Layer-Wise Mixed Models for Accurate Vibrations Analysis of Multilayered Plates

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This paper presents the dynamic analysis of multilayered plates using layer-wise mixed theories. With respect to existing two-dimensional theories at the displacement formulated, the proposed models a priori fulfill the continuity of transverse shear and normal stress components at each interface between two adjacent layers. A Reissner's mixed variational equation is employed to derive the differential equations, in terms of the introduced stress and displacement variables, that govern the dynamic equilibrium and compatibility of each layer. The continuity conditions at the interfaces are used to write corresponding equations at multilayered level. Related standard displacement formulations, based on the principle of virtual displacements, are given for comparison purposes. Numerical results are presented for the free-vibration response (fundamental and higher order frequencies are calculated) of symmetrically and unsymmetrically laminated cross-ply plates. Several comparisons to three-dimensional elasticity analysis and to some available results, related to both layer-wise and equivalent single-layer theories, have shown that the presented mixed models: (1) match the exact three-dimensional results very well and (2) lead to a better description in comparison to results related to other available analysis.

Introduction

Classical plate theories originally developed for traditional isotropic one-layer plates and based on Poisson-Kirchooff assumptions (Poisson, 1829; Kirchhoff, 1850) (CLT–Classical lamination Theory), have been proven to be inadequate in predicting the behavior of anisotropic plates (Yang et al., 1966). Furthermore, Reissner-Mindlin (Reissner, 1945; Mindlin, 1951) type models (FSDT—First Shear Deformation Theory) can lead to large errors in the prediction of local response of moderately thick laminated plates. In fact, three-dimensional solutions (Pagano, 1969; Srinivas et al., 1970; Noor, 1973; Noor and Burton, 1989a) have shown that both the displacement and transverse stress fields, for compatibility and equilibrium reasons, have a piecewise continuous distribution in the thickness plate direction of the multilayered structures: They reveal discontinuous derivatives in correspondence to each interface which cannot be described by CLT or FSDT type analyses. The so-called *zig-zag* form of displacement fields and the *interlaminar continuity* for the transverse stresses were summarized by Carrera (1995) with the acronym C_z^0-requirements: Displacement and transverse stress fields are C^0-continuous functions in the plate thickness direction z.

Attempts to partially introduce the C_z^0-requirements have been made both in the field of the Equivalent Single Layer Model (ESLM) and Layer-Wise Models (LWM). In comparison to LWM analyses, the ESLM ones preserve the independence of the number of the unknowns from the numbers of the layers, permitting advantageous extension to computational mechanics. Many review papers of exhaustive literature have appeared on that topic; among these one can mention those by Sun and Whitney (1973), Librescu and Reddy (1986), Reddy (1987), Kapania and Raciti (1989), and Noor and Burton (1989b). A more comprehensive and detailed overview has been provided in a recent book by Reddy (1997). A few contributions that have been considered useful by the author and which are related overall, to dynamic analyses, are discussed in the following text.

ELSM analyses were considered by Reddy and Phan (1985) who extended an original plate model by Vlasov (1957) to the dynamics of laminate structures. The used cubic in-plane displacement fields permitted the fulfillment of top and bottom plate homogeneous conditions. Further discussion and analyses were provided by Librescu and Kheider (1988). Di Sciuva (1987) introduced a linear zig-zag in-plane displacement field which accounted for interlaminar continuity. Top and bottom plate conditions were introduced in this last work by Bhaskar and Varadan (1989) while the generalization to arbitrarily laminated structures was presented by Cho and Parmerter (1993). These two last articles were restricted to static analysis. An almost equivalent approach, based on early works by Lekhnitskii (1935), was presented by Ren and Owen (1989). Due to the mechanical coupling between transverse and in-plane normal stresses, all the proposed ESLM studies could not accurately describe the transverse stress and related effects (Carrera, 1997). As a consequence the analyses have shown severe limitations in the study of the vibration of thick plates with arbitrary layouts (Nosier et al., 1992; Cho et al., 1992). The so-called predictor-corrector methods should be mentioned to complete the picture. These methods improve the ESLM results by implementing a post-processing technique to the three-dimensional equations (Noor and Burton, 1989a).

A much better description can be obtained by the use of layer-wise models. The first attempts to consider each layer in a sandwich structures as a separate beam were made by Kao and Ross (1968) and Shift and Heller (1974). Seide (1980) expanded the idea to laminated plates by considering each layer as an individual Reissner-Mindlin plate. Reddy (1987) and Nosier et al., (1992) treated each layer separately using in-plane displacements linear in the thickness direction z. Valisetty and Rehfield (1988) treated each layer separately, by employing a higher-order displacement field for flexural wave propagation

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analysis in laminated plates. Cho et al. (1991) suggested using the Hildebrand et al. (1949) (applied to laminate plates by Lo et al., 1974) higher-order displacement field in each individual layer. All the mentioned LW models, in particular those by Cho et al. (1991) and Nosier et al. (1992), showed good agreement with respect to exact three-dimensional analysis to predict free-vibration response. However, all these layer-wise models did not a priori fulfill the transverse shear stress continuity: The fulfillment of the C_z^0-requirement was restricted to the displacement variables.

The limitations of the analyses based on the displacement formulation motivated the works by Reissner (1984, 1986) who proposed a mixed variational theorem for the displacements and transverse stresses. This equation was employed by Murakami (1985) and Toledano and Murakami (1987a) in the field of ESLM. Even though the transverse normal stress was considered by Toledano and Murakami (1987a), the obtained results showed severe limitations in the analysis of thick generally laminated plates. Such a conclusion motivated another work by the same authors (Toledano and Murakami, 1987b) in which a layer-wise description was introduced. The normal stress was unfortunately discarded in this work.

The author (Carrera 1995, 1997) has recently proposed mixed layer-wise models based on Reissner's equation that are able to completely fulfill the C_z^0-requirements. This means that with respect to available models, Carrera's theory a priori fulfills the interlaminar equilibrium of the transverse shear and normal stresses and permits their evaluation without requiring any post-processing process as those used in other LWM analysis or in the predictor-corrector procedures by Noor and Burton (1989a). This theory was subsequently applied to several problems related to the static analysis of multilayered thick arbitrarily laminated plates (Carrera 1998a, 1998b). The obtained results showed the superiority of the mixed models, with respect to available layer-wise analyses, to evaluate both global and local characteristics. Based on these positive responses this paper extends the mixed layer-wise models to linear dynamics. Numerical results are given for the free-vibration response of cross-ply laminated plates; it has been anticipated that the good performance of the new models will be confirmed for the evaluation of fundamental and higher-order frequencies. Bold letters are here used for arrays. Indicial notations, subscripts and superscripts, have been used extensively in order to handle, in a concise manner, the presented derivations. The Appendix quotes the explicit forms of the introduced basic arrays.

Basic Assumptions

Geometry and Materials. The geometry and Cartesian coordinate system x, y, z of the multilayered plates made of N_l layers are shown in Fig. 1. The lamina are considered homogeneous and to operate in the linear elastic range. Stiffness coefficients of Hooke's law for the anisotropic k-lamina are employed in standard form. This reads $\sigma_i^k = \tilde{C}_{ij}^k \epsilon_j^k$ where subindices i and j, ranging from 1 to 6, stand for the index couples 11, 22, 33, 13, 23, and 12, respectively. The material is assumed to be orthotropic, as specified by: $\tilde{C}_{14} = \tilde{C}_{24} = \tilde{C}_{34} = \tilde{C}_{64} = \tilde{C}_{15} = \tilde{C}_{25} = \tilde{C}_{35} = \tilde{C}_{65} = 0$. This implies that σ_{xz}^k and σ_{yz}^k depend only on ϵ_{xz}^k and ϵ_{yz}^k. In matrix form,

$$\boldsymbol{\sigma}_{pH_d}^k = \tilde{\mathbf{C}}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{pn}^k \boldsymbol{\epsilon}_{nG}^k$$

$$\boldsymbol{\sigma}_{nH_d}^k = \tilde{\mathbf{C}}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \tilde{\mathbf{C}}_{nn}^k \boldsymbol{\epsilon}_{nG}^k \qquad (1)$$

in which subscripts p, n denote in-plane and out-of-plane (normal) components ($\boldsymbol{\sigma}_p^k = \{\sigma_{xx}^k, \sigma_{yy}^k, \sigma_{xy}^k\}$, $\boldsymbol{\sigma}_n^k = \{\sigma_{xz}^k, \sigma_{yz}^k, \sigma_{zz}^k\}$); H and G indicate values from Hooke's law and from geometrical Eq. (3) relations, respectively; and the further subscript d denotes values related to the standard displacement formulation.

For the adopted mixed solution procedure, the stress-strain relationships are conveniently put in the following mixed form:

$$\boldsymbol{\sigma}_{pH}^k = \mathbf{C}_{pp}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{pn}^k \boldsymbol{\sigma}_{nM}^k$$

$$\boldsymbol{\epsilon}_{nH}^k = \mathbf{C}_{np}^k \boldsymbol{\epsilon}_{pG}^k + \mathbf{C}_{nn}^k \boldsymbol{\sigma}_{nM}^k \qquad (2)$$

where both stiffness and compliance coefficients are employed. M denotes stress from the assumed model Eq. (4). The relations between the arrays of coefficients at the two forms of Hooke's law are easily found,

$$\mathbf{C}_{pp}^k = \tilde{\mathbf{C}}_{pp}^k - \tilde{\mathbf{C}}_{pn}^k (\tilde{\mathbf{C}}_{nn}^k)^{-1} \tilde{\mathbf{C}}_{np}^k, \quad \mathbf{C}_{pn}^k = \tilde{\mathbf{C}}_{pn}^k (\tilde{\mathbf{C}}_{nn}^k)^{-1}$$

$$\mathbf{C}_{np}^k = -(\tilde{\mathbf{C}}_{nn}^k)^{-1} \tilde{\mathbf{C}}_{np}^k, \quad \mathbf{C}_{nn}^k = (\tilde{\mathbf{C}}_{nn}^k).$$

Superscripts -1 indicates the inverse of a square array.

