Elastodynamic Behavior of Relatively Thick, Symmetrically Laminated, Anisotropic Circular Cylindrical Shells

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Introduction

In a previous paper (Di Sciuva, 1987), the first author developed the general linear equations governing the elastodynamic behavior of moderately thick, multilayered anisotropic shells by making use of a displacement field which allows piecewise linear variation of the in-plane displacements u and v, and constant value of the transverse displacement w through the thickness of the laminate (discrete-layer model). The major advantage of the proposed displacement field rests upon the fact that it fulfills *a priori* the static and geometric continuity conditions at the interfaces between the layers and that only five generalized displacements are required to describe the deformation of the laminate, as in first-order shear deformation theories (F.S.D.T.). No numerical results were given. In the present paper the approach developed by Di Sciuva (1987) is applied to the analysis of the elastodynamic behavior of circular cylindrical shells. To avoid geometrical nonlinear effects, that can be present in the unsymmetric laminates even if deflection is in the so-called small deflection range (Sun and Chin, 1988), only symmetric laminates will be considered in the sequel.

Exact solutions for cross-ply lamination schemes are developed. Numerical results show how the various material and geometrical parameters influence the undamped frequencies of cylindrical panels and circular cylindrical shells. To illustrate the accuracy and reliability of the proposed approach, some comparisons with results from the open literature are made.

Governing Equations

Consider a moderately thick, laminated, circular cylindrical shell composed of N generally orthotropic layers, symmetrically located with respect to the midsurface of the shell. The equations of motion can be derived from those given by Di Sciuva (1987). In terms of displacement components these read

$$\sum_{j=1}^{5} L_{ij}U_j=\sum_{j=1}^{5} M_{ij}\ddot{U}_j \qquad i=1,5. \tag{1}$$

For the differential operators L_{ij} the following expressions hold:

$$L_{11}=L_3^A;\ L_{12}=L_4^A;\ L_{13}=-L_1^A/R;\ L_{14}=L_{15}=0;$$
$$L_{22}=L_5^A+L_5^D/R^2;\ L_{23}=(L_4^D/R)_{,x}+(L_5^D/R)_{,y}-L_2^A;$$
$$L_{24}=-(L_4^D+L_4^{Da}+L_5^{Dd})/R;\ L_{25}=-(L_5^D+L_5^{Dd}+L_4^{Dc})/R;$$
$$L_{33}=L^D_{3,xx}+2L^D_{4,xy}+L^D_{5,yy}+A_{22}/R^2$$
$$L_{34}=-[(L_3^D+L_3^{Da}+L_4^{Dd})_{,x}+(L_4^D+L_4^{Da}+L_5^{Dd})_{,y}]$$
$$L_{35}=-[(L_4^D+L_4^{Dd}+L_3^{Dc})_{,x}+(L_5^D+L_5^{Db}+L_4^{Dc})_{,y}]$$
$$L_{44}=L_3^D+2L_3^{Da}+L_3^{Daa}+2L_4^{Dd}+2L_4^{Dad}+L_5^{Ddd}-A_{44}-A_{44}^a-A_{45}^d$$
$$L_{45}=L_4^D+L_3^{Dc}+L_4^{Da}+L_4^{Db}+L_5^{Dd}+L_3^{Dac}+L_4^{Dab}+L_4^{Dcd}+L_5^{Dbd}-A_{45}-A_{45}^a-A_{55}^d$$
$$L_{54}=L_4^D+L_3^{Dc}+L_4^{Da}+L_4^{Db}+L_5^{Dd}+L_3^{Dac}+L_4^{Dab}+L_4^{Dcd}+L_5^{Dbd}-A_{45}-A_{45}^b-A_{44}^c$$
$$L_{55}=L_5^D+2L_4^{Dc}+2L_5^{Db}+L_3^{Dcc}+2L_4^{Dbc}+L_5^{Dbb}-A_{55}-A_{55}^b-A_{45}^c$$

