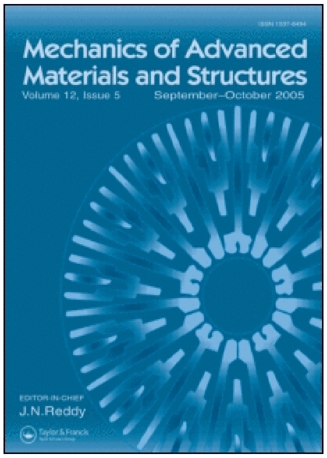

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Assessment of Plate Elements on Bending and Vibrations of Composite Structures

E. CARRERA
L. DEMASI
M. MANGANELLO

Department of Aerospace Engineering, Politecnico di Torino, Torino, Italy

ABSTRACT

This article assesses classical and refined finite plate elements on bending and vibrations of layered composites and sandwich structures. To this purpose, recent authors' findings have been extended to dynamics. About 20 plate finite elements have been implemented and compared: classical ones based on displacement assumptions are compared to advanced mixed elements which are formulated on the basis of Reissner's mixed variational theorem. Finite elements which preserve the independence of the number of independent variables from the numbers of the N_l layers (equivalent single-layer models) as well as those elements in which the number of the unknown variables remains N_l-dependent (layer-wise models) are both considered. Linear up to fourth-order expansions in the thickness direction are treated for the unknown stress and displacement variables. Sandwich beams and cross-ply as well as angle-ply composites plates have been analyzed. Simply supported as well as clamped edges have been considered. Finite-element results have been implemented and compared, where available, to analytical closed form solutions. Mostly the fundamental circular frequency has been used as a test bed to assess the whole implemented multilayered elements.

Composite structures combine light weight, high stiffness, high structural efficiency, and durability, and therefore have been used to build large portions of aerospace as well automotive and ship vehicles. Examples of multilayered anisotropic structures are sandwich constructions, composite structures made by orthotropic laminae, layered structures made of different isotropic layers (such as those employed for thermal protection), as well as intelligent structures embedding piezo layers. As far as two-dimensional modeling of multilayered flat structures is concerned (to which developments this article is devoted), there are a number of requirements that should be considered for an accurate description of their stress and strain fields.

First, anisotropic multilayered structures often possess higher transverse shear and normal flexibility than traditional isotropic one-layer ones. In fact, classical two-dimensional analyses of plates and shells based on Cauchy-Poisson-Kirchhoff [1–3] assumptions, namely, classical lamination theory (CLT) [4], are inadequate to predict the global response of thick plates; furthermore, Reissner-Mindlin [5, 6] types of theories, namely, first-order shear

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Address correspondence to Dr. Erasmo Carrera, Professor of Aerospace Structures and Aeroelasticity, Department of Aeronautics and Aerospace Engineering, Politecnico di Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy. E-mail: carrera@polito.it

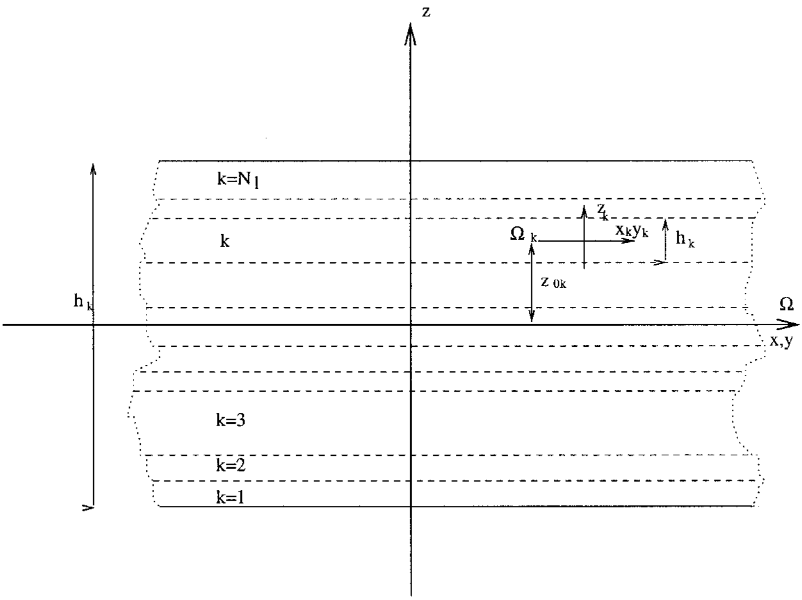

Figure 1. *Geometry and notation of layered plates.*

deformation theory (FSDT) [7], even though accounting for transverse shear deformations, can lead to very inaccurate conclusions as far as local response of thick layered structures are concerned.

Second, the intrinsic discontinuity of the thermomechanical properties in the thickness plate/shell direction puts further difficulties on two-dimensional modeling of layered structures. Exact three-dimensional solutions [8–11] have shown that displacement $\mathbf{u} = (u_x, u_y, u_z)$ (variables are measured in a tri-orthogonal Cartesian system x, y, z, which has z as through-the-thickness, normal coordinate, see Figure 1) and transverse stress $\boldsymbol{\sigma}_n = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$ distributions, due to compatibility and equilibrium reasons, respectively, are C^0-continuous functions in the thickness z direction, see also [12, 13]. Furthermore, $\mathbf{u}$ and $\boldsymbol{\sigma}_n$ have, in the most general case, discontinuous first derivatives with correspondence to each interface in which the mechanical properties change. In [12, 14] these facts were referred to as C_z^0 requirements. Literature often marks these requirements as zigzag ZZ form for $\mathbf{u}$ and interlaminar continuity (equilibrium) IC for $\boldsymbol{\sigma}_n$. ZZ and IC are strongly connected to each other by the physical behavior of a multilayered structures as in solids for compatibility and equilibrium.

Many refinements of classical models have been proposed directed to overcome the limitation of classical CLT and FSDT theories and to include partially or completely the above-mentioned C_z^0 requirements in the formulations. The first and most relevant works belong to the Russian literature. Among these one should mention the pioneering article by Lekhnitskii [15], the translated books by the same author [16] and by Ambartsumian [17], as well as the article by Vlasov [18]. The very elegant Lekhnitskii approach, originally developed for beams, which describes interlaminar continuous transverse shear stress as well as ZZ effects, was extended to plates by Ren [19]; Ambartsumian's theory, describing both IC and ZZ effects, was first extended to the unsymmetric case by Whitney [7] and then to shell geometry by Rath and Das [20] and lately reelaborated by many authors [21–24]; Vlasov [18], as well as Hildebrand et al. [25] types of theories were considered by Sun and Whitney [26], Lo et al. [27], Reddy and Phan [28], and Librescu and Khdeir [29]. Early approaches which were based on Reissner's mixed theorem [30, 31] were considered first by Murakami [32] and then by Toledano and Murakami [33]. Most of the above-mentioned works belong to the ESLM equivalent single-layer model categories. Following Reddy [13], it is understood that ESLMs preserve the independence of the number of independent variables from the numbers of the N_l layers while the number of the unknown variables remains N_l-dependent in the LWM layer-wise models. Relevant LW works in which classical models are considered at the layer level are those by Srinivas [34], Bert et al. [35], Nosier et al. [36], Robbins and Reddy [37], and Carrera [38–45]. However, a complete discussion of the several contributions which have appeared in the literature has been covered by recent exhaustive state-of-the-art articles. Among these one can mention the articles by Librescu and Reddy [46], Kapania and Raciti [47], Noor and Burton [48], Yu [49], Reddy and Robbins [50], Carrera [51], and the books by Librescu [52] and Reddy [13].

Most of the articles mentioned above deal with analytical formulations and closed-form solutions. Practical problems often involve various laminate layouts and different geometric as well as mechanical boundary conditions. The use of approximate solutions becomes mandatory in such cases. Many computational methods and techniques were developed in the second half of the past century [53–59]. Among these the finite-element method has played and continues to play a predominant role in both academic and industrial environments. Many finite elements have been proposed which were based on the approaches overviewed above. Others, based on some special finite-element techniques (such as hybrid) have also been proposed. A review follows.