The strain components $\boldsymbol{\epsilon}_p^k = \{\epsilon_{xx}^k, \epsilon_{yy}^k, \epsilon_{xy}^k\}$, $\boldsymbol{\epsilon}_n^k = \{\epsilon_{xz}^k, \epsilon_{yz}^k, \epsilon_{zz}^k\}$ are linearly related to the displacements $\mathbf{u}^k = \{u_x^k, u_y^k, u_z^k\}$ according to the following geometrical relations:

$$\boldsymbol{\epsilon}_{pG}^k = \mathbf{D}_p \mathbf{u}^k, \quad \boldsymbol{\epsilon}_{nG}^k = \mathbf{D}_n \mathbf{u}^k. \qquad (3)$$

$\mathbf{D}_p$ and $\mathbf{D}_n$ denote in-plane and out-of-plane differential operators (see the Appendix), respectively.

Displacement and Transverse Stress Models. In order to *completely* and a priori fulfill the C_z^0-requirements a layer-wise model is assumed for both displacement $\mathbf{u}^k$ and transverse stress fields $\boldsymbol{\sigma}_n^k$. In comparison on previous works by the author, the developments are here restricted to parabolic expansion. The following expansions, in the thickness direction of each k-layer ($k = 1, \ldots, N_l$) are used:

$$\mathbf{u}^k = F_t \mathbf{u}_t^k + F_b \mathbf{u}_b^k + F_2 \mathbf{u}_2^k = F_\tau \mathbf{u}_\tau^k$$

$$\boldsymbol{\sigma}_{nM}^k = F_t \boldsymbol{\sigma}_{nt}^k + F_b \boldsymbol{\sigma}_{nb}^k + F_2 \boldsymbol{\sigma}_{n2}^k = F_\tau \boldsymbol{\sigma}_{n\tau}^k. \qquad (4)$$

Subscripts t and b denote values related to the top and bottom surface of the layer, respectively. They consist of the linear part of the expansion while the F_2 polynomials permit quadratic displacement and stress fields. Contracted forms have been written by means of τ-index ($\tau = t, b, 2$); repeated indexes are summed over their ranges.

The thickness functions $F_\tau(\zeta_k)$ ($\zeta = 2z_k/h_k$ not dimensional k-layer thickness coordinates, z_k local thickness coordinate, and h_k the layer thickness, see Fig. 1) are defined by

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_2 = P_2 - P_0,$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of the j-order ($j = 0, 1, 2$). The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1 : F_t = 1; \quad F_b = 0; \quad F_2 = 0 \\ -1 : F_t = 0; \quad F_b = 1; \quad F_2 = 0. \end{cases}$$

That is, the top and bottom values have been used as unknowns. Based on this fact, the C_z^0-requirements can be easily linked. In fact, the compatibility of the displacement and the equilibrium for the transverse stress components are written simply as

$$\mathbf{u}_t^k = \mathbf{u}_b^{(k+1)}, \quad k = 1, N_l - 1$$

$$\boldsymbol{\sigma}_{nt}^k = \boldsymbol{\sigma}_{nb}^{(k+1)}, \quad k = 1, N_l - 1. \qquad (5)$$

In those cases in which the top/bottom-plate stress values are prescribed (homogeneous or nonhomogeneous conditions), the following additional C_z^0-requirements must be accounted for

$$\boldsymbol{\sigma}_{nb}^1 = \bar{\boldsymbol{\sigma}}_{nb}, \quad \boldsymbol{\sigma}_{nt}^{N_l} = \bar{\boldsymbol{\sigma}}_{nt}. \qquad (6)$$

The bar denotes imposed values.

Equations (4) refer to the same order of expansion of the three components for both displacements and transverse

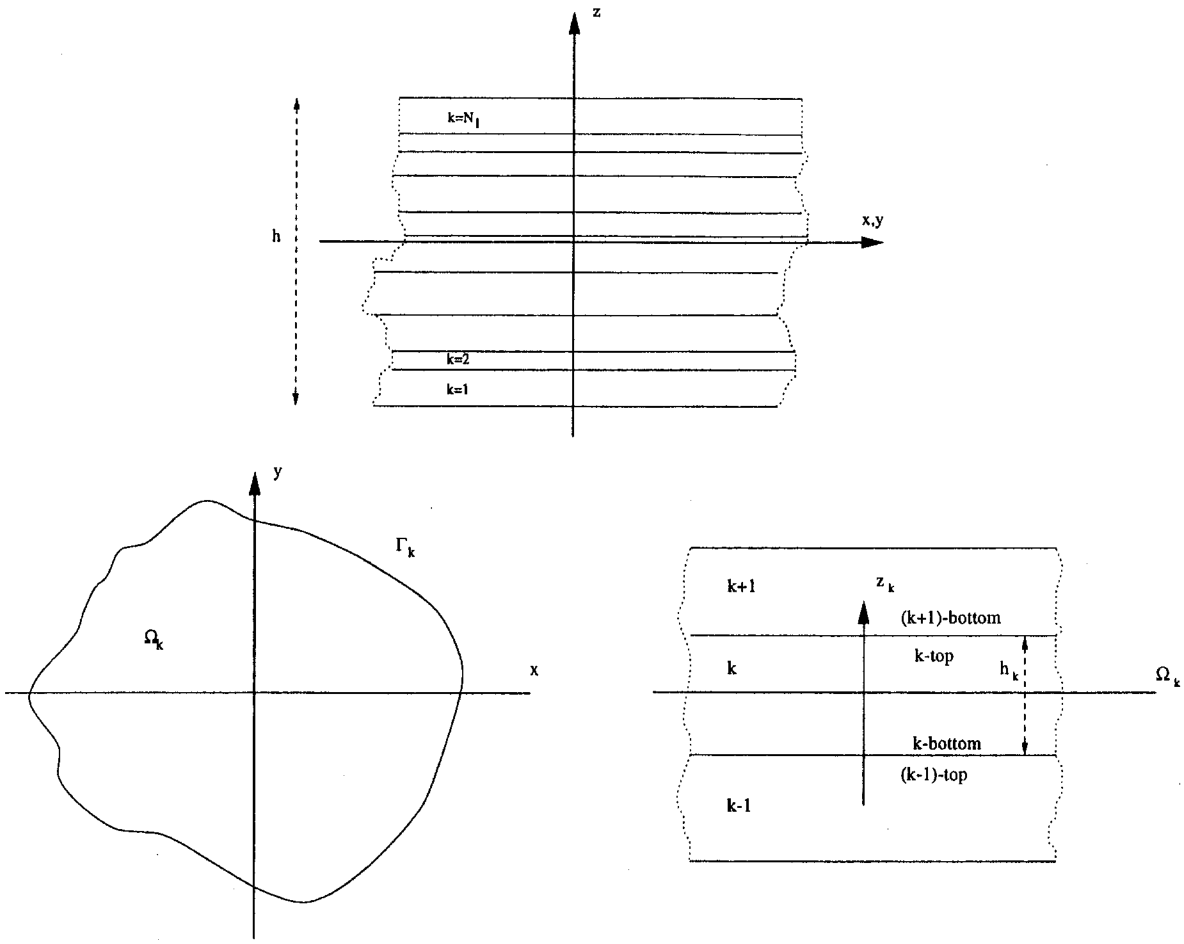

Fig. 1 Geometry and notation of multilayered plates

stresses. In order to encounter the well-known results from the literature, see, for example, Reissner (1985) or in the author's discussion reported in Carrera (1997)), different polynomial orders should be used in developments that will be presented in the subsequent sections. Trace operators that would lead to a shortening of the resulting arrays at Eqs. (15), (18), (26), and (29), could be introduced for this aims. For the sake of brevity, the results related to these aspects have not been discussed in the numerical parts as they do not change the obtained conclusions. These will be reported in future works.

Governing Equations for the k-Layer

Displacement Formulation. The classical displacement approach is formulated in terms of $\mathbf{u}^k$ via the principle of virtual displacements. In the dynamic case this states

$$\sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} (\delta \boldsymbol{\epsilon}^{k^T}_{pG} \boldsymbol{\sigma}^k_{pH_d} + \delta \boldsymbol{\epsilon}^{k^T}_{nG} \boldsymbol{\sigma}^k_{nH_d}) d\Omega^k dz = \sum_{k=1}^{N_l} \int_{\Omega^k} \int_{A_k} \rho^k \delta \mathbf{u}^k \ddot{\mathbf{u}}^k dV + \delta L^e. \quad (7)$$

δ is the variational symbol and subscript T denotes transposition of arrays. A_k and V denote the layer-thickness domain and volume; Ω^k is the layer middle surface bounded by Γ^k (Γ^k_g, Γ^k_m denotes those parts of Γ^k on which the geometrical and mechanical boundary conditions are prescribed, respectively). ρ is the mass density and double dots denote acceleration. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's Law and strain from geometrical relations. δL_e is the virtual variation of the work made by the external layer-forces $\mathbf{p}^k = \{p^k_x, p^k_y, p^k_z\}$.

In order to handle the next developments in a concise manners, it is convenient to do as follows:

1 The displacement model Eq. (4) is substituted in the strains

$$\boldsymbol{\epsilon}^k_{pG} = \mathbf{D}_p F_\tau \mathbf{u}^k_\tau,$$

$$\boldsymbol{\epsilon}^k_{nG} = (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz})\mathbf{u}^k = \mathbf{D}_{n\Omega} F_\tau \mathbf{u}^k_\tau + F_{\tau_z} \mathbf{u}^k_\tau. \quad (8)$$

The subscript z denotes differentiation with respect to z.

