A Reissner's Mixed Variational Theorem Applied to Vibration Analysis of Multilayered Shell

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A comprehensive model of anisotropic multilayered double curved shells fulfilling a priori the interlaminar continuity requirements for the transverse shear and transverse normal stress as well as the static conditions on the bounding surfaces of the shell is developed in this paper. To this end, Reissner's mixed variational theorem is employed to derive the equations governing the dynamic equilibrium and compatibility of each layer, while the interlaminar continuity conditions are used to drive the equations at the multilayered level. No assumptions have been made concerning the terms of type thickness to radii shell ratio h/R. *Classical displacement formulations and related equivalent single layer equations have been derived for comparison purposes. Comparison of frequency predictions based upon the presented structural model with a number of results spread throughout the specialized literature and obtained via other models reveals that this advanced model provides results in excellent agreement with the ones based on three-dimensional elasticity theory, and better as compared to the ones violating the interlaminar stress continuity requirements and/or transverse normal stress and related effects.*

1 Introduction

Due to the noncontinuous distribution of thermomechanical material properties in the thickness direction, the multilayered shells exhibits low values of the transverse properties relative to the in-plane ones and a directional nature of the elastic behavior which is exploited to design material as well as structural properties. Examples of multilayered shell structures used in modern aerospace, automotive, and ship vehicles are laminated constructions made of anisotropic composite materials, sandwich panels, layered structures used as thermal protection or intelligent structural systems embedding piezo-layers. In the recent years, considerable attention is being paid to the development of appropriate two-dimensional shell theories that can accurately describe the static and dynamic response of multilayered anisotropic thick shells. Exhaustive overviews were given in the articles by Kapania (1989) and by Noor and Burton (1990) (see also the recent article by Timarci and Soldatos; 1995).

The directional nature of multilayered structures leads to higher transverse shear and normal stress deformability with respect to traditional isotropic cases. Therefore, further requisites become essential towards a reliable modeling of layered structures. Among them, the fulfillment of both continuity of displacement and transverse shear and normal stresses at the interface between two adjacent layers are such a necessary desideratum. In Carrera, (1995) these requisites were referred to by the acronyms C_z^0-requirements which state that both displacements and transverse stress components are C^0-continuous functions in the thickness shell coordinate z. Approximated and exact three-dimensional solutions by Mirsky (1964), Chau and Achenbach (1972), Srinivas (1974), Noor and co-author (Noor and Rarig, 1974; Noor and Peters, 1989a, 1989b), Ren (1987), Varadan and Bhaskar (1991), and Ye and Soldatos (1994) have numerically confirmed the need of the above-mentioned refinements for static and dynamic analysis of shell problems. In particular, the fundamental role played by transverse normal stress σ_{zz} was underlined. Exact elasticity solutions are known for particular cases only, concerning simple configurations. In the most general cases and to minimize the computational effort, two-dimensional models are preferred in practice. Attempts to introduce C_z^0 requirements in a two-dimensional model have been done by many investigators in the last three decades. Both layer-wise models (LWMs) (it is intended the number of the unknown variables remains dependent on the number of layers) and equivalent single layer models (ESLMs) (these models preserve the independence of the number of the independent variables from the numbers of the layers) have been proposed. Exhaustive overview can be read in the aforementioned review articles by Kapania (1989) and Noor and Burton (1990). More recent works have been reviewed by Soldatos and Timarci (1993) and Timarci and Soldatos (1995). A few of the most important contributions are reviewed in what follows. These are related to works addressing dynamic analysis.

The free-vibration analysis of layered anisotropic shells has been studied by many investigators. Classical ESLM theories were employed by Weingarten (1964), Dong (1968), Bert, Baker, and Egle (1969), and Stavsky and Loewy (1971). Shear deformable models were employed by Soldatos (1985), Reddy (1984), and Carrera (1991). Refined models were implemented by Sun and Whitney (1974). Top and bottom shell surface homogeneous conditions were considered by Reddy and Liu (1985) and by Librescu, Kheider, and Frederick (1989). Rath and Das (1973) fulfilled a priori the interlaminar transverse shear continuity and described the so called zig-zag form for the in-plane displacements in a ESL-theory; the particular case of symmetrical through the thickness response of a layered shell was considered by Di Sciuva (1987) and Di Sciuva and Carrera (1992). A refinement of these last models were proposed by Cho and Parmerter (1993) for the plates case. Further refinements with application to laminated shells were provided by Soldatos and Timarci (1993), Timarci and Soldatos (1995), and Librescu and Lin (1996).

Based on the theory by Ambartsumian (1958), Hsu and Wang (1971) have derived the equations for the free-vibration

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analysis of layered orthotropic cylindrical shell considering each layer separately; e.g., a LW description was referred to. Other LW approaches were adopted by Cheung and Wu (1972), Alam and Asnani (1984), and Barbero, Reddy, and Teply (1990). All the mentioned theories being based on the displacement approach cannot describe completely and a priori the C_z^0-requirements.

For the case of plates, the author in recent articles (Carrera 1995, 1997, 1998a, b) has shown the convenience to refer to a Reissner's mixed variational theorem (RMVT) (Reissner, 1986) as a tool to furnish quasi-three-dimensional evaluations of stresses, displacements, and vibration response of multilayered thick structures. Similar discussion and conclusions were reported in the overview paper by Grigolyuk and Kulikov (1988). RMVT was also applied by Murakami (1986) and Toledano and Murakami (1986; 1987a, b) to the static analysis of plates and extended to shell by Jing and Tzeng (1993). In particular the mentioned author's works have shown that the use of RMVT require a layer-wise description in thick plate analyses. On the bases of the good results obtained for plates (Carrera 1998a, b), this paper extends the dynamic analysis to double curved shell structures. The equations governing the dynamic equilibrium of theories which are based on classical displacement formulation are given, in the Appendix, for comparison purposes.

2 Preliminary

The salient features of shell geometry are shown in Fig. 1. A laminated shell composed of N_l layers is considered. The integer k, used as superscript or subscript, denotes the layer number which starts from the shell bottom. The layer geometry is denoted by the same symbols as those used for the whole multilayered shell and vice versa. α_k and β_k are the curvilinear orthogonal coordinates (coinciding with principal curvature lines) on the layer reference surface Ω_k (middle surface of the k-layer). z_k denotes the rectilinear coordinate in the direction normal to Ω_k. Γ_k is the Ω_k boundary: Γ_k^g and Γ_k^m are those parts of Γ_k on which geometrical and mechanical boundary conditions are imposed, respectively; these boundary are herein considered parallel to α_k or β_k. The further dimensionless thickness coordinate is introduced, $\zeta_k = 2z_k/h_k$; where h_k denotes the thickness in the A_k domain. The following relation holds in the given orthogonal system of curvilinear coordinates for the square of line element, for the area of an infinitesimal rectangle on Ω_k, and for an infinitesimal volume, respectively (Kraus, 1967):

$$ds_k^2 = H_\alpha^k d\alpha_k^2 + H_\beta^k d\beta_k^2 + H_z^k dz_k^2$$
$$d\Omega_k = H_\alpha^k H_\beta^k d\alpha_k d\beta_k$$
$$dV = H_\alpha^k H_\beta^k H_z^k d\alpha_k d\beta_k dz_k \qquad (1)$$

where $H_\alpha^k = A^k(1 + z_k/R_\alpha^k)$, $H_\beta^k = B^k(1 + z_k/R_\beta^k)$, $H_z^k = 1$. R_α^k and R_β^k are the radii of curvature in the directions of α_k and β_k, respectively. A^k and B^k are the coefficients of the first fundamental form of Ω_k. For the sake of simplicity attention is herein restricted to a shell with a constant curvature, i.e., doubly curved shell (cylindrical, spherical, toroidal geometries) for which $A^k = B^k = 1$.

The laminae are considered to be homogeneous and to operate in the linear elastic range. By employing stiffness coefficients, Hooke's law for the anisotropic k-lamina is written in the form $\sigma_i = \tilde{C}_{ij}\epsilon_j$ where the subindices i and j, ranging from 1 to 6, stand for the index couples 11, 22, 33, 13, 23, and 12, respectively. The material is assumed to be orthotropic, as specified, by $\tilde{C}_{14} = \tilde{C}_{24} = \tilde{C}_{34} = \tilde{C}_{64} = \tilde{C}_{15} = \tilde{C}_{25} = \tilde{C}_{35} = \tilde{C}_{65} = 0$. This implies that $\sigma_{\alpha z}^k$ and $\sigma_{\beta z}^k$ depend only on $\epsilon_{\alpha z}^k$ and $\epsilon_{\beta z}^k$. In matrix form

$$\boldsymbol{\sigma}_{pH_d}^k = \tilde{\mathbf{C}}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{pn}^k \boldsymbol{\epsilon}_{nG}^k$$
$$\boldsymbol{\sigma}_{nH_d}^k = \tilde{\mathbf{C}}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{nn}^k \boldsymbol{\epsilon}_{nG}^k \qquad (2)$$

where

$$\tilde{\mathbf{C}}_{pp}^k = \begin{bmatrix} \tilde{C}_{11}^k & \tilde{C}_{12}^k & \tilde{C}_{16}^k \\ \tilde{C}_{12}^k & \tilde{C}_{22}^k & \tilde{C}_{26}^k \\ \tilde{C}_{16}^k & \tilde{C}_{26}^k & \tilde{C}_{66}^k \end{bmatrix}, \quad \tilde{\mathbf{C}}_{pn}^k = \tilde{\mathbf{C}}_{np}^{kT} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13}^k \\ 0 & 0 & \tilde{C}_{23}^k \\ 0 & 0 & \tilde{C}_{36}^k \end{bmatrix},$$

$$\tilde{\mathbf{C}}_{nn}^k = \begin{bmatrix} \tilde{C}_{44}^k & \tilde{C}_{45}^k & 0 \\ \tilde{C}_{45}^k & \tilde{C}_{55}^k & 0 \\ 0 & 0 & \tilde{C}_{66}^k \end{bmatrix}.$$