Early articles concerning laminated plate elements including transverse shear effects (SDT) have been developed by Pryor and Barker [60], and Reddy [61]. Many refinements of SDT-type elements have been proposed (see the overview articles by and Pandya and Kant [62] and Barboni and Gaudenzi [63]. Dozens of finite elements have been proposed that are based on the Ambartsumian-Whitney-Rath-Das type of theory. Among these the recent works by Cho and Averill [64] and by Polit and Touratier [65] should be mentioned. Layer-wise finite elements have been discussed by Pinsky and Kim [66], Gaudenzi et al. [67], and more recently by Botello et al. [68].

The finite-element models based on ESL or LW approaches have their own advantages/disadvantages in terms of solution accuracy and/or solution economy. However, these approaches can be combined to lead to the so-called multiple-method or global/local technique: an LW description is used in those zones of the structures in which an accurate description is required, while ESL is used for the remaining parts. Examples of these approaches can be found in Reddy [13]. Similar techniques, denoted sublaminate approaches, have recently been developed in the already-mentioned Cho and Averill [64] article, in the framework of zigzag-type theories.

Hybrid stress finite elements based on a modified complementary energy statement in which equilibrating intraelement stresses, and independently, intraelement or element boundary displacements, are interpolated in terms of stress parameter and nodal diplacement, respectively, have been developed in [69–74]. Exhaustive review can be found in the articles by Pian and Mau [75], Reddy [76], and Noor et al. [77].

Application of the Reissner mixed variational theorem to develop standard finite elements was proposed by Rao and Meyer-Piening [78]. Applications were quoted for laminate and sandwich plates and were related to eight-noded plate isoparametric elements. RMVT has been extensively employed by Carrera and co-authors [79–84] for linear and nonlinear problems in the framework of an extension of the standard Reissner-Mindlin model to multilayered structures. A review of this topic can be found in the mentioned article [51].

On the basis of recent analytical developments of RMVT which were documented in [12, 38–45], Carrera and Demasi have developed [85, 86] a comprehensive finite-element modeling of multilayered plates. A number of axiomatic plate elements have been proposed and discussed in these last articles: classical models formulated on the basis of principle of virtual displacements (PVD) and mixed models based on Reissner's mixed variational theorem (RMVT); layer-wise (LW) and equivalent single-layer (ESL) models related to linear to fourth-order expansion N in the plate thickness z direction; ESL cases in which zigzag effects or interlaminar continuity can be forced or discarded; ESL cases in which transverse normal strains are considered or discarded. More than 42 models were implemented and compared for sample problems related to simply supported, cross-ply laminated plates.

The present article extends the finite element developed in [85, 86], to linear dynamic problems. Results are given for most of the implemented elements for bending and fundamental circular frequency response. Layered beams and plates related to several layouts as well as various boundary conditions are considered in the numerical investigation. The article has been organized as follows. Mechanics of the two-dimensional models as well as finite-element descriptions are briefly recalled in Section 1; description of the mixed and classical plate theories considered is given in Section 2. Governing equations and finite-element matrices are presented in Section 3. Results and discussion are provided in Section 4. Readers who are interested in details of described formulations are referred to the already-mentioned articles.

§1. Mechanics of the layered plate elements

The plate geometries and notations have been depicted in Figure 1. The integer k, which is used extensively as both subscript and superscript, denotes the layer number, which starts from the bottom layer. x and y are the plate middle surface Ω^k coordinates. Ω will be also used to denote the reference surface. Γ^k is the layer boundary on Ω^k. z and z_k are the plate and layer thickness coordinates; h and h_k denote plate and layer thickness, respectively. $\zeta_k = 2z_k/h_k$ is the nondimensioned local plate coordinate; A_k will denote the k-layer thickness domain. Symbols not affected by k subscript/superscripts refer to the whole plate.

This article develops two-dimensional models of layered plates by making assumptions in the thickness plate directions z. Displacements $\mathbf{u}$ and transverse stresses $\boldsymbol{\sigma}_n$ are the variables expanded. Such expansions are made according to the following formulas:

$$\mathbf{u} = F_t \mathbf{u}_t + F_b \mathbf{u}_b + F_r \mathbf{u}_r = F_\tau \mathbf{u}_\tau \qquad \tau = t, b, r, \quad r = 2, \ldots, N \tag{1}$$

$$\boldsymbol{\sigma}_n = F_t \boldsymbol{\sigma}_{n_t} + F_b \boldsymbol{\sigma}_{n_b} + F_r \boldsymbol{\sigma}_{n_r} = F_\tau \boldsymbol{\sigma}_{n_\tau} \qquad \tau = t, b, r, \quad r = 2, \ldots, N \tag{2}$$

Bold letters denote arrays ($\mathbf{u} = \{u_x, u_y, u_z\}$, $\boldsymbol{\sigma}_n = \{\sigma_{n_x}, \sigma_{n_y}, \sigma_{n_z}\}$, and so on; x, y, z coordinates denote a system of Cartesian coordinates; see Figure 1). F_t, F_b, and F_r are the base functions used for the z expansions; the first two polynomials are related to the linear part of such expansions, while F_r introduces the $N - 1$ higher-order terms (power of z and Legendre polynomials are used to build up F_r, as detailed in Section 2). The same meaning is assumed by the related introduced variables $\mathbf{u}_t$, $\mathbf{u}_b$, $\mathbf{u}_r$, $\boldsymbol{\sigma}_{nt}$, $\boldsymbol{\sigma}_{nb}$, $\boldsymbol{\sigma}_{nr}$.

Governing equations in weak form of the introduced 2-D models are written according to two variational statments: the principle of virtual displacements (PVD) and the Reissner mixed variation theorem (RMVT) [30, 31]. The first one, which states the static case,

$$\int_V \left(\delta\boldsymbol{\epsilon}_{pG}^T \boldsymbol{\sigma}_{pH} + \delta\boldsymbol{\epsilon}_{nG}^T \boldsymbol{\sigma}_{nH}\right) dV = \int_V \rho\delta\mathbf{u}\ddot{\mathbf{u}}\, dV + \delta L_e \tag{3}$$

is used to derive governing equations if only displacement assumptions are made. ρ_k denotes mass density, while double dots signify accelerations. The superscript T signifies an array transposition, V denotes the 3-D multilayered body volume, while the subscript p denotes in-plane components $\boldsymbol{\sigma}_p = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$, $\boldsymbol{\epsilon}_p = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}\}$. The subscript H underlines that stresses are computed via Hooke's law. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometric relations (subscript G). δL_e is the virtual variation of the work made by the external layer force $\mathbf{p}$.

The Reissner mixed variational theorem, which states,

$$\int_V \left[\delta\boldsymbol{\epsilon}_{pG}^T\boldsymbol{\sigma}_{pH} + \delta\boldsymbol{\epsilon}_{nG}^T\boldsymbol{\sigma}_{nM} + \delta\boldsymbol{\sigma}_{nM}^T(\boldsymbol{\epsilon}_{nG} - \boldsymbol{\epsilon}_{nH})\right] dV = \int_V \rho\delta\mathbf{u}\ddot{\mathbf{u}}\, dV + \delta L_e \tag{4}$$

is employed, in which both displacement and stress variables are assumed. The third "mixed" term variatianally enforces the compatibility of the transverse strain components. Subscript M underlines that transverse stresses are those of the assumed model.