2 The following stress layer-resultants are defined:

$$(\mathbf{R}^{k\tau}_{pH_d}, \mathbf{R}^{k\tau}_{nH_d}, \mathbf{R}^{k\tau_z}_{nH_d}) = \int_{A_k} (F_\tau \boldsymbol{\sigma}^k_{pH_d}, F_\tau \boldsymbol{\sigma}^k_{nH_d}, F_{\tau_z} \boldsymbol{\sigma}^k_{nH_d}) dz. \quad (9)$$

3 The following array formulas for the integration by parts is introduced:

$$\int_{\Omega^k} (\mathbf{D}_\Omega \boldsymbol{\phi})^T \boldsymbol{\varphi} d\Omega^k = -\int_{\Omega^k} \boldsymbol{\phi}^T \mathbf{D}^T_\Omega \boldsymbol{\varphi} d\Omega^k + \int_{\Gamma^k} \boldsymbol{\phi}^T \mathbf{I}^T_\Omega \boldsymbol{\varphi} d\Gamma^k. \quad (10)$$

For the sake of simplicity, it is intended that the boundary Γ^k is parallel to the direction x, y; $\boldsymbol{\phi}$, and $\boldsymbol{\varphi}$ are two generic columns of displacements or stresses; $\mathbf{D}_\Omega$ denotes a generic array including only first-order partial differential operators with respect to the in-plane coordinates x, y. The algebraic array $\mathbf{I}^T_\Omega$ is built as follows: the unit number 1 is set in correspondence to those elements of $\mathbf{D}_\Omega$ which are different from zero (see the Appendix).

Based on this, Eq. (7) takes on the following form:

$$\sum_{k=1}^{N_l}\left(\int_{\Omega^k}\delta\mathbf{u}_\tau^{k^T}(-\mathbf{D}_p^T\mathbf{R}_{pH_d}^{k\tau}+\mathbf{R}_{nH_d}^{k\tau_z}-\mathbf{D}_{n\Omega}^T\mathbf{R}_{nH_d}^{k\tau})d\Omega^k\right.$$

$$\left.+\int_{\Gamma^k}\delta\mathbf{u}_\tau^{k^T}(\mathbf{I}_p^T\mathbf{R}_{pH_d}^{k\tau}+\mathbf{I}_{n\Omega}^T\mathbf{R}_{nM}^{k\tau})d\Gamma^k\right)$$

$$=\sum_{k=1}^{N_l}\int_{\Omega^k}\delta\mathbf{u}_\tau^{k^T}(E_{\tau s}\mathbf{I}\ddot{\mathbf{u}}^k+\mathbf{p}_\tau^k)d\Omega^k \quad (11)$$

where $\mathbf{p}_\tau^k=\{p_{x\tau}^k, p_{y\tau}^k, p_{z\tau}^k\}$ are the variationally consistent load vectors due to $\mathbf{p}^k$. By imposing the definition of virtual variations for the unknown displacements, the differential system of governing equations and related boundary conditions are derived in terms of the introduced stress resultants. For the k-layer, the equilibrium equations on Ω^k are

$$\delta\mathbf{u}_\tau^k\colon -\mathbf{D}_p^T\mathbf{R}_{pH_d}^{k\tau}+\mathbf{R}_{nHd}^{k\tau_z}-\mathbf{D}_{n\Omega}^T\mathbf{R}_{nH_d}^{k\tau}=E_{\tau s}\mathbf{I}\ddot{\mathbf{u}}^k+\mathbf{p}_\tau^k \quad (12)$$

while the boundary conditions on Γ^k are

geometrical on Γ_g^k mechanical on Γ_m^k

$$\mathbf{u}_\tau^k=\bar{\mathbf{u}}_\tau^k \quad \text{or} \quad \mathbf{I}_p^T\mathbf{R}_{pH_d}^{k\tau}+\mathbf{I}_{n\Omega}^T\mathbf{R}_{nH_d}^{k\tau}=\mathbf{I}_p^T\bar{\mathbf{R}}_{pH_d}^{k\tau}+\mathbf{I}_{n\Omega}^T\bar{\mathbf{R}}_{nH_d}^{k\tau}. \quad (13)$$

The bar denotes imposed values at the boundary. The complete set of equations for the N_l-layers could be written simply by expanding over their ranges the introduced subscripts and superscripts.

In order to express the governing equations in terms of the displacement variables the stress resultants of Eq. (9) are written in terms of unknown variables $\mathbf{u}^k$ by substituting the Eqs. (2), (4), and (8) as follows:

$$\mathbf{R}_{pH_d}^{k\tau}=\tilde{\mathbf{Z}}_{pp}^{k\tau s}\mathbf{D}_p\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{pn}^{k\tau s}\mathbf{D}_{n\Omega}\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{pn}^{k\tau s_z}\mathbf{D}_{n\Omega}\mathbf{u}_s^k,$$

$$\mathbf{R}_{nH_d}^{k\tau}=\tilde{\mathbf{Z}}_{np}^{k\tau s}\mathbf{D}_p\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{nn}^{k\tau s}\mathbf{D}_{n\Omega}\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{nn}^{k\tau s_z}\mathbf{u}_s^k,$$

$$\mathbf{R}_{nH_d}^{k\tau_z}=\tilde{\mathbf{Z}}_{np^z}^{k\tau^s}\mathbf{D}_p\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{nn^z}^{k\tau^s}\mathbf{D}_{n\Omega}\mathbf{u}_s^k+\tilde{\mathbf{Z}}_{nn^z}^{k\tau^s z}\mathbf{u}_s^k. \quad (14)$$

The further subscript/superscript $s=t,b,2$ and the following layer stiffnesses and integrals have been introduced:

$$(\tilde{\mathbf{Z}}_{pp}^{k\tau s},\tilde{\mathbf{Z}}_{pn}^{k\tau s},\tilde{\mathbf{Z}}_{np}^{k\tau s},\tilde{\mathbf{Z}}_{nn}^{k\tau s})=(\tilde{\mathbf{C}}_{pp}^k,\tilde{\mathbf{C}}_{pn}^k,\tilde{\mathbf{C}}_{np}^k,\tilde{\mathbf{C}}_{nn}^k)E_{\tau s}$$

$$(\tilde{\mathbf{Z}}_{pn}^{k\tau s_z},\tilde{\mathbf{Z}}_{np^z}^{k\tau_z s},\tilde{\mathbf{Z}}_{nn}^{k\tau s_z},\tilde{\mathbf{Z}}_{nn^z}^{k\tau_z s},\tilde{\mathbf{Z}}_{nn^z}^{k\tau_z s_z})$$

$$=(\tilde{\mathbf{C}}_{pn}^kE_{\tau s_z},\tilde{\mathbf{C}}_{np}^kE_{\tau_z s},\tilde{\mathbf{C}}_{nn}^kE_{\tau s_z},\tilde{\mathbf{C}}_{nn}^kE_{\tau_z s},\tilde{\mathbf{C}}_{nn}^kE_{\tau_z s_z})$$

$$(E_{\tau s}^k,E_{\tau_z s},E_{\tau s_z},E_{\tau_z s_z}^k)$$

$$=\int_{A_k}(F_\tau F_s,F_{\tau_z}F_s,F_\tau F_{s_z},F_{\tau_z}F_{s_z})dz \quad (15)$$

The explicit values of the integrals are given in the Appendix.

The obtained strain and stress resultants Eq. (14) can be introduced in Eqs. (12), (22), and (23) leading to equilibrium equations expressed in terms of the displacement variables

$$\delta\mathbf{u}_\tau^k:\mathbf{K}_d^{k\tau s}\mathbf{u}_s^k=\mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_s^k+\mathbf{p}_\tau^k \quad (16)$$

the related boundary conditions are

$$\mathbf{u}_\tau^k=\bar{\mathbf{u}}_\tau^k \quad \text{or} \quad \mathbf{\Pi}_d^{k\tau s}\mathbf{u}_s^k=\mathbf{\Pi}_d^{k\tau s}\bar{\mathbf{u}}_s^k \quad (17)$$

where

$$\mathbf{K}_d^{k\tau s}=-\mathbf{D}_p^T(\tilde{\mathbf{Z}}_{pp}^{k\tau s}\mathbf{D}_p+\tilde{\mathbf{Z}}_{pn}^{k\tau s}\mathbf{D}_{n\Omega}+\tilde{\mathbf{Z}}_{pn}^{k\tau s_z}\mathbf{D}_{n\Omega})-\mathbf{D}_{n\Omega}^T(\tilde{\mathbf{Z}}_{np}^{k\tau s}\mathbf{D}_p$$

$$+\tilde{\mathbf{Z}}_{nn}^{k\tau s}\mathbf{D}_{n\Omega}+\tilde{\mathbf{Z}}_{nn}^{k\tau s_z})+\tilde{\mathbf{Z}}_{np^z}^{k\tau_z s}\mathbf{D}_p+\tilde{\mathbf{Z}}_{nn^z}^{k\tau_z s}\mathbf{D}_{n\Omega}+\tilde{\mathbf{Z}}_{nn^z}^{k\tau_z s_z}$$

$$\mathbf{\Pi}_d^{k\tau s}=\mathbf{I}_p^T(\tilde{\mathbf{Z}}_{pp}^{k\tau s}\mathbf{D}_p+\tilde{\mathbf{Z}}_{pn}^{k\tau s}\mathbf{D}_{n\Omega}+\tilde{\mathbf{Z}}_{pn}^{k\tau s_z}\mathbf{D}_{n\Omega})+\mathbf{I}_{n\Omega}^T(\tilde{\mathbf{Z}}_{np}^{k\tau s}\mathbf{D}_p$$

$$+\tilde{\mathbf{Z}}_{nn}^{k\tau s}\mathbf{D}_{n\Omega}+\tilde{\mathbf{Z}}_{nn}^{k\tau s_z})$$

$$\mathbf{M}^{k\tau s}=E_{\tau s}\mathbf{I}. \quad (18)$$

Mixed Formulation. In the mixed case, equilibrium and compatibility are both formulated in terms of the $\mathbf{u}^k$ and $\boldsymbol{\sigma}_n^k$ unknowns via Reissner's variational equation (Reissner, 1984, 1986; Murakami, 1986),

$$\sum_{k=1}^{N_l}\int_{\Omega^k}\int_{A_k}(\delta\boldsymbol{\epsilon}_{pG}^{k^T}\boldsymbol{\sigma}_{pH}^k+\delta\boldsymbol{\epsilon}_{nG}^{k^T}\boldsymbol{\sigma}_{nM}^k+\delta\boldsymbol{\sigma}_{nM}^{k^T}(\boldsymbol{\epsilon}_{nG}^k-\boldsymbol{\epsilon}_{nH}^k))d\Omega^kdz$$

$$=\sum_{k=1}^{N_l}\int_{\Omega^k}\int_{A_k}\rho^k\delta\mathbf{u}^k\ddot{\mathbf{u}}^kdV+\delta L^e. \quad (19)$$

The L.H.S. includes the variations of the internal work in the plate: the first two terms come from the displacement formulation and lead to variationally consistent equilibrium conditions; the third "mixed" term variationally enforces the compatibility of the transverse strains components.

