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An assessment of mixed and classical theories on global and local response of multilayered orthotropic plates

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Abstract

This article assesses two-dimensional theories to evaluate global and local response of orthotropic, multilayered plates. The introductory discussion outlines five, relevant key-points that should be to be addressed in the modeling of multilayered structures. Classical theories formulated on the basis of Principle of Virtual Displacements (PVD) and mixed theories based on the Reissner Mixed Variational Theorem (RMVT) are presented. Theories which preserve the independence of the number of the independent variables from the numbers of the N_l-layers (ESLM, Equivalent Single Layer Models) and theories in which the number of the unknown variables remains N_l-dependent (LWM, Layer-Wise Models) are both considered. Modelings related to linear up-to-fourth order variations of the unknown variables in the thickness direction are treated. Sub-cases have been implemented which permits one to evaluate explicitly the effect of the so-called zig-zag (ZZ) effect, interlaminar continuity (IC) and transverse normal stresses σ_{zz}. As a result more than 40 theories have been implemented and compared for both local (displacement and stress distribution in the thickness plate direction) and global (mainly free vibration) response. A number of conclusions have been outlined in the numerical part. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Mixed formulation; Multilayered plates; 2D theories; C_z^0-requirements

1. Introduction

Multilayered made structures are increasingly used in aerospace, ship as well as in automotive vehicles. Examples of multilayered made anisotropic structures are sandwich constructions, composite structures made by orthotropic laminae, layered structures made of different isotropic layers (as those employed for thermal protection) as well as intelligent structures embedding piezo-layers. As far as two-dimensional modelings of multilayered flat structures is concerned (to which developments this paper is devoted) there are a number of requirements that should be considered for an accurate description of their stress and strain fields.

First, anisotropic multilayered structures often possess higher transverse shear and normal flexibility than traditional isotropic one-layer ones. In fact, classical two-dimensional analyses of plates and shells based on Cauchy–Poisson–Kirchhoff [1–3] assumptions, namely Classical Lamination Theory (CLT) [4], are inadequate to predict the global response of thick plates; furthermore, Reissner–Mindlin [5,6] type theories, namely First Shear Deformation Theory (FSDT) [7], even though account for transverse shear deformations, can lead to very inaccurate conclusions as far as local response of thick layered structures is concerned.

Second, the intrinsic discontinuity of the thermo-mechanical properties in the thickness plate/shell direction puts further difficulties on two-dimensional modelings of layered structures. Fig. 1 shows, from qualitative point of view, what can be the scenario of displacement $\boldsymbol{u} = (u_x, u_y, u_z)$ (variables are measured in a triorthogonal Cartesian system x, y, z which has z as through-the-thickness, normal coordinate, see Fig. 2) and transverse stress $\boldsymbol{\sigma}_n = (\sigma_{xz}, \sigma_{yz}, \sigma_{zz})$ distributions in a multilayered made structures as it could appear in the exact solution and/or experiments. This figure makes evident that both displacement and transverse stresses, due to compatibility and equilibrium reasons, respectively, are C^0-continuous function in the thickness z direction, see also [8–10], Furthermore, $\boldsymbol{u}$ and $\boldsymbol{\sigma}_n$ have in the most general case, discontinuous first derivatives with correspondence to each interface where the mechanical properties change. In [10] these facts were referred to as C_z^0-requirements. Literature often marks these requirements as zig-zag (ZZ) form for $\boldsymbol{u}$ and IC (equilibrium) for $\boldsymbol{\sigma}_n$. ZZ and IC are strongly connected

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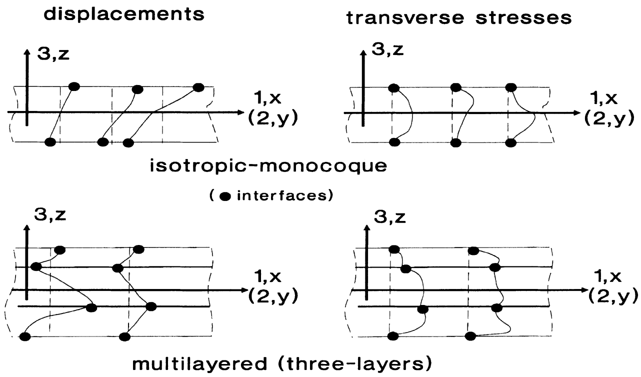


Fig. 1. Displacement and stress distribution in the plate thickness direction z. Comparison between a one-layered and a three-layered structures.

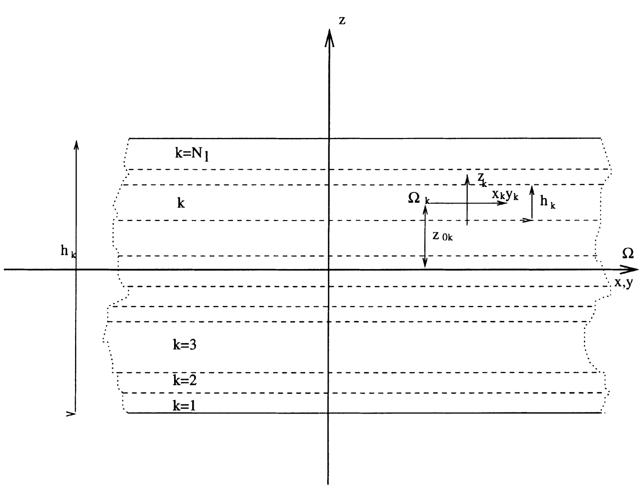


Fig. 2. Multilayered plate.

to each other by the physical behavior of a multilayered structures as it is in solids for compatibility and equilibrium.

Many refinements of classical models have been proposed directed to overcome limitation of classical CLT and FSDT theories and to include partially or completely the above-mentioned C_z^0-requirements in the formulations. The first and most relevant works belong to the Russian literature. Among these one should mention the pioneering paper by Lekhnitskii [11], the translated books by the same author [12] and by Ambartsumian [13], as well as the article by Vlasov [14] and the recent survey paper by Grigolyuk and Kulikov [15]. The very elegant Lekhnitskii's approach, originally developed for beams, which describe interlaminar continuous transverse shear stress as well as ZZ effects, was extended to plate by Ren [16]; Ambartsumian's theory, describing both IC and ZZ effects, has been first extended to unsymmetric case by Whitney [7] and then to shell geometry by Rath and Das [17] and lately re-elaborated by many authors [18–24]; Vlasov's [14], as well as Hildebrand et al.'s [25] type theories, were considered by Sun and Whitney [26] Lo et al. [27], Reddy and Phan [28], Kheider and Librescu and [29]. Approaches similar to those discussed by Grigolyuk and Kulikov and based on Reissner's Mixed theorem [30,31] have been considered first by Murakami [32] and then in [33–41]. The above mentioned works belong to the ESLM Equivalent Single Layer Model categories. Following Reddy [9] it is understood that ESLMs preserve the independence of the number of the independent variables from the numbers of the N_l-layers while the number of the unknown variables remains N_l-dependent in the LWMs. Relevant LW works in which classical models are considered at the layer level are those by Srinivas [42], Bert et al. [43], Nosier et al. [45] Robbins and Reddy [44] and Carrera [46,47]. A complete discussion of the several contributions appeared in literature is not the aim of the present work. Such a topic has been covered by recent exhaustive state-of-art articles. Among these one can mention the papers by Librescu and Reddy [48], Kapania and Raciti [49], Noor et al. [50–53] Reddy and Robbins [52], and by the recent book by Reddy [9]. Indeed the present paper mainly has the two following purposes: (i) to mark possible KPs keypoints that are relevant for accurate two-dimensional modeling of multilayered structures; (ii) to give a numerical assessment in which theories are compared in view of the fulfillment of the traced KPs. With respect to that at point (i), this author believes that last three decades literature has shown that relevant features that could be chosen as KPs to evaluate and assess are those in the following list:

*KP*1: *Simulation of ZZ forms of displacement fields* $\boldsymbol{u}$ *with the thickness plate directions, including through the thickness components* u_z.
*KP*2: *Fulfillment of interlaminar continuity* (*IC*) *for the transverse shear and normal stresses at each layer interface.*
*KP*3: *Accurate description of transverse normal stress* σ_{zz} *and related consequences.*

Of further interest are the following additional features.

*KP*4: *Possibility of a given theory to furnish accurate transverse stresses a priori* (*i.e. without requiring any post-processing procedure such as integration of indefinite equilibrium equations of three-dimensional elasticity*).
*KP*5: *To keep the the total number of degree of freedoms as less as possible for a given level of accuracy, in particular to keep the number of the independent variables independent by the number of the constitutive layers.*

In order to meet the two mentioned proposals this paper reconsiders and extends recent author's articles on two-dimensional modeling of multilayered plate struc-

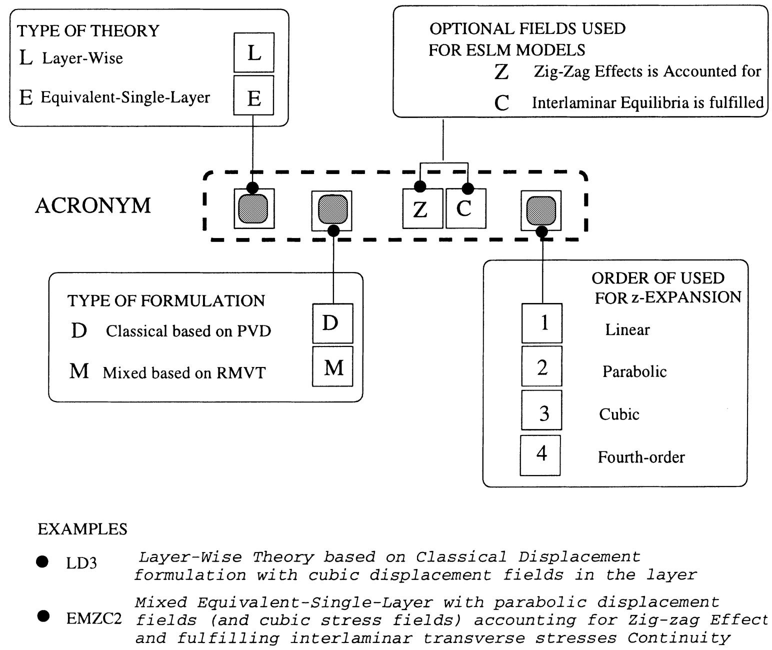


Fig. 3. Meanings of the introduced acronyms.

tures [10,41,46,47,54]. A number of axiomatic plate theories have been proposed and discussed in these articles: classical models formulated on the basis of Principle of Virtual Displacements (PVD) and mixed models based on the Reissner's Mixed Variational Theorem (RMVT); Layer-Wise (LW) and Equivalent Single Layer (ESL) Models related to linear to fourth order expansion N in the plate thickness z-direction; ESL cases in which ZZ effects or IC can be forced or discarded; ESL cases in which transverse normal strains are considered or discarded.