As far as Hooke's law and geometric relations are concerned, reference is made to the following formulas. The material of the layers of skins and core are considered to be homogeneous and to operate in the linear elastic range. By employing stiffness coefficients, Hooke's law for the anisotropic k lamina is written in the form $\sigma_i = \tilde{C}_{ij}\varepsilon_j$, where the sub-indices i and j, ranging from 1 to 6, stand for the index couples 11, 22, 33, 13, 23, and 12, respectively. The material is assumed to be orthotropic, as specified, by $\tilde{C}_{14} = \tilde{C}_{24} = \tilde{C}_{34} = \tilde{C}_{64} = \tilde{C}_{15} = \tilde{C}_{25} = \tilde{C}_{35} = \tilde{C}_{65} = 0$. This implies that σ_{13}^k and σ_{23}^k depend only on ε_{13}^k and ε_{23}^k. In matrix form,

$$\begin{aligned} \boldsymbol{\sigma}_{pH}^k &= \tilde{\mathbf{C}}_{pp}^k\boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{pn}^k\boldsymbol{\epsilon}_{nG}^k \\ \boldsymbol{\sigma}_{nH}^k &= \tilde{\mathbf{C}}_{np}^k\boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{nn}^k\boldsymbol{\epsilon}_{nG}^k \end{aligned} \tag{5}$$

In explicit form the matrices are

$$\tilde{\mathbf{C}}_{pp}^k = \begin{bmatrix} \tilde{C}_{11}^k & \tilde{C}_{12}^k & \tilde{C}_{16}^k \\ \tilde{C}_{12}^k & \tilde{C}_{22}^k & \tilde{C}_{26}^k \\ \tilde{C}_{16}^k & \tilde{C}_{26}^k & \tilde{C}_{66}^k \end{bmatrix} \quad \tilde{\mathbf{C}}_{pn}^k = \tilde{\mathbf{C}}_{np}^{k^T} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13}^k \\ 0 & 0 & \tilde{C}_{23}^k \\ 0 & 0 & \tilde{C}_{36}^k \end{bmatrix} \quad \tilde{\mathbf{C}}_{nn}^k = \begin{bmatrix} \tilde{C}_{44}^k & \tilde{C}_{45}^k & 0 \\ \tilde{C}_{45}^k & \tilde{C}_{55}^k & 0 \\ 0 & 0 & \tilde{C}_{66}^k \end{bmatrix}$$

The above formulas are employed in the framework of PVD applications, while the application of the RMVT Eq. (4) requires transverse strains from Hooke's law. The stress–strain relationships are therefore put in the following mixed form (see also [52]):

$$\begin{aligned} \boldsymbol{\sigma}_{pH}^k &= \mathbf{C}_{pp}^k\boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{pn}^k\boldsymbol{\sigma}_{nM}^k \\ \boldsymbol{\epsilon}_{nH}^k &= \mathbf{C}_{np}^k\boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{nn}^k\boldsymbol{\sigma}_{nM}^k \end{aligned} \tag{6}$$

where both stiffness and compliance coefficients are employed. The subscript M states that the transverse stresses are those of the assumed model. The relation between the arrays of coefficients in the two forms of Hooke's law is simply found:

$$\begin{aligned} \mathbf{C}_{pp}^k &= \tilde{\mathbf{C}}_{pp}^k - \tilde{\mathbf{C}}_{pn}^k\tilde{\mathbf{C}}_{nn}^{k^{-1}}\tilde{\mathbf{C}}_{np}^k & \qquad \mathbf{C}_{pn}^k &= \tilde{\mathbf{C}}_{pn}^k\tilde{\mathbf{C}}_{nn}^{k^{-1}} \\ \mathbf{C}_{np}^k &= -\tilde{\mathbf{C}}_{nn}^{k^{-1}}\tilde{\mathbf{C}}_{np}^k & \qquad \mathbf{C}_{nn}^k &= \tilde{\mathbf{C}}_{nn}^{k^{-1}} \end{aligned}$$

Superscript -1 denotes an inversion of the array.

Within the small deformation theory, the strain components $\boldsymbol{\epsilon}_p, \boldsymbol{\epsilon}_n$ are linearly related to the displacements $\mathbf{u}$ according to the differential geometric relations. These can be formally written as

$$\boldsymbol{\epsilon}_{pG} = \mathbf{D}_p \mathbf{u} \qquad \boldsymbol{\epsilon}^k_{nG} = \mathbf{D}_n \mathbf{u} \tag{7}$$

The explicit forms of differential arrays $\mathbf{D}_p$ and $\mathbf{D}_n$ are

$$\mathbf{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_x & \partial_x & 0 \end{bmatrix} \qquad \mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix} \tag{8}$$

In-plane finite-element approximations are used to express the displacement and stress variables in terms of their nodal values, via shape functions, according to a standard isoparametric description [13, 86]:

$$\mathbf{u}^k_\tau = N_i \mathbf{q}^k_{\tau i} \qquad (i = 1, 2, \ldots, N_n) \tag{9}$$

$$\boldsymbol{\sigma}^k_{n\tau} = N_i \mathbf{g}^k_{\tau i} \qquad (i = 1, 2, \ldots, N_n) \tag{10}$$

where N_n denotes the numbers of the nodes in the element while N_i are the shape functions. $\mathbf{q}$ and $\mathbf{g}$ denote the displacement and transverse stress values at the nodes. Four-, eight-, and nine-noded finite elements were considered in [86]. These all have been extended here to dynamics, although the numerical investigation will be mostly restricted to the eight-node element, which will be denoted as $Q9$.

§2. Implemented finite elements

The thickness assumptions made at Eqs. (1)–(2) permits one to develop a large variety of two-dimensional theories. Depending on the variational statement used (PVD or RMVT), the description of the variables (LWM or ESLM), the order of the expansion used N, a number of two-dimensional theories can be constructed. Such a variety of sandwich theories fits very well with the assessment proposed in this article. In fact, these theories are able to cover a large part of the known classical and refined modelings of sandwich plates. The richest ones, such as LM4, lead to quasi three-dimensional descriptions of sandwich plates; the poorest, such as ED1, lead to results very close to Kirchhoff-type approximation theories [12–86].

In order to express different theories in a concise manner, acronyms are introduced. These acronyms have here been built as illustrated in Figure 2. Extensive use of such acronyms will be made in the subsequent sections. A few details on the assumptions related to the different plate theories follow.

2.1. *Plate elements with only displacement variables*

2.1.1. *ESLM classical elements, ED1–ED4*

The following Taylor-type expansion is used for the displacement of the whole plate. This is here written in the following unified notation:

$$\mathbf{u} = F_t \mathbf{u}_t + F_b \mathbf{u}_b + F_r \mathbf{u}_r = F_\tau \mathbf{u}_\tau \qquad \tau = t, b, t, \quad t = 2, \ldots, N-1 \tag{11}$$

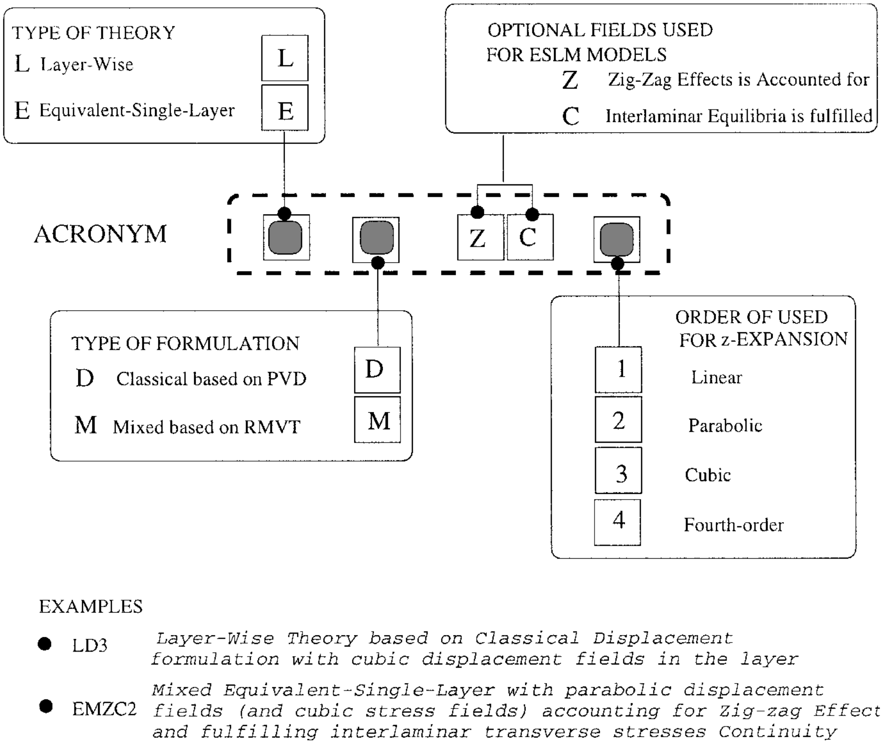

Figure 2. *Acronyms used to denote the implemented finite elements.*

in which subscript b denotes values with correspondence to Ω ($\mathbf{u}_b = \mathbf{u}_0$) while subscript t refers to the highest-order term ($\mathbf{u}_t = \mathbf{u}_N$). The F_τ polynomials assume the following explicit form:

$$F_b = 1 \qquad F_t = z^N \qquad F_r = z^r \qquad r = 2, \dots, N-1 \tag{12}$$

b and t subscripts will also signify, see below, values of the displacement and/or stress variables with correspondence to layer bottom and layer top surfaces, respectively.