By using what has been done in the previous subsection and by introducing the following further stress and strain layer-resultants

$$(\mathbf{R}_{pH}^{k\tau},\mathbf{R}_{nM}^{k\tau},\mathbf{R}_{n\tilde{M}}^{kt_z},\mathbf{S}_{nG}^{k\tau},\mathbf{S}_{nH}^{k\tau})$$

$$=\int_{A_k}(F_\tau\boldsymbol{\sigma}_{pH}^k,F_\tau\boldsymbol{\sigma}_M^k,F_{\tau_z}\boldsymbol{\sigma}_M^k,F_\tau\boldsymbol{\epsilon}_{nG}^k,F_\tau\boldsymbol{\epsilon}_{nH}^k)dz, \quad (20)$$

the Eq. (19) leads to

$$\sum_{k=1}^{N_l}\left(\int_{\Omega^k}(\delta\mathbf{u}_\tau^{k^T}(-\mathbf{D}_p^T\mathbf{R}_{pH}^{k\tau}+\mathbf{R}_{n\tilde{M}}^{k\tau_z}-\mathbf{D}_{n\Omega}^T\mathbf{R}_{nM}^{k\tau})+\delta\boldsymbol{\sigma}_{n\tau}^{k^T}(\mathbf{S}_{nG}^{k\tau}\right.$$

$$\left.-\mathbf{S}_{nH}^{k\tau}))d\Omega+\int_{\Gamma^k}\delta\mathbf{u}_\tau^{k^T}(\mathbf{I}_p^T\mathbf{R}_{pH}^{k\tau}+\mathbf{I}_{n\Omega}^T\mathbf{R}_{nM}^{k\tau})d\Gamma^k\right)$$

$$=\sum_{k=1}^{N_l}\int_{\Omega^k}\delta\mathbf{u}_\tau^{k^T}(E_{\tau s}\mathbf{I}\ddot{\mathbf{u}}^k+\mathbf{p}_\tau^k)d\Omega. \quad (21)$$

By imposing the definition of virtual variations for the unknown stress and displacement variables, the differential system of governing equations and related boundary conditions for the k-layer are obtained in terms of the introduced stress and strain resultants.

The equilibrium equations on Ω^k are

$$\delta\mathbf{u}_\tau^k\colon -\mathbf{D}_p^T\mathbf{R}_{pH}^{k\tau}+\mathbf{R}_{n\tilde{M}}^{k\tau_z}-\mathbf{D}_{n\Omega}^T\mathbf{R}_{nM}^{k\tau}=E_{\tau s}\mathbf{I}\ddot{\mathbf{u}}^k+\mathbf{p}_\tau^k. \quad (22)$$

The constitutive equations on Ω^k are

$$\delta\boldsymbol{\sigma}_{n\tau}^k:\mathbf{S}_{nG}^{k\tau}-\mathbf{S}_{nH}^{k\tau}=0. \quad (23)$$

The boundary conditions on Γ^k are

geometrical on Γ_g^k $\mathbf{u}_\tau^k=\bar{\mathbf{u}}_\tau^k$

or

mechanical on Γ_m^k $\mathbf{I}_p^T\mathbf{R}_{pH}^{k\tau}+\mathbf{I}_{n\Omega}^T\mathbf{R}_{nM}^{k\tau}$

$$=\mathbf{I}_p^T\bar{\mathbf{R}}_{pH}^{k\tau}+\mathbf{I}_{n\Omega}^T\bar{\mathbf{R}}_{nM}^{k\tau}. \quad (24)$$

In order to express the governing equation in terms of stress and displacement variables, the extra resultants in Eq. (9) are written in terms of the unknown variables by using Eqs. (2), (4), and (8),

$$\mathbf{R}_{pH}^{k\tau}=\mathbf{Z}_{pp}^{k\tau s}\mathbf{D}_p\mathbf{u}_s^k+\mathbf{Z}_{pn}^{k\tau s}\boldsymbol{\sigma}_{ns}^k,\quad \mathbf{R}_{nM}^{k\tau}=E_{\tau s}^k\boldsymbol{\sigma}_{ns}^k,\quad \mathbf{R}_{n\tilde{M}}^{k\tau_z}=E_{\tau_z s}\boldsymbol{\sigma}_{ns}^k,$$

$$\mathbf{S}_{nG}^{k\tau}=E_{\tau s}^k\mathbf{D}_{n\Omega}\mathbf{u}_s^k+E_{\tau s_z}\mathbf{u}_s^k,\quad \mathbf{S}_{nH}^{k\tau}=\mathbf{Z}_{np}^{k\tau s}\mathbf{D}_p\mathbf{u}_s^k+\mathbf{Z}_{nn}^{k\tau s}\boldsymbol{\sigma}_{ns}^k. \quad (25)$$

Further layer stiffnesses and integrals have been introduced:

$$(\mathbf{Z}_{pp}^{k\tau s},\mathbf{Z}_{pn}^{k\tau s},\mathbf{Z}_{np}^{k\tau s},\mathbf{Z}_{nn}^{k\tau s})=(\mathbf{C}_{pp}^k,\mathbf{C}_{pn}^k,\mathbf{C}_{np}^k,\mathbf{C}_{nn}^k)E_{\tau s}. \quad (26)$$

The governing equations expressed in terms of displacement and stress variables are

$$\delta \mathbf{u}_\tau^k : \mathbf{K}_{uu}^{k\tau s}\mathbf{u}_s^k + \mathbf{K}_{u\sigma}^{k\tau s}\boldsymbol{\sigma}_{ns}^k = \mathbf{M}^{k\tau s}\ddot{\mathbf{u}}_s^k + \mathbf{p}_\tau^k$$
$$\delta \boldsymbol{\sigma}_{n\tau}^k : \mathbf{K}_{\sigma u}^{k\tau s}\mathbf{u}_s^k + \mathbf{K}_{\sigma\sigma}^{k\tau s}\boldsymbol{\sigma}_{ns}^k = 0 \quad (27)$$

with boundary conditions

$$\mathbf{u}_\tau^k = \bar{\mathbf{u}}_\tau^k \quad \text{or} \quad \mathbf{\Pi}_u^{k\tau s}\mathbf{u}_s^k + \mathbf{\Pi}_\sigma^{k\tau s}\boldsymbol{\sigma}_{ns}^k = \mathbf{\Pi}_u^{k\tau s}\bar{\mathbf{u}}_s^k + \mathbf{\Pi}_\sigma^{k\tau s}\bar{\boldsymbol{\sigma}}_{ns}^k \quad (28)$$

where

$$\mathbf{K}_{uu}^{k\tau s} = -\mathbf{D}_p^T\mathbf{Z}_{pp}^{k\tau s}\mathbf{D}_p \quad \mathbf{K}_{u\sigma}^{k\tau s} = -\mathbf{D}_p^T\mathbf{Z}_{pn}^{k\tau s} + E_{\tau_z s}\mathbf{I} - E_{\tau s}\mathbf{D}_{n\Omega}^T$$
$$\mathbf{K}_{\sigma u}^{k\tau s} = E_{\tau s}\mathbf{D}_{n\Omega} + E_{\tau s_z}\mathbf{I} - \mathbf{Z}_{np}^{k\tau s}\mathbf{D}_p\mathbf{K}_{\sigma\sigma}^{k\tau s} = -\mathbf{Z}_{nn}^{k\tau s}$$
$$\mathbf{\Pi}_u^{k\tau s} = \mathbf{I}_p^T\mathbf{Z}_{pp}^{k\tau s}\mathbf{D}_p \quad \mathbf{\Pi}_\sigma^{k\tau s} = \mathbf{Z}_{pn}^{k\tau s} + E_{\tau s}\mathbf{I}_{n\Omega}^T. \quad (29)$$

Fulfillment of C_z^0-Requirements: Multilayered Equations

The C_z^0-requirements Eqs. (5) and (6) can be accounted for by simply writing the governing equations at the multilayered level. The necessary steps can be summarized as follows (see Carrera, 1995, for more details on similar techniques):

1 The governing equations for the N_l layers are written.
2 The continuity conditions at Eqs. (5) are linked through index contractions.