As a result a considerable number of theories are considered in the present work in view of features at KP1–KP5. More than 40 models have been implemented and compared for sample problems related to simply supported, cross-ply laminated plates. Dynamic and static response are discussed as well as local and global characteristic are compared. Fig. 3 introduces a technique to build acronyms related to the whole possible theories. It is shown that theories can be considered in both frameworks of PVD and RMVT. In the same way, LW and ESL cases are discussed. ZZ form can be discarded or included in ESL approaches. And finally, the order of the used expansion in the layer and/or in the whole plate can vary from linear to the fourth order.

The paper has been organized as follows. Section 2 illustrates how variational statements PVD and RMVT are used to derive plate theories. Section 3 gives details of the considered plate theories. Results and final remarks are given in Section 4.

Results have been restricted to problem for which closed form solution of Navier type is available. As a consequence more realistic problems, as those coming from different lay-out configuration and by considering different boundary conditions, have been omitted. This limitation can be overcome by considering computational implementation of the considered theories. Work in this direction is in progress and results will be submitted to the scientific community in the near future. Furthermore, more popular ESL approaches as those coming by the fundamental work by Whitney [7] and Ren [16] have not been implemented. Where possible, results related to these approaches are taken from the literature. In any case already mentioned author's works have proven that Whitney's or Ren's type analyses lead to results which accuracy is comparable to those of the EMZ and EMZC theories discussed in the present article.

2. Use of variational statements to develop plate theories

In the field of so-called axiomatic approach [55], where a certain displacement or stress fields are postulated in the plate z-direction, two-dimensional theories are usually constructed accordingly to the following four steps:

1. Material behavior is assigned, i.e. Hooke's law is given.
2. A geometrical relation, i.e. strain–displacement relation is assumed.
3. Displacement and or stress distributions in the thickness z plate direction are *postulated* by referring to a certain set of base functions.

4. An appropriate variational statements (PVD or RMVT) is used to establish governing equations and boundary conditions which are variationally consistent with the hypothesis introduced in the previous points 1–3.

Our goal is to discuss theories in the framework of PVD and RMVT. Displacement and transverse shear and normal stress can be assume in such a framework. Details of the previous steps are given in the following.

2.1. Hooke's law

The geometry and Cartesian coordinate system x, y, z of the multilayered plates made of N_l layers are those of Fig. 2. The lamina are considered homogeneous and to operate in the linear elastic range. Stiffness coefficients of Hooke's law for the anisotropic k-lamina are employed in standard form.

The Hooke's law reads $\sigma_i = \tilde{C}_{ij}\epsilon_j$ where sub-indices i and j, ranging from 1 to 6, stand for the index couples 11, 22, 33, 13, 23 and 12, respectively. The material is assumed to be orthotropic as specified by: $\tilde{C}_{14} = \tilde{C}_{24} = \tilde{C}_{34} = \tilde{C}_{64} = \tilde{C}_{15} = \tilde{C}_{25} = \tilde{C}_{35} = \tilde{C}_{65} = 0$. This implies that σ^k_{xz} and σ^k_{yz} depend only on ϵ^k_{xz} and ϵ^k_{yz}. In matrix form

$$\begin{aligned} \boldsymbol{\sigma}^k_{\mathrm{pH}} &= \tilde{\boldsymbol{C}}^k_{\mathrm{pp}}\boldsymbol{\epsilon}^k_{\mathrm{pG}} + \tilde{\boldsymbol{C}}^k_{\mathrm{pn}}\boldsymbol{\epsilon}^k_{\mathrm{nG}}, \\ \boldsymbol{\sigma}^k_{\mathrm{nH}} &= \tilde{\boldsymbol{C}}^k_{\mathrm{np}}\boldsymbol{\epsilon}^k_{\mathrm{pG}} + \tilde{\boldsymbol{C}}^k_{\mathrm{nn}}\boldsymbol{\epsilon}^k_{\mathrm{nGb}}, \end{aligned} \tag{1}$$

where

$$\tilde{\boldsymbol{C}}^k_{\mathrm{pp}} = \begin{bmatrix} \tilde{C}^k_{11} & \tilde{C}^k_{12} & \tilde{C}^k_{16} \\ \tilde{C}^k_{12} & \tilde{C}^k_{22} & \tilde{C}^k_{26} \\ \tilde{C}^k_{16} & \tilde{C}^k_{26} & \tilde{C}^k_{66} \end{bmatrix}, \quad \tilde{\boldsymbol{C}}^k_{\mathrm{pn}} = \tilde{\boldsymbol{C}}^{k\mathrm{T}}_{\mathrm{np}} = \begin{bmatrix} 0 & 0 & \tilde{C}^k_{13} \\ 0 & 0 & \tilde{C}^k_{23} \\ 0 & 0 & \tilde{C}^k_{36} \end{bmatrix},$$

$$\tilde{\boldsymbol{C}}^k_{\mathrm{nn}} = \begin{bmatrix} \tilde{C}^k_{44} & \tilde{C}^k_{45} & 0 \\ \tilde{C}^k_{45} & \tilde{C}^k_{55} & 0 \\ 0 & 0 & \tilde{C}^k_{66} \end{bmatrix}.$$

Bold letters denote arrays. The superscript T signifies array transposition. The subscripts n and p denote transverse (out-of-plane, normal) and in-plane values, respectively. Therefore

$$\boldsymbol{\sigma}^k_{\mathrm{p}} = \{\sigma^k_{xx}, \sigma^k_{yy}, \sigma^k_{xy}\}, \boldsymbol{\sigma}^k_{\mathrm{n}} = \{\sigma^k_{xz}, \sigma^k_{yz}, \sigma^k_{zz}\},$$

$$\boldsymbol{\epsilon}^k_{\mathrm{p}} = \{\epsilon^k_{xx}, \epsilon^k_{yy}, \epsilon^k_{xy}\}, \boldsymbol{\epsilon}^k_{\mathrm{n}} = \{\epsilon^k_{xz}, \epsilon^k_{yz}, \epsilon^k_{zz}\}.$$

Subscript H denotes stresses evaluated by Hooke's law, while subscript G denotes strain from the geometrical relation (3).

Eq. (1) is used in conjunction with a standard displacement formulation on the cases of PVD, while, for the adopted mixed solution procedure, the stress–strain relationships are conveniently put in the following mixed form:

$$\begin{aligned} \boldsymbol{\sigma}^k_{\mathrm{pH}} &= \boldsymbol{C}^k_{\mathrm{pp}}\boldsymbol{\epsilon}^k_{\mathrm{pG}} + \boldsymbol{C}^k_{\mathrm{pn}}\boldsymbol{\sigma}^k_{\mathrm{nM}}, \\ \boldsymbol{\epsilon}^k_{\mathrm{nH}} &= \boldsymbol{C}^k_{\mathrm{np}}\boldsymbol{\epsilon}^k_{\mathrm{pG}} + \boldsymbol{C}^k_{\mathrm{nn}}\boldsymbol{\sigma}^k_{\mathrm{nM}}, \end{aligned} \tag{2}$$

where both stiffness and compliance coefficients are employed. The subscript M indicates that the transverse stresses are those of the assumed model in Eq. (25) (see the next sections). The relation between the arrays of coefficients in the two forms of Hooke's law is simply found

$$\begin{aligned} &\boldsymbol{C}^k_{\mathrm{pp}} = \tilde{\boldsymbol{C}}^k_{\mathrm{pp}} - \tilde{\boldsymbol{C}}^k_{\mathrm{pn}}\tilde{\boldsymbol{C}}^{k^{-1}}_{nn}\tilde{\boldsymbol{C}}^k_{\mathrm{np}}, \quad \boldsymbol{C}^k_{\mathrm{pn}} = \tilde{\boldsymbol{C}}^k_{\mathrm{pn}}\tilde{\boldsymbol{C}}^{k^{-1}}_{\mathrm{nn}}, \\ &\boldsymbol{C}^k_{\mathrm{np}} = -\tilde{\boldsymbol{C}}^{k^{-1}}_{\mathrm{nn}}\tilde{\boldsymbol{C}}^k_{\mathrm{np}}, \quad \boldsymbol{C}^k_{\mathrm{nn}} = \tilde{\boldsymbol{C}}^{k^{-1}}_{\mathrm{nn}}. \end{aligned}$$

Superscript -1 denotes an inversion of the array.

2.2. Geometrical relation

The strain components $\boldsymbol{\epsilon}^k_{\mathrm{p}}, \boldsymbol{\epsilon}^k_{\mathrm{n}}$ are linearly related to the displacements $\boldsymbol{u}^k$ ($\{u^k_x, u^k_y, u^k_z\}$), according to the following geometrical (subscript G) relations:

$$\boldsymbol{\epsilon}^k_{\mathrm{pG}} = \boldsymbol{D}_{\mathrm{p}}\boldsymbol{u}^k, \quad \boldsymbol{\epsilon}^k_{\mathrm{nG}} = \boldsymbol{D}_{\mathrm{n}}\boldsymbol{u}^k. \tag{3}$$

$\boldsymbol{D}_{\mathrm{p}}$ and $\boldsymbol{D}_{\mathrm{n}}$ denotes in-plane and out-of-plane differential operators, respectively.

$$\boldsymbol{D}_{\mathrm{p}} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}; \quad \boldsymbol{D}_{\mathrm{n}} = \begin{bmatrix} \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \\ 0 & 0 & \partial_z \end{bmatrix}.$$

2.3. Displacement and transverse assumption

The behavior of a displacement and/or stress variable f are postulated in the thickness plate z-directions according to a given expansion

$$f(x, y, z) = F_i(z) f_i(x, y), \quad i = 1, N^{\star}. \tag{4}$$

The repeated indexes i are summed over their ranges. The polynomials $F_i(z)$ constitute a set of independent functions. Such a base can be arbitrarily chosen: power of z and Legendre polynomials will be considered in this paper. $N^{\star}$ denotes the number of the introduced terms.

Displacement and transverse normal stress assumptions will lead to the following formula:

$$\begin{aligned} \boldsymbol{u}(x, y, z) &= F_i(z)\boldsymbol{u}_i(x, y), \\ \boldsymbol{\sigma}_{\mathrm{nM}}(x, y, z) &= F_i(z)\boldsymbol{\sigma}_{\mathrm{n}_i}(x, y) i = 1, N^{\star}. \end{aligned} \tag{5}$$

Subscript M (as model) has been introduced to distinguish assumed stresses by stress computed by Hooke's law. $N^{\star} \leqslant 5$ will be considered in the numerical investigation. In the most general case $N^{\star}$ can be different for the different variables, see the discussion reported in [10]. The assumptions at Eq. (5) can be made at layer or multilayered level. LW and ESLM descriptions correspond to first and second case, respectively.