Boldface letters denote arrays. The superscript T signifies array transposition. The subscripts n and p denote transverse (out-of-plane, normal) and in-plane values, respectively. Therefore $\boldsymbol{\sigma}_p^k = \{\sigma_{\alpha\alpha}^k, \sigma_{\beta\beta}^k, \sigma_{\alpha\beta}^k\}$, $\boldsymbol{\sigma}_n^k = \{\sigma_{\alpha z}^k, \sigma_{\beta z}^k, \sigma_{zz}^k\}$ and $\boldsymbol{\epsilon}_p^k = \{\epsilon_{\alpha\alpha}^k, \epsilon_{\beta\beta}^k, \epsilon_{\alpha\beta}^k\}$, $\boldsymbol{\epsilon}_n^k = \epsilon_{\alpha z}^k, \epsilon_{\beta z}^k, \epsilon_{zz}^k\}$. Subscript H denotes stresses evaluated by Hooke's law while subscript G denotes strain from the geometrical relation Eq. (4). The further subscript d signifies values employed in the displacement formulation (see Appendix). For the adopted mixed solution procedure, the stress-strain relationships are conveniently put in the following mixed form (Librescu, 1975):

$$\boldsymbol{\sigma}_{pH}^k = \mathbf{C}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{pn}^k \boldsymbol{\sigma}_{nM}^k$$
$$\boldsymbol{\epsilon}_{nH}^k = \mathbf{C}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{nn}^k \boldsymbol{\sigma}_{nM}^k \qquad (3)$$

where both stiffness and compliance coefficients are employed. The subscript M states that the transverse stresses are those of the assumed model in Eq. (5) (see the next section). The relation between the arrays of coefficients in the two forms of Hooke's law is simply found,

$$\mathbf{C}_{pp}^k = \tilde{\mathbf{C}}_{pp}^k - \tilde{\mathbf{C}}_{pn}^k \tilde{\mathbf{C}}_{nn}^{k^{-1}} \tilde{\mathbf{C}}_{np}^k, \quad \mathbf{C}_{pn}^k = \tilde{\mathbf{C}}_{pn}^k \tilde{\mathbf{C}}_{nn}^{k^{-1}}$$
$$\mathbf{C}_{np}^k = -\tilde{\mathbf{C}}_{nn}^{k^{-1}} \tilde{\mathbf{C}}_{np}^k, \quad \mathbf{C}_{nn}^k = \tilde{\mathbf{C}}_{nn}^{k^{-1}}.$$

Superscript -1 denotes an inversion of the array.

As one remains within the small deformation field, the strain components $\boldsymbol{\epsilon}_p^k$, $\boldsymbol{\epsilon}_n^k$ are linearly related to the displacements $\mathbf{u}^k$ ($\mathbf{u}^k = u_\alpha^k, u_\beta^k, u_z^k$), according to the following geometrical relations (Kraus, 1967):

$$\boldsymbol{\epsilon}_{pG}^k = \mathbf{D}_p \mathbf{u}^k + \mathbf{A}_p \mathbf{u}^k, \quad \boldsymbol{\epsilon}_{nG}^k = \mathbf{D}_{n\Omega} \mathbf{u}^k + \lambda_D \mathbf{A}_n \mathbf{u}^k + \mathbf{D}_{nz} \mathbf{u}^k \qquad (4)$$

where

$$\mathbf{D}_p = \begin{bmatrix} \dfrac{\partial_\alpha}{H_\alpha^k} & 0 & 0 \\ 0 & \dfrac{\partial_\beta}{H_\beta^k} & 0 \\ \dfrac{\partial_\beta}{H_\beta^k} & \dfrac{\partial_\alpha}{H_\alpha^k} & 0 \end{bmatrix}, \quad \mathbf{A}_p = \begin{bmatrix} 0 & 0 & \dfrac{1}{H_\alpha^k R_\alpha^k} \\ 0 & 0 & \dfrac{1}{H_\beta^k R_\beta^k} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \dfrac{\partial_\alpha}{H_\alpha^k} \\ 0 & 0 & \dfrac{\partial_\beta}{H_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_n = \begin{bmatrix} -\dfrac{1}{H_\alpha^k R_\alpha^k} & 0 & 0 \\ 0 & -\dfrac{1}{H_\beta^k R_\beta^k} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}.$$

λ_D is a trace operator which can assume the values 0 or 1.

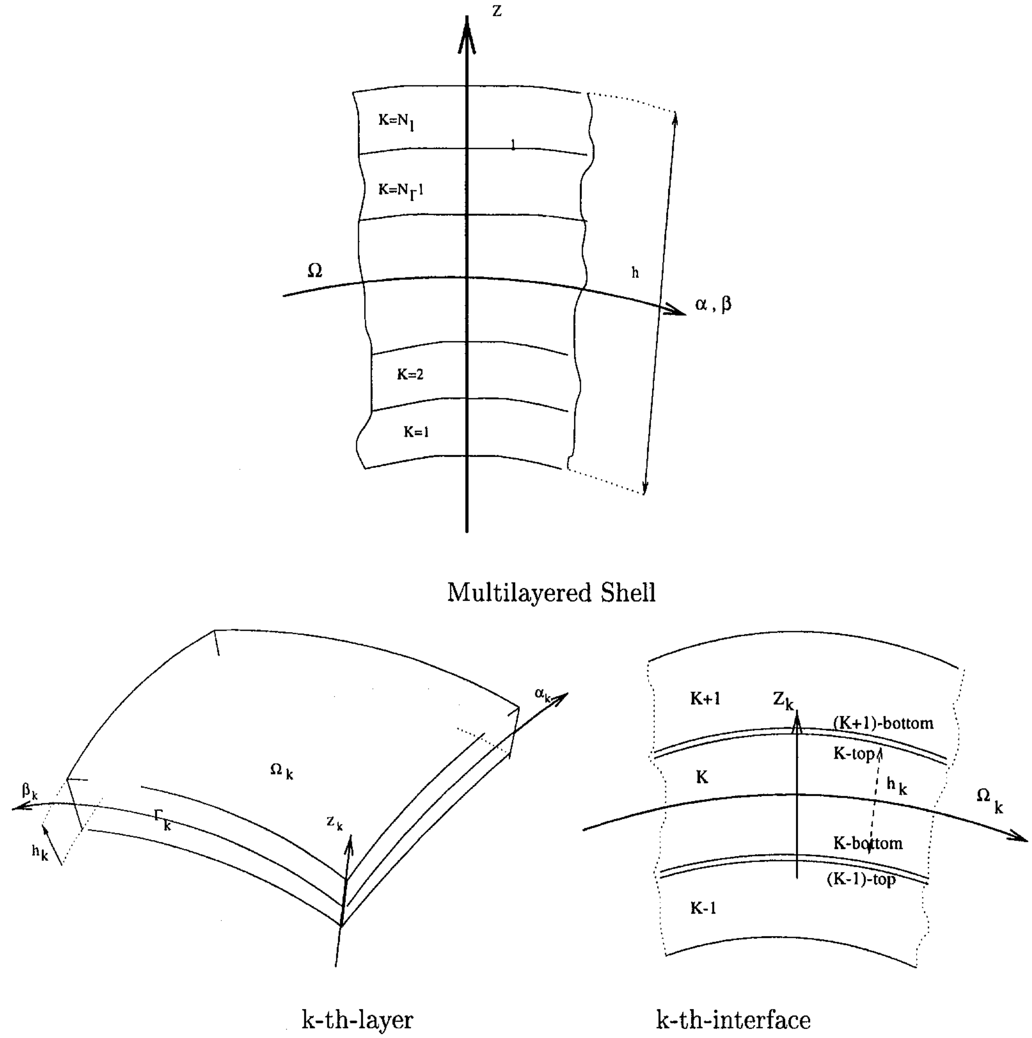

Fig. 1 Geometry and notation of multilayered shells

This has been introduced in order to identify terms which are neglected in the Donnell-type shallow shell theory (Kraus, 1967) which is therefore obtained by putting $\lambda_D = 0$. No assumption have been made for those terms which are divided by H_α^k and H_β^k which are not expanded as Taylor series (Kraus, 1967; Soldatos, 1985; Carrera, 1991). That is, curvature terms have been entirely retained in the following developments.