By choosing the top and 2-resultant values as unknowns, the arrays of the unknown displacements for the whole multilayer holds:

$$\mathbf{u}^T = \{\mathbf{u}_b^{1^T}, \mathbf{u}_t^{1^T}, \mathbf{u}_2^{1^T}; \mathbf{u}_t^{2^T}, \mathbf{u}_2^{2^T}; \ldots \mathbf{u}_t^{k^T}, \mathbf{u}_2^{k^T}; \mathbf{u}_t^{(k+1)^T}, \mathbf{u}_2^{(k+1)^T}; \ldots \mathbf{u}_t^{(N_l-1)^T}, \mathbf{u}_2^{(N_l-1)^T}; \mathbf{u}_{t_l}^{N^T}, \mathbf{u}_{2^l}^{N^T}\}. \quad (30)$$

The governing equations for the displacement formulation are formally written

$$\mathbf{K}_d\mathbf{u} = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{p}$$
$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{or} \quad \mathbf{\Pi}_d\mathbf{u} = \mathbf{\Pi}_d\bar{\mathbf{u}}. \quad (31)$$

What has been done for the displacement formulation can be extended to the transverse stress components and this leads to the complete fulfillment of the C_z^0-requirements. The following array for the whole multilayer is introduced

$$\boldsymbol{\sigma}_n^T = \{\boldsymbol{\sigma}_{nt}^{1^T}, \boldsymbol{\sigma}_{n2}^{1^T}; \boldsymbol{\sigma}_{nt}^{2^T}, \boldsymbol{\sigma}_{n2}^{2^T}; \ldots \boldsymbol{\sigma}_{nt}^{k^T}, \boldsymbol{\sigma}_{n2}^{k^T}; \boldsymbol{\sigma}_{nt}^{(k+1)^T}, \boldsymbol{\sigma}_{n2}^{(k+1)^T}; \ldots \boldsymbol{\sigma}_{nt}^{(N_l-1)^T}, \boldsymbol{\sigma}_{n2}^{(N_l-1)^T}; \boldsymbol{\sigma}_{n2}^{N_l^T}\}. \quad (32)$$

The governing system of differential equations at the multilayered level is formally written in the following final form:

$$\mathbf{K}_{uu}\mathbf{u} + \mathbf{K}_{u\sigma}\boldsymbol{\sigma}_n = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{p} + \mathbf{p}_u^{1N_l}$$
$$\mathbf{K}_{\sigma u}\mathbf{u} + \mathbf{K}_{\sigma\sigma}\boldsymbol{\sigma}_n = \mathbf{p}_\sigma^{1N_l}. \quad (33)$$

$\mathbf{p}_u^{1N_l}$, $\mathbf{p}_\sigma^{1N_l}$ are loading due to the nonhomogeneous conditions in Eq. (6). The boundary conditions are

$$\mathbf{u} = \bar{\mathbf{u}} \quad \text{or} \quad \mathbf{\Pi}_u\mathbf{u} + \mathbf{\Pi}_\sigma\boldsymbol{\sigma}_n = \mathbf{\Pi}_u\bar{\mathbf{u}} + \mathbf{\Pi}_\sigma\bar{\boldsymbol{\sigma}}_n + \mathbf{q}_\sigma^{1N_l}. \quad (34)$$

The expanded expressions of the multilayered governing equations have been omitted for the sake of brevity. Exact closed-form solutions of the derived system of differential Eqs. (31), (33), and (34), that govern the dynamic response of a generally laminated multilayered plate, are not available. Approximated solution procedures could be conveniently implemented for this purpose (Carrera 1995). The particular case in which the material has the following properties $\tilde{C}_{16} = \tilde{C}_{26} = \tilde{C}_{36} = \tilde{C}_{45} = 0$ has her been considered. In such a case, Navier-type closed-form solutions can be found by assuming the following harmonic forms for the applied loadings and unknown variables,

Table 1 List of the acronyms used to denote three-dimensional and two-dimensional plate theories, in alphabetic order.

3D	Three Dimensional Solution
3D-1	3D Solution by Noor & Burton (1989a)
3D-2	3D Solution by Noor (1973)
3D-3	3D Solution in Kheider and Librescu (1988)
3D-4	3D Solution by Srinivas et al. (1970)
3D-5	3D Solution by Nosier et al. (1993)
CLT	Classical Lamination Theory
D-p	Present LWM at the Displacements formulated with parabolic fields in the k-layer
D-l	Present LWM at the Displacements formulated with linear fields in the k-layer
ESLM	Equivalent Single Layer Model
FSDT	First order Shear Deformation Theory
HSDT	Higher order Shear Deformation Theory of ESLM-type
HSDT-1	HSDT by Kant & Kommineni (1985)
HSDT-2	HSDT by Ren & Owen (1989)
HSDT-3	HSDT by Kheider & Librescu (1988)
HSDT-4	further HSDT by Kheider & Librescu (1988)
HSDT-5	HSDT by Reddy & Phan (1985)
LWM	Layer Wise Model
LWM-1	LWM by Cho et al. (1981)
LWM-2	LWM by Nosier et al. (1993)
M-p	Present Mixed LWM model with parabolic fields in the k-layer
M-l	Present Mixed LWM model with linear fields in the k-layer

$$(u_{x_\tau}^k, \sigma_{xz_\tau}^k, p_{x_\tau}^k) = \sum_{m,n} (U_{x_\tau}^k e^{i\omega_{mn}\hat{t}}, S_{xz_\tau}^k e^{i\omega_{mn}\hat{t}}, P_{x_\tau}^k(\hat{t})) \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$
$$(u_{y_\tau}^k, \sigma_{yz_\tau}^k, p_{y_\tau}^k) = \sum_{m,n} (U_{y_\tau}^k e^{i\omega_{mn}\hat{t}}, S_{yz_\tau}^k e^{i\omega_{mn}\hat{t}}, P_{y_\tau}^k(\hat{t})) \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b}$$
$$(u_{z_\tau}^k, \sigma_{zz_\tau}^k, p_{z_\tau}^k) = \sum_{m,n} (U_{z_\tau}^k e^{i\omega_{mn}\hat{t}}, S_{zz_\tau}^k e^{i\omega_{mn}\hat{t}}, P_{z_\tau}^k(\hat{t})) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}, \quad (35)$$

which correspond to simply supported boundary conditions. a and b are the plate lengths in the x and y directions, respectively, while m and n are the correspondent wave numbers; $i = \sqrt{-1}$, $\hat{t}$ is the time and ω_{mn} is the circular frequency. Capital letters at R.H.S. denote correspondent maximum amplitudes. Upon substitution of Eqs. (35), the governing equations assume the form of a linear system of ordinary differential equations in the time domain. The free-vibration response leads to an eigenvalue problem: For the displacement formulation one obtains

$$\|\hat{\mathbf{K}}_d - \omega_{mn}^2\hat{\mathbf{M}}\| = 0, \quad (36)$$

while, upon eliminations of the the stress unknowns, the mixed case leads to

$$\|\hat{\mathbf{K}}_{uu} - (\hat{\mathbf{K}}_{u\sigma}(\hat{\mathbf{K}}_{\sigma\sigma})^{-1}\hat{\mathbf{K}}_{\sigma u}) - \omega_{mn}^2\hat{\mathbf{M}}\| = 0. \quad (37)$$

The double bar denotes the determinant, while the caret indicates arrays constituted by real numbers.

Numerical Illustrations

In order to evaluate the accuracy of the proposed models to analyze the free vibrational response of laminated plates, a numerical investigation related to simply supported, cross-ply laminated thick and thin plates has been conducted. Table 1 gives a list of the acronyms used to denote the theories compared in the subsequent analysis.

As a preliminary assessment, related to static bending of a cross-ply plate, Table 2 compares the results of the mixed models with the three-dimensional analysis of Noor and Burton (1989a). In-plane and transverse stresses and displacements, related to displacements and mixed solutions, are given for both

Table 2 Comparison of present Analysis to three-dimensional exact solution on stress and displacement fields. Simply supported square bended plates. Cross-ply skew-symmetric laminate with $N_l = 10$, $G_{LT}/E_T = G_{Lz}/E_T = .50$, $G_{TT}/E_T = .35$, $\nu_{LT} = \nu_{Lz} = .30$, $\nu_{TT} = 0.49$.

$\frac{a}{h}$	$\frac{E_L}{E_T}$	z	3D–1	Present Layer-Wise Analysis M-p	M-l	D-p	D-l
			Transverse displacement $u_z \times \frac{E_T}{P_{z_t}^{N_l} h}$.				
	3	t	195.2	195.2	195.2	195.2	194.8
10		b	194.8	194.8	194.8	194.8	194.4
	30	t	52.31	52.31	52.27	52.31	52.13
		b	51.92	51.92	51.88	51.99	51.71
	3	t	14.59	14.59	14.59	14.59	14.55
5		b	14.17	14.17	14.17	14.17	14.13
	30	t	6.227	6.226	6.215	6.226	6.183
		b	5.838	5.838	5.827	5.838	5.794
			In-plane displacement $u_y \times \frac{E_T}{P_{z_t}^{N_l} h}$.				
10	3	t	-30.45	-30.45	-30.45	-30.45	-30.38
		b	27.40	27.40	27.40	27.40	27.33
	30	t	-7.017	-7.017	-7.011	-7.171	-6.985
		b	5.561	5.561	5.553	5.561	5.534
5	3	t	-3.784	-3.784	-3.784	-3.784	-3.769
		b	3.499	3.499	3.498	3.499	3.485
	30	t	-1.092	-1.092	-1.089	-1.092	-1.077
		b	0.767	0.767	0.763	0.767	0.754
			In-plane stress $\sigma_{yy} \times \frac{1}{P_{z_t}^{N_l}}$.				
10	3	t	13.02	13.02	13.02	13.02	13.71
		b	-29.62	-29.62	-29.62	-29.62	-30.17
	30	t	3.225	3.225	3.227	3.230	3.357
		b	-53.24	-53.24	-53.17	-53.24	-53.11
5	3	t	3.631	3.631	3.631	3.637	3.821
		b	-7.573	-7.576	-7.571	-7.578	-7.713
	30	t	1.326	1.326	1.323	1.331	1.372
		b	-14.71	-14.71	-14.63	-14.71	-14.51
			Transverse shear stress $\sigma_{yz} \times \frac{1}{P_{z_t}^{N_l}}$.				
10	3	c	2.380	2.380	2.370	2.380	2.375
	30	c	2.344	2.344	2.341	2.344	2.343
5	3	c	1.185	1.185	1.179	1.185	1.182
	30	c	1.141	1.141	1.383	1.141	1.142

cases of parabolic and linear fields in Eqs. (4). $P_{zt}^{N_l}$ denotes the amplitude of the applied z-transverse pressure applied in correspondence to the t-top surface of the N_l-layer. Two values of thickness ratio a/h (h is the plate thickness) and orthotropic ratio E_L/E_T are investigated (standard notations are used to denote mechanical properties of the lamina, see Jones, 1975); t, b, c (plate-top, bottom, and center, respectively) denote the location in the z-direction of the quoted values. The obtained results show the excellent agreement between the present mixed parabolic (M-p) solutions and the three-dimensional analysis. Some minor comments can be made: (1) in some cases the results related to the mixed linear (M-1) cases can be more accurate than the parabolic case related to the displacement formulation (D-p); (2) the D-1 analysis leads to the least accurate description; (3) as expected as a/h and E_L/E_T decrease, the discrepancies between the different modelings increase.