2.1.2. *ESLM classical elements with zigzag functions, EDZ1–EDZ3*

It is possible to introduce zigzag effects in the previous expansion and in the PVD framework by referring to Murakami's idea, which was originally introduced in the framework of the RMVT. Murakami [32] proposed to add a zigzag-type function in a Taylor-type expansion. According to our notation, one can assume $F_t = (-1)^k \zeta_k \zeta_k = 2z_k/h_k$ is a not-dimensioned layer coordinate (z_k is the physical coordinate of the k layer, whose thickness is h_k). The exponent k changes the sign of the zigzag term in each layer. Such a trick permits one to reproduce the discontinuity of the first derivative of the diplacement variables in the z directions which comes physically from the intrinsic transverse anisotropy (TA) of multilayered structures (as depicted in Figure 3). By employing unified notation the previous model becomes

$$\mathbf{u} = F_t \mathbf{u}_t + F_b \mathbf{u}_b + F_r \mathbf{u}_r = F_\tau \mathbf{u}_\tau \qquad \tau = t, b, r, \quad r = 2, \dots, N \tag{13}$$

Subscript t has been chosen to denote the zigzag term [$\mathbf{u}_t = \mathbf{u}_Z$, $F_t = (-1)^k \zeta_k$].

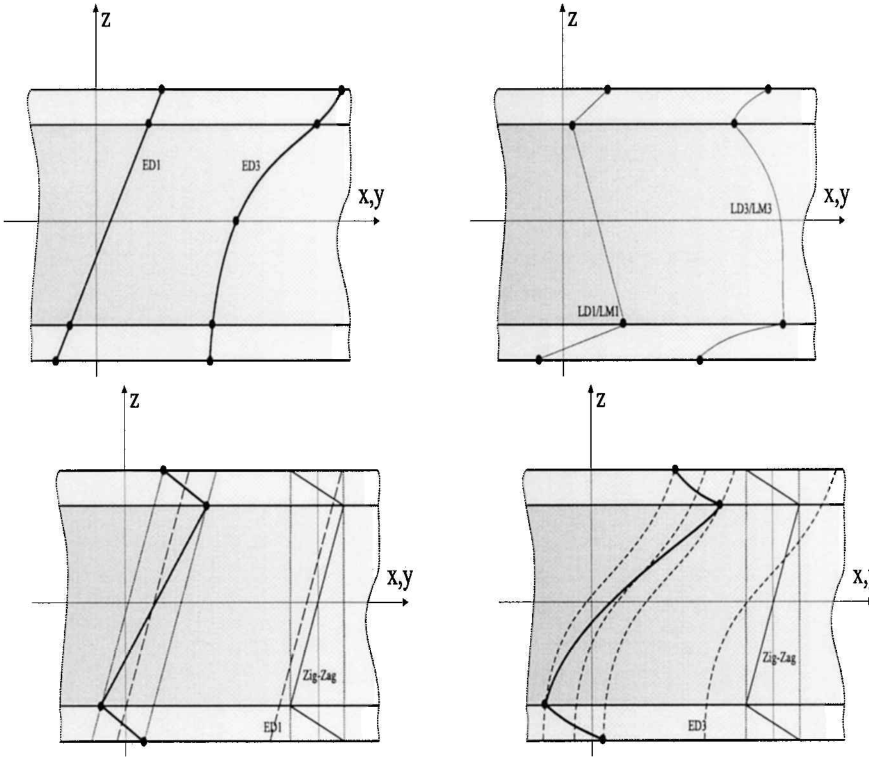

Figure 3. *Examples of ESL and LW assumptions in a three-layered plate. Linear and cubic cases of zigzag functions are drawn in the lower part.*

2.1.3. *Layer-wise cases, LD1–LD4*

A layer-wise description is simply obtained by assuming the displacement expansion of the previous section in each layer. Nevertheless, Taylor-type expansion is not convenient for a layer-wise description. In fact, the fulfillment of continuity requirements for the displacement at interfaces could be easily introduced by using the interface variables as unknowns. A convenient combination of Legendre polynomials [12] could be used as base functions:

$$\mathbf{u}^k = F_t\mathbf{u}_t^k + F_b\mathbf{u}_b^k + F_r\mathbf{u}_r^k = F_\tau\mathbf{u}_\tau^k \qquad \tau = t, b, r, \quad r = 2, 3, \dots, N, \quad k = 1, 2, \dots, N_l \tag{14}$$

It is now intended that the subscripts t and b denote values related to the layer top and bottom surface, respectively. These two terms consist of the linear part of the expansion. The thickness functions $F_\tau(\zeta_k)$ have now been defined at the k-layer level,

$$F_t = \frac{P_0 + P_1}{2} \qquad F_b = \frac{P_0 - P_1}{2} \qquad F_r = P_r - P_{r-2} \qquad r = 2, 3, \dots, N \tag{15}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of j order defined in the ζ_k domain: $-1 \le \zeta_k \le 1$. The continuity of the displacement at each interface is easily linked:

$$\mathbf{u}_t^k = \mathbf{u}_b^{(k+1)} \qquad k = 1, N_l - 1 \tag{16}$$

Examples of LW descriptions are given in Figure 3.

2.2. *Mixed elements with displacement and transverse stress variables*

The RMVT consists of a variational tool designed for multilayered structures. Appropriate applications of the RMVT demand displacement fields which describe the zigzag effect and transverse stresses which are continuous at the interfaces.

2.2.1. *ESLM cases, EMZC1–EMZC3*

Taylor-type expansion is not appropriate for ESL description of transverse stresses. Its use would require additional constraints in order to fulfill transverse shear and normal stress continuity. The use of the RMVT, further, demands layer-wise description of transverse stresses even though ESLM expansions are used for displacements (it is intended that in the presented derivations the ESLM description is related only to the displacement fields in RMVT applications). Transverse stresses are assumed independent in each layer. The layer-wise description already used for displacements is extended to transverse stresses:

$$\boldsymbol{\sigma}^k_{nM} = F_t \boldsymbol{\sigma}^k_{nt} + F_b \boldsymbol{\sigma}^k_{nb} + F_r \boldsymbol{\sigma}^k_{nr} = F_\tau \boldsymbol{\sigma}^k_{n\tau}$$
$$\tau = t, b, r, \quad r = 2, 3, \ldots, N, \quad k = 1, 2, \ldots, N_l \tag{17}$$

The interlaminar transverse shear and normal stress continuity can therefore be linked by simply writing:

$$\boldsymbol{\sigma}^k_{nt} = \boldsymbol{\sigma}^{(k+1)}_{nb} \qquad k = 1, N_l - 1 \tag{18}$$

In those cases in which the top/bottom plate stress values are prescribed (zero or imposed values), the following additional equilibrium conditions must be accounted for:

$$\boldsymbol{\sigma}^1_{nb} = \bar{\boldsymbol{\sigma}}_{nb} \qquad \boldsymbol{\sigma}^{N_l}_{nt} = \bar{\boldsymbol{\sigma}}_{nt} \tag{19}$$

where the overbar denotes the imposed values in correspondence to the plate boundary surfaces.

2.2.2. *LW cases, LM1–LM4*

Full layer-wise description can be introduced by simply extending the stress assumptions of the previous paragraph to displacement variables:

$$\begin{aligned} \mathbf{u}^k &= F_t \mathbf{u}^k_t + F_b \mathbf{u}^k_b + F_r \mathbf{u}^k_r = F_\tau \mathbf{u}^k_\tau & \tau &= t, b, r \\ & & r &= 2, 3, \ldots, N \\ \boldsymbol{\sigma}^k_{nM} &= F_t \boldsymbol{\sigma}^k_{nt} + F_b \boldsymbol{\sigma}^k_{nb} + F_r \boldsymbol{\sigma}^k_{nr} = F_\tau \boldsymbol{\sigma}^k_{n\tau} & k &= 1, 2, \ldots, N_l \end{aligned} \tag{20}$$

It is to be noticed that the LW description does not require any zigzag function for the simulation of zigzag effects.