where

$$L_1^\tau=\tau_{12}(\bullet)_{,x}+\tau_{26}(\bullet)_{,y};\ L_3^\tau=\tau_{11}(\bullet)_{,xx}+2\tau_{16}(\bullet)_{,xy}+\tau_{66}(\bullet)_{,yy}$$
$$L_2^\tau=\tau_{26}(\bullet)_{,x}+\tau_{22}(\bullet)_{,y};\ L_5^\tau=\tau_{66}(\bullet)_{,xx}+2\tau_{26}(\bullet)_{,xy}+\tau_{22}(\bullet)_{,yy}$$
$$L_4^\tau=\tau_{16}(\bullet)_{,xx}+(\tau_{12}+\tau_{66})(\bullet)_{,xy}+\tau_{26}(\bullet)_{,yy}$$

and $U_1=u$; $U_2=v$; $U_3=w$; $U_4=\phi_x$; $U_5=\phi_y$. Here, u, v, and w denote the displacement components of a point belonging to the shell reference surface; ϕ_x and ϕ_y denote the values of the shear rotations in the (x, z) and (y, z) planes, respectively. R is the curvature radius of the shell reference cylindrical surface; x and y are the orthogonal coordinates on the reference surface (Fig. 1). The symbol $(\bullet)_{,x}$ stands for partial derivative with respect to x; the overdot indicates differentiation with respect to time t.

The differential operators which appear in the expressions for L_{ij} are obtained by replacing τ in L_i with the superscripts which appear in L_{ij}. These superscripts stand for the extensional and bending stiffnesses whose expressions are given below:

$$[A_{ij}\ D_{ij}]=\langle[1\ z^2]Q_{ij}(s)\rangle \text{ for } i,j=1,2,6$$
$$D_{ij}^\delta=\langle zQ_{ij}(s)\delta_k\rangle;\ D_{ij}^\delta=\langle Q_{ij}(s)\delta_k\delta_r\rangle$$
$$A_{ij}=hQ_{ij}(1) \text{ and } A_{ij}^\delta=Q_{ij}(1)\langle\Sigma_k\delta_k\rangle \text{ for } i,j=4,5.$$

In the above definitions, the Q_{ij} are the coefficients of the stiffness matrix of the sth layer; a_k, b_k, c_k, and d_k are known constants depending on the coefficients of the transverse shear stiffness matrix of the constituent layers only (Di Sciuva, 1987). For notational convenience, $\langle..\rangle$ is defined by

$$\langle..\rangle=\sum_{s=1}^{N}\int_{z_{s-1}}^{z_s}(..)dz \text{ and } \Sigma_k \text{ stands for } \sum_{k=1}^{s-1}.$$

For the inertia operators M_{ij}, the following expressions hold

$$M_{11}=M(\bullet);\ M_{12}=M_{13}=M_{14}=M_{15}=0;$$
$$M_{22}=(M+J/R^2)(\bullet);\ M_{23}=(J/R)(\bullet)_{,y};$$
$$M_{24}=-(J^d/R)(\bullet);\ M_{25}=-[(J+J^b)/R](\bullet);$$
$$M_{33}=[-M(\bullet)+J(\bullet)_{,xx}+J(\bullet)_{,yy};$$
$$M_{34}=[-(J+J^a)(\bullet)_{,x}-J^d(\bullet)_{,y}];$$
$$M_{35}=[-J^c(\bullet)_{,x}-(J+J^d)(\bullet)_{,y};$$
$$M_{44}=(J+2J^a+J^{aa}+J^{dd})(\bullet)$$
$$M_{45}=(J^c+J^d+J^{ac}+J^{bd})(\bullet);\ M_{55}=(J+2J^b+J^{bb}+J^{cc})(\bullet)$$

where the following definitions hold:

$$[M\ J]=\langle[1\ z^2]\mu(s)\rangle$$
$$[J^a\ J^b\ J^c\ J^d]=\langle z[A_k\ B_k\ C_k\ D_k]\mu(s)\rangle$$
$$[J^{aa}\ J^{bb}\ J^{cc}\ J^{dd}]=\langle[A_kA_r\ B_kB_r\ C_kC_r\ D_kD_r]\mu(s)\rangle$$

with $(A_k, B_k, C_k, D_k)=\Sigma_k(a_k, b_k, c_k, d_k)(z-z_k)$ and $\mu(s)$ being the material mass density of the sth layer.