3 Displacement and Transverse Stress Fields

According to what was carried out for plates (Carrera, 1995, 1998a, b), the C_z^0-requirements are fulfilled completely and a priori by assuming a layer-wise model for both displacement $\mathbf{u}^k$ and transverse stress fields $\boldsymbol{\sigma}_n^k$. The following N-order expansions has been chosen in the thickness direction of each k-layer:

$$\mathbf{u}^k = F_t \mathbf{u}_t^k + F_b \mathbf{u}_b^k + F_r \mathbf{u}_r^k = F_\tau \mathbf{u}_\tau^k$$

$$\boldsymbol{\sigma}_{nM}^k = F_t \boldsymbol{\sigma}_{nt}^k + F_b \boldsymbol{\sigma}_{nb}^k + F_r \boldsymbol{\sigma}_{nr}^k = F_\tau \boldsymbol{\sigma}_{n\tau}^k$$

$$\tau = t, b, r \quad r = 2, 3, \ldots, N \quad k = 1, 2, \ldots, N_l. \quad (5)$$

N is a free parameter of the model. Subscripts t and b denote values related to the layer top and bottom surface, respectively. They consist of the linear part of the expansion. Higher order distributions in the z-direction (parabolic, cubic, etc.) are introduced by the r-polynomials. The repeated indexes τ are summed over their ranges. The thickness functions $F_\tau(\zeta_k)$ have been defined by

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2},$$

$$F_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the j-order defined in the ζ_k-domain: $-1 \le \zeta_k \le 1$. The top and bottom values have been used as unknown variables. The C_z^0-requirements can therefore be easily linked: The compatibility of the displacement and the equilibrium for the transverse stress components read

$$\mathbf{u}_t^k = \mathbf{u}_b^{(k+1)}, \quad k = 1, N_l - 1$$

$$\boldsymbol{\sigma}_{nt}^k = \boldsymbol{\sigma}_{nb}^{(k+1)}, \quad k = 1, N_l - 1$$

$$\boldsymbol{\sigma}_{nb}^1 = \bar{\boldsymbol{\sigma}}_{nb}, \quad \boldsymbol{\sigma}_{nt}^{N_l} = \bar{\boldsymbol{\sigma}}_{nt} \quad (6)$$

where the overbar denotes imposed values in correspondence to static conditions on the bounding surfaces of the shells.

Equations (5) refers to the same order of expansion as the three components for both displacements and transverse stresses. In order to meet well-known results from literature (Kraus, 1967; Sun and Whitney, 1974) or the author's discussion reported in Carrera (1995, 1997), different polynomial orders should be used in developments that are presented in the subsequent sections. Trace operators that would lead to a shortening of the resulting arrays in Eqs. (10), (23) could be introduced for this aim. For the sake of brevity, the results related to these aspects have not been discussed. These could be more conveniently reported in future works.

4 Governing Equations

The equilibrium and compatibility are formulated in terms of the $\mathbf{u}^k$ and $\boldsymbol{\sigma}_n^k$ unknowns via Reissner's mixed variational theorem (Reissner, 1986) that in the shell dynamic case is herein written as

$$\sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}(\delta\boldsymbol{\epsilon}_{pG}^{k^T}\boldsymbol{\sigma}_{pH}^k + \delta\boldsymbol{\epsilon}_{nG}^{k^T}\boldsymbol{\sigma}_{nM}^k + \delta\boldsymbol{\sigma}_{nM}^{k^T}(\boldsymbol{\epsilon}_{nG}^k - \boldsymbol{\epsilon}_{nH}^k))d\Omega_k dz$$

$$= \sum_{k=1}^{N_l}\int_{\Omega_k}\int_{A_k}\rho^k\delta\mathbf{u}^k\ddot{\mathbf{u}}^k dV + \delta L^e. \quad (7)$$

δ signifies variational symbol. ρ_k denotes mass density. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress coming from Hooke's Law and strain coming from geometrical relations. δL_e is the virtual variation of the work done by external surface forces $\mathbf{p}^k = \{p_\alpha^k,\ p_\beta^k,\ p_z^k\}$. The L.H.S. includes the variations of the internal work in the shell: The first two terms come from the displacement formulation, they will lead to variationally consistent equilibrium conditions; the third "mixed" term variationally enforces the compatibility of the transverse strains components.

4.1 Equilibrium and Constitutive Equations for the k-layers. In contrast with most of the available shell literature and to the previous author's works related to plates, in the present analysis the definition of stress or strain resultants in the shell thickness direction has been omitted. Such a choice is mainly due to the author's wish of preserving the terms H_α^k, H_β^k in the strain Eqs. (4). In fact, if the Love's approximations $H_\alpha^k = H_\beta^k = 1$ are not introduced, as is the case in the present article, the definition of stress and strain resultants still remains possible, but according to the author's opinion, not convenient. As a result, along with this paper, the governing equations will be directly written in terms of the introduced stress and displacement variables. By using the array formula for the integration by parts similar to those introduced in Carrera (1998), the RMVT work equations Eq. (7) assumes the following form:

$$\sum_{k=1}^{N_l}\Bigg(\int_{\Omega_k}\{\delta\mathbf{u}_\tau^{k^T}[(-F_\tau\mathbf{D}_p^T + F_\tau\mathbf{A}_p^T)\mathbf{C}_{pp}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)\mathbf{u}_s^k$$

$$+ (-F_\tau\mathbf{D}_p^T + F_\tau\mathbf{A}_p^T)\mathbf{C}_{pn}F_s\boldsymbol{\sigma}_{ns}^k$$

$$+ (-F_\tau\mathbf{D}_{n\Omega}^T + \lambda_D F_\tau\mathbf{A}_n^T + F_{\tau_z})F_s\boldsymbol{\sigma}_{ns}^k]\delta\boldsymbol{\sigma}_\tau^{k^T}[(F_\tau F_s\mathbf{D}_{n\Omega}$$

$$+ \lambda_D F_\tau F_s\mathbf{A}_n + F_\tau F_{s_z} - F_\tau\mathbf{C}_{np}(F_s\mathbf{D}_p + F_s\mathbf{A}_p))\mathbf{u}_s^k$$

$$- F_\tau F_s\mathbf{C}_{nn}\boldsymbol{\sigma}_{ns}^k]\}d\Omega_k$$

$$+ \int_{\Gamma_k}\int_{A_k}\delta\mathbf{u}_\tau^{k^T}[F_\tau\mathbf{I}_p^T\mathbf{C}_{pp}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)\mathbf{u}_s^k$$

$$+ F_\tau F_s\mathbf{I}_p^T\mathbf{C}_{pn}\boldsymbol{\sigma}_{ns}^k + F_\tau F_s\mathbf{I}_{n\Omega}^T\boldsymbol{\sigma}_{ns}^k]d\Gamma_k\Bigg)$$

$$= \sum_{k=1}^{N_l}\int_{\Omega_k}\delta\mathbf{u}_\tau^{k^T}\mathbf{p}_\tau^k d\Omega_k + \sum_{k=1}^{N_l}\int_{\Omega_k}\delta\mathbf{u}_\tau^{k^T}\rho^k F_\tau F_s\ddot{\mathbf{u}}^k d\Omega_k \quad (8)$$

where

$$\mathbf{I}_p = \begin{bmatrix}\frac{1}{H_\alpha^k} & 0 & 0\\ 0 & \frac{1}{H_\beta^k} & 0\\ \frac{1}{H_\beta^k} & \frac{1}{H_\alpha^k} & 0\end{bmatrix};\quad \mathbf{I}_{n\Omega} = \begin{bmatrix}0 & 0 & \frac{1}{H_\alpha^k}\\ 0 & 0 & \frac{1}{H_\beta^k}\\ 0 & 0 & 0\end{bmatrix}$$

and $\mathbf{p}_\tau^k = \{p_{x\tau}^k,\ p_{y\tau}^k,\ p_{z\tau}^k\}$ are the variatianally consistent load vectors coming from the applied loadings $\mathbf{p}^k$. Of practical interest could be the case in which both shearing ($p_{\alpha t}^k$, $p_{\beta t}^k$, $p_{\alpha b}^k$, $p_{\beta b}^k$) and normal (p_{zt}^k, p_{zb}^k) surface forces are applied with correspondence to the top and or bottom surface of the layer, $d\Omega_k^p = d\Omega_k^t = (1 + (h_k/2R_\alpha^k))(1 + (h_k/2R_\beta^k))d\Omega_k$ and $d\Omega_k^p = d\Omega_k^b = (1 - (h_k/2R_\beta^k))d\Omega_k$. By imposing the definition of virtual variations for the unknown stress and displacement variables, the differential system of governing equations and related boundary conditions for the $N_l k$-layers in each Ω_k domain are found. The equilibrium and compatibility equations are

$$\delta\mathbf{u}_\tau^k:\quad \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_s^k + \mathbf{K}_{u\sigma}^{k\tau s}\boldsymbol{\sigma}_{ns}^k = \mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_s^k + \mathbf{p}_\tau^k$$

$$\delta\boldsymbol{\sigma}_{n\tau}^k:\quad \mathbf{K}_{\sigma u}^{k\tau s}\mathbf{u}_s^k + \mathbf{K}_{\sigma\sigma}^{k\tau s}\boldsymbol{\sigma}_{ns}^k = 0 \quad (9)$$

with boundary conditions

$$\text{geometrical on } \Gamma_k^g \quad \text{mechanical on } \Gamma_k^m$$

$$\mathbf{u}_\tau^k = \bar{\mathbf{u}}_\tau^k \quad \text{or} \quad \boldsymbol{\Pi}_u^{k\tau s}\mathbf{u}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s}\boldsymbol{\sigma}_{ns}^k = \boldsymbol{\Pi}_u^{k\tau s}\bar{\mathbf{u}}_s^k + \boldsymbol{\Pi}_\sigma^{k\tau s}\bar{\boldsymbol{\sigma}}_{ns}^k \quad (10)$$

in which the bar denotes the assigned. The introduced differential arrays are given by the following relations:

$$\mathbf{K}_{uu}^{k\tau s} = \int_{A_k}(-F_\tau\mathbf{D}_p^T + F_\tau\mathbf{A}_p^T)\mathbf{C}_{pp}^k(F_s\mathbf{D}_p + F_s\mathbf{A}_p)H_\alpha^k H_\beta^k dz_k$$