2.4. Governing equations via PVD and RMVT

PVD puts in a variational form the 3D indefinite equilibrium equations that in the dynamic case are

$$\sigma_{ij}, i - \rho \ddot{u}_i = p_i \quad i,j = x,y,z. \tag{6}$$

Corresponding PVD integral equation for a multilayered plate is (see [56] as example)

$$\sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} (\delta\, \boldsymbol{\epsilon}_{pG}^{k^T} \boldsymbol{\sigma}_{pH_d}^k + \delta \boldsymbol{\epsilon}_{nG}^{k^T} \boldsymbol{\sigma}_{nH_d}^k)\, d\Omega^k\, dz = \sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} \rho^k \delta \boldsymbol{u}^k \ddot{\boldsymbol{u}}^k\, dV + \delta L_e, \tag{7}$$

where δ is the variational symbol, A_k and V denote the layer-thickness domain and volume; Ω^k is the layer middle surface bounded by Γ^k (Γ_g^k, Γ_m^k denotes those parts of Γ^k on which the geometrical and mechanical boundary conditions are prescribed, respectively), ρ is the mass density and double dots denote acceleration. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometrical relations. δL_e is the virtual variation of the work made by the external layer-forces $\boldsymbol{p}^k = \{p_x^k, p_y^k, p_z^k\}$. Upon substitution of that in Eqs. (1), (3) and the first of Eq. (5), such a variational statement will lead to a set of equilibrium equations and boundary conditions. The equilibrium equations can be formally put in the following compact form:

$$\delta \boldsymbol{u}_\tau^k : \quad \boldsymbol{K}_d^{k\tau s} \boldsymbol{u}_s^k = \boldsymbol{M}^{k\tau s} \ddot{\boldsymbol{u}}_s^k + \boldsymbol{p}_\tau^k \tag{8}$$

the related boundary conditions are

$$\boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k \quad \text{or} \quad \boldsymbol{\Pi}_d^{k\tau s} \boldsymbol{u}_s^k = \boldsymbol{\Pi}_d^{k\tau s} \bar{\boldsymbol{u}}_s^k. \tag{9}$$

The number of the obtained equations coincides with the number of the introduced variables: τ and s vary from 1 to $N^{\star}$ and k range from 1 to N_l. $\boldsymbol{K}, \boldsymbol{M}, \boldsymbol{\Pi}$ are arrays constitute by differential operators.

Displacements and transverse stresses are assumed if Reissner's mixed method is employed. The RMVT forces both 3D indefinite equilibrium equations (6) and compatibility of transverse strains. These strains can be in fact computed by Hooke's law and by a geometrical relations. In each point, i.e. the strong form of such a compatibility condition is

$$\boldsymbol{\epsilon}_{nH}^k - \boldsymbol{\epsilon}_{nG}^k = 0 \quad \text{or} \quad \boldsymbol{\epsilon}_{nH}^k - \boldsymbol{D}_n \boldsymbol{u}^k = 0. \tag{10}$$

Reissner's mixed theorem [30,31], formulates both equilibrium equations (6) and compatibility equation (10) in terms of the $\boldsymbol{u}^k$ and $\boldsymbol{\sigma}_n^k$ unknowns via the following variational equation:

$$\sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} (\delta\, \boldsymbol{\epsilon}_{pG}^{k^T} \boldsymbol{\sigma}_{pH}^k + \delta \boldsymbol{\epsilon}_{nG}^{k^T} \boldsymbol{\sigma}_{nM}^k + \delta \boldsymbol{\sigma}_{nM}^{k^T} (\boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\epsilon}_{nH}^k))\, d\Omega^k\, dz = \sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} \rho^k \delta \boldsymbol{u}^k \ddot{\boldsymbol{u}}^k\, dV + \delta L^e. \tag{11}$$

The L.H.S. includes the variations of the internal work in the plate: the first two terms come from the displacement formulation and lead to variationally consistent equilibrium conditions; the third 'mixed' term variationally enforces the compatibility of the transverse strains components. The governing equations expressed in terms of displacement and stress variables are, in compact form

$$\begin{aligned} \delta \boldsymbol{u}_\tau^k : &\quad \boldsymbol{K}_{uu}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{u\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{M}^{k\tau s} \ddot{\boldsymbol{u}}_s^k + \boldsymbol{p}_\tau^k, \\ \delta \boldsymbol{\sigma}_{n\tau}^k : &\quad \boldsymbol{K}_{\sigma u}^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{K}_{\sigma\sigma}^{k\tau s} \boldsymbol{\sigma}_{ns}^k = 0, \end{aligned} \tag{12}$$

with boundary conditions

$$\boldsymbol{u}_\tau^k = \bar{\boldsymbol{u}}_\tau^k \quad \text{or} \quad \boldsymbol{\Pi}_u^{k\tau s} \boldsymbol{u}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \boldsymbol{\sigma}_{ns}^k = \boldsymbol{\Pi}_u^{k\tau s} \bar{\boldsymbol{u}}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s} \bar{\boldsymbol{\sigma}}_{ns}^k. \tag{13}$$

In those cases in which LW descriptions are employed the written governing equations are first derived at the layer level. Multilayer equations are than written by imposing the continuity requirements for stresses and displacements. Examples of the described methodologies are reported in the already cited author's works.

3. Plate theories considered in this article

3.1. Classical theories on the basis of PVD

Classical plate theories variationally formulated on the basis of PVD, assume a certain expansion in terms of introduced displacement variables in the z-direction. This can be done in two ways as discussed in the following two sections.

3.1.1. Equivalent single layer models (ED1, ..., ED4)

According to ESLM formulation the displacement variables assume the same values in each layers. If Taylor type expansion is employed, the displacement fields is

$$\boldsymbol{u} = \boldsymbol{u}_0 + z^r \boldsymbol{u}_r, \quad r = 1,2,\ldots,N, \tag{14}$$

where N is a free parameter of the model. In contrast to $N^{\star}$, N does not denote the number of terms but the order of the expansion. The repeated indexes r are summed over their ranges. Subscript 0 denotes values related to the plate reference surface Ω. $\boldsymbol{u}_r$ are linear and higher order terms.

If a generic set of base function $F^r(z)$ is introduced, the previous expansion can be rewritten as follows:

$$\boldsymbol{u} = F^r(z) \boldsymbol{u}_r, \quad r = 0,1,2,\ldots,N. \tag{15}$$

In order to handle all the modelings in a unified manner we will use the following notations:

$$\boldsymbol{u} = F_t \boldsymbol{u}_t + F_b \boldsymbol{u}_b + F_r \boldsymbol{u}_r = F_\tau \boldsymbol{u}_\tau, \quad \tau = t, b, r, \quad r = 1, 2, \ldots, N. \tag{16}$$

Subscript b denotes values related to the plate reference surface Ω ($\boldsymbol{u}_b = \boldsymbol{u}_0$) while subscript t refers to highest order terms. For the case of Taylor type expansion the introduced base function holds

$$F_b = 1, \quad F_t = z^{N+1}, \quad F_r = z^r, \quad r = 1, 2, \ldots, N. \tag{17}$$

For instance, FSDT corresponds to $r = 0$ case with $u_{zt} = 0$. Shear correction factors χ can be used for FSDT cases, in particular CLT results can be numerically achieved by implementing a penalty technique on χ. In fact, $\chi = \infty$ corresponds to CLT analysis. Three cases of shear correction factors will be considered in the numerical analysis. Other examples of Eq. (14) type theory are those discussed by Sun and Whitney [26] or by Lo et al. [27].

Comments: The above discussed theories do not include ZZ effects KP1 neither fulfill interlaminar equilibria KP2.

3.1.2. ESLMs Including ZZ (EDZ1, …, EDZ3)

The ZZ form of the displacements fields can be reproduced in an equivalent Single-Layer description upon generalization of the Murakami idea [32]. Such idea simply consists in adding a ZZ term into a classical Taylor type expansion. According to [32,33], the displacement model is generalized in the form

$$\boldsymbol{u} = \boldsymbol{u}_0 + (-1)^k \zeta_k \boldsymbol{u}_Z + z^r \boldsymbol{u}_r, \quad r = 1, 2, \ldots, N. \tag{18}$$

Subscript Z refers to the introduced ZZ term. Higher order distributions in the z-direction are introduced by the r-polynomials. In a unified form the displacement model is therefore rewritten

$$\boldsymbol{u} = F_t \boldsymbol{u}_t + F_b \boldsymbol{u}_b + F_r \boldsymbol{u}_r = F_\tau \boldsymbol{u}_\tau, \quad \tau = t, b, r, \quad r = 1, 2, \ldots, N. \tag{19}$$

Subscript b denotes values related to the plate reference surface Ω ($\boldsymbol{u}_b = \boldsymbol{u}_0$) while subscript t now refers to the introduced ZZ term ($\boldsymbol{u}_t = \boldsymbol{u}_Z$). The functions F_τ assume the following explicit form:

$$F_b = 1, \quad F_t = (-1)^k \zeta_k, \quad F_r = z^r, \quad r = 1, 2, \ldots, N. \tag{20}$$

To notice that the F_t assumes the values ± 1 in correspondence to the bottom and the top interface of the k-layer (see Fig. 4). Transverse normal stress effect KP3 will be outlined by forcing u_z constant. This will be denoted by adding a letter d at the end of the corresponding acronyms.

Comments: These theories discard interlaminar equilibria KP2 but take into account ZZ effects KP1.