Numerical Results and Concluding Remarks

We assume for the spatial variations of the unknown generalized displacements the following expressions:

$$U_i=\sum_{m=1}^{R}\sum_{n=1}^{T} C_{i,mn}F_{i,mn}(x,y) \tag{2}$$

where $C_{i,mn}$ are arbitrary parameters to be determined and $F_{i,mn}$ are a priori chosen functions. The set

$$F_{1,mn}=F_{4,mn}=\cos\alpha_m\sin\beta_n\ F_{2,mn}=F_{5,mn}=\sin\alpha_m\cos\beta_n$$
$$F_{3,mn}=\sin\alpha_m\sin\beta_n\ (\alpha_m=m\pi x/a;\ \beta_n=n\pi y/b)$$

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Table 1 Comparison of fundamental frequency parameter $\bar{\omega} = \omega a^2(M/E_T h^3)^{1/2}$. Simply-supported cylindrical shells (square panel and complete ring). The HSDT values are from Reddy and Liu (1985). Symmetric cross-ply 0 deg/90 deg/90 deg/0 deg.

R/a	a/h = 10				a/h = 100			
	CLT	FSDT X=5/6	HSDT	ZIG-ZAG	CLT	FSDT X=5/6	HSDT	ZIG-ZAG
square panel m=n=1								
5	15.123	12.265	11.830	11.880	20.396	20.364	20.360	20.356
10	15.107	12.235	11.790	11.847	16.672	16.633	16.630	16.623
20	15.103	12.227	11.780	11.839	15.599	15.555	15.555	15.546
50	15.102	12.225	11.780	11.837	15.284	15.242	15.230	15.231
100	15.101	12.225	11.780	11.837	15.239	15.196	15.190	15.148
Sq.plate	15.101	12.225	11.780	11.837	15.227	15.184	15.170	15.140
complete ring m=1, n in apexes								
5	13.704^{14}	10.958^{14}		10.462^{14}	20.035^{36}	20.000^{36}		19.990^{36}
10	13.499^{18}	10.698^{22}		10.187^{22}	16.544^{56}	16.506^{56}		16.496^{56}
20	13.401^{24}	10.579^{28}		10.063^{28}	14.827^{86}	14.787^{86}		14.776^{86}
50	13.345^{10}	10.496^{16}		9.996^{16}	13.881^{70}	13.840^{70}		13.827^{70}
100	13.336^{4}	10.496^{4}		9.971^{4}	13.600^{200}	13.559^{200}		13.547^{200}
Strip	13.326	10.490		9.971	13.380	13.338		13.326

Table 2 Comparison of fundamental (m = 1) frequency parameter $\bar{\omega} = \omega R^2(M/E_T h^3)^{1/2}$. Simply-supported cylindrical shells. Symmetric cross-ply 0 deg/90 deg/0 deg. * strip; ** beam theory.