$$\mathbf{K}_{u\sigma}^{k\tau s} = \int_{A_k}[(-F_\tau\mathbf{D}_p^T + F_\tau\mathbf{A}_p^T)\mathbf{C}_{pn}^k F_s + F_{\tau_z}F_s\mathbf{I}$$

$$+ \lambda_D F_\tau F_s\mathbf{A}_n - F_\tau F_s\mathbf{D}_{n\Omega}^T]H_\alpha^k H_\beta^k dz_k$$

$$\mathbf{K}_{\sigma u}^{k\tau s} = \int_{A_k}\{F_\tau F_s\mathbf{D}_{n\Omega} + \lambda_D F_\tau F_s\mathbf{A}_n + F_\tau F_{s_z}\mathbf{I}$$

$$- \mathbf{C}_{np}^{k\tau s}(F_\tau F_s\mathbf{D}_p + F_\tau F_s\mathbf{A}_p)\}H_\alpha^k H_\beta^k dz_k$$

$$\mathbf{K}_{\sigma\sigma}^{k\tau s} = -\int_{A_k}F_\tau F_s\mathbf{C}_{nn}^{k\tau s}H_\alpha^k H_\beta^k dz_k$$

$$\boldsymbol{\Pi}_u^{k\tau s} = \int_{A_k}F_\tau\mathbf{I}_p^T\mathbf{C}_{pp}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)\mathbf{I}_p^T H_\alpha^k H_\beta^k dz_k$$

$$\boldsymbol{\Pi}_\sigma^{k\tau s} = \int_{A_k}(F_\tau F_s\mathbf{I}_p^T\mathbf{C}_{pn} + F_\tau F_s\mathbf{I}_{n\Omega}^T)H_\alpha^k;\quad H_\beta^k dz_k$$

$$\mathbf{M}^{k\tau s} = \int_{A_k}\rho^k F_\tau F_s\mathbf{I}H_\alpha^k H_\beta^k dz_k. \quad (11)$$

$\mathbf{I}$ is the unit array. As usual in two-dimensional modelings, the integration in the thickness direction can be made a priori by introducing the following layer-integrals (the further integrals related to the displacement formulation, (see Appendix) are also introduced),

$$(J^{k\tau s},\ J_\alpha^{k\tau s},\ J_\beta^{k\tau s},\ J_{\alpha/\beta}^{k\tau s},\ J_{\beta/\alpha}^{k\tau s},\ J_{\alpha\beta}^{k\tau s})$$

$$= \int_{A_k}F_\tau F_s\left(1,\ H_\alpha^k,\ H_\beta^k,\ \frac{H_\alpha^k}{H_\beta^k},\ \frac{H_\beta^k}{H_\alpha^k},\ H_\alpha^k H_\beta^k\right)dz$$

$$(J^{k\tau_z s},\ J_\alpha^{k\tau_z s},\ J_\beta^{k\tau_z s},\ J_{\alpha\beta}^{k\tau_z s}) = \int_{A_k}F_{\tau_z}F_s(1,\ H_\alpha^k,\ H_\beta^k,\ H_\alpha^k H_\beta^k)dz$$

$$(J^{k\tau s_z}, J_{\alpha}^{k\tau s_z}, J_{\beta}^{k\tau s_z}, J_{\alpha\beta}^{k\tau s_z}) = \int_{A_k} F_\tau F_{s_z}(1, H_\alpha^k, H_\beta^k, H_\alpha^k H_\beta^k) dz$$

$$(J^{k\tau_z s_z}, J_{\alpha\beta}^{k\tau_z s_z}) = \int_{A_k} F_{\tau_z} F_{s_z}(1, H_\alpha^k H_\beta^k) dz. \quad (12)$$

As a further step, the differential and algebraic operators can be conveniently split in the two terms related to the H_α^k and H_α^k, respectively,

$$(\mathbf{D}_p, \mathbf{A}_p, \mathbf{D}_{n\Omega}, \mathbf{A}_n, \mathbf{I}_p, \mathbf{I}_{n\Omega}) = \frac{1}{H_\alpha}(\mathbf{D}_p^\alpha, \mathbf{A}_p^\alpha, \mathbf{D}_{n\Omega}^\alpha, \mathbf{A}_n^\alpha, \mathbf{I}_p, \mathbf{I}_{n\Omega})$$
$$+ \frac{1}{H_\beta}(\mathbf{D}_p^\beta, \mathbf{A}_p^\beta, \mathbf{D}_{n\Omega}^\beta, \mathbf{A}_n^\beta, \mathbf{I}_p^\beta, \mathbf{I}_{n\Omega}^\beta). \quad (13)$$

Therefore, the differential operators of Eqs. (11) are written

$$\mathbf{K}_{uu}^{k\tau s} = (-\mathbf{D}_p^{\alpha^T} + \mathbf{A}_p^{\alpha^T})\mathbf{C}_{pp}$$
$$\times [J_{\beta/\alpha}^{k\tau s}(\mathbf{D}_p^\alpha + \mathbf{A}_p^\alpha) + J^{k\tau s}(\mathbf{D}_p^\beta + \mathbf{A}_p^\beta)]$$
$$+ (-\mathbf{D}_p^{\beta^T} + \mathbf{A}_p^{\beta^T})\mathbf{C}_{pp}[J_{\alpha/\beta}^{k\tau s}(\mathbf{D}_p^\alpha + \mathbf{A}_p^\alpha) + J^{k\tau s}(\mathbf{D}_p^\beta + \mathbf{A}_p^\beta)]$$
$$\mathbf{K}_{u\sigma}^{k\tau s} = (-J_\beta^{k\tau s}\mathbf{D}_p^{\alpha^T} - J_\alpha^{k\tau s}\mathbf{D}_p^{\beta^T} + J_\alpha^{k\tau s}\mathbf{A}_p^{\beta^T} + J_\beta^{k\tau s}\mathbf{A}_p^{\alpha^T})\mathbf{C}_{pn}^k$$
$$+ J_{\alpha\beta}^{k\tau_z s}\mathbf{I} + \lambda_D(J_\beta^{k\tau s}\mathbf{A}_n^{\alpha^T} + J_\alpha^{k\tau s}\mathbf{A}_n^{\beta^T}) - J_\beta^{k\tau s}\mathbf{D}_{n\Omega}^{\alpha^T} - J_\alpha^{k\tau s}\mathbf{D}_{n\Omega}^{\beta^T}$$
$$\mathbf{K}_{\sigma u}^{k\tau s} = -\mathbf{C}_{np}^k(J_\beta^{k\tau s}\mathbf{D}_p^\alpha + J_\alpha^{k\tau s}\mathbf{D}_p^\beta + J_\alpha^{k\tau s}\mathbf{A}_p^\beta + J_\beta^{k\tau s}\mathbf{A}_p^\alpha)$$
$$+ J_{\alpha\beta}^{k\tau s_z}\mathbf{I} + \lambda_D(J_\beta^{k\tau s}\mathbf{A}_n^\alpha + J_\alpha^{k\tau s}\mathbf{A}^{\beta_n}) + J_\beta^{k\tau s}\mathbf{D}_{n\Omega}^\alpha + J_\alpha^{k\tau s}\mathbf{D}_{n\Omega}^\beta$$
$$\mathbf{K}_{\sigma\sigma}^{k\tau s} = -J_{\alpha_\beta}^{k\tau s}C_{nn}^{k\tau s}$$
$$\mathbf{\Pi}_u^{k\tau s} = (J_{\beta/\alpha}^{k\tau s}\mathbf{I}_p^{\alpha^T} + J^{k\tau s}\mathbf{I}_p^{\beta^T})\mathbf{C}_{pp}(\mathbf{D}_p^\alpha + \mathbf{A}_p^\alpha)$$
$$+ (J^{k\tau s}\mathbf{I}_p^{\alpha^T} + J_{\alpha/\beta}^{k\tau s}\mathbf{I}_p^{\beta^T})\mathbf{C}_{pp}(\mathbf{D}_p^\beta + \mathbf{A}_p^\beta)$$
$$\mathbf{\Pi}_\sigma^{k\tau s} = (J_\beta^{k\tau s}\mathbf{I}_p^{\alpha^T} + J_\alpha^{k\tau s}\mathbf{I}_p^{\beta^T})\mathbf{C}_{pn} + J_\beta^{k\tau s}\mathbf{I}_{n\Omega}^{\alpha^T} + J_\alpha^{k\tau s}\mathbf{I}_{n\Omega}^{\beta^T}. \quad (14)$$

The inertia array is found

$$\mathbf{M}_{ij}^{k\tau s} = J_{\alpha\beta}^{k\tau s}\delta_{ij}, \quad i, j = 1, 3 \quad (15)$$

where the Kroneker symbol δ_{ij} has been introduced. Cylindrical shell equations are simply obtained by enforcing $R_\alpha = \infty$ (or $R_\beta = \infty$) while spherical shell geometries correspond to the case $R_\alpha = R_\beta$. Discarding the terms multiplied by the trace operators λ_D, the corresponding Donnell's shallow shell equations are obtained. Neglecting all the curvature terms the governing equation written for multilayered plates (Carrera 1998b) are given as particular cases.