3.1.2.1. Layer-wise theories (LD1, …, LD4).

Layer-wise description requires assuming independent displacement variables in each k-layer. The thickness expansion used for ESLM cases (Eq. (19)) is not convenient for layer-wise description. IC conditions can be more conveniently imposed by employing interface values in the thickness expansion. Therefore, layer-wise description is written according to the following expansion:

$$\boldsymbol{u}^k_{\text{nM}} = F_t \boldsymbol{u}^k_{\text{n}t} + F_b \boldsymbol{u}^k_{\text{n}b} + F_r \boldsymbol{u}^k_{\text{n}r} = F_\tau \boldsymbol{u}^k_{\text{n}\tau}, \quad \tau = t, b, r, \quad r = 2, 3, \ldots, N; \; k = 1, 2, \ldots, N_l. \tag{21}$$

In contrast to that in Eq. (19), it is now intended that the subscripts t and b denote values related to the layer top and bottom surface, respectively. In fact, they consist of the linear part of the expansion. The thickness functions $F_\tau(\zeta_k)$ have been defined by

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N, \tag{22}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the j-order defined in the ζ_k-domain: $-1 \leqslant \zeta_k \leqslant 1$. Fourth order case will be used in the numerical investigations; related polynomials are

$$P_0 = 1, \; P_1 = \zeta_k, \quad P_2 = (3\zeta_k^2 - 1)/2,$$
$$P_3 = \frac{5\zeta_k^3}{2} - \frac{3\zeta_k}{2}, \; P_4 = \frac{35\zeta_k^4}{8} - \frac{15\zeta_k^2}{4} + \frac{3}{8}.$$

The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1: & F_t = 1; \; F_b = 0; \; F_r = 0 \\ -1: & F_t = 0; \; F_b = 1; \; F_r = 0. \end{cases} \tag{23}$$

The top and bottom values have been used as unknown variables. The interlaminar compatibility of displacement can be therefore easily linked

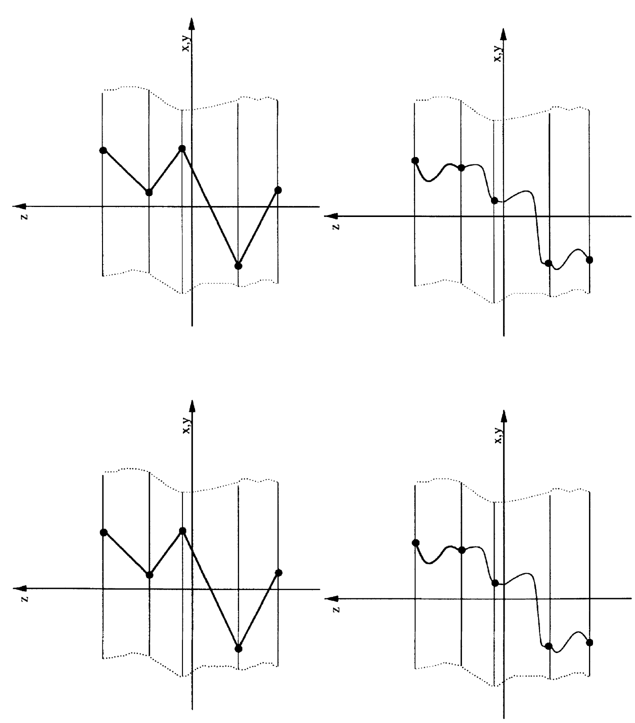
$$\boldsymbol{u}^k_t = \boldsymbol{u}^{(k+1)}_b, \quad k = 1, \quad N_l - 1. \tag{24}$$

Examples of linear and higher order fields have been plotted in Fig. 2b.

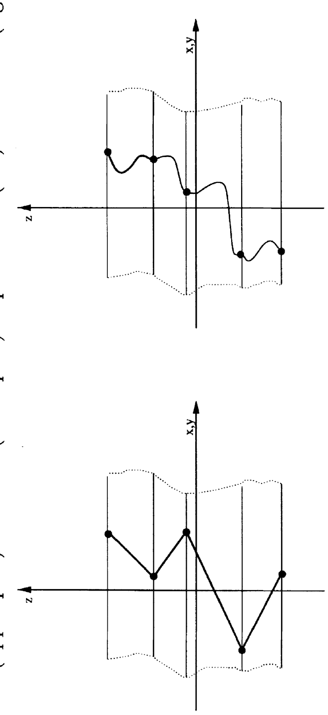
Comments: These types of theories do not address interlaminar equilibria but include ZZ description of both in-plane and out-of-plane displacements.

3.2. Mixed theories on the basis of RMVT

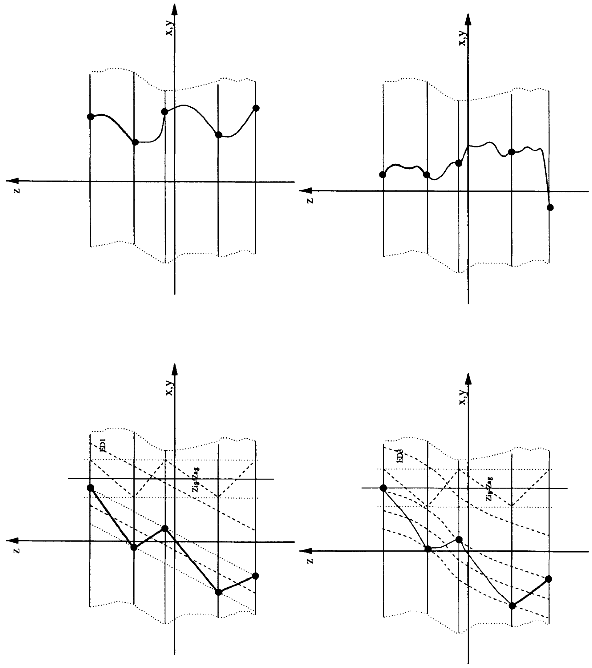
As stated above the advantage of using RMVT consists of the possibility of assuming two independent fields for displacement and transverse shear fields leads to a priori and complete fulfillment of the C^0_z-requirements. As for classical case, this can be done for both ESLM and LW cases.



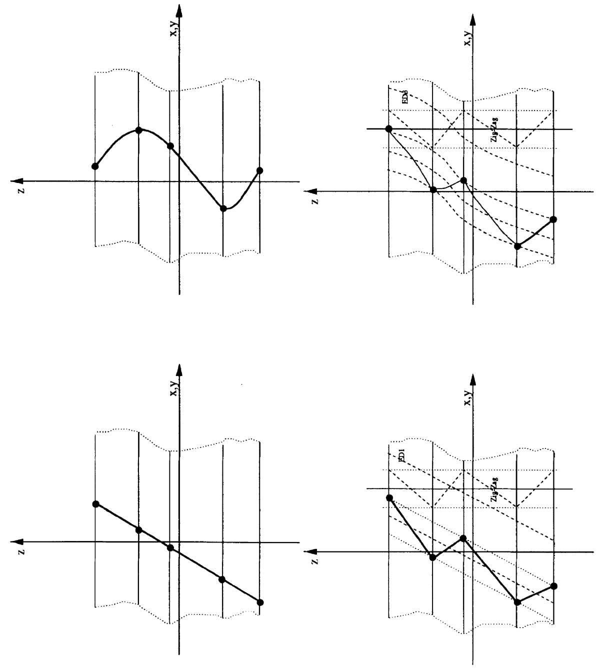
LM1 (upper part) and LM3 (lower part) displacement (left) and stress (right) fields .



LD1 (left) and LD3 (right) displacement fields.



EMZC1 (upper part) and EMZC3 (lower part) displacement (left) and stress (right) fields.



ED1, ED3 (upper part) and EDZ1 and EDZ3 (lower part) displacement fields.

Fig. 4. Examples of assumed fields in the thickness plate direction in a four-layered plate.

3.2.1. Layer-wise cases: LM1, ..., LM4

The layer-wise description used for displacements is certainly suitable certainly for transverse stresses

$$\begin{aligned} \boldsymbol{u}^k &= F_t\boldsymbol{u}_t^k + F_b\boldsymbol{u}_b^k + F_r\boldsymbol{u}_r^k = F_\tau\boldsymbol{u}_\tau^k, \\ &\quad \tau = t, b, r, \quad r = 2, 3, \ldots, N, \\ \boldsymbol{\sigma}_{\mathrm{nM}}^k &= F_t\boldsymbol{\sigma}_{\mathrm{n}t}^k + F_b\boldsymbol{\sigma}_{\mathrm{n}b}^k + F_r\boldsymbol{\sigma}_{\mathrm{n}r}^k = F_\tau\boldsymbol{\sigma}_{\mathrm{n}\tau}^k, \\ &\quad k = 1, 2, \ldots, N_l. \end{aligned} \tag{25}$$

The top and bottom values have also been used as unknown variables. The interlaminar transverse shear and normal stress continuity can be therefore easily linked

$$\boldsymbol{\sigma}_{\mathrm{n}t}^k = \boldsymbol{\sigma}_{\mathrm{n}b}^{(k+1)}, \quad k = 1, N_l - 1. \tag{26}$$

In those cases in which top/bottom-shell stress values are prescribed (zero or imposed values), the following additional equilibrium conditions must be accounted for:

$$\boldsymbol{\sigma}_{\mathrm{n}b}^1 = \bar{\boldsymbol{\sigma}}_{\mathrm{n}b}, \boldsymbol{\sigma}_{\mathrm{n}t}^{N_l} = \bar{\boldsymbol{\sigma}}_{\mathrm{n}t}, \tag{27}$$

Table 1
Circular frequency parameter $\omega h\sqrt{\rho/E_T}$ of simply supported square plates[a]