Dynamic responses are treated in the subsequent tables. Table 3 compares the values of the fundamental ($m = n = 1$) frequency parameter of different theories. The plate analyzed in

Table 3 Comparison of present analysis to three-dimensional exact and ESLM available solution on fundamental circular frequency parameter $\bar{\omega}_1 = \omega h\sqrt{\rho/E_T}$ simply supported square plates $a/h = 5$. Cross-ply skew-symmetric and symmetric laminates. The same material of Table 2 is used.

N_l	$\frac{E_L}{E_T}$	3D–1 [23]	Present Layer-Wise Analysis M-p	M-l	D-p	D-l	ESLM (Kant et alii 1989) FSDT	HSDT–1
2	3	0.2392	0.2394	0.2417	0.2394	0.2478	0.2379	0.2388
	30	0.3117	0.3142	0.3134	0.3168	0.3209	0.3165	0.3117
10	3	0.2530	0.2530	0.2530	0.2530	0.2534	0.2527	0.2531
	30	0.4027	0.4027	0.4031	0.4027	0.4042	0.4086	0.4028
3	3	0.2516	0.2516	0.2511	0.2517	0.2556	-	-
	30	0.3739	0.3749	0.3721	0.3763	0.3807	-	-
9	3	0.2535	0.2535	0.2535	0.2535	0.2540	0.2531	0.2536
	30	0.4040	0.4040	0.4044	0.4040	0.4058	0.4067	0.4067

Table 4 Effect fictitious layers on fundamental circular frequency parameter $\bar{\omega}_1 = \omega h\sqrt{\rho/E_T}$. Simply supported square plates $a/h = 5$. Cross-ply skew-symmetric with $N_l = 2$ laminates. The same material of Table 2 is used.

$\frac{E_L}{E_T}$	N_l^*	3D–1	Present Analysis M-p	M-l	D-p	D-l
3	2 × 1	0.2392	0.2394	0.2417	0.2394	0.2478
	2 × 1		0.2392	0.2396	0.2392	0.2416
	2 × 4		0.2392	0.2392	0.2392	0.2398
	2 × 5		0.2392	0.2392	0.2392	0.2395
30	2 × 1	0.3117	0.3142	0.3134	0.3168	0.3209
	2 × 2		0.3119	0.3129	0.3121	0.3179
	2 × 4		0.3117	0.3118	0.3117	0.3135
	2 × 5		0.3117	0.3117	0.3117	0.3128

Table 2 is referred to. Two symmetric ($N_l = 3, 9$) and two antisymmetric ($N_l = 2, 10$) cross-ply plates are investigated with two different values of the orthotropic ratio. It has been confirmed that the M-p analysis leads to the best description, with respect to three-dimensional solutions. The results related to the ESLM analysis by Kant and Kommineni (1989) are also compared (the HSDT-1 model includes a cubic displacement field but it does not include the zig-zag effects nor fulfill the interlaminar equilibrium). As expected the accuracy of the ESLM analyses are very much subordinate to the layouts and material properties, while present layer-wise models are not affected by these factors.

Layer-wise results can be improved by introducing fictitous interfaces. Such a possibility is discussed in Table 4, where two cases of the previous analysis are treated. N_l^* is the number of the layer used in the calculations. The same number of fictitious interfaces has been considered in each layer. For instance, $N_l^* = N_l \times 2$ signifies that each phisical layer has been simulated by means of two fictitious layers. N_l^* increasing the three-dimensional analysis is approached. Mixed models show higher convergence rates with respect to those of the displacement models.

Further comparisons with available three-dimensional and to ESLM solutions are considered in Table 5. HSDT-2 describes the zig-zag form of the in-plane displacements and fulfills transverse shear continuity. HSDT-3 and HSDT-4 consist of the two different higher-order models described by Kheider and Librescu, (1989). The FSDT and CLT values are also those quoted in this last paper. The three-dimensional-2 and three-dimensional-3 solutions are based on approximated finite difference technique and exact closed-form solutions, respectively. The conclusions reached for the previous analysis have been confirmed.

Table 6 compares the present analysis with that by Reddy and Phan, (1985). Thickness ratios from 2 to 1000 are considered. HSDT-5 fulfills the top and bottom zero-transverse shear plate conditions. The ESLM analysis leads to very large errors for the very thick plate cases. CLT solution is obtained for thin

Table 5 Comparison of present analysis to three-dimensional solutions and available ESLM analyses on fundamental circular frequency parameter $\bar{\omega}_2 = 10 \times \omega h\sqrt{\rho/E_T}$ simply supported square plates $a/h = 5$. Cross-ply skew-symmetric and symmetric laminates. $G_{LT}/E_T = G_{Lz}/E_T = .60$, $G_{TT}/E_T = .50$, $\nu_{LT} = \nu_{Lz} = \nu_{TT} = 0.25$.

Model	N_l 2		10		3		9	
$\frac{E_L}{E_T}$	3	30	3	30	3	30	3	30
3D–2	2.5031	3.2705	2.6583	4.4011	2.6474	4.1089	2.6640	4.4210
3D–3	-	-	-	-	2.6293	4.1087	2.6392	4.4193
M–p	2.4950	3.2826	2.6382	4.3670	2.6315	4.0919	2.6438	4.3827
M–l	2.5174	3.2788	2.6383	4.3711	2.6358	4.0738	2.6434	4.3916
D–p	2.4954	3.3042	2.6382	4.3670	2.6312	4.1032	2.6438	4.3867
D–l	2.5350	3.3255	2.6407	4.3825	2.6492	4.1526	2.6468	4.4058
HSDT-2	2.4128	3.2529	2.6308	4.3631	2.5560	3.9311	2.6356	4.3582
HSDT-3	-	-	-	-	2.6292	3.9387	2.6436	4.3847
HSDT-4	-	-	-	-	2.6223	3.9301	2.6375	4.3788
FSDT	-	-	-	-	2.6252	3.9540	2.6376	4.3818
CLT	-	-	-	-	3.0143	6.6419	3.0143	6.6419

Table 6 Comparison of present analysis to ESLM solution by Reddy and Phan (1985) on fundamental circular frequency parameter $\bar{\omega}_3 = \omega\sqrt{a^4\rho/E_T h^2}$ simply supported square plates. Cross-ply skew-symmetric and symmetric laminates. $E_L/E_T = 40$, $G_{LT}/E_T = G_{Lz}/E_T = .50$, $G_{TT}/E_T = .60$, $\nu_{LT} = \nu_{Lz} = \nu_{TT} = 0.25$.

a/h	*Present Layer-Wise Analysis*				ESLM (Reddy et alii, 1985)		
	M-p	M-l	D-p	D-l	CLT	FSDT	HSDT–5
			stacking sequences 0/90				
2	4.774	4.772	4.804	4.848	8.499	5.191	5.699
4	7.450	7.424	7.528	7.562	10.292	7.975	8.294
5	8.288	8.261	8.355	8.397	10.584	8.757	9.010
10	10.141	10.128	10.178	10.215	11.011	10.355	10.449
20	10.876	10.872	10.888	10.921	11.125	10.941	10.968
50	11.116	11.115	11.118	11.150	11.158	11.127	11.132
100	11.152	11.152	11.152	11.184	11.163	11.155	11.156
1000	11.164	11.164	11.164	11.164	-	-	-
			stacking sequence 0/90/90/0				
2	5.270	5.320	5.277	5.414	15.830	5.492	5.576
4	9.230	9.343	9.235	9.472	17.907	9.369	9.497
5	10.753	10.876	10.756	11.005	18.215	10.820	10.989
10	15.150	15.248	15.152	15.334	18.652	15.083	15.270
20	17.626	17.668	17.627	17.702	18.767	17.583	17.668
50	18.600	18.607	18.600	18.617	18.799	18.590	17.606
100	18.753	18.755	18.753	18.761	18.804	18.751	18.755
1000	18.805	18.805	18.805	18.805	-	-	-

plates. As already observed by Cho et al., (1992), HSDT results can result to be less accurate than FSDT solutions. However, this last theory requires the use of a shear correction factor whose value is not a priori known (Noor and Burton 1989a). A comparison with present layer-wise analysis confirms that the accuracy of ESLM results are layout dependent. They are in particular more accurate in the case of symmetric laminates.

Higher frequencies related to higher modes are considered in Table 7 for an isotropic plate. A comparison with the aforementioned ESLM results and with LWM-1 by Cho et al. (1992) has been performed.

The effect of fictitous interfaces has also been evaluated. As the plate is isotropic, the mixed and displacement formulations lead to the same results. One should notice that the symmetric modes are better evaluated than antisymmetric ones. A slower convergence rate is found for higher antisymmetric modes. N_l^* increasing the present layer-wise model shows a better convergence rate than LWM-1 analysis.