4.2 Fulfillment of C_z^0-Requirements and Governing Equations at a Multilayered Level. In the preceding sections the governing shell equations have been considered to be independent of the N_l layers. Interlaminar continuity conditions, i.e., C_z^0-requirements in Eqs. (6) must be used to drive the governing equations for the whole multilayered shell. This can be done by following the same steps described in previous works (Carrera, 1995, 1998a, b). Here it is only remarked that in contrast to plates, it is essential to have the same product of type $d\alpha^* d\beta^*$ in each k-layer equations for shell geometry due to the curvature changes. This can be done by choosing the value on the reference shell surface $\Omega d\alpha d\beta$ as $d\alpha^* d\beta^*$. Each $d\alpha_k d\beta_k$ defined on Ω_k is therefore written as $d\alpha_k d\beta_k = (R_\alpha R_\beta / R_\alpha^k R_\beta^k) d\alpha\, d\beta$. The following shell arrays for the displacement and stress unknowns are introduced at the very end,

$$\mathbf{u} = \{\mathbf{u}_b^{1^T}, \mathbf{u}_t^{1^T}, \mathbf{u}_r^{1^T};\ \ \mathbf{u}_t^{2^T}, \mathbf{u}_r^{2^T};\ \ \ldots\ \mathbf{u}_t^{k^T}, \mathbf{u}_r^{k^T};$$
$$\mathbf{u}_t^{(k+1)^T}, \mathbf{u}_r^{(k+1)^T};\ \ \ldots\ \mathbf{u}_t^{(N_l-1)^T}, \mathbf{u}_r^{(N_l-1)^T};\ \mathbf{u}_t^{N_l^T}, \mathbf{u}_r^{N_l^T}\}$$
$$\boldsymbol{\sigma}_n = \{\boldsymbol{\sigma}_{nt}^{1^T}, \boldsymbol{\sigma}_{nr}^{1^T};\ \ \boldsymbol{\sigma}_{nt}^{2^T}, \boldsymbol{\sigma}_{nr}^{2^T};\ \ \ldots\ \boldsymbol{\sigma}_{nt}^{k^T}, \boldsymbol{\sigma}_{nr}^{k^T};$$
$$\boldsymbol{\sigma}_{nt}^{(k+1)^T}, \boldsymbol{\sigma}_{nr}^{(k+1)^T};\ \ \ldots\ \boldsymbol{\sigma}_{nt}^{(N_l-1)^T}, \boldsymbol{\sigma}_{nr}^{(N_l-1)^T};\ \ \boldsymbol{\sigma}_{nr}^{N_l^T}\}, \quad (16)$$

in which the top and r-variables have been chosen as layer unknowns. The dimensions M_u and M_σ of these two arrays depend on the number of layers N_l as well as on the order of the used expansion N according to the following formula:

$$M_u = 3(N_l + 1) + 3\,NN_l, \quad M_\sigma = M_u - 6.$$

Therefore, the governing system of differential equations at a multilayered level is formally written in the following final form:

$$\mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\sigma}\boldsymbol{\sigma}_n = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{p} + \mathbf{p}_u^{1N_l}$$
$$\mathbf{K}_{\sigma u}\mathbf{u} + \mathbf{K}_{\sigma\sigma}\boldsymbol{\sigma}_n = \mathbf{p}_\sigma^{1N_l}, \quad (17)$$

while the boundary conditions are

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{or} \quad \mathbf{\Pi}_u\mathbf{u} + \mathbf{\Pi}_\sigma\boldsymbol{\sigma}_n = \mathbf{\Pi}_u\bar{\mathbf{u}} + \mathbf{\Pi}_\sigma\bar{\boldsymbol{\sigma}}_n + \mathbf{q}_\sigma^{1N_l}. \quad (18)$$

$\mathbf{p}_u^{1N_l}$, $\mathbf{p}_\sigma^{1N_l}$, and $\mathbf{q}_\sigma^{1N_l}$ are arrays coming from transverse stress values imposed at the top/bottom of the shell surfaces.

4.3 Closed-Form Solutions for Shells Composed of Orthotropic Layers. The boundary value problem governed by Eqs. (17), (18) in the most general case of geometry, boundary conditions, and layouts, could be solved by only implementing approximate solution procedures. In order to assess the proposed models these equations are herein solved for a special case in which closed-form solutions are given. The particular case in which the material has the following properties (as it is the case of cross-ply shells) $\tilde{C}_{16} = \tilde{C}_{26} = \tilde{C}_{36} = \tilde{C}_{45} = 0$ has been considered, for which Navier-type closed-form solutions can be found by assuming the following harmonic forms for the applied loadings $\mathbf{p}^k = \{p_{\alpha_\tau}^k, p_{\beta_\tau}^k, p_{z_\tau}^k\}$ and unknown displacement $\mathbf{u}^k = \{u_{\alpha_\tau}^k, u_{\beta_\tau}^k, u_{z_\tau}^k\}$ and stress $\boldsymbol{\sigma}_n^k = \{\sigma_{\alpha z_\tau}^k, \sigma_{\beta z_\tau}^k, \sigma_{zz_\tau}^k\}$ variables in each k-layer:

$$(u_{\alpha_\tau}^k, \sigma_{\alpha z_\tau}^k, p_{\alpha_\tau}^k)$$
$$= \sum_{m,n}(U_\alpha^k, S_{\alpha z_\tau}^k, P_{\alpha_\tau}^k)\cos\frac{m\pi\alpha_k}{a_k}\sin\frac{n\pi\beta_k}{b_k}e^{i\omega_{mn}\hat{t}} \quad k = 1, N_l$$
$$(u_{\beta_\tau}^k, \sigma_{\beta z_\tau}^k, p_{\beta_\tau}^k)$$
$$= \sum_{m,n}(U_\beta^k, S_{\beta z_\tau}^k, P_{\beta_\tau}^k)\sin\frac{m\pi\alpha_k}{a_k}\cos\frac{n\pi\beta_k}{b_k}e^{i\omega_{mn}\hat{t}} \quad \tau = t, b, r$$
$$(u_{z_\tau}^k, \sigma_{zz_\tau}^k, p_{z_\tau}^k)$$
$$= \sum_{m,n}(U_z^k, S_{zz_\tau}^k, P_{z_\tau}^k)\sin\frac{m\pi\alpha_k}{a_k}\sin\frac{n\pi\beta_k}{b_k}e^{i\omega_{mn}\hat{t}} \quad r = 2, N \quad (19)$$

which correspond to simply supported boundary conditions. a_k and b_k are the shell lengths in the α_k and β_k directions, respectively, while m and n are the correspondent wave numbers; $i = \sqrt{-1}$, $\hat{t}$ is the time and ω_{mn} is the circular frequency. Capital letters at R.H.S. denote correspondent maximum amplitudes. Upon substitution of Eqns. (19), the governing equations assume the form of a linear system of ordinary differential equations in the time domain. The free-vibration response leads to an eigenvalue problem. Upon eliminations of the stress unknowns, the mixed case leads to

$$\|\hat{\mathbf{K}}_{uu} - (\hat{\mathbf{K}}_{u\sigma}(\hat{\mathbf{K}}_{\sigma\sigma})^{-1}\hat{\mathbf{K}}_{\sigma u}) - \omega_{mn}^2\hat{\mathbf{M}}\| = 0. \quad (20)$$

The double bar denotes determinant, while the hat indicates arrays constituted by real numbers.

5 Results and Discussion

As an attempt to test and evaluate the foregoing mixed models, the theories were applied to a large number of homogeneous and laminated cylindrical and spherical shell problems. The most significant results are described in the following analysis.

Table 1 List of the acronyms used to denote shell theories, in alphabetic order

D-l$^{N_l^*}$, D-p$^{N_l^*}$	Present LWMs based on displacements formulation, p and l refer to parabolic and linear expansion, N_l^* is the numbers of mathematical interfaces
ESLM	Present Equivalent Single Layer Model
E-l, E-p	Present ESLM with linear and parabolic displacements fields.
E-l.b	E-l neglecting transverse normal stress
3D	Three dimensional solution
LWM	Layer Wise Model
M-p$^{N_l^*}$,M-l$^{N_l^*}$	Present Mixed LWMs.
.Δ	Donnell-Type Approximations

The free-vibrational response of simply supported, cross-ply laminated cylindrical and spherical shells has been analyzed. Three-dimensional solutions as well as related analyses based on displacement formulations (of layer-wise LW and equivalent single-layer ESLM type), is compared. A compendium of the used acronyms to denote the theories have been given in Table 1. Continuous reference to these acronyms is made in the subsequent discussion. In a few cases the reported tables quote results related to Donnell's type approximations. Static conditions on the bounding surfaces of the shell could not be imposed for the mixed linear expansion (M-1) of Eqs. (5). In fact, their linkage would require discarding the stiffness and/or compliance contributions related to the top and bottom shell layers. These conditions have therefore not been considered in all the quoted M-1 results. Fictitious mathematical interfaces have been introduced in the calculation in order to both improve the displacement and stress fields and to reduce the curvature approximations related to the thickness to radii shell ratio h/R. The total number of interfaces is denoted by N_l^*. If it is different to N_l, it will be put as a superscript in correspondence to the acronyms of the considered layer-wise results. Mostly the case $N_l^* = 2 \times N_l$ has been implemented.

As a preliminary result a static problem has been solved. The effectiveness of the layer-wise mixed analysis to describe a priori the transverse stress field has been shown in Fig. 2. A cross-ply laminated cylinder, loaded by harmonic distribution of transverse pressure of amplitude $P_{z_b}^1$, applied at the inner surface has been considered. Exact solutions were given by Varadan and Bhaskar (1991). A comparison has been made with further results by the same authors (Bhaskar and Varadan, 1992) in which the Reissner's theorem was used in the field of ESLM analysis. An excellent agreement between the exact studies and the present M-p a priori evaluations can be seen. A poor description was found for the ESLM analysis by Bhaskar and Varadan (1992) even though the transverse shear stresses were evaluated via integration of the three-dimensional indefinite equilibrium equations (H.b-B&V). Furthermore, the related transverse shear stresses evaluated a priori (H.a-B&V) demonstrate, as it was found for plates (Carrera, 1998a), that such an evaluation requires a layer-wise description.