N_l	2		10		3		9	
E_L/E_T	3	30	3	30	3	30	3	30
3D [62]	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
HSDT [63]	0.2388	0.3117	0.2527	0.4028	–	–	0.2536	0.4027
FSDT [63]	0.2379	0.3165	0.2527	0.4086	–	–	0.2531	0.4067
LM4	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LM3	0.2392	0.3115	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LM2	0.2392	0.3115	0.2530	0.4027	0.2516	0.3738	0.2525	0.4040
LM1	0.2312	0.2354	0.2527	0.4011	0.2466	0.3354	0.2532	0.4019
LM4ni	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LM3ni	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LM2ni	0.2394	0.3143	0.2530	0.4027	0.2517	0.3749	0.2535	0.4040
LM1ni	0.2417	0.3134	0.2530	0.4031	0.2511	0.3721	0.2535	0.4044
EMZC3	0.2392	0.3144	0.2531	0.4042	0.2517	0.3758	0.2536	0.4052
EMZC2	0.2408	0.3133	0.2565	0.4035	0.2523	0.3782	0.2578	0.4242
EMZC1	0.2436	0.3131	0.2758	0.4276	0.2701	0.3789	0.2765	0.4260
EMZC3d	0.2610	0.3251	0.2713	0.4063	0.2700	0.3793	0.2717	0.4073
EMZC2d	0.2641	0.3247	0.2768	0.4269	0.2711	0.3819	0.2769	0.4273
EMZC1d	0.2625	0.3223	0.2766	0.4277	0.2701	0.3789	0.2765	0.4260
EMZC3ni	0.2392	0.3151	0.2531	0.4042	0.2517	0.3758	0.2535	0.4053
EMZC2ni	0.2418	0.3169	0.2568	0.4250	0.2525	0.3791	0.2570	0.4252
EMZC1ni	0.2455	0.3181	0.2766	0.4305	0.2711	0.3822	0.2771	0.4284
EMC4	0.2393	0.3129	0.2539	0.4075	0.2518	0.3739	0.2771	0.4089
EMC3	0.2393	0.3141	0.2538	0.4072	0.2518	0.3739	0.2543	0.4087
EMC2	0.2400	0.3117	0.2570	0.4258	0.2555	0.3739	0.2575	0.4275
EMC1	0.2566	0.3179	0.2754	0.4230	0.2700	0.3354	0.2756	0.4224
LD4	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LD3	0.2392	0.3117	0.2530	0.4027	0.2516	0.3739	0.2535	0.4040
LD2	0.2395	0.3168	0.2530	0.4027	0.2517	0.3763	0.2535	0.4040
LD1	0.2478	0.3210	0.2534	0.4042	0.2556	0.3808	0.2540	0.4058
EDZ3	0.2392	0.3156	0.2532	0.4042	0.2517	0.3761	0.2536	0.4058
EDZ2	0.2418	0.3180	0.2568	0.4250	0.2527	0.3803	0.2570	0.4254
EDZ1	0.2478	0.3210	0.2767	0.4305	0.2717	0.3842	0.2772	0.4286
EDZ3d	0.2610	0.3261	0.2713	0.4064	0.2700	0.3795	0.2717	0.4074
EDZ2d	0.2659	0.3300	0.2772	0.4285	0.2717	0.3842	0.2772	0.4286
EDZ1d	0.2659	0.3300	0.2773	0.4306	0.2717	0.3842	0.2772	0.4286
ED4	0.2394	0.3133	0.2539	0.4078	0.2518	0.3764	0.2544	0.4093
ED3	0.2394	0.3167	0.2540	0.4080	0.2519	0.3766	0.2545	0.4095
ED2	0.2418	0.3198	0.2540	0.4291	0.2569	0.4031	0.2581	0.4308
ED1	0.2662	0.3367	0.2540	0.4340	0.2778	0.4082	0.2787	0.4343
ED4d	0.2612	0.3241	0.2540	0.4102	0.2703	0.3800	0.2727	0.4116
ED3d	0.2613	0.3269	0.2723	0.4102	0.2703	0.3800	0.2727	0.4116
ED2d	0.2659	0.3316	0.2783	0.4327	0.2778	0.4082	0.2787	0.4343
ED1d	0.2662	0.3367	0.2783	0.4340	0.2778	0.4082	0.2787	0.4343
ED1d$^{\chi=5/6}$	0.2610	0.3264	0.2723	0.4118	0.2717	0.3871	0.2726	0.4118
ED1d$^{\chi=2/3}$	0.2537	0.3123	0.2640	0.3839	0.2632	0.3613	0.2642	0.3838
ED1d$^{\chi=\infty}$	0.2972	0.4066	0.3150	0.6435	0.3157	0.6519	0.3157	0.6519

[a] Data: $a/h = 5$; Cross-ply skew-symmetric and symmetric laminates (the total thickness of layers 90° and 0° oriented is the same); $G_{LT}/E_T = G_{Lz}/E_T = 0.50$, $G_{TT}/E_T = 0.35$, $v_{LT} = v_{Lz} = 0.3$, $v_{TT} = 0.49$.

where the over-bar is the imposed values in correspondence to the plate boundary surfaces (these will be zero in what follows). Examples of linear and higher order fields have been plotted in Fig. 4. For comparison purpose the case in which top–bottom zero transverse shear is not imposed will be considered. This will be denoted by adding the suffix ni to the acronyms of the considered theory: for instance LM4ni corresponds to the LM4 theory in which top–bottom homogeneous conditions are not imposed.

Comments: LM theories include ZZ effects and fulfill a priori interlaminar equilibria, that is C_z^0-requirements are completely and a priori fulfilled.

3.2.2. Equivalent single layer cases: EMC1, ..., EMC4; EMZC1, ..., EMZC3

In order to include ZZ function in the framework of ESLM analysis, the displacement field of EDZ theories can be adopted

Table 2
Circular frequency parameter $\omega\sqrt{a^4\rho/E_T h^2}$ of simply supported square plates[a]

a/h	0/90					0/90/90/0				
	2	4	10	20	100	2	4	10	20	100
CLT [28]	8.499	10.292	11.011	11.125	11.163	15.830	17.907	18.652	18.767	18.804
FSDT [28]	5.191	7.975	10.335	10.941	11.155	5.492	9.379	10.820	17.583	18.751
HSDT [28]	5.699	5.699	9.010	10.499	11.132	5.576	9.947	15.270	17.668	18.755
LM4	4.703	7.345	10.088	10.859	11.151	5.260	9.224	15.148	17.626	18.753
LM3	4.680	7.332	10.087	10.859	11.151	5.259	9.224	15.148	17.626	18.753
LM2	4.668	7.329	10.087	10.859	11.151	5.247	9.220	15.148	17.626	18.753
LM1	4.136	5.660	6.666	6.874	6.947	5.143	9.103	15.087	17.604	18.752
LM4ni	4.706	7.345	10.088	10.859	11.151	5.260	9.224	15.148	17.626	18.753
LM3ni	4.709	7.345	10.088	10.859	11.151	5.261	9.224	15.148	17.626	18.753
LM2ni	4.774	7.450	10.141	10.875	11.152	5.270	9.231	15.150	17.667	18.755
LM1ni	4.772	7.424	10.128	10.871	11.152	5.320	9.343	15.248	17.667	18.755
EMZC3	4.685	7.444	10.144	10.877	11.152	5.370	9.371	15.224	17.625	18.754
EMZC2	4.727	7.395	10.119	10.896	11.152	5.847	9.846	15.455	17.737	18.758
EMZC1	4.672	7.340	10.106	10.878	11.170	5.782	9.768	15.423	17.759	18.814
EMZC3d	4.763	7.490	10.229	10.976	11.257	5.374	9.377	15.263	17.714	18.827
EMZC2d	4.752	7.447	10.205	10.969	11.257	5.850	9.863	15.504	18.801	18.831
EMZC1d	4.685	7.375	10.174	10.959	11.256	5.782	9.771	15.433	17.773	18.830
EMZC3ni	4.772	7.473	10.156	10.881	11.152	5.389	9.387	15.232	17.655	18.754
EMZC2ni	4.820	7.513	10.173	10.886	11.152	5.919	9.936	15.521	17.763	18.759
EMZC1ni	4.792	7.508	10.182	10.902	11.172	5.923	9.952	15.560	17.814	18.817
EMC4	4.731	7.408	10.124	10.871	11.152	5.361	9.367	15.224	17.653	18.754
EMC3	4.799	7.527	10.181	10.889	11.153	5.365	9.364	15.221	17.652	18.754
EMC2	4.711	7.352	10.092	10.861	11.152	5.781	9.757	15.387	17.709	18.757
EMC1	11.259	7.822	10.367	11.018	11.259	5.572	9.522	15.260	17.703	18.826
LD4	4.707	7.345	10.088	10.859	11.152	5.260	9.224	15.148	17.626	18.753
LD3	4.710	7.346	10.088	10.859	10.152	5.262	9.224	15.148	17.626	18.753
LD2	4.803	7.519	10.178	10.888	11.153	5.277	9.236	15.152	17.626	18.753
LD1	4.848	7.562	10.215	10.921	11.184	5.414	9.473	15.335	17.703	18.761
EDZ3	4.780	7.490	10.165	10.884	11.152	5.390	9.388	15.232	17.655	18.754
EDZ2	4.838	7.545	10.189	11.891	11.153	5.920	9.938	15.522	17.763	18.760
EDZ1	4.848	7.562	10.215	10.921	11.184	5.926	9.956	15.563	17.817	18.819
EDZ3d	4.799	7.535	10.250	10.983	11.257	5.393	9.394	15.271	17.717	18.827
EDZ2d	4.874	7.605	10.279	10.992	11.257	5.927	9.960	15.572	17.829	18.833
EDZ1d	4.874	7.605	10.279	10.992	11.257	5.927	9.595	15.572	17.829	18.833
ED4	4.745	7.425	10.132	10.874	11.152	5.380	9.384	15.232	17.655	18.754
ED3	4.883	7.647	10.235	10.906	11.154	5.392	9.389	15.232	17.655	18.754
ED2	4.968	7.701	10.254	10.911	11.154	5.920	9.938	15.522	17.763	18.759
ED1	5.544	8.314	10.545	11.072	11.261	5.927	9.960	15.573	17.829	18.833
ED4d	4.766	7.471	10.217	10.972	11.258	5.393	9.394	15.271	17.717	18.827
ED3d	4.896	7.688	10.319	11.005	11.258	5.393	9.394	15.271	17.717	18.827
ED2d	4.999	7.756	10.543	11.012	11.258	5.927	9.960	15.573	17.829	18.833
ED1d	5.544	8.314	10.544	11.072	11.261	5.927	9.960	15.573	17.829	18.833
ED1d$^{\chi=5/6}$	5.205	8.019	10.441	11.041	11.259	5.498	9.389	15.132	17.648	18.824
ED1d$^{\chi=2/3}$	4.791	7.625	10.290	10.994	11.258	5.000	8.704	14.545	17.388	18.810
ED1d$^{\chi=\infty}$	8.576	10.388	11.115	11.230	11.267	15.892	17.977	18.725	18.840	18.877

[a] Cross-ply skew-symmetric and symmetric laminates (layers of equal thickness); $E_L/E_T = 40$, $G_{LT}/E_T = G_{Lz}/E_T = 0.50$, $G_{TT}/E_T = 0.60$, $\nu_{LT} = \nu_{Lz} = \nu_{TT} = 00.25$.

$$\boldsymbol{u} = F_t \boldsymbol{u}_t + F_b \boldsymbol{u}_b + F_r \boldsymbol{u}_r = F_\tau \boldsymbol{u}_\tau, \quad \tau = t, b, r, \quad r = 1, 2, \ldots, N. \tag{28}$$

While interlaminar continuous transverse shear and normal stresses in the RMVT framework require to refer to LW description for transverse stresses:

$$\boldsymbol{\sigma}^k_{\mathrm{nM}} = F_t \boldsymbol{\sigma}^k_{\mathrm{n}t} + F_b \boldsymbol{\sigma}^k_{\mathrm{n}b} + F_r \boldsymbol{\sigma}^k_{\mathrm{n}r} = F_\tau \boldsymbol{\sigma}^k_{\mathrm{n}\tau}, \quad \tau = t, b, r, \quad r = 2, 3, \ldots, N; \quad k = 1, 2, \ldots, N_l. \tag{29}$$

Note that ESLM description has herein restricted to displacement variables. In practice stress variables can be eliminated by static condensation or by employing technique described in [10]. EMC cases are obtained by omitting ZZ function.

Comments: From analytical point of view EMZC type models fulfill complete and a priori C^0_z-requirements. In practice, the assumption that u_Z is k-independent makes the obtained theories unable to accurately describe that at KP3.