§3. Governing FEM equations

The assumed displacement field is first introduced in the expression for the strains; then, finite-element approximations are used to express the diplacement in terms of their nodal values, via shape functions. The finite-element equations in the case of PVD applications can be written as

$$\delta \mathbf{q}^{kT}_{\tau i}: \qquad \mathbf{K}^{k\tau s i j} \mathbf{q}^k_{sj} = \mathbf{M}^{k\tau s i j} \ddot{\mathbf{q}}^k_{sj} + \mathbf{P}^k_{\tau i} \tag{21}$$

where the following finite-element stiffness matrix,

$$\begin{aligned}\mathbf{K}^{k\tau sij} &= \lhd \mathbf{D}_p^T(N_i\mathbf{I})\big[\tilde{\mathbf{Z}}_{pp}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s,_z}N_j\big] \\ &\quad + \mathbf{D}_{n\Omega}^T(N_i\mathbf{I})\big[\tilde{\mathbf{Z}}_{np}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau s,_z}N_j\big] \\ &\quad + N_i\big[\tilde{\mathbf{Z}}_{np}^{k\tau,_z s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau,_z s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau,_z s,_z}N_j\big]\rhd_\Omega \end{aligned} \tag{22}$$

and mass matrix,

$$\mathbf{M}^{k\tau s} = \lhd \rho^k F_\tau F_s (N_i\mathbf{I})(N_j\mathbf{I})\rhd_\Omega \tag{23}$$

have been introduced.

It is to be noted once again that subscripts τ and i have been used for the finite values of unknown variables, while subscripts s and j have been introduced for their variations. The symbols $\lhd \cdots \rhd_\Omega$ have been introduced to denote integrals on Ω.

It is noted that the matrix $\mathbf{K}^{k\tau sij}$ is made by triplicate products of 3×3 arrays, so that $\mathbf{K}^{k\tau sij}$ is itself a 3×3 array. Such an array consists of the fundamental nucleus of finite-element matrices related to PVD applications [86]. By varying N and N_n, the finite-element matrices of the k layer, corresponding to the implemented two-dimensional theories and number of nodes, are obtained.

For RMVT applications, transverse stress variables are expressed in terms of shape functions, as is done for the displacement ones. By imposing the definition of virtual variations, the RMVT leads to the following equilibrium and compatibility equations:

$$\begin{aligned} \delta\mathbf{q}_{\tau i}^{kT}: &\qquad \mathbf{K}_{uu}^{k\tau sij}\mathbf{q}_{sj}^k + \mathbf{K}_{u\sigma}^{k\tau sij}\mathbf{g}_{sj}^k = \mathbf{M}^{k\tau sij}\ddot{\mathbf{q}}_{sj}^k + \mathbf{P}_{\tau i}^k \\ \delta\mathbf{g}_{\tau i}^{kT}: &\qquad \mathbf{K}_{\sigma u}^{k\tau sij}\mathbf{q}_{sj}^k + \mathbf{K}_{\sigma\sigma}^{k\tau sij}\mathbf{g}_{sj}^k = \mathbf{0} \end{aligned} \tag{24}$$

Four additional 3×3 fundamental nuclei have been obtained in this case, where

$$\begin{aligned} \mathbf{K}_{uu}^{k\tau sij} &= \lhd\big[\mathbf{D}_p^T(N_i\mathbf{I})\mathbf{Z}_{pp}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I})\big]\rhd_\Omega \\ \mathbf{K}_{u\sigma}^{k\tau sij} &= \lhd\big[\mathbf{D}_p^T(N_i\mathbf{I})\mathbf{Z}_{pn}^{k\tau s}N_j + \mathbf{D}_{n\Omega}^T(N_i\mathbf{I})E_{\tau s}N_j + E_{\tau,_z s}N_iN_j\mathbf{I}\big]\rhd_\Omega \\ \mathbf{K}_{\sigma u}^{k\tau sij} &= \lhd\big[N_iE_{\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + E_{\tau s,_z}N_iN_j\mathbf{I} - N_i\mathbf{Z}_{np}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I})\big]\rhd_\Omega \\ \mathbf{K}_{\sigma\sigma}^{k\tau sij} &= \lhd\big[-N_i\mathbf{Z}_{nn}^{k\tau s}N_j\big]\rhd_\Omega \end{aligned} \tag{25}$$

§4. Results and discussion

An extensive investigation has been conducted in order to evaluate the implemented multilayered plate elements. Selected results are presented in this section. Bending response analysis has been restricted to sandwich beams. Plate geometry cases have already been treated in previous authors' works. Both beam and plate geometries have been addresses in the dynamic cases. Fundamental frequency parameters have been used as test beds to compare the whole implemented elements. Wherever available, analytical solutions that were given in [43] have been compared. $Q9$, a nine-noded plate element, has been used in the analysis.

4.1. *Data on the treated problems*

Beam and plate problems have been treated in the numerical investigations.

4.1.1. *Sandwich beam*

Top and bottom skins have the same geometric and mechanical data. The skin thickness is $h_s = 0.1\,h$. The core thickness is $h_c = 0.8\,h$. The skin material consists of unidirectional lamina whose orientation coincides with the longitudinal beam axis. The mechanical properties of the laminae which are used as skins are

$$E_L = 25 \times 10^6 \text{ psi} \qquad E_T = 1 \times 10^6 \text{ psi} \qquad G_{LT} = 0.5 \times 10^6 \text{ psi}$$
$$G_{TT} = 0.2 \times 10^6 \text{ psi} \qquad \nu_{LT} = \nu_{TT} = 0.25$$

where, following usual notations, L signifies the fiber direction, T the transverse direction, and ν_{LT} is the major Poisson ratio; see [13] as example. The core material used for the sandwich plates is transversely isotropic with respect the z axis and is characterized by the following elastic properties:

$$E_{xx} = E_{yy} = 0.04 \times 10^6 \text{ psi} \qquad E_{zz} = 0.5 \times 10^6 \text{ psi} \qquad G_{xz} = G_{yz} = 6.0 \times 10^4 \text{ psi}$$
$$G_{xy} = 1.6 \times 10^4 \text{ psi} \qquad \nu_{xy} = \nu_{zy} = \nu_{zx} = 0.25$$

The principal material directions of the core always coincide with the geometric axes of the beam.

Four boundary conditions are considered for the sandwich beams. Clamped and simply supported edges are combined to lead to the four cases depicted in Figure 4. Transverse distribution of constant pressure has been applied with correspondence to the top skin surface. Stresses and displacement locations given in the subsequent tables change by changing the boundary conditions. Locations are given below, along with the introduced nondimensionalization.

1. Clamped—Clamped edges, CC

$$\bar{w} = u_z(L/2, 0)\frac{E_T h^3}{pL^4} \qquad \sigma_{zz} = \sigma_{zz}(,) \quad \sigma_{yz} = \sigma_{yz}(L/4,)$$

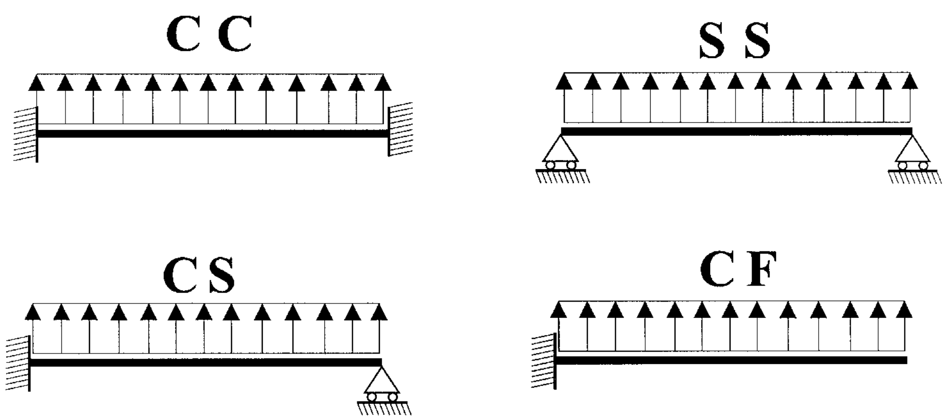

Figure 4. *Sketch of the boundary conditions considered for beams.*

p is the value of tranverse pressure which is uniformly applied with correspondence to the top surface of the beam.