a/R	R/h=100			R/h=50			R/h=20		
	CLT	FSDT X=1	ZIG-ZAG	CLT	FSDT X=1	ZIG-ZAG	CLT	FSDT X=1	ZIG-ZAG
0.10	1409.90	1160.00	1075.00	1120.50	815.53	715.25	448.31	384.29	352.93
	1395.00*	1142.00*	1056.00*	1111.70*	808.85*	707.14*	444.69*	382.69*	349.23*
0.25	248.06	240.75	236.20	233.17	205.48	194.86	220.01	132.46	116.98
	223.90*	215.59*	211.53*	223.53*	194.92*	183.81*	177.87*	129.42*	113.94*
0.5	83.45	82.96	82.74	70.07	68.04	67.15	59.24	50.02	46.96
1	38.72	38.69	38.68	27.31	27.18	27.14	18.82	18.15	17.87
2.5	17.52	17.51	17.51	11.49	11.48	11.48	6.55	6.53	6.52
5	9.37	9.37	9.37	6.28	6.28	6.28	3.70	3.70	3.70
10	4.76	4.76	4.76	3.53	3.53	3.53	1.73	1.73	1.73
25	1.74	1.74	1.74	1.27	1.27	1.27	0.74	0.74	0.74
50	1.08	1.08	1.08	0.54	0.54	0.54	0.22	0.22	0.22
	1.18**	1.18**	1.18**	0.59**	0.59**	0.59**	0.24**	0.24**	0.24**
100	0.28	0.28	0.28	0.14	0.14	0.14	0.06	0.06	0.06
	0.30**	0.30**	0.30**	0.15**	0.15**	0.15**	0.06**	0.06**	0.06**

(m and n are the half-waves number in the axial and circumferential direction, respectively) leads to an exact solution for symmetric cross-ply laminates with the following simple-support boundary conditions (hinged edges, free in the in-plane normal direction)

$$x=0,a: \quad N_{xx}=M_{xx}=U_2=U_3=U_5=0$$

$$y=0,b: \quad N_{yy}=M_{yy}=U_1=U_3=U_5=0.$$

Upon substitution of Eq. (2) into Eqs. (1), the required system of linear algebraic equations is obtained. Assuming a time dependence of harmonic type with frequency ω, Eqs. (1) give rise to a classical eigenvalue problem.

To illustrate the accuracy and reliability of the proposed approach, we present exact solutions for cross-ply circular cylindrical shells and compare these with other numerical results from the literature.

The numerical results refer to the following lamina mechanical properties:

$$E_L/E_T=25;\ G_{LT}/E_T=.5;\ G_{TT}/E_T=.2;\ \nu_{LT}=\nu_{TT}=.25;\ \mu=1$$

where E_L and E_T are major and minor Young's moduli in the principal material directions (L, T); ν_{LT} and ν_{TT} are Poisson's ratios, G_{LT} and G_{TT} are the shear moduli.

For purposes of comparison, Table 1 gives some numerical results from ESDT (X is the shear correction factor) and HSDT (third-order shear deformation theory, Reddy and Liu, 1985). In the tables and the figure we will denote present theory as zig zag. Simply-supported circular cylindrical shells (square panel and complete ring) are considered. Notice that the proposed model gives results very close to those obtained by Reddy and Liu, irrespective of the values of the radius-to-length ratio R/a and of the length-to-thickness ratio a/h.

It is very interesting to note that for a complete ring, the CLT fails to predict the fundamental mode. In effect, at low values of a/h and intermediate values of R/a ratios, the fundamental mode predicted by CLT has a number of circumferential half-waves n lower than those predicted by FSDT and by the present study. The limiting values for a square panel and for a complete ring are those relative to the square plate and to strip of infinite length, respectively. Further, the values of the fundamental frequency parameter for a square plate of side length a and for a strip are approached in a different manner for increasing values of R/a, depending on the values of a/h. In particular, moderately thick, symmetrically laminated shells (in the numerical example, a/h = 10) are less sensitive to R/a variations than thin shells (in the numerical example, a/h = 100).