3.3. Summary of the considered theories

Depending on the used variational statement (PVD or RMVT), variables description (LW or ESL), order of the used expansion N, inclusion or not of σ_{zz} effects etc., a number of two-dimensional theories can be constructed on the basis of modelings described in the above sections. Acronyms will be extensively used in the numerical part to denote shortly these theories. Fig. 3 has shown how such acronyms have been built. Examples of displacement and stress fields related to few particular-case theories have been plotted in Fig. 4. Few comments on these plots follow. Transverse stress and displacement z-fields have z-distribution for mixed cases: LM1 (Layer-wise Mixed, linear) and LM3 (Layer-wise Mixed, cubic). Only displacement assumptions are made for LD1 (Layer-wise Displacement, linear) and LD3 (Layer-wise Displacement, cubic) cases. Parabolic transverse stress field in each layer is associated to linear ZZ diplacement field for EMZC1 case (ESLM including ZZ and IC, linear) and fourth order transverse stress field in each layer is associated to cubic ZZ displacement field for EMZC3 case (ESLM including ZZ and IC, cubic). Note the geometrical meaning of the use of the ZZ function: a linear through the thickness function of constant amplitude and opposite sign for two adjacent layers has been added to a Taylor type, linear (or cubic for EMZC3) ESL z-distribution to force discontinuous derivatives at each interface for the displacement components.

The following main comments can be made with respect to Key-points 1–5.

- LM and EM type theories fulfill completely KP1-2, these furnish a priori transverse normal stresses without requiring any post-processing procedures. Worst description is expected in the EM cases due to the assumed, non-physical independence of the ZZ function from the k-layer.
- LD and ED type theories do not fulfill a priori interlaminar continuous transverse stresses KP2 while KP1 can be included in ED-cases by considering the related EDZ type theories.

Table 3
First and second circular frequency parameters $\omega h\sqrt{\rho/E_T}$ related to fundamental and higher order modes[a]

	$m=n=1$		$m=n=2$	
	First	Second	First	Second
3D [45]	0.06715	0.50350	0.20798	0.97517
HSDT [45]	0.06839	0.50897	0.21526	1.0179
FSDT [45]	0.06931	0.50897	0.22055	1.0179
CLT [45]	0.07769	0.50897	0.31077	1.0179
LM4	0.06715	0.50350	0.20798	0.97517
LM3	0.06715	0.50350	0.20798	0.97517
LM2	0.06715	0.50333	0.20796	0.97403
LM1	0.06517	0.50343	0.19940	0.97333
LM4ni	0.06715	0.50349	0.20798	0.97517
LM3ni	0.06715	0.50349	0.20798	0.97517
LM2ni	0.06716	0.50349	0.20809	0.97515
LM1ni	0.06718	0.50541	0.20822	0.98888
EMZC3	0.06715	0.50586	0.20806	0.99268
EMZC2	0.06743	0.59575	0.20993	0.99189
EMZC1	0.06729	0.50891	0.20852	1.0170
EMZC3d	0.06722	0.50609	0.20810	0.99288
EMZC2d	0.06752	0.50597	0.21003	0.99195
EMZC1d	0.06729	0.50933	0.20852	1.0187
EMZC3ni	0.06715	0.50586	0.20806	0.99482
EMZC2ni	0.06752	0.50613	0.21066	0.99492
EMZC1ni	0.06758	0.50891	0.21066	1.0171
EMC4	0.06806	0.50396	0.21327	0.97853
EMC3	0.06801	0.50576	0.21291	0.99174
EMC2	0.06691	0.50508	0.22525	0.98665
EMC1	0.06641	0.50893	0.20157	1.0172
LD4	0.06715	0.50349	0.20798	0.97517
LD3	0.06715	0.50349	0.20798	0.97517
LD2	0.06716	0.50350	0.20817	0.97521
LD1	0.06758	0.50560	0.21100	0.99073
EDZ3	0.06715	0.50615	0.20807	0.99498
EDZ2	0.06757	0.50615	0.21096	0.99512
EDZ1	0.06766	0.50891	0.21109	1.0171
EDZ3d	0.06722	0.50638	0.20812	0.99514
EDZ2d	0.06766	0.50638	0.21109	0.99513
EDZ1d	0.06765	0.50933	0.21109	1.0187
ED4	0.06825	0.50409	0.21443	0.97945
ED3	0.06825	0.50617	0.21446	0.99500
ED2	0.07050	0.50617	0.22995	0.99514
ED1	0.07063	0.50896	0.23032	1.0174
ED4d	0.06833	0.50430	0.21451	0.97987
ED3d	0.06833	0.50638	0.21451	0.99514
ED2d	0.07063	0.50638	0.23033	0.99514
ED1d	0.07063	0.50933	0.23033	1.0186
ED1d$^{\chi=\infty}$	0.07721	0.50933	0.30161	1.0187

[a] Simply supported square plates $a/h = 10$. Cross-ply symmetric laminates 0/90/0 (layers of equal thickness); $E_L = 25.1 \times 10^6$ psi, $E_T = 4.8 \times 10^6$ psi, $E_z = 0.75 \times 10^6$ psi, $G_{LT} = 1.36 \times 10^6$ psi, $G_{Lz} = 1.2 \times 10^6$ psi, $G_{Tz} = 0.47 \times 10^6$ psi, $\nu_{LT} = 0.036$, $\nu_{Lz} = 0.25$, $\nu_{TT} = 0.171$.

- LM and LD type models are computationally expensive KP5, the number of independent variables is in fact dependent on the number of the constitutive layers N_l.
- Theories with suffix d, for instance EDd and EMZCd, force constant distribution of transverse displacement u_z, i.e. transverse normal strain effects KP3 are discarded.

4. Comparison and remarks

About 40 theories have been implemented and compared in this article. Classical problems for which three-dimensional exact solutions are available have been considered in the numerical investigation. These are related to simply supported orthotropic plates, subjected to harmonic distribution of transverse pressure. Navier type closed form solutions have been obtained by implementing the procedures reported in [41,54,57].

It has been confirmed that the accuracy of the considered theories is very much subordinate to the following plate geometrical and mechanical parameters: thickness ratio a/h, orthotropic ratio E_l/E_t, number of layers N_l, lamination schemes, i.e. symmetrical and unsymmetrical lay-outs. Furthermore, local and global characteristics are differently approximate by the different theories. Such accuracy can be different

Table 4
In-plane and out-of-plane stress amplitudes[a]

z	0/90/0			0/90/0/90		
	σ_{xx}/p +0.5	σ_{xz}/p 0	σ_{zz}/p 0	σ_{xx}/p −0.5	σ_{xz}/p 0	σ_{zz}/p 0
LM4	15.52	1.374	0.4987	−15.02	1.293	0.4956
LM3	15.35	1.374	0.4987	−14.86	1.293	0.4985
LM2	15.03	1.375	0.5250	−14.59	1.294	0.5078
LM1	10.29	1.723	0.5843	−9.046	1.437	0.6477
LM4ni	15.52	1.374	0.4984	−15.02	1.293	0.4955
LM3ni	15.52	1.373	0.4984	−15.02	1.293	0.4954
LM2ni	15.43	1.367	0.5023	−15.00	1.293	0.4970
LM1ni	15.54	1.386	0.5030	−14.50	1.301	0.4937
EMZC3	15.71	1.358	0.4984	−14.67	1.307	0.4938
EMZC2	14.27	1.370	0.5096	−11.02	1.356	0.5129
EMZC1	13.86	1.362	0.5043	−12.61	1.386	0.5157
EMZC3d	15.28	1.355	0.5012	−14.81	1.313	0.4987
EMZC2d	14.12	1.367	0.5092	−11.15	1.367	0.5180
EMZC1d	14.17	1.363	0.5241	−12.60	1.391	0.5320
EMZC3ni	15.75	1.358	0.4982	−14.71	1.308	0.4953
EMZC2ni	14.48	1.369	0.5010	−11.17	1.356	0.4943
EMZC1ni	14.14	1.366	0.4814	−12.58	1.388	0.4848
EMC4	15.40	1.366	0.4968	−15.21	1.305	0.4941
EMC3	15.46	1.362	0.4983	−14.13	1.324	0.4973
EMC2	11.25	1.436	0.5309	−10.15	1.373	0.5287
EMC1	10.32	1.373	0.6121	−12.26	1.386	0.5855
LD4	15.52	1.374	0.4982	−15.02	1.293	0.4955
LD3	15.52	1.373	0.4982	−15.02	1.293	0.4955
LD2	15.33	1.357	0.4982	−14.99	1.290	0.4954
LD1	14.27	1.369	0.4945	−13.86	1.301	0.4952
EDZ3	15.69	1.357	0.4973	−14.70	1.307	0.4955
EDZ2	14.28	1.367	0.4999	−11.16	1.356	0.4953
EDZ1	13.85	1.363	0.4808	−12.62	1.386	0.4839
EDZ3d	15.31	1.354	0.5000	−14.85	1.314	0.5000
EDZ2d	14.14	1.363	0.5000	−11.29	1.368	0.5000
EDZ1d	14.14	1.363	0.5000	−12.61	1.391	0.5000
ED4	15.39	1.366	0.4977	−15.05	1.307	0.4953
ED3	15.51	1.360	0.4978	−14.22	1.325	0.4959
ED2	11.35	1.447	0.4977	−10.80	1.375	0.4956
ED1	10.98	1.443	0.4840	−12.25	1.385	0.4839
ED4d	15.36	1.358	0.5000	−15.30	1.320	0.5000
ED3d	15.36	1.358	0.5000	−14.38	1.345	0.5000
ED2d	11.17	1.443	0.5000	−10.93	1.395	0.5000
ED1d	11.17	1.443	0.5000	−12.22	1.402	0.5000

[a] Simply supported square plates $a/h = 5$. Cross-ply skew-symmetric and symmetric laminates as in Table 1. $E_L/E_T = 30$, $G_{LT}/E_T = G_{Lz}/E_T = 0.50$, $G_{TT}/E_T = 0.35$, $v_{LT} = v_{Lz} = 0.3$, $v_{TT} = 0.49$.

for the description of in-plane or out-of-plane responses.

Natural frequency has been chosen as global parameter to compare several theories. Results are given in Tables 1–3. Where available 3D exact solutions and other analysis based on classical and refined theories are reported. Symmetrically and unsymmetrically laminated plates are considered . Higher order modes as well as fundamental and second frequency parameters are compared in Table 3. Local stress description is considered in Table 4 and Figs. 5–14. ED1 case presents three sub-cases related to three values of the shear correction factor: $\chi = 5/6,\ 2/3$ and ∞; the last one corresponds to the CLT analysis.

The following main comments and remarks can be made on the reported analysis.