2. Simply supported—Simply supported edges, SS

$$\sigma_{zz} = \sigma_{zz}(,) \qquad \sigma_{xz} = \sigma_{yz}(L/4, z)$$

3. Simply supported—Clamped, SC

$$\sigma_{zz} = \sigma_{zz}(,) \qquad \sigma_{xz} = \sigma_{yz}(L/2, z)$$

4. Cantilever (clamped free), CF

$$\sigma_{zz} = \sigma_{zz}(,) \qquad \sigma_{xz} = \sigma_{yz}(L/2, z)$$

4.1.2. *Plate problems*

1. Isotropic plate. This consists of a simply supported plate made of aluminum alloy having the following mechanics and geometry:

$$E = 73G \text{ Pa} \qquad \nu = 0.34 \qquad a = b = 0.5 \text{ m} \qquad h = 0.002 \text{ m}$$

2. Cross-ply skew-symmetric and symmetric simply supported plates, SSSS. The two following mechanical properties have been considered for the lamina.

$$\text{MAT1:} \quad \frac{E_L}{E_T} = 40 \qquad \frac{G_{LT}}{E_T} = \frac{G_{Lz}}{E_T} = 0.50 \qquad \frac{G_{TT}}{E_T} = 0.60$$
$$\nu_{LT} = \nu_{Lz} = \nu_{TT} = 0.25$$

$$\text{MAT2:} \quad \frac{E_L}{E_T} = 30 \qquad \frac{G_{LT}}{E_T} = \frac{G_{Lz}}{E_T} = 0.50 \qquad \frac{G_{TT}}{E_T} = 0.35$$
$$\nu_{LT} = \nu_{Lz} = 0.3 \qquad \nu_{TT} = 0.49$$

3. Angle-ply simply supported plates, SSSS.
4. Cross-ply skew-symmetric, simply supported clamped plates, CCSS. Two opposites edges are simply supported, while the two others are clamped.

4.2. *Results on bending*

Convergence rates on transverse displacements are given in Figures 5 and 6. $a/h = 20$ thickness ratio case have been considered for CC and SS sandwich beams, respectively. Two significant plate elements are compared. *NE* is the number of the element in the longitudinal beam direction. Higher convergence rate is shown by these analyses. A comprehensive comparison of different theories on bending sandwich plates is given in Tables 1 and 2. Very thick $a/h = 2$, thick $a/h = 4$, moderately thick $a/h = 10$, moderately thin $a/h = 20$, and thin $a/h = 100$ cases are considered. Evaluations of transverse displacements and trasnverse shear and normal stresses are given. Eight significant theories have been compared in Table 1: highest- and lowest-order theories related to mixed M and classical D formulation as well as equivalent single-layer E and layer-wise L variable description. LM4 and LM1 results located at sandwich interfaces have been given for transverse shear

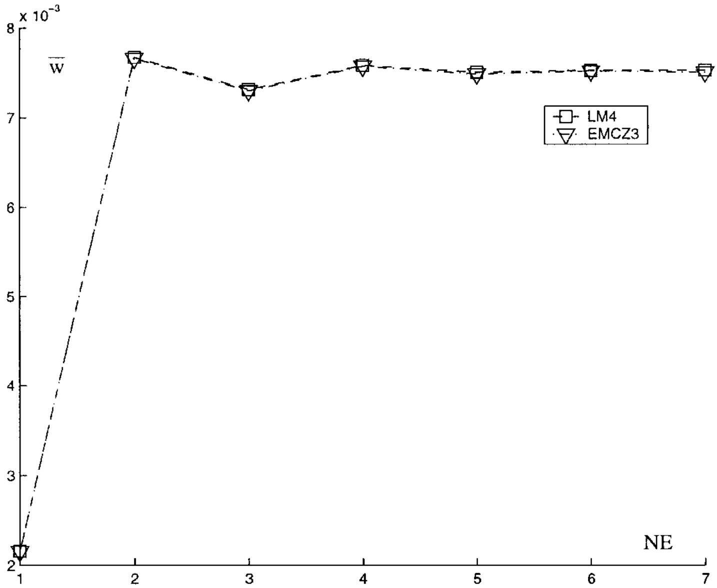

Figure 5. *Convergence rate in a CC beam.*

and normal stresses. 3-D solutions are not available for the beam problems considered. Previous authors' works have concluded that LM4 results give a quasi-3-D description of stress states in laminated structures. LM4 should be therefore seen as a reference solution for the whole presented analyses.

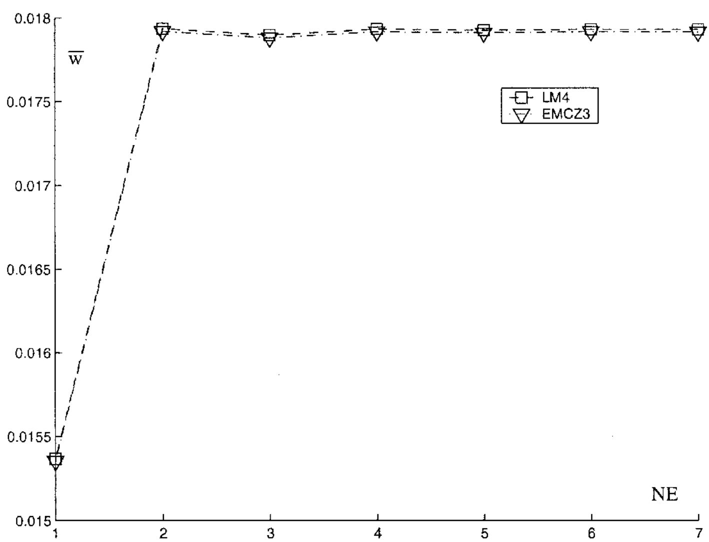

Figure 6. *Convergence rate in a SS beam.*

Table 1
Transverse displacement $\bar{w}$ of sandwich beams: comparison of different elements

a/h	2	4	10	20	100
			CC case		
LM4	.33943	.10655	.021702	.007505	.027531
LM1	.34689	.10723	.021658	.0075025	.027534
LD4	.33889	.10639	.021696	.0075038	.027530
LD1	.33693	.10594	.021564	.0074813	.027524
EMCZ3	.33479	.10603	.021589	.0074856	.027493
EMCZ1	.34101	.10655	.021598	.0074844	.027464
ED2	.32034	.099008	.019950	.0070331	.027313
ED1	.21360	.055301	.010985	.046548	.026291
			SS case		
LM4	.46616	.13736	.033282	.017926	.012988
LM1	.46826	.13739	.033279	.017926	.012988
LD4	.46616	.13736	.033281	.017926	.012988
LD1	.46616	.13730	.033235	.017914	.012987
EMZC3	.49625	.13907	.033282	.017911	.012977
EMZC1	.46978	.13742	.033231	.017891	.012958
ED2	.51103	.13173	.031448	.017430	.012965
ED1	.22355	.065441	.021180	.014858	.012834
			CS case		
LM4	.39847	.12301	.027178	.010962	.00053519
LM1	.40363	.12343	.027144	.010957	.00053519
LD4	.39814	.13736	.027175	.010962	.00053519
LD1	.39786	.12292	.02706	.010937	.00053512
EMZC3	.40816	.12338	.027094	.01094	.00053460
EMZC1	.40146	.12302	.033231	.010934	.00053393
ED2	.40545	.11603	.025232	.010409	.00053254
ED1	.22234	.062125	.014844	.0075813	.00051954
			CF case		
LM4	1.2291	.38096	.10242	.058549	.044069
LM1	1.2348	.3818	.1023	.058527	.044068
LD4	1.2279	.38065	.10241	.058549	.044069
LD1	1.2157	.37897	.10207	.058479	.044066
EMZC3	1.2168	.37946	.27094	.058448	.044014
EMZC1	1.2244	.38015	.33231	.058397	.043959
ED2	1.1465	.35333	.9687	.057068	.043976
ED1	.67713	.20177	.68683	.049671	.043588

The following comments can be made on Tables 1–3 results.

1. Different plate elements merge in the case of thin beams.
2. Differences among different plate elements decrease as a/h is increased.
3. Layer-wise analyses lead to better description with respect to the corresponding equivalent single-layer one.