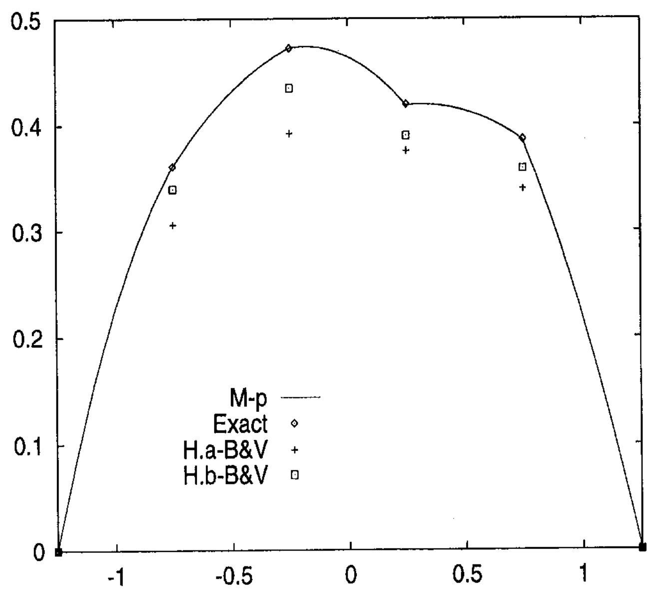

Fig. 2 Amplitude of transverse shear stress $\bar{S}_{\alpha z} = |S_{\alpha z} \times 10h/P_{z_b}^1 R_\beta|$ versus z. Varadan and Bhaskar's (1991) cylindrical shells with $R_\beta/h = 4$, $h = 2.5$, $m = 1$, $n = 8$. Five layers case, $E_L/E_T = 25$, $G_{LT}/E_T = .5$, $G_{TT} = .2$, $\nu_{LT} = \nu_{TT} = .25$.

Table 2 Effect of degree of orthotropy of the individual layers E_L/E_T on $\omega \times 10\sqrt{\rho h^2/E_T}$. Comparison to three-dimensional solutions by Noor and Rarig (1974). $h/R_\beta = 0.2$, $a/R_\beta = 1$, $m = 1$, $n = 4$.

	N_l	E_L/E_T 3	10	40
3D	2	2.3141	2.5464	2.9262
M-p		2.3139	2.5461	2.9248
M-p^4		2.3141	2.5464	2.9260
M-l		2.3217	2.5538	2.9328
D-p		2.3150	2.5514	2.9594
D-l		2.3292	2.5626	2.9666
E-p		2.3219	2.5589	2.9822
E-l		2.3450	2.5826	3.0367
E-l.b		2.3812	2.6019	3.0756
3D	3	2.3173	2.6542	3.1675
M-p		2.3173	2.6541	3.1663
M-p^6		2.3173	2.6542	3.1673
M-l		2.3153	2.6472	3.1358
D-p		2.3181	2.6589	3.1979
D-l		2.3522	2.6665	3.2087
E-p		2.3266	2.6842	3.3205
E-l		2.3577	2.7066	3.3322
E-l.b		2.3798	2.7186	3.3497
3D	5	2.3767	2.8245	3.4631
M-p		2.3768	2.8245	3.7506
M-p^{10}		2.3767	2.8245	3.4633
M-l		2.3768	2.8264	3.7538
D-p		2.3768	2.8248	3.7507
D-l		2.3808	2.8364	3.7611
E-p		2.3908	2.8805	3.8972
E-l		2.4201	2.8981	3.9198
E-l.b		2.5761	3.2320	4.0927
3D	10	2.4357	2.9856	3.7506
M-p		2.4357	2.9856	3.7506
M-p^{20}		2.4357	2.9856	3.7506
M-l		2.4358	2.9863	3.7538
D-p		2.4357	2.9856	3.7507
D-l		2.4366	2.9885	3.7611
E-p		2.4488	3.0281	3.8972
E-l		2.4768	3.0467	3.9198
E-l.b		2.4975	3.0548	3.9281

A few parametric studies have been presented to provide some insight into the effects of variation in material and geometric characteristics of laminated shells on their vibration characteristic. Tables 2–5 consider laminated circular cylinders having both symmetric and skew-symmetric lamination with respect to the middle surface for which approximate three-dimensional solution were given by Noor and Rarig (1974) and Noor and Peters (1989b). The fibers of the different layers alternate between the longitudinal α and circumferential β directions, with the fibers of the top layer running in the circumferential direction. The total thickness of the circumferential and longitudinal layers in each shell was the same. The material characteristics of the individual layers were taken to be those of typical of high-modulus fibrous composites, namely $G_{LT}/E_T = G_{Lz}/E_T = .6$, $G_{TT}/E_T = .5$, $\nu_{LT} = \nu_{TT} = .25$, where the subscripts L refer to the direction of fibers and subscript T refers to the transverse

Table 3 Effect of thickness to radii ratio h/R_β on $\omega \times 10\sqrt{\rho h^2/E_T}$. Comparison to three-dimensional solutions by Noor and Rarig (1974). $E_L/E_T = 30$, $a/R_\beta = 1$, $m = 1$.

	h/R_β	n = 2	4	6
		N_l=2		
3D	.05	0.8165	0.5385	0.4218
M-p		0.8165	0.5385	0.4218
D-p		0.8166	0.5387	0.4220
E-l		0.8169	0.5394	0.4238
M-p.Δ		0.8213	0.5450	0.4309
3D	.25	4.4910	3.8047	4.1584
M-p		4.4903	3.8023	4.1539
D-p		4.5194	3.8498	4.2196
E-l		4.6020	3.9952	4.4293
M-p.Δ		4.5772	3.9556	4.4368
3D	.40	7.5953	6.9568	7.9209
M-p		7.5873	6.9410	7.8976
M-p[4]		7.5948	6.9537	7.9156
M-p[8]		7.5970	6.9572	7.9202
D-p		7.6780	7.0766	8.0744
D-p[4]		7.6051	6.9690	7.9353
D-p[8]		7.5979	6.9584	7.9218
E-l		7.9928	7.5598	8.6944
M-p.Δ		7.7572	7.2678	8.3377
		N_l=10		
3D	.05	0.8410	0.5832	0.5027
M-p		0.8410	0.5832	0.5027
D-p		0.8410	0.5832	0.5027
E-l		0.8414	0.5844	0.5055
M-p.Δ		0.8422	0.5881	0.5148
3D	.25	5.1280	4.7990	5.4710
M-p		5.1281	4.7992	5.4712
D-p		5.1283	4.7993	5.4714
E-l		5.2651	5.0361	5.8226
M-p.Δ		5.1604	4.9648	5.7233
3D	.40	8.6111	8.4732	9.8040
M-p		8.6116	8.4736	9.8043
D-p		8.6119	8.4742	9.8053
E-l		8.9576	9.0508	10.0629
M-p.Δ		8.6928	8.8329	10.2779

Table 4 Effect of length to radii ratio a/R_β on $\omega \times 10\sqrt{\rho h^2/E_T}$. Comparison to three-dimensional solutions by Noor and Rarig (1974). $N_l = 10$, $E_L/E_T = 30$, $h/R_\beta = .2$, $m = 1$.

	a/R_β	n = 2	4	6
3D	.5	8.5280	8.0598	8.1694
M-p		8.5280	8.0597	8.1693
D-p		8.5285	8.0602	8.1699
E-l		8.9135	8.5952	8.7780
M-p.Δ		8.5440	8.1194	8.2645
3D	1.	3.9756	3.5961	4.0428
M-p		3.9757	3.5962	4.0429
D-p		3.9758	3.5962	4.0430
E-l		4.0538	3.7355	4.2559
M-p.Δ		3.9966	3.7062	4.2239
3D	5.	0.6627	1.0499	2.3511
M-p		0.6632	1.0503	2.3512
D-p		0.6632	1.0503	2.3513
E-l		0.6634	1.0877	2.4932
M-p.Δ		0.7226	1.3530	2.6408
3D	20.	0.1039	0.9663	2.3294
M-p		0.1065	0.9667	2.3295
D-p		0.1066	0.9668	2.3296
E-l		0.1066	1.0061	2.4719
M-p.Δ		0.3073	1.2882	2.6215

direction; ν_{LT} is the major Poisson's ratio. Three parameters were varied in these tables, namely the degree of orthotropy of the individual layers E_L/E_T, the thickness ratio h/R and the

Table 5 Effect of longitudinal and circumferential wave numbers on $\omega \times \sqrt{\rho h^2/E_T}$. Comparison to three-dimensional solutions by Noor and Peters (1989b). $N_l = 40$, $E_L/E_T = 15$, $G_{LT}/E_T = .5$, $G_{TT}/E_T = .35$, $\nu_{LT} = \nu_{TT} = .3 a/R_\beta = 5.$, $h/R_\beta = .2$.