Table 7 Comparison of present analysis to three-dimensional exact and other LWM and ESLM solutions on the lowest five circular frequency parameter $\bar{\omega}_4 = \omega h\sqrt{\rho/G}$. Simply supported square isotropic plates ($\nu = 0.3$). A and S denote modes which are antisymmetric and symmetric about the midplane. I-S and II-S are both thickness-twist modes.

Model	N_l^*	I–A	I–S	II–S	II–A	III–A
			(mh/a =0.1, nh/a =0.1)			
3D–4	-	0.0931	0.4443	0.7498	3.1729	3.2465
LWM-1	1	0.0930	0.4443	0.7416	3.1933	3.2756
LWM-1	4	0.0931	0.4443	0.7499	3.1736	3.2496
HSDT–5	-	0.0931	-	-	-	-
FSDT	-	0.0930	-	-	3.1729	3.2538
M–p & D–p	1	0.0934	0.4443	0.7498	3.4925	3.5699
M–l & D–l	1	0.1029	0.4443	0.7502	3.4985	3.5855
M–p & D–p	2	0.0932	0.4443	0.7498	3.1845	3.2587
M–l & D–l	2	0.0959	0.4443	0.7499	3.4925	3.5744
M–p & D–p	4	0.0931	0.4443	0.7498	3.1736	3.2437
M–l & D–l	4	0.0939	0.4443	0.7499	3.2533	3.4291
			(mh/a =0.2, nh/a =0.2)			
3D–4	-	0.3421	0.8886	1.4923	3.2648	3.5298
LWM-1	1	0.3404	0.8886	1.5066	3.2847	3.5831
LWM-1	4	0.3416	0.8886	1.4932	3.2656	3.5398
HSDT–5	-	0.3411	-	-	-	-
FSDT	-	0.3402	-	-	3.2648	3.5580
M–p & D–p	1	0.3456	0.8886	1.4925	3.5763	3.8589
M–l & D–l	1	0.3763	0.8886	1.4959	3.7627	3.9257
M–p & D–p	2	0.3423	0.8886	1.4923	3.2762	3.5436
M–l & D–l	2	0.3537	0.8886	1.4934	3.5762	3.8748
M–p & D–p	4	0.3421	0.8886	1.4923	3.2656	3.5307
M–l & D–l	4	0.3452	0.8886	1.4925	3.3431	3.6172

Table 8 Comparison of present mixed analysis to three-dimensional exact and other LWM and ESLM solutions on the lowest ten circular frequency parameters $\bar{\omega}_1 = \omega h\sqrt{\rho/E_T}$ simply supported square plates $a/h = 10$. Cross-ply symmetric laminates 0/90/0. $E_L = 35.1 \times 10^6$ psi, $E_T = 4.8 \times 10^6$ psi, $E_z = 0.75 \times 10^6$ psi, $G_{LT} = 1.36 \times 10^6$ psi, $G_{Lz} = 1.2 \times 10^6$ psi, $G_{Tz} = 0.47 \times 10^6$ psi, $\nu_{LT} = 0.036$, $\nu_{Lz} = 0.25$, $\nu_{TT} = 0.171$.

3D–5	Present M–p		LWM–2	ESLM (Nosier et alii, 1993)		
	$N_l \times 1$	$N_l \times 3$	$N_l \times 6$	HSDT–5	FSDT	CLT
			($m=n=1$)			
0.06715	0.06716	0.06715	0.06716	0.06839	0.06931	0.07769
0.50350	0.50349	0.50349	0.50355	0.50897	0.50897	0.50897
0.63775	0.63775	0.63775	0.63782	0.64174	0.64174	0.64174
1.2429	1.2438	1.2429	1.2445	1.4106	1.2719	
1.2790	1.2804	1.2790	1.2812	1.4638	1.5845	
1.3292	1.3305	1.3292	1.3301			
2.1533	2.1459	2.1533	2.1621			
2.4894	2.4943	2.4895	2.5022			
2.7419	2.7556	2.7422	2.7612			
3.5416	3.7728	3.5437	3.5917			
			($m=n=2$)			
0.20798	0.20809	0.20798	0.20808	0.21526	0.22055	0.31077
0.97517	0.97516	0.97517	0.97560	1.0179	1.0179	1.0179
1.2034	1.2040	1.2035	1.2047	1.2835	1.2835	1.2835
1.2916	1.2920	1.2916	1.2927	1.5695	1.4343	
1.4326	1.4336	1.4326	1.4348	1.9300	2.0309	
1.8037	1.8065	1.8037	1.8048			
2.3653	2.3560	2.3653	2.3740			
2.4877	2.4938	2.4878	2.5008			
2.9047	2.9176	2.9051	2.9237			
3.6361	3.7688	3.6382	3.6864			

Finally, Table 8 compares the present layer-wise mixed models with those by Nosier et al. (1993). Fundamental and higher frequencies related to two half-waves modes are considered for a symmetrically cross-ply laminated plate. The effects of a fictitious interfaces is also analyzed. The present mixed analysis shows results that are very close to the three-dimensional exact solutions. The quoted frequencies, especially for the higher mode cases, are much better evaluated using present models than those by Nosier et al. (1993). This results once again confirms the effectiveness of the a priori fulfillment of the transverse stress continuity, i.e., the superiority of the present mixed analysis with respect to other layer-wise theories based on displacement formulations.

Concluding Remarks

By employing a Reissner's mixed variational equation, this paper has presented the differential equations that govern the dynamics of multilayered plates modeled using mixed layer-wise theories. Classical displacement formulations are also derived for comparison purposes. A numerical investigation has been conducted for the free-vibration response of cross-ply laminated, simply supported, thick and thin plates for which closed-form solutions are given.

From a two-dimensional modeling point of view, one can notice that the proposed mixed theory consists of the unique layer-wise analysis which permits one to completely and a priori fulfill the C_z^0-continuity conditions at the interfaces for both displacement and transverse stress components (transverse displacement and transverse normal stress included).

Nevertheless, the conducted numerical analyses have shown, by means of several comparisons, that the mixed models lead to a better description than to standard displacement ones. Furthermore, the related results match the three-dimensional elasticity solutions very well and their accuracy has been confirmed for symmetrically, unsymmetrically, as well as arbitrarily laminated plates.

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References

Bhashar, B., and Varadan, T. K., 1989, "Refinement of Higher-Order laminated plate theories," *American Institute of Aeronautics and Astronautics Journal*, Vol. 27, pp. 1830–1831.

Carrera, E., 1995, "A class of Two Dimensional Theories for Multilayered Plates Analysis," *Atti Academia delle Scienze di Torino, Mem Sci. Fis.*, Vol. 19–20, pp. 49–87.

Carrera, E., 1997, "C_z^0-Requirements: Models for the Two-Dimensional Analysis of Multilayered Structures," *Composite Structures*, Vol. 37, pp. 373–384.

Carrera, E., 1998a, "Mixed Layer-Wise Models for Multilayered Plates Analyses," preprint in DIAS Report No. 1/98, Politecnico di Torino, Torino, Italy.

Carrera, E., 1998b, "Evaluation of Mixed Layer-Wise Models for Multilayered Plates Analyses," *American Institute of Aeronautics and Astronautics Journal*, Vol. 36, No. 5, pp. 830–839.

Cho, K. N., Bert, C. W., and Striz, A. G., 1991, "Free Vibrations of Laminated Rectangular Plates Analyzed by Higher order Individual-Layer Theory," *Journal of Sound and Vibration*, Vol. 145, pp. 429–442.

Cho, M., and Parmerter, R. R., 1993, "Efficient higher order composite plate theory for general lamination configurations," *American Institute of Aeronautics and Astronautics Journal*, Vol. 31, pp. 1299–1305.

Di Sciuva, M., 1987, "An Improved Shear Deformation Theory for Moderately Thick Multilayered Anisotropic Shells and Plates," ASME Journal of Applied Mechanics, Vol. 54, pp. 589–586.

Hildebrand, F. B., Reissner, E., and Thomas, G. B., "Notes on the foundations of the theory of small displacement of orthotropic shells," NACA TN-1833, Washington, DC.

Jones, R. M., 1975, *Mechanics of Composite Materials*, McGraw-Hill, New York.

Kant, T., and Kommineni, J. R., 1989, "Large Amplitude Free Vibration Analysis of Cross-Ply Composite and Sandwich Laminates with a Refined Theory and C^0 Finite Elements," *Computer & Structures*, Vol. 50, pp. 123–134.

Kapania, R. K., and Raciti, S., 1989, "Recent advances in analysis of laminated beams and plates," *American Institute of Aeronautics and Astronautics Journal*, Vol. 27, pp. 923–946.

Kao, J., and Ross, R. J., 1968, "Bending of Multilayer sandwich beams," *American Institute of Aeronautics and Astronautics Journal*, Vol. 6, pp. 1583–1585.

Kheider, A. A., and Librescu, L., 1988, "Analysis of Symmetric Cross-ply Laminated Elastic Plates Using a Higher order Theory: Part II—Buckling and Free Vibration," *Composite Structures*, Vol. 9, pp. 259–277.

Kirchhoff, G., 1850, "Über das Gleichwich und die Bewegung einer Elastischen Sheibe," *J. Angew. math.*, Vol. 40, pp. 51–88.

Lekhnitskii, S. G., 1935, "Strength Calculation of Composite Beams," *Vestnik inzhen. i tekhnikov*, No. 9.

Librescu, L., and Reddy, J. N., 1987 "A Critical Review and Generalization of Transverse Shear Deformable Anisotropic Plates," *Refined dynamical Theories of Beams, Plates and Shells and their Applications*, Euromech Colloquium 219, Kassel, Sept. 1996, I. Ellshakoff and H. Irretier, eds., Springer-Verlag, Berlin, pp. 32–43.

Lo, K. H., Christensen, R. M., and Wu, E. M., 1977, "A Higher-Order Theory of Plate Deformation, Part II: Laminated Plates," ASME Journal of Applied Mechanics, Vol. 44, pp. 669–676.