Table 2
Transverse normal stress σ_{zz} in sandwich beams, located at the skin interfaces: comparison of different elements

a/h	2	4	10	20	100
			CC case		
LM4 ($z = +.4$)	.14756	.066613	.073793	.18050	.31993
LM4 ($z = -.4$)	.86633	.93090	.94855	.83843	.67604
LM1 ($z = +.4$)	.17203	.061091	.90182	.18954	.35177
LM1 ($z = -.4$)	.87582	.91558	.92411	.86011	.75170
			SS case		
LM4 ($z = +.4$)	.12717	.064350	.063761	.12489	.091646
LM4 ($z = -.4$)	.89370	.93336	.94288	.87934	.87636
LM1 ($z = +.4$)	.14298	.055745	.077640	.14141	.13332
LM1 ($z = -.4$)	.89659	.91819	.92491	.88701	.89275
			CS case		
LM4 ($z = +.4$)	.10052	.061534	.054746	.057180	.058832
LM4 ($z = -.4$)	.92762	.93643	.93005	.93191	.94144
LM1 ($z = +.4$)	.10517	.049081	.060053	.065658	.054700
LM1 ($z = -.4$)	.92170	.92140	.92565	.92888	.93199
			CF case		
LM4 ($z = +.4$)	.15807	.066709	.062600	.16771	.25055
LM4 ($z = -.4$)	.84737	.93081	.97095	.88514	.88873
LM1 ($z = +.4$)	.18643	.062198	.10008	.20166	.20757
LM1 ($z = -.4$)	.86272	.91514	.92034	.84743	.86923

4. Mixed finite plate elements formulated on the basis of the RMVT give better results than corresponding ones which are based on PVD.
5. The RMVT is very effective in the ESL cases: the differences between EMZC3-ED3 finite elements as well as EMZC1-ED1 results are more relevant than corresponding LM4-LD4 or LD1-LM1 ones.

Further analyses and comments as well as results on stress evaluations can be found in [86].

4.3. *Results on vibrations*

Most of the investigations conducted concerned free vibrations of beams and plates. Results are given in Tables 4–11.

Table 4 concerns a classical thin, isotropic plate problem for which the closed-form solution for simply supported boundary conditions is given by the following well-known formula:

$$f = \sqrt{\frac{D}{\rho t}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]} \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{26}$$

where $f = 2\pi/\omega$ is the frequency, which is herein measured in hertz. m and n are the number of waves in the x and y plate directions, respectively. Values computed by this formula have been used to give a preliminary assessment of the finite-element models to analyze free

Table 3
Tranverse shear stress σ_{xz} in sandwich beams, located at the skins interfaces: comparison of different elements

a/h		CC	CS	SS	CF
2	LM4 ($z = +.4$)	−.34333	.066486	−.41653	1.0534
	LM4 ($z = -.4$)	−.34009	.067882	−.40286	1.0502
	LM1 ($z = +.4$)	−.33142	.061563	−.40344	1.0458
	LM1 ($z = -.4$)	−.34420	.063661	−.40863	1.0451
4	LM4 ($z = +.4$)	−.84407	.20410	−.87527	2.1981
	LM4 ($z = -.4$)	−.84295	.20427	−.87403	2.1980
	LM1 ($z = +.4$)	−.83606	.20083	−.87528	2.1992
	LM1 ($z = -.4$)	−.83571	.20084	−.87459	2.1992
10	LM4 ($z = +.4$)	−2.1461	.94469	−2.2041	5.5031
	LM4 ($z = -.4$)	−2.1504	.94477	−2.2061	5.5051
	LM1 ($z = +.4$)	−2.1495	.94808	−2.2062	5.5165
	LM1 ($z = -.4$)	−2.1557	.94744	−2.2126	5.5181
20	LM4 ($z = +.4$)	−4.3142	2.4133	−4.4160	10.956
	LM4 ($z = -.4$)	−4.3150	2.4142	−4.4149	10.957
	LM1 ($z = +.4$)	−4.3186	2.4187	−4.4228	10.980
	LM1 ($z = -.4$)	−4.3210	2.4186	−4.4243	10.982
100	LM4 ($z = +.4$)	−22.054	13.649	−22.094	55.083
	LM4 ($z = -.4$)	−22.054	13.650	−22.094	55.083
	LM1 ($z = +.4$)	−22.093	13.668	−22.136	55.176
	LM1 ($z = -.4$)	−22.093	13.668	−22.136	55.176

response of plates. Acceptable agreement has been found between the FE and analytical results.

Table 5 addresses the four beam problems already treated in Tables 1–3. The comments already made on bending response should be confirmed.

A more comprehensive assessment of different plate theories to trace vibrational response of laminated plates has been given in Tables 6–9. Analytical solutions given by

Table 4
First four frequencies f *Eq. (26) in anisotropic simply supported plate: comparison between analytical (thin-plate theories) and finite-element solutions (ED4, mesh 5 × 5, nine-node element* Q9*)*

m	n	Anal.	FEM
1	1	39.3	39.0
2	1	98.2	98.5
1	2	98.2	98.5
2	2	157.2	158.2

Table 5

First frequency ω of sandwich beams: comparison of different elements

a/h		CC	CS	SS	CF
2	LM4	.4772	.4382	.2013	.2078
	LM1	.4742	.4381	.2276	.2079
	LD4	.4772	.4384	.2015	.2079
	LD1	.4822	.4417	.2309	.2092
	EMZC3	.4800	.3798	.1302	.2087
	EMZC1	.4791	.4404	.3616	.2088
	ED4	.4813	.4312	.2108	.2157
	ED1	.5223	.5041	.4249	.2404
4	LM4	.2139	.1984	.1568	.09292
	LM1	.2136	.1983	.1680	.09290
	LD4	.2141	.1985	.1568	.09295
	LD1	.2150	.1991	.1705	.09315
	EMZC3	.2146	.1982	.1737	.09308
	EMZC1	.2144	.1988	.1879	.09306
	ED4	.2216	.1838	.1916	.09644
	ED1	.2535	.2327	.2143	.09748
10	LM4	.07587	.06768	.04252	.02659
	LM1	.07591	.06771	.05343	.02659
	LD4	.07587	.06768	.04255	.02659
	LD1	.07607	.06781	.05344	.02661
	EMZC3	.07602	.06776	.04892	.02661
	EMZC1	.07601	.06778	.05380	.02662
	ED4	.07907	.06554	.04863	.02717
	ED1	.08563	.07025	.05383	.03082
100	LM4	.001516	.001023	.0006444	.0003535
	LM1	.001516	.001023	.0006446	.0003535
	LD4	.001516	.001023	.0006444	.0003535
	LD1	.001516	.001023	.0006446	.0003535
	EMZC3	.001517	.001023	.0006441	.0003537
	EMZC1	.001517	.001023	.0006446	.0003539
	ED4	.001517	.001023	.0006438	.0003537
	ED1	.001518	.001024	.0006452	.0003547

Carrera [43] are also cited. The agreement between FE and analytical results demonstrates the effectiveness of the dynamic finite-element formulation presented here. Different thickness ratios and lamination schemes, as well as orthotropic ratios, have been considered.

About 20 different theories are compared. These start from the best one, which coincides with LM4, and finishes with $ED1^{\infty}$, which coincides with plate elements formulated on the basis of the classical lamination theory, CLT (apexes denote the values of the shear correction factor χ; a penalty technique on χ is used to derive CLT results from ED1 ones).

