear theory of laminated orthotropic sloping shells, *Prikl. Mekh.* **24**, 32–37.
[89] Grikolyuk EI and Vasilenko AT (1981), Theory of shells of variables thickness, *Methods of Shell Design* **4**, Nauk Dumka, Kiev.
[90] Grikolyuk EI and Vasilenko AT (1987), Statics of anisotropic shells with finite shear stiffness, Nauk Dumka, Kiev.
[91] Vasilenko AT (1986), Allowance of transverse compression in problems of the static of laminated shells, *Prikl. Mekh.* **13**, 28–36.
[92] Whitney JM (1969), The effects of transverse shear deformation on the bending of laminated plates, *J. Compos. Mater.* **3**, 534–547.
[93] Rath BK and Das YC (1973), Vibration of layered shells, *J. Sound Vib.* **28**, 737–757.
[94] Sun CT and Whitney JM (1973), On the theories for the dynamic response of laminated plates, *AIAA J.* **11**, 372–398.
[95] Hsu TM and Wang JTS (1970), A theory of laminated cylindrical shells consisting of layers of orthotropic laminae, *AIAA J.* **8**, 2141–2146.
[96] Hsu T and Wang JTS (1971), Rotationally symmetric vibrations of orthotropic layered cylindrical shells, *J. Sound Vib.* **16**, 473–487.
[97] Yi-Yuan Y (1959), A new theory of elastic sandwich plates one dimensional case, *ASME J. Appl. Mech.* **37**, 1031–1036.
[98] Chou PC and Carleone J (1973), Transverse shear in laminated plates theories, *AIAA J.* **11**, 1333–1336.
[99] Disciuva M (1985), A refinement of the transverse shear deformation theory for multilayered plates, *l'Aerotecnica Missili e Spazio* **63**, 84–92.
[100] Disciuva M, Cicorello A, and Dalle Mura E (1985), A class of multilayered anisotropic plate elements including the effects of transverse shear deformability, AIDAA Conf, Torino, 877–892.
[101] Disciuva M (1987), An improved shear deformation theory for moderately thick multilayered anisotropic shells and plates, *ASME J. Appl. Mech.* **54**, 589–596.
[102] Cho M and Parmerter RR (1993), Efficient higher order composite plate theory for general lamination configurations, *AIAA J.* **31**, 1299–1305.
[103] Bhashar B and Varadan TK (1989), Refinement of higher-order laminated plate theories, *AIAA J.* **27**, 1830–1831.
[104] Savithri S and Varadan TK (1990), Refinement of higher-order laminated plate theories, *AIAA J.* **28**, 1842–1843.
[105] Lee KH, Senthilnathan NR, Lim SP, and Chow ST (1990), An improved zig-zag model for the bending analysis of laminated composite plates, *Compos. Struct.* **15**, 137–148.
[106] Vlasov BF (1957), On the equations of bending of plates, *Dokla Ak Nauk Azerbeijanskoi-SSR* **3**, 955–979.
[107] Reddy JN (1984), A simple higher order theory for laminated composite plates, *ASME J. Appl. Mech.* **51**, 745–752.
[108] Soldatos KP and Timarci T (1993), A unified formulation of laminated composites, shear deformable, five-degrees-of-freedom cylindrical shell theories, *Compos. Struct.* **25**, 165–171.
[109] Timarci T and Soldatos KP (1995), Comparative dynamic studies for symmetric cross-ply circular cylindrical shells on the basis a unified shear deformable shell theories, *J. Sound Vib.* **187**, 609–624.
[110] Lee D and Waas AM (1996), Stability analysis of rotating multilayer annular plate with a stationary frictional follower load, *Int. J. Mech. Sci.* **39**, 1117–1138.
[111] Lee D, Waas AM, and Karno BH (1997), Analysis of rotating multilayer annular plate modeled via a layer-wise zig-zag theory: free vibration and transient analysis, *Compos. Struct.* **66**, 313–335.
[112] Iblidi A, Karama M, and Touratier M (1997), Comparison of various laminated plate theories, *Compos. Struct.* **37**, 173–184.
[113] Aitharaju VR and Averill RC (1999), C^0 zig-zag kinematic displacement models for the analysis of laminated composites, *Mech. of Compos. Mat. and Struct.* **6**, 31–56.
[114] Cho YB and Averill RC (2000), First order zig-zag sub-laminate plate theory and finite element model for laminated composite and sandwich panels, *Compos. Struct.* **50**, 1–15.