	m	n = 0	4	10
3D	1	0.08886	0.08365	0.4489
M-p		0.08886	0.08365	0.4489
D-p		0.08886	0.08365	0.4489
E-l		0.08886	0.08596	0.4789
3D	2	0.1777	0.1175	0.4545
M-p		0.1777	0.1175	0.4545
D-p		0.1777	0.1175	0.4545
E-l		0.1777	0.1199	0.4847
3D	3	0.2666	0.1695	0.4681
M-p		0.2666	0.1695	0.4681
D-p		0.2666	0.1695	0.4681
E-l		0.2666	0.1728	0.4987

length to thickness ratio a/R as well as circumferential and longitudinal modes m and n. As a main fact one remarks that the present layer-wise mixed analyses are, in general, in very good agreement with three-dimensional results even though very thick shells are considered. It is of further interest to notice that stress and displacement formulations lead to a lower and higher estimate of the shell stiffness as far as the three-dimensional solutions are concerned. In fact, D-p and D-1 frequency values are always higher than the exact ones. In the mixed cases a unique conclusion cannot be drawn: This very much depends on the relative role played by the in-plane and out-of-plane energy and the results obviously become problem-dependent. For most of the considered problems it has been found that mixed and displacements solutions lead to lower and higher approximations for the fundamental and higher order frequencies, respectively. The following further comments can be made about the results of these tables. Thickness, E_L/E_t and N_l decreasing the ESLMs results improve. Lower accuracy should be registered for both higher circumferential and longitudinal modes. In some cases ESLM results related to parabolic expansion (E-p) could be better than those related to LW analysis with linear expansion in each layer (D-1). Transverse normal stress effects can remarked by comparing E-1 to E-1.b results. No shear correction factors have been used for the quoted ESLM results. As demonstrated by Noor and Peters (1989a, b), the values of exact shear correction factors is problem-dependent. Therefore, the use of a fixed value could be meaningless as long as thick shells are analyzed. On the other hand, the accuracy of LW results increase by N_l increasing (the numbers of the unknown variables increases). In fact, fictitious interfaces (Tables 2, 3) show major benefits for laminates composed by two or three layers as well as for thicker shells. The increasing accuracy obtained by using fictitious interfaces confirm the convergence of the proposed layer-wise models to the three-dimensional solutions. One-dimensional effects along the longitudinal directions become predominant in long cylinders and M-p and D-p merge in such cases (Table 4). This is confirmed by the results related to the $n = 0$ analysis, e.g., pure longitudinal vibrations (Table 5). Donnell's shallow shell approximations are not affected by h/R_β and in general the related curvature terms should be retained for thicker as well as thin closed cylinders. The errors introduced by such approximations could be not conservative. In other words, two-dimensional refinements of the classical model could be meaningless unless curvature terms are discarded (see also Carrera, 1991).

A comparison to the recent three-dimensional exact solution by Ye and Soldatos (1994) and to several refined models quoted in Timarci and Soldatos (1995) and Di Sciuva and Carrera (1992) have been provided in Table 6. A three-layered moderately thick cylindrical shell has been considered. An excellent

Table 6 Effect of radii to length ratio R/a on $\omega \times a^2\sqrt{\rho/h^2E_T}$. Comparison to three-dimensional exact solution by Ye and Soldatos (1994) and to other refined ESLM analyses (Timarci and Soldatos, 1995; Di Sciuva and Carrera, 1992). $a/h = 10$, $m = 1$, $n = 2$ unless given in brackets. $N_l = 3$, $h_1 = h_3 = h_2/2$. $E_L/E_T = 25$, $G_{LT}/E_T = .5$, $G_{TT}/E_T = .2$, $\nu_{LT} = \nu_{TT} = .25$.

R_β/a	5	10	20	50	100
3D	10.305	10.027	9.902	9.834	9.815
	(14)	(22)	(30)	(24)	
M-p	10.305	10.027	9.902	9.834	9.815
	(14)	(22)	(30)	(24)	
D-p	10.306	10.028	9.903	9.835	9.816
	(14)	(22)	(30)	(24)	
PAR_{ds}	10.496	10.223	10.099	10.032	10.013
			(28)	(26)	
HYP_{ds}	10.496	10.226	10.103)	10.036	10.018
			(28)	(26)	
UNI_{cs}	10.462	10.187	10.063	9.996	9.977
			(28)	(26)	
PAR_{cs}	10.329	10.051	9.927	9.859	9.840
				(26)	
HYP_{cs}	10.328	10.050	9.925	9.858	9.839
				(26)	
ZZL	10.462	10.187	10.063	9.996	9.971
	(14)	(24)	(28)	(16)	(4)

agreement between present mixed analysis M-p and an exact solution has to be registered. Better results with respect to standard displacement formulation D-p are found. The value quoted in brackets accompanying some of the numerical results in Table 6 indicates the circumferential wave number, n, for which the fundamental frequency was detected. All the theories considered in the above-mentioned references belong to the ESLM family. Further, they all do neglect transverse normal stress effects. Uniform UNI, parabolic PAR, and hyperbolic HYP transverse shear stress distribution in the thickness shell direction where considered as well as the two cases of interlaminar discontinuous ds and continuous cs shear stresses (see Timarci and Soldatos, 1995, for further details). Uniform distribution cases do not fulfill static conditions at the top and bottom shell surfaces. ZZL results refer to Di Sciuva and Carrera's (1992) analysis which is the same as that of UNI_{cs}. The fundamental mode, e.g., n-values can be erroneously predicted by ESLM analysis. The importance of fulfilling the interlaminar transverse shear stress continuity has been confirmed by the present analysis (see the interesting discussion in Timarci and Soldatos (1995)). Nevertheless, the role played by the transverse normal stress should be underlined. Such a role could become more predominant for thicker shells. The numerical analysis of this aspect could be the subject of future investigations.

Only a few results for square spherical shell panels have been considered in Tables 7–8. Table 7 gives a comparison of the present studies with the first-order shear deformation theories (FSDT) by Reddy (1984) the higher-order shear deformation Theory (HSDT) by Reddy and Liu (1984), for three-layer cross-plied panels. Classical lamination theories (CLT) results were taken from (Carrera, 1991). Increasing $R = R_\alpha = R_\beta$

Table 7 Effect of radii to length ratio R/a on $\omega \times \sqrt{a^4\rho/h^2E_T}$. Spherical, square cross-plied panels. $a/h = 10$, $m = n = 1$. $N_l = 3$, $a/R_\beta = 5.$, $h/R_\beta = .2$, material of Table 6.

R/a	1	2	5	10	plate
M-p	16.043	12.889	11.720	11.531	11.457
D-p	16.048	12.896	11.727	11.538	11.464
E-l	16.271	13.714	12.785	12.641	12.593
M-p.Δ	16.682	13.122	11.762	11.542	11.457
D-p.Δ	16.688	13.129	11.790	11.549	11.464
E-l.Δ	17.010	13.975	12.832	12.653	12.593
HSDT	-	-	12.060	11.860	11.790
FSDT	16.115	13.382	12.372	12.215	12.162
CLT	17.820	15.878	15.233	15.136	15.104

Table 8 Effect of radii to length ratio R/a on $\omega \times \sqrt{\rho h^2/E_T}$. Spherical, square cross-plied panels. Material data are those of Tables 2–4 with $El/Et = 40$. $a/h = 5$, $m = n = 1$.

	N_l	R/a .5	1	10
M-p	2	.5478	.4232	.3441
M-p[4]		.5471	.4233	.3441
D-p		.5505	.4290	.3510
E-l		.4828	.4062	.3662
M-p.Δ		.6081	.4487	.3445
D-p.Δ		.6142	.4553	.3513
E-l.Δ		.5315	.4315	.3666
M-p	3	.5546	.4683	.4291
M-p[6]		.5539	.4681	.4292
D-p		.5557	.4712	.4323
E-l		.4978	.4650	.4588
M-p.Δ		.6205	.4999	.4296
D-p.Δ		.6229	.5031	.4327
E-l.Δ		.5446	.4989	.4592

values are considered. Quite different results were found from the referenced ESLM analyses. As R/a is increased better results are furnished by Donnell's descriptions. Thicker, symmetric, and unsymmetric cross-ply laminated spherical shells are considered in Table 8. Larger difference between LW and ESLM analysis have been found with respect to Table 7. To confirm the importance of the curvature terms in LW analysis one should observe the lower discrepancy of the layer-wise results related to the different N_l^* when R/a (i.e., R/h) increases (Table 8).

6 Concluding Remarks

Layer-wise theories have been presented for the dynamic analysis of multilayered shells made by orthotropic layers. The theories have been formulated on the bases of a Reissner's mixed variational theorem. Classical displacement formulations and related equivalent single-layer equations have been also considered for comparison purposes. With respect to existing theories, the proposed mixed models permit fulfilling completely and a priori the continuity conditions at the interfaces for both displacement and transverse stress components (transverse displacement and transverse normal stress included). That is, no post-processing techniques are requested to compute transverse stresses. No assumptions have been made concerning the terms related to the thickness-to-radii h/R shell ratio. The equations related to the Donnell's shallow shell-type approximations have been derived for all the considered theories.

Parametric studies were made of the effects of variations in the lamination and geometric parameters of cylindrical and spherical simply supported shells on their free-vibration characteristics. Solutions obtained by the proposed mixed models were compared with those related to available three-dimensional elasticity analysis as well as with those of available and implemented ESLM and LW results. The conducted analysis confirm that the proposed mixed models furnish an excellent agreement to exact three-dimensional even though very thick anisotropic shells were considered.

As a drawback the layer-wise description requires the use of unknown variables, the number of which depends on the number of layers. Such a drawback should be taken into account wherever computational models, e.g., finite element applications have to be developed. Nevertheless, the most expensive layer-wise analysis performed in this article (which is based on simple closed-form solution procedures) took only a few c.p.u. seconds on a standard personal computer. However, the mixed layer-wise theories preserve the advantages of two-dimensional modeling and lead to an excellent agreement with exact three-dimensional results. This fact should encourage the use of the proposed theories to design vibrational response of multilayered structures or their use to provide reference solutions in order to assess simplified models.