Table 6

Circular frequency parameter $\omega h\sqrt{\rho/E_T}$ of simply supported square plate: cross-ply skew-symmetric and symmetric laminates (mesh 5 × 5, a/h = 5, MAT2)

N_l	2				3			
E_L/E_T	3		30		3		30	
	Anal.	FEM	Anal.	FEM	Anal.	FEM	Anal.	FEM
LM4	.2392	.2393	.3117	.3118	.2516	.2526	.3739	.3747
LM3	.2392	.2393	.3115	.3118	.2516	.2526	.3739	.3747
LM2	.2392	.2396	.3115	.3114	.2516	.2526	.3738	.3756
LM1	.2312	.2419	.2354	.3135	.2466	.2521	.3354	.3728
LD4	.2392	.2393	.3117	.3118	.2516	.2526	.3739	.3747
LD3	.2392	.2393	.3117	.3118	.2516	.2526	.3739	.3747
LD2	.2395	.2396	.3168	.3169	.2517	.2527	.3763	.3770
LD1	.2478	.2479	.3210	.3211	.2556	.2566	.3808	.3815
EMZC3	.2392	.2206	.3144	.2977	.2517	.2415	.3758	.3737
EMZC2	.2408	.2230	.3133	.2922	.2523	.2423	.3782	.3767
EMZC1	.2436	.2323	.3131	.3104	.2701	.2628	.3789	.3811
EDZ3	.2392	.2213	.3156	.2995	.2517	.2415	.3761	.3739
EDZ2	.2418	.2240	.3180	.3021	.2527	.2426	.3803	.3779
EDZ1	.2478	.2348	.3210	.3138	.2717	.2634	.3842	.3831
ED4	.2394	.2244	.3133	.3020	.2518	.2409	.3764	.3719
ED3	.2394	.2254	.3167	.3080	.2519	.2418	.3766	.3745
ED2	.2418	.2277	.3198	.3107	.2569	.2467	.4031	.3987
ED1	.2662	.2551	.3367	.3322	.2778	.2696	.4082	.4070
ED1$^{\chi=\infty}$	.2972	.2999	.4066	.4117	.3157	.3247	.6519	.4446

Table 7

Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ of simply supported square plates: cross-ply skew-symmetric laminates 0/9 (MAT1)

a/h	2		4		10	
(3 × 3)	Anal.	FEM	Anal.	FEM	Anal.	FEM
LM4	4.703	4.713	7.345	7.363	10.088	10.107
LM3	4.680	4.716	7.332	7.364	10.087	10.107
LM2	4.668	4.781	7.329	7.468	10.087	10.160
LM1	4.136	4.446	5.660	7.443	6.666	10.147
LD4	4.707	4.715	7.345	7.363	10.088	10.107
LD3	4.710	4.717	7.346	7.364	10.088	10.107
LD2	4.803	4.811	7.519	7.537	10.178	10.197
LD1	4.848	4.446	7.562	7.537	10.215	10.235
EMZC3	4.685	4.576	7.444	7.581	10.144	10.014
EMZC2	4.727	4.454	7.395	6.875	10.119	10.037
EMZC1	4.672	4.446	7.340	6.789	10.106	10.085
EDZ3	4.780	4.780	7.490	7.343	10.165	10.028
EDZ2	4.838	4.790	7.545	6.999	10.189	10.059
EDZ1	4.848	4.446	7.562	7.048	10.215	10.121
ED4	4.745	4.176	7.425	7.113	10.132	10.014
ED3	4.883	4.489	7.647	7.421	10.235	10.128
ED2	4.968	4.542	7.701	7.465	10.254	10.152
ED1	5.544	4.446	8.314	8.259	10.545	10.463
ED1$^{\chi=\infty}$	8.576	8.869	10.388	8.893	11.115	11.060

Table 8
Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ *of simply supported square plate: cross-ply skew-symmetric 0/90 (MAT1)*

a/h (3 × 3)	20		100	
	Anal.	FEM	Anal.	FEM
LM4	10.859	10.883	11.151	11.178
LM3	10.859	10.883	11.151	11.178
LM2	10.859	10.900	11.151	11.178
LM1	6.874	10.896	6.947	11.178
LD4	10.859	10.883	11.152	11.178
LD3	10.859	10.883	10.152	11.178
LD2	10.888	10.912	11.153	11.179
LD1	10.921	10.945	11.184	11.210
EMZC3	10.877	10.801	11.152	11.091
EMZC2	10.896	10.810	11.152	11.092
EMZC1	10.878	10.833	11.152	11.115
EDZ3	10.884	10.804	11.152	11.091
EDZ2	11.891	10.815	11.153	11.092
EDZ1	10.921	10.852	11.184	11.127
ED4	10.874	10.797	11.154	11.092
ED3	10.906	10.831	11.154	11.094
ED2	10.911	10.840	11.154	11.095
ED1	11.072	11.006	11.261	11.205
$ED1^{\chi=\infty}$	11.230	11.175	11.267	11.212

Seventeen theories are included in between. These results give a quite exhaustive assessment of two-dimensional refined finite plate elements for the dynamic analysis of layered plates. The comments at points 1–5 made for bending analysis are confirmed for free vibrational analysis of laminated plates. Further comments are listed below.

1. N_l increasing results in L theories becoming independent by the order N used, or by M (RMVT) or D (PVD) cases. This is caused by the intrinsic increment of the number of degrees of freedom.
2. The order of the expansion N used plays a very important role, especially as far as unsymmetric laminates are concerned. Note that the quadratic expansions are much more effective for unsymmetric laminates. N increasing makes the differences between LM and LD disappear.
3. The accuracy of classical ED type results decreases with N_l increasing.
4. The zigzag function improves the related results very much: EDZ analyses are more accurate than ED ones. Advantages come also by imposing interlaminar continuity, i.e., EM results are more accurate than ED ones.

Further results for which analytical closed-form solutions are either not available or difficult to find are given in Tables 10 and 11. These concern an angle-ply simply supported plate and a cross-ply, simply supported, clamped plate, respectively. The analyses conducted confirm the previous comments. Tables 10 and 11 could be used to

Table 9

Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ *of simply supported square plate: cross-ply symmetric laminates 0/90/90/0 (MAT1)*

a/h	2		4		10		20		100	
(3 × 3)	Anal.	FEM	Anal.	FEM	Anal.	FEM	Anal.	FEM	Anal.	FEM
LM4	5.260	5.2985	9.224	9.178	15.148	15.031	17.626	17.593	18.753	18.793
LD4	5.260	5.2985	9.224	9.178	15.148	15.031	17.626	17.593	18.753	18.793
EMZC3	5.370	5.1249	9.371	9.137	15.224	14.987	17.655	17.535	18.754	18.725
EDZ3	5.390	5.133	9.388	9.1448	15.232	14.990	17.655	17.535	18.754	18.725
ED4	5.380	4.8599	9.384	9.0183	15.232	14.974	17.655	17.535	18.754	18.725
ED1	5.927	5.9097	9.960	9.9385	15.573	15.532	17.829	17.797	18.833	18.808
ED1$^{\chi=\infty}$	15.892	15.804	17.977	18.6403	18.725	18.697	18.840	18.815	18.877	18.853

Table 10

Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ *of simply supported square plate: angle ply +45/−45 (MAT1)*

a/h (3 × 3)	2 FEM	4 FEM	10 FEM	20 FEM	100 FEM
LM4	5.587	9.308	14.408	16.811	18.258
LD4	5.589	9.312	14.410	16.812	18.258
EMZC3	5.360	8.252	12.494	15.406	18.043
EDZ3	5.529	8.415	12.568	15.514	18.060
ED4	4.493	8.211	13.582	16.396	18.143
ED1	5.992	9.977	15.057	17.187	18.272
$ED1^{\chi=\infty}$	6.381	17.118	18.137	18.295	18.347

assess future plate elements in those cases for which an analytical closed-form solution is not available.

§5. Concluding remarks

This article has presented an assessment of plate elements for bending and vibration analyses of multilayered plate structures. Classical and mixed formulations have been implemented in the framework of both layer-wise and equivalent single-layer theories. Results have been given for several benchmark problems which were able to highlight in a critical sense the performance of the plate elements considered. Most of the conclusions already known from static analyses have been confirmed for the finite-element vibration analyses presented here.

The authors believe that the numerical assessment given here could serve as a tool to assess new contributions to multilayered finite plate modelings.

Table 11

Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ *of clamped, clamped-simply supported, simply supported square plate: cross-ply skew-symmetric 0/90 (MAT1)*

a/h (3 × 3)	2 FEM	4 FEM	10 FEM	20 FEM	100 FEM
LM4	5.3805	9.0357	14.647	17.420	18.853
LD4	5.3832	9.038	14.647	17.420	18.853
EMZC3	4.9323	8.9664	14.805	17.513	18.860
EDZ3	4.9691	9.0031	14.837	17.529	18.861
ED4	5.1640	8.9869	14.716	17.465	18.860
ED1	5.9374	8.8928	15.675	17.970	18.997
$ED1^{\chi=\infty}$	13.4164	17.3693	18.766	18.990	19.063

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