[115] Polit O and Touratier M (2000), Higher order triangular sandwich plate finite elements for linear and nonlinear analyses, *Comput. Methods Appl. Mech. Eng.* **185**, 305–324.
[116] Reissner E (1986), On a mixed variational theorem and on a shear deformable plate theory, *Int. J. Numer. Methods Eng.* **23**, 193–198.
[117] Murakami H (1985), Laminated composite plate theory with improved in-plane responses, *ASME Proc of PVP Conf*, New Orleans, June 24–26, PVP-Vol 98-2, 257–263.
[118] Murakami H (1986), Laminated composite plate theory with improved in-plane response, *ASME J. Appl. Mech.* **53**, 661–666.
[119] Toledano A and Murakami H (1987), A high-order laminated plate theory with improved in-plane responses, *Int. J. Solids Struct.* **23**, 111–131.
[120] Carrera E (1996), C^0 Reissner-Mindlin multilayered plate elements including zig-zag and interlaminar stresses continuity, *Int. J. Numer. Methods Eng.* **39**, 1797–1820.
[121] Carrera E (1997), An improved Reissner-Mindlin-Type model for the electromechanical analysis of multilayered plates including piezo-layers, *J. Intell. Mater. Syst. Struct.* **8**, 232–248.
[122] Carrera E and Kröplin B (1997), Zig-Zag and interlaminar equilibria effects in large deflection and post-buckling analysis of multilayered plates, *Mech. of Compos. Mat. and Struct.* **4**, 69–94.
[123] Carrera E (1998), Evaluation of layer-wise mixed theories for laminated plates analysis, *AIAA J.* **36**, 830–839.
[124] Carrera E (1998), Layer-wise mixed models for accurate vibration analysis of multilayered plates, *ASME J. Appl. Mech.* **65**, 820–828.
[125] Carrera E (1999), A Reissner's mixed variational theorem applied to vibrational analysis of multilayered shell, *ASME J. Appl. Mech.* **66**(1), 69–78.
[126] Carrera E (1999), Multilayered shell theories accounting for a layer-wise mixed description. Part I Governing equations, Part II Numerical evaluations, *AIAA J.* **37**, 1117–1124.
[127] Carrera E (1999), Transverse normal stress effects in multilayered plates, *ASME J. Appl. Mech.* **66**, 1004–1012.
[128] Carrera E (1999), A study of transverse normal stress effects on vibration of multilayered plates and shells, *J. Sound Vib.* **225**, 803–829.
[129] Carrera E (2000), Single-layer vs multi-layers plate modelings on the basis of Reissner's mixed theorem, *AIAA J.* **38**, 342–343.
[130] Carrera E (2000), *A priori* vs a posteriori evaluation of transverse stresses in multilayered orthotropic plates, *Compos. Struct.* **48**, 245–260.
[131] Bhaskar K and Varadan TK (1992), Reissner's new mixed variational principle applied to laminated cylindrical shells, *ASME J. Pressure Vessel Technol.* **114**, 115–119.
[132] Jing H and Tzeng KG (1993), Refined shear deformation theory of laminated shells, *AIAA J.* **31**, 765–773.
[133] Rao KM and Meyer-Piening HR (1990), Analysis of thick laminated anisotropic composites plates by the finite element method, *Compos. Struct.* **15**, 185–213.
[134] Carrera E and Demasi L (2002), Classical and advanced multilayered plate elements based upon PVD and RMVT, Part I. Derivation of finite element matrices, *Int. J. Numer. Methods Eng.* **55**, 191–231, Part II. Numerical implementations, 253–291.
[135] Bhaskar K and Varadan TK (1991), A higher-order theory for bending analysis of laminated shells of revolution, *Compos. Struct.* **40**, 815–819.

[136] Brank B and Carrera E (2000), Multilayered shell finite element with interlaminar continuous shear stresses: A refinement of the Reissner-Mindlin formulation, *Int. J. Numer. Methods Eng.* **48**, 843–874.
[137] Toledano A and Murakami H (1987), A composite plate theory for arbitrary laminated configurations, *ASME J. Appl. Mech.* **54**, 181–189.
[138] Gurtovoi AG and Piskunov VG (1998), Comparative analysis of refined models for layered orthotropic plate, *Int., Appl. Mech.* **34**, 69–74.

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