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APPENDIX

Related Theories Based on Displacement Formulation Layer-Wise Models

The displacement approach is formulated in terms of $\mathbf{u}^k$ by variatianally imposing the equilibrium via classical principle of virtual displacements,

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} (\delta \boldsymbol{\epsilon}_{p_G}^{k^T} \boldsymbol{\sigma}_{p_{H_d}}^k + \delta \boldsymbol{\epsilon}_{n_G}^{k^T} \boldsymbol{\sigma}_{n_{H_d}}^k) d\Omega_k dz = \sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \rho^k \delta \mathbf{u}^k \ddot{\mathbf{u}}^k dV + \delta L^e. \quad (21)$$

Upon introducing Eqs. (3), (5), and (4) and following the same procedure developed for the mixed case, the Eq. (21) leads to

$$\sum_{k=1}^{N_l} \Bigg(\int_{\Omega_k} \int_{A_k} \delta \mathbf{u}_\tau^{k^T} \{ (-F_\tau \mathbf{D}_p^T + F_\tau \mathbf{A}_p^T) [\tilde{\mathbf{C}}_{pp} (F_s \mathbf{D}_p + F_s \mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{pp} (F_s \mathbf{D}_{n\Omega} + \lambda_D F_s \mathbf{A}_n + F_{s_z})]$$
$$+ (-F_\tau \mathbf{D}_{n\Omega}^T + \lambda_D F_\tau \mathbf{A}_n^T + F_{\tau_z}) [\tilde{\mathbf{C}}_{np} (F_s \mathbf{D}_p + F_s \mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{nn} (F_s \mathbf{D}_{n\Omega} + \lambda_D F_s \mathbf{A}_n + F_{s_z})] \} \mathbf{u}_s d\Omega_k$$
$$+ \int_{\Gamma_k} \int_{A_k} \delta \mathbf{u}_\tau^{k^T} \{ F_\tau \mathbf{I}_p^T [\tilde{\mathbf{C}}_{pp} (F_s \mathbf{D}_p + F_s \mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{pp} (F_s \mathbf{D}_{n\Omega} + \lambda_D F_s \mathbf{A}_n + F_{s_z})]$$
$$+ F_\tau \mathbf{I}_{n\Omega}^T [\tilde{\mathbf{C}}_{np} (F_s \mathbf{D}_p + F_s \mathbf{A}_p)$$

$$+ \tilde{\mathbf{C}}_{nn}(F_s\mathbf{D}_{n\Omega} + \lambda_D F_s \mathbf{A}_n + F_{s_z})]\}\mathbf{u}_s d\Gamma_k\Big)$$

$$= \sum_{k=1}^{N_l} \int_{\Omega_k} \delta\mathbf{u}_\tau^{k^T} \mathbf{p}_\tau^k d\Omega_k^p + \sum_{k=1}^{N_l} \int_{\Omega_k} \delta\mathbf{u}_\tau^{k^T} \rho^k F_\tau F_s \ddot{\mathbf{u}}^k. \quad (22)$$

The differential system of governing equations and related boundary conditions follows.

$$\delta\mathbf{u}_\tau^k: \quad \mathbf{K}_d^{k\tau s}\mathbf{u}_s^k = \mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_s^k + \mathbf{p}_\tau^k$$

$$\text{geometrical on } \Gamma_k^g \quad \text{mechanical on } \Gamma_k^m$$

$$\mathbf{u}_\tau^k = \bar{\mathbf{u}}_\tau^k \quad \text{or} \quad \mathbf{\Pi}_{1d}^{k\tau s}\mathbf{u}_s^k = \mathbf{\Pi}_d^{k\tau s}\bar{\mathbf{u}}_s^k \quad (23)$$

The introduced differential arrays are

$$\mathbf{K}_d^{k\tau s} = \int_{A_k} \{(-F_\tau \mathbf{D}_p^T + F_\tau \mathbf{A}_p^T)[\tilde{\mathbf{C}}_{pp}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{pn}(F_s\mathbf{D}_{n\Omega} + \lambda_D F_s\mathbf{A}_n + F_{s_z})]$$
$$+ (-F_\tau \mathbf{D}_{n\Omega}^T + \lambda_D F_\tau \mathbf{A}_n^T + F_{\tau_z})[\tilde{\mathbf{C}}_{np}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{nn}(F_s\mathbf{D}_{n\Omega} + F_s\mathbf{A}_n + F_{s_z})]\} H_\alpha^k H_\beta^k dz_k$$

$$\mathbf{\Pi}_d^{k\tau s} = \int_{A_k} \{F_\tau \mathbf{I}_p^T[\tilde{\mathbf{C}}_{pp}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{pp}(F_s\mathbf{D}_{n\Omega} + \lambda_D F_s\mathbf{A}_n + F_{s_z})]$$
$$+ F_\tau \mathbf{I}_{n\Omega}^T[\tilde{\mathbf{C}}_{np}(F_s\mathbf{D}_p + F_s\mathbf{A}_p)$$
$$+ \tilde{\mathbf{C}}_{nn}(F_s\mathbf{D}_{n\Omega} + \lambda_D F_s\mathbf{A}_n + F_{s_z})]\} H_\alpha^k H_\beta^k dz_k. \quad (24)$$

The definitions given by Eqs. (12) and (13) can be introduced in the previous arrays in a similar to what done for the mixed case. For the sake of brevity the resulting formula are not given.

Equivalent Single-Layer Models. As in classical shell analysis, the assumed model Eq. (5) can also be written at a multilayered level,

$$\mathbf{u} = F_t\mathbf{u}_t + F_b\mathbf{u}_b + F_r\mathbf{u}_r = F_\tau\mathbf{u}_\tau$$

$$\boldsymbol{\sigma}_{nM} = F_t\boldsymbol{\sigma}_{nt} + F_b\boldsymbol{\sigma}_{nb} + F_r\boldsymbol{\sigma}_{nr} = F_\tau\boldsymbol{\sigma}_{n\tau}, \quad (25)$$

in which the functions F_τ are expressed in terms of the shell coordinate $\zeta = 2z/h$ and correspondent ESLM theories can be derived. The use of top and bottom values as variables is not mandatory for standard displacement formulation. Classical Taylor expansions are more convenient for this purpose. In such a case $F_t = 1$, $F_b = z$, $F_r = z^r$; $\mathbf{u}^t$ becomes the displacement vector of a shell point lying on Ω and $\mathbf{u}^b$, $\mathbf{u}^r$ their higher order derivatives.

By repeating what has been done for the layer-wise case, the governing equations are formally written in the same manner. In fact, it is sufficient to introduce the following multilayered 3×3 stiffness arrays:

$$(\mathbf{E}_{ij}^{\tau s}, \mathbf{E}_{\alpha_{ij}}^{\tau s}, \mathbf{E}_{\beta_{ij}}^{\tau s}, \mathbf{E}_{\alpha/\beta_{ij}}^{\tau s}, \mathbf{E}_{\beta/\alpha_{ij}}^{\tau s}, \mathbf{E}_{\alpha\beta_{ij}}^{\tau s})$$
$$= \sum_{k=1}^{N_l} \tilde{\mathbf{C}}_{ij}^k (J^{k\tau s}, J_\alpha^{k\tau s}, J_\beta^{k\tau s}, J_{\alpha/\beta}^{k\tau s}, J_{\beta/\alpha}^{k\tau s}, J_{\alpha\beta}^{k\tau s})$$

$$(\mathbf{E}_{ij}^{\tau_z s}, \mathbf{E}_{\alpha_{ij}}^{\tau_z s}, \mathbf{E}_{\beta_{ij}}^{\tau_z s}, \mathbf{E}_{\alpha\beta_{ij}}^{\tau_z s}) = \sum_{k=1}^{N_l} \tilde{\mathbf{C}}_{ij}^k (J^{k\tau_z s}, J_\alpha^{k\tau_z s}, J_{z\beta}^{k\tau_z s}, J_{\alpha\beta}^{k\tau_z s})$$

$$(\mathbf{E}_{ij}^{\tau s_z}, \mathbf{E}_{\alpha_{ij}}^{\tau s_z}, \mathbf{E}_{\beta ij}^{\tau s_z}, \mathbf{E}_{\alpha\beta_{ij}}^{\tau s_z}) = \sum_{k=1}^{N_l} \tilde{\mathbf{C}}_{ij}^k (J^{k\tau s_z}, J_\alpha^{k\tau s_z}, J_\beta^{k\tau s_z}, J_{\alpha\beta}^{k\tau s_z})$$

$$(\mathbf{E}_{ij}^{\tau_z s_z}, \mathbf{E}_{\alpha\beta_{ij}}^{\tau_z s_z}) = \sum_{k=1}^{N_l} \tilde{\mathbf{C}}_{ij}^k (J^{k\tau_z s_z}, J_{\alpha\beta}^{k\tau_z s_z})$$

$$\mathbf{E}_\rho^{\tau s} = \sum_{k=1}^{N_l} \mathbf{I}\rho_k J_{\alpha\beta}^{k\tau s}. \quad (26)$$

They consist of a 3×3 array for each τ, s couple and they result in a four index array. The index coupled ij can assume the four expressions pp, pn, np, nn and are related to Hooke's law in Eq. (3). It is intended that both function F_τ, F_s and terms H_α^k and H_β^k in J^k-type integrals, have to be considered as functions of the whole shell coordinate z for ESLM cases.