- LWMs descriptions are more accurate than corresponding ESLMs;
- M Mixed descriptions based on RMVT are more accurate than corresponding D formulation based on PVD;
- EM cases are more accurate than ED ones, in other words, RMVT is much more effective for ESLM formulations;
- M Mixed analyses do not require any post-processing procedure at KP-4;

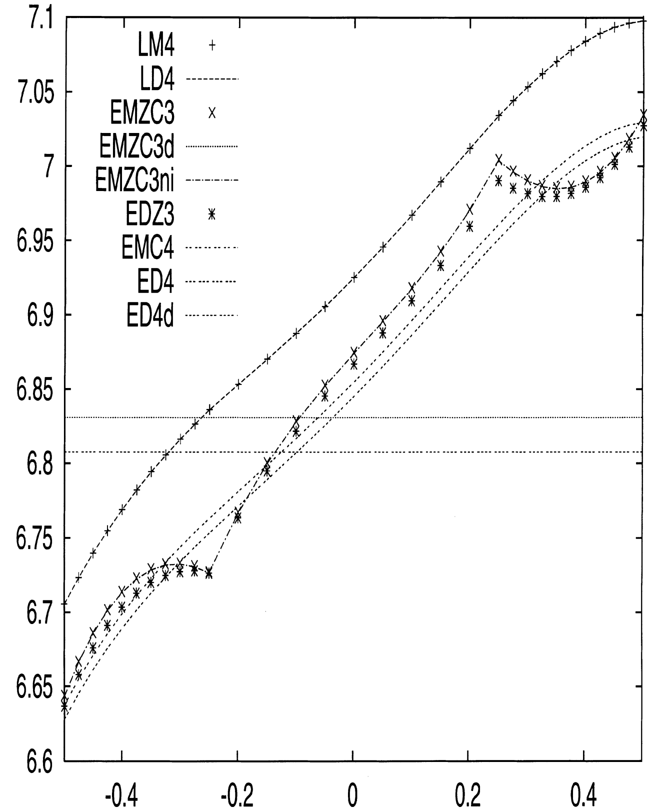


Fig. 6. U_z/h vs z/h. Three layers case of Table 4. $a/h = 5$.

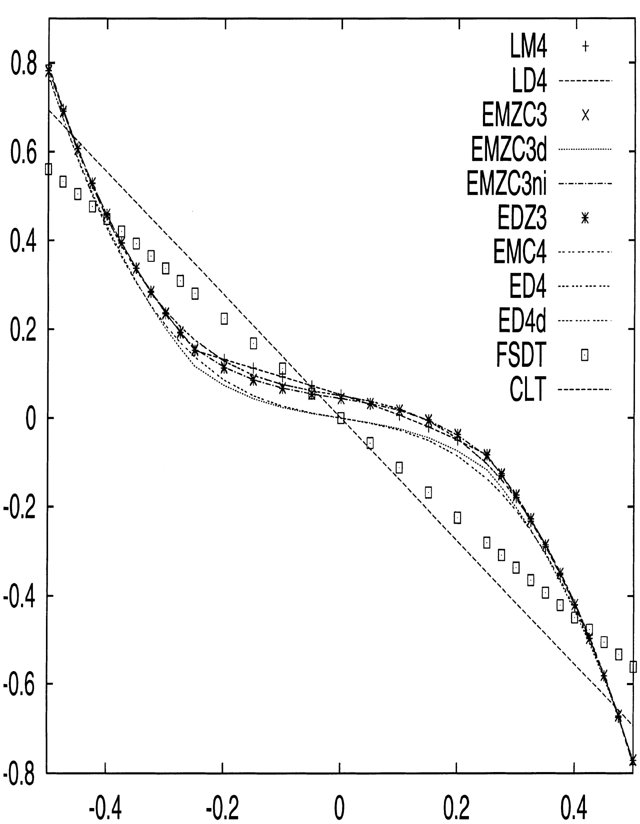


Fig. 5. U_x/h vs z/h. Three layers case of Table 4. $a/h = 5$.

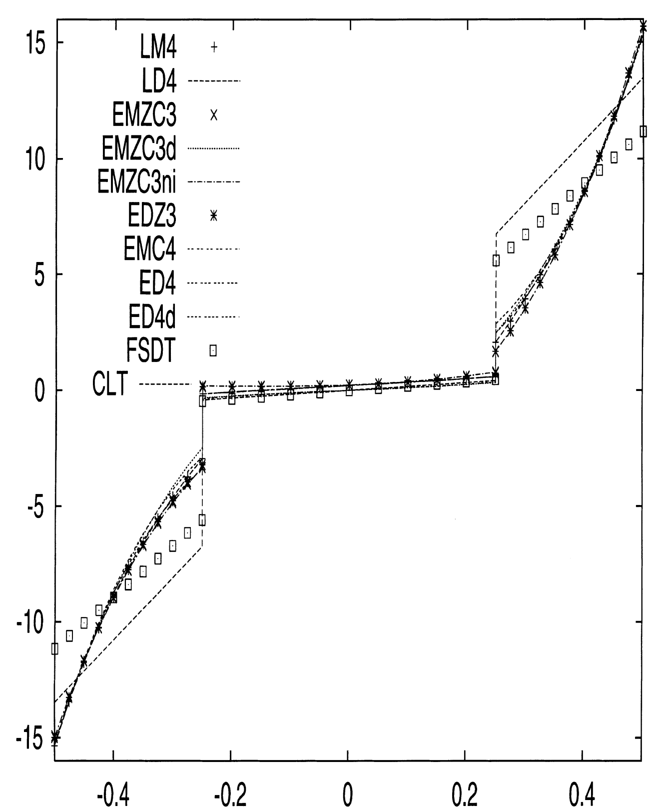


Fig. 7. $S_{xx}/p_{z_t}^{N_l}$ vs z/h. Three layers case of Table 4. $a/h = 5$.

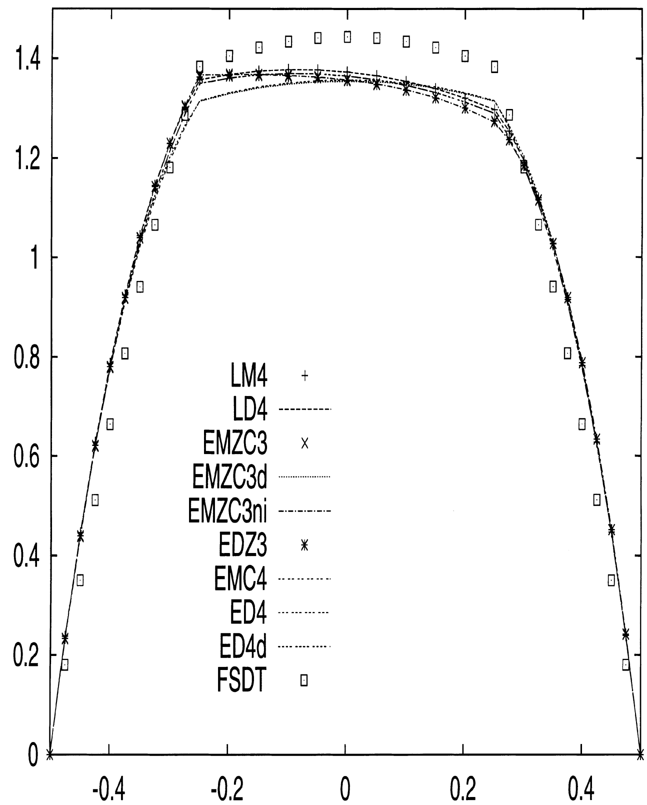


Fig. 8. $S_{xz}/p_{z_t}^{N_l}$ vs z/h. Three layers case of Table 4. $a/h = 5$.

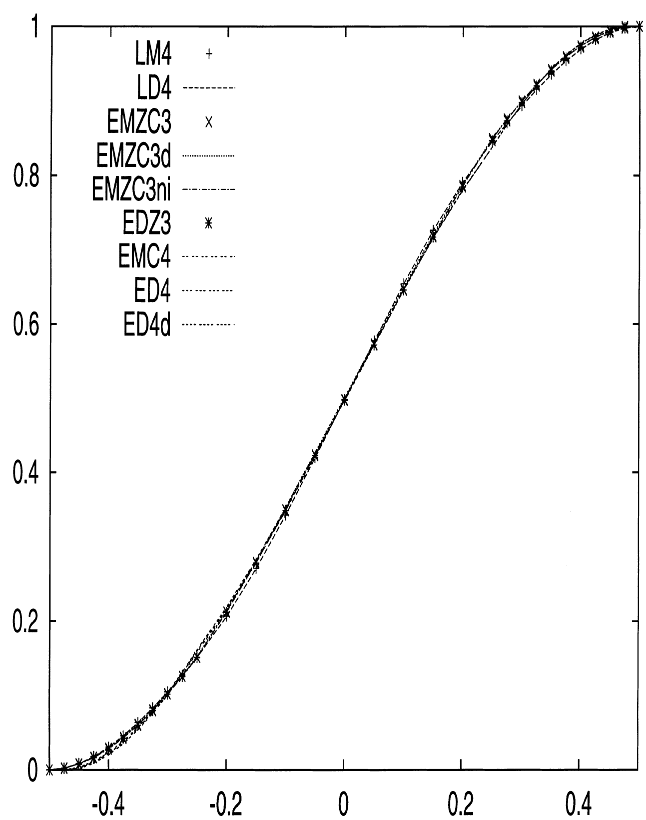


Fig. 9. $S_{zz}/p_{z_t}^{N_l}$ vs z/h. Three layers case of Table 4. $a/h = 5$.

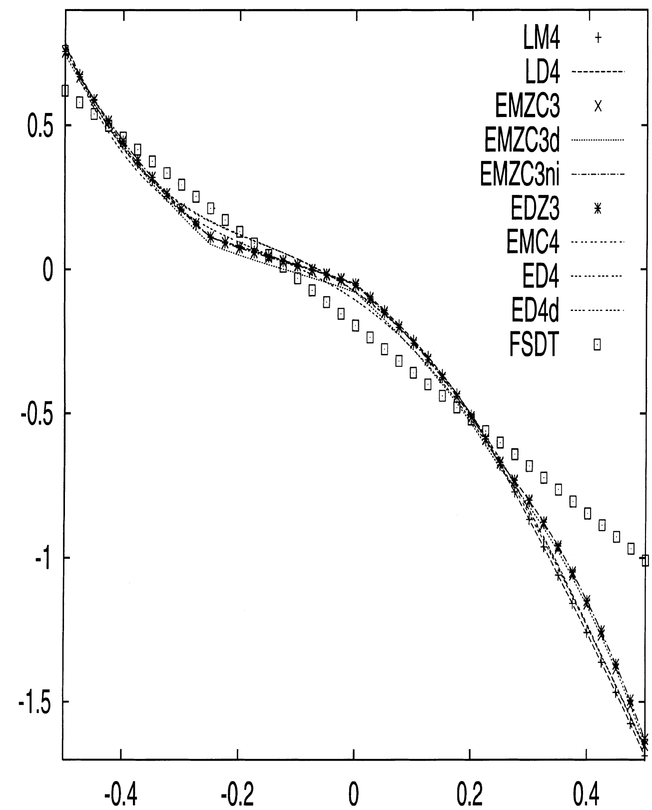


Fig. 10. U_x/h vs z/h. Four layers case of Table 4. $a/h = 5$.