Mindlin, R. D., 1951, "Influence of Rotatory Inertia and Shear on Flexural Motions of Isotropic Elastic Plates," ASME Journal of Applied Mechanics, Vol. 18, pp. 1031–1036.

Mindlin, R. D., Schacknow, A., and Deresiewicz, H., 1956, "An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates," *Journal of Sound and Vibration*, Vol. 12, pp. 187–195.

Murakami, H., 1986, "Laminated Composite Plate Theory With Improved In-Plane Response," ASME Journal of Applied Mechanics, Vol. 53, pp. 661–666.

Noor, A. K., 1973, "Free Vibration of Multilayered Plates," *American Institute of Aeronautics and Astronautics Journal*, Vol. 11, pp. 1038–1039.

Noor, A. K., and Burton, W. S., 1989a, "Stress and Free Vibration Analyses of Multilayered Composite Plates," *Composite Structures*, Vol. 11, pp. 183–204.

Noor, A. K., and Burton, W. S., 1989b, "Assessment of shear deformation theories for multilayered composite plates," ASME *Applied Mechanics Reviews*, Vol. 41, pp. 1–18.

Nosier, A., Kapania, R. K., and Reddy, J. N., 1993, "Free Vibration Analysis of Laminated Plates Using a Layer-Wise Theory," *American Institute of Aeronautics and Astronautics Journal*, Vol. 31, pp. 2335–2346.

Pagano, N. J., 1969, "Exact solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, pp. 398–411.

Poisson, S. D., 1829, "Memoire sur l'equilibre et le mouvement des corps elastique," *Mem. Acad. Sci.*, Vol. 8, p. 357.

Reddy, J. N., and Phan, N. D., 1985, "Stability and Vibration of Isotropic, Orthotropic, and Laminated Plates According to a Higher order Shear Deformation Theory," *Journal of Sound and Vibration*, Vol. 98, pp. 157–170.

Reddy, J. N., 1987, "A generalization of Two-Dimensional Theories of Laminated Composite Plates," *Communication in Applied Numerical Methods*, Vol. 3, pp. 173–180.

Reddy, J. N., 1997, *Mechanics of Laminated Composite Plates: Theory and Analysis*, CRC Press, Boca Raton, FL.

Reissner, E., 1945, "The Effects of Transverse Shear Deformation on the Bending of Elastic Plates," ASME Journal of Applied Mechanics, Vol. 12, pp. 69–76.

Reissner, E., 1984, "On a certain mixed variational theory and a proposed applications," *International Journal of Numerical Methods in Engineering*, Vol. 20, pp. 1366–1368.

Reissner, E., 1985, "Reflections on the theory of elastic plates," ASME *Applied Mechanics Reviews*, Vol. 38, pp. 1433–1464.

Reissner, E., 1986, "On a mixed variational theorem and on a shear deformable plate theory," *International Journal of Numerical Methods in Engineering*, Vol. 23, pp. 193–198.

Ren, J. G., and Owen, D. R. J., 1989, "Vibration and Buckling of Laminated Plates," *International Journal of Solids and Structures*, Vol. 23, pp. 111–131.

Seide, P., 1980, "An approximate theory for the bending of laminated plates," *Mechanics Today*, Vol. 5, pp. 451–466.

Srinivas, S., Joga Rao, C. V., and Rao, A. K., 1970, "Flexural Vibration of Rectangular Plates," ASME Journal of Applied Mechanics, Vol. 23, pp. 430–436.

Srinivas, S., 1973, "A refined analysis of composite laminates," *Journal of Sound and Vibration*, Vol. 30, pp. 495–50.

Sun, C. T., and Whitney, J. M., 1973, "On the theories for the dynamic response of laminated plates," *American Institute of Aeronautics and Astronautics Journal*, Vol. 11, pp. 372–398.

Swift, G. W., and Heller, R. A., 1974, "Layered beams analysis," *ASCE Journal of Engineering Mechanics Division*, Vol. 100, pp. 267–282.

Toledano, A., and Murakami, H., 1987a, "A Higher-Order Laminated Plate Theory with Improved In-Plane Responses," *International Journal of Solids and Structures*, Vol. 23, pp. 111–131.

Toledano, A., and Murakami, H., "A composite plate theory for arbitrary laminated configurations," ASME Journal of Applied Mechanics, Vol. 54, pp. 181–189.

Valisetty, R. R., and Rehfield, L. W., 1988, "Application of ply level analysis to flexure wave vibration," *Journal of Sound and Vibration*, Vol. 126, pp. 183–194.

Vlasov, B. F., 1957, "Ob uravnieniakh isgiba palstinok (On the equations of bending of plates)," Dokla Ak. Nauk Azerbeijanskoi-SSR, Vol. 3, pp. 955–979.

Yang, P. C., Norris, C. H., and Stavsky, Y., 1966, "Elastic Wave Propagation in Heterogenous Plates," *International Journal of Solids and Structures*, Vol. 2, pp. 85–97.

APPENDIX

Explicit Form of Basic Arrays

The arrays in the mixed Hooke's law referred to a generic fiber orientation with respect to the x-axis are

$$\tilde{\mathbf{C}}^k_{pp} = \begin{bmatrix} \tilde{C}^k_{11} & \tilde{C}^k_{12} & \tilde{C}^k_{16} \\ \tilde{C}^k_{12} & \tilde{C}^k_{22} & \tilde{C}^k_{26} \\ \tilde{C}^k_{16} & \tilde{C}^k_{26} & \tilde{C}^k_{66} \end{bmatrix}; \quad \tilde{\mathbf{C}}^k_{pn} = \tilde{\mathbf{C}}^{k^T}_{np} \begin{bmatrix} 0 & 0 & \tilde{C}^k_{13} \\ 0 & 0 & \tilde{C}^k_{23} \\ 0 & 0 & \tilde{C}^k_{36} \end{bmatrix};$$

$$\tilde{\mathbf{C}}^k_{nn} = \begin{bmatrix} \tilde{C}^k_{44} & \tilde{C}^k_{45} & 0 \\ \tilde{C}^k_{45} & \tilde{C}^k_{55} & 0 \\ 0 & 0 & \tilde{C}^k_{33} \end{bmatrix}.$$

The introduced arrays of differential operators on Ω^k are

$$\mathbf{D}_p = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ \partial_y & \partial_x & 0 \end{bmatrix}; \quad \mathbf{D}_n = \begin{bmatrix} \partial_z & 0 & \partial_x \\ 0 & \partial_z & \partial_y \\ 0 & 0 & \partial_z \end{bmatrix};$$

$$\mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \partial_x \\ 0 & 0 & \partial_y \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{D}_{nz} = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}$$

related not differential operators are

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad \mathbf{I}_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix};$$

$$\mathbf{I}_{n\Omega} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The integrals at the Eqs. (15) related to a linear field in the k-layer hold,

$$E^k_{tt} = h_k\tfrac{1}{3}, \quad E^k_{tb} = h_k\tfrac{1}{6}, \quad E^k_{bt} = h_k\tfrac{1}{6}, \quad E^k_{bb} = h_k\tfrac{1}{3}$$

$$E_{t_z t} = \tfrac{1}{2}, \quad E_{t_z b} = \tfrac{1}{2}, \quad E_{b_z t} = -\tfrac{1}{2}, \quad E_{b_z b} = -\tfrac{1}{2}$$

$$E_{tt_z} = \tfrac{1}{2}, \quad E_{tb_z} = -\tfrac{1}{2}, \quad E_{bt_z} = \tfrac{1}{2}, \quad E_{bb_z} = -\tfrac{1}{2}$$

$$E^k_{t_z t_z} = \frac{1}{h_k}, E^k_{t_z b_z} = -\frac{1}{h_k}, \quad E^k_{b_z t_z} = -\frac{1}{h_k}, \quad E^k_{b_z b_z} = \frac{1}{h_k}.$$

For the parabolic case the F_2 functions are also involved,

$$E^k_{tt} = \frac{h_k}{3}, \quad E^k_{t2} = -\frac{h_k}{2}, \quad E^k_{tb} = \frac{h_k}{6},$$

$$E^k_{2t} = -\frac{h_k}{2}, \quad E^k_{22} = \frac{6h_k}{5}, \quad E^k_{2b} = -\frac{h_k}{2},$$

$$E^k_{bt} = \frac{h_k}{6}, \quad E^k_{b2} = -\frac{h_k}{2}, \quad E^k_{bb} = \frac{h_k}{3}$$

$$E_{t_z t} = \tfrac{1}{2}, \quad E_{t_z 2} = -1, \quad E_{t_z b} = \tfrac{1}{2},$$

$$E_{2_z t} = 1, \quad E_{2_z 2} = 0, \quad E_{2_z b} = -1,$$

$$E_{b_z t} = -\tfrac{1}{2}, \quad E_{b_z 2} = 1, \quad E_{b_z b} = -\tfrac{1}{2},$$

$$E_{tt_z} = \tfrac{1}{2}, \quad E_{t2_z} = 1, \quad E_{tb_z} = -\tfrac{1}{2},$$

$$E_{2t_z} = -1, \quad E_{22_z} = 0, \quad E_{2b_z} = 1,$$

$$E_{bt_z} = \tfrac{1}{2}, \quad E_{b2_z} = -1, \quad E_{bb_z} = -\tfrac{1}{2},$$

$$E^k_{t_z t_z} = \frac{1}{h_k}, \quad E^k_{t_z 2_z} = 0, \quad E^k_{t_z b_z} = \frac{-1}{h_k},$$

$$E^k_{2_z t_z} = 0, \quad E^k_{2_z 2_z} = \frac{12}{h_k}, \quad E^k_{2_z b_z} = 0.$$

$$E^k_{b_z t_z} = \frac{-1}{h_k}, \quad E^k_{b_z 2_z} = 0, \quad E^k_{b_z b_z} = \frac{1}{h_k}$$