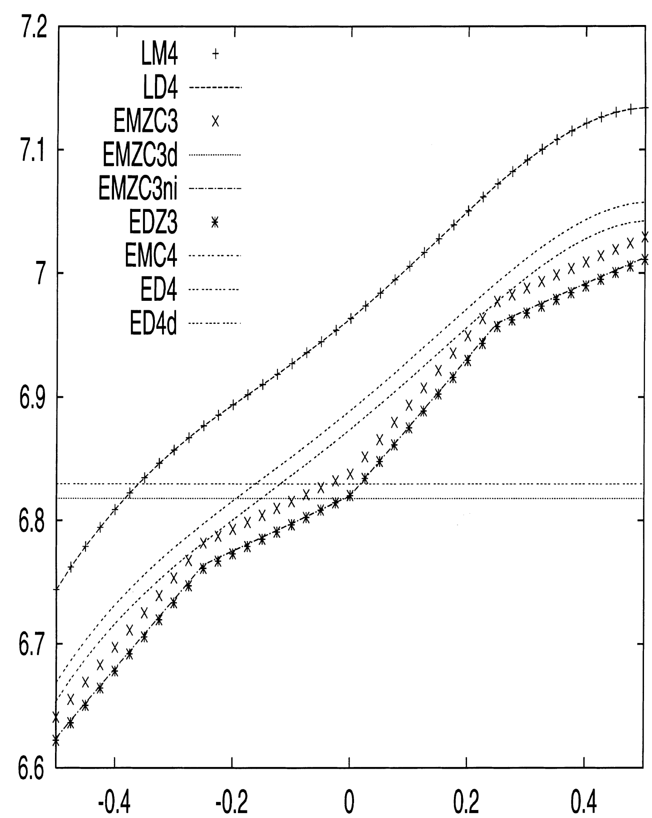


Fig. 11. U_z vs z/h. Four layers case of Table 4. $a/h = 5$.

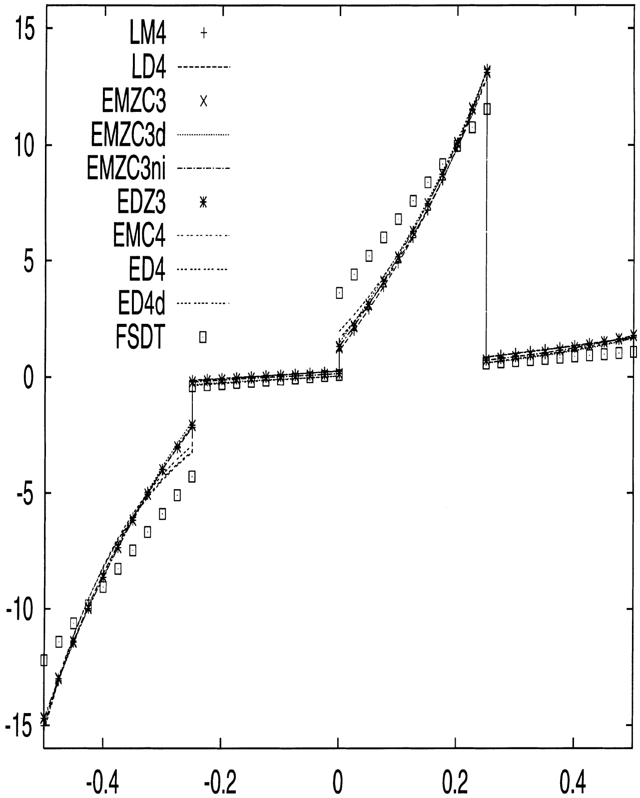


Fig. 12. $S_{xx}/p_{z_t}^{N_l}$ vs z/h. Four layers case of Table 4. $a/h = 5$.

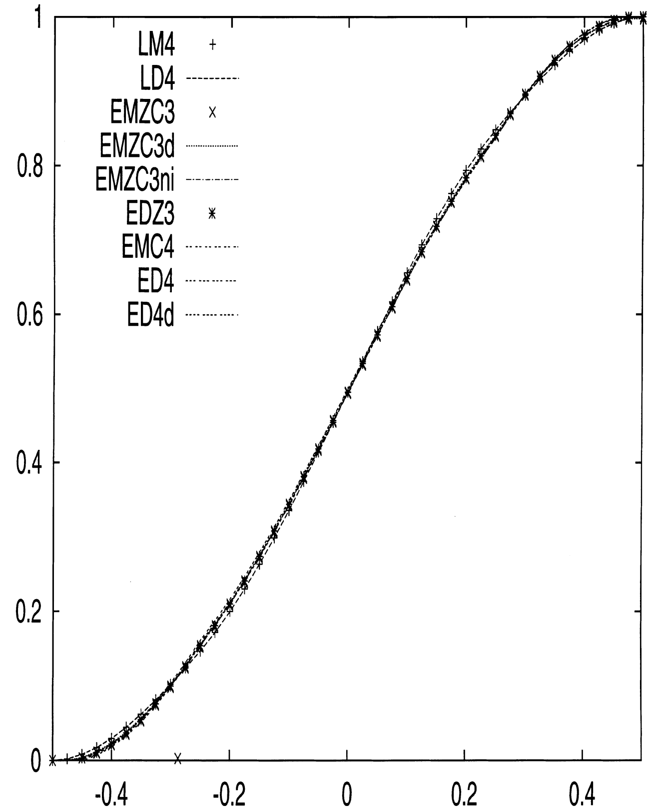


Fig. 14. $S_{zz}/p_{z_t}^{N_l}$ vs z/h. Four layers case of Table 4. $a/h = 5$.

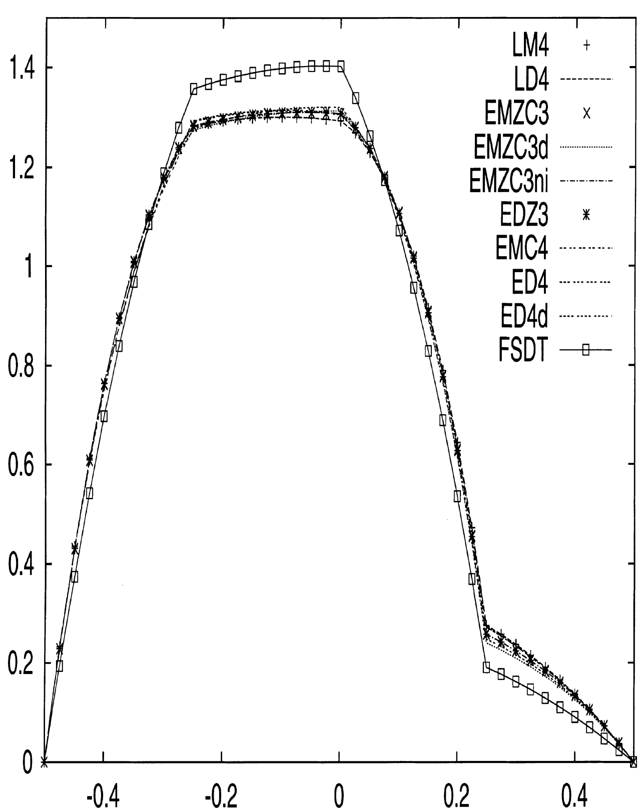


Fig. 13. $S_{xz}/p_{z_t}^{N_l}$ vs z/h. Four layers case of Table 4. $a/h = 5$.

- LWMs are computationally expensive with respect to ESL theories; on the other hand, the use of Layer-wise description is essential for those cases in which an accurate description of σ_{zz} and related effects is required. In other words, E type theories experience difficulties to fulfill KP3.

Further comments are listed below.

1. N_l increasing results of L theories become independent by the used order N, or by M (RMVT) or D (PVD) cases. This is caused by the intrinsic increment of the number of degrees of freedom.
2. The order of the used expansion N plays a very important role especially as far as unsymmetric laminates is concerned. Note that the quadratic expansions are much more effective for unsymmetric laminates. N increasing the differences between LM and LD disappear.
3. The accuracy of classical ED type results decreases by N_l increasing.
4. The ZZ function improves very much the related results: EDZ analysis are more accurate than ED ones. Advantages come also by imposing interlaminar continuity, i.e. EM results are more accurate than ED ones.
5. σ_{zz} can play a predominant role, for instance EMZC3d results can be much worst than EMZC1 ones. σ_{zz} is very much influenced by a/h. Thick plates

show that σ_{zz} leads to unsymmetric distribution of, through the thickness distribution of, local parameters. Such an unsymmetry cannot be described by any two-dimensional which neglects σ_{zz}. See also the 3D solution by Pagano [58] and Pagano and Hatfield [59]. As was pointed out by Koiter [60] one underlines that the order of magnitude of σ_{zz} is the same as that of the transverse shear stresses.

6. N and/or N_l increasing the fulfillment of homogeneous conditions at the top–bottom plate surfaces on transverse stresses in LM and EM type theories, becomes irrelevant. Note that these conditions cannot be imposed in LM1 cases where only top and bottom stress variables are used to force them would signify neglecting of stiffnesses and/or compliance of the top and bottom layers.
7. The use of shear correction factor χ is questionable. As it was demonstrated, see [61] for example, the correct value of shear correction, being related to the form of the distribution of shear stresses, is problem dependent. That is, the χ-value is not known a priori. It can happen that a shear corrected theory which neglects σ_{zz} can lead to better results than analysis in which such a stress is preserved. See also Table 3 were first and higher order modes are considered. Furthermore, shear correction factors can only improve global response they are completely ineffective to improve local response.
8. Plots show that the major discrepancies between the different theories exist with correspondence to the layer interfaces.

The author believes that the numerical assessment given above could serve as a tool to assess new contributions to multilayered plate modelings. In fact, theories able to furnish a three-dimensional description (as it is the case of LM4 and LD4 models) and early classical models (ED1d, ED1d$^{\infty}$ namely CLT, FSDT) have been both considered in this paper. Furthermore, the author recommend that articles proposing improved theories should clearly state: which one of the KP1–KP5 are addressed by the proposed theory and what is the accuracy of obtained results. Such accuracy should be established in both local and global responses.

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