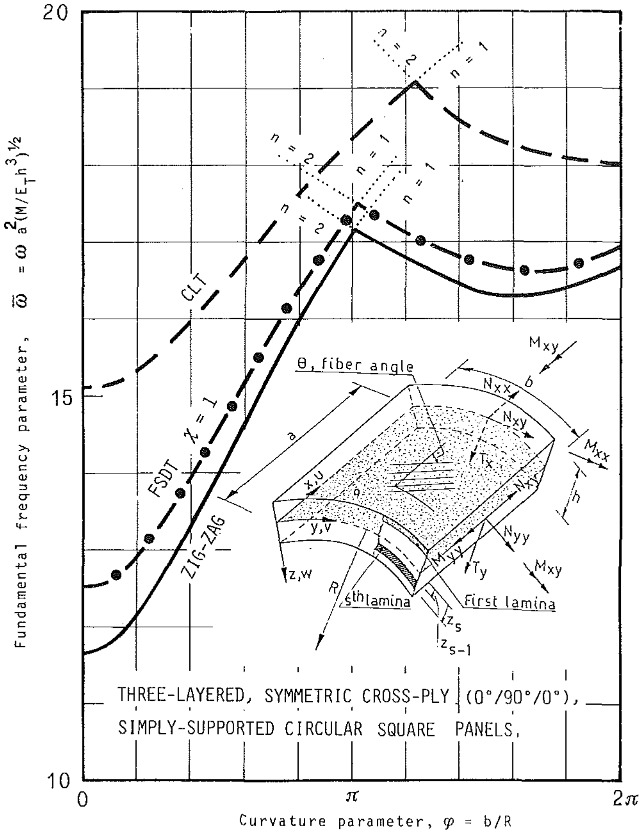

Fig. 1 Influence of the curvature parameter φ on the fundamental frequency parameter; $R/h = 10$, $m = 1$

Table 2 gives the comparison with the results from the CLT and FSDT theories for three different values of the shell radius-to-thickness ratio a/h. The starred and double-starred values are those of the fundamental frequency parameter as given by the strip of infinite length and beam solutions, respectively, for the same value of the ratio a/h. Some remarks should be made about the results given in this table. As before, for the same value of the ratios a/R and R/h, the fundamental mode predicted by the three theories, is not always for the same value of n. Moreover, for decreasing values of R/h, the strip and beam solutions are closer to the corresponding shell solution.

The influence of the curvature is shown in Fig. 1 for three-layered, symmetric cross-ply (0 deg/90 deg/0 deg), simply-supported circular cylindrical square panels. As expected, the results indicate that an increase in the curvature produces an increase in the frequency parameter, thereby countering the effects of the transverse shear deformability terms which tend to reduce the frequency parameter. Notice that the values of the curvature parameter ϕ are obtained by changing R for a fixed value of b. Then, $\phi = 0$ ($R \rightarrow \infty$) corresponds to the flat square plate, while $\phi = 2\pi$ corresponds to the circular cylindrical shells.

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Circumferential Stresses in Curved Beams

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We consider circumferential stress σ_θ in the curved beam of Fig. 1. From mechanics of materials approximations, the accepted formula for σ_θ at arbitrary θ, obtained by superposing bending stress and direct stress, is

$$\sigma_\theta = \frac{M(r_n - r)}{Aer} + \frac{N}{A} \tag{1}$$

where A is the area of the cross-section, r_n locates the neutral axis of pure bending, and

$$e = R - r_n \qquad R = \frac{1}{A}\int r\,dA \qquad \frac{1}{r_n} = \frac{1}{A}\int \frac{dA}{r}. \tag{2}$$

The centroid of A is at $r = R$. In Eq. (1), $M = M_o + PR\sin\theta$ and $N = P\sin\theta$. The term that contains M in Eq. (1) is rationally derived from the assumptions that stress σ_θ is uniaxial and that plane cross-sections remain plane under bending. The appended N/A term has almost no rationale: Its form is familiar and it satisfies statics, but it is intended for use with straight bars. If $N = 0$, Eq. (1) is in excellent agreement with exact results from theory of elasticity for rectangular cross-sections (Cook and Young, 1985; Timoshenko and Goodier, 1970). For example, at $r = b$ and with $N = 0$, σ_θ from Eq. (1) is 0.3 percent high for $a/b = 4$ and 5.0 percent high for $a/b = 8$. However, if $N \neq 0$, errors are of greater magnitude, and σ_θ is underestimated rather than overestimated. The offending term, N/A in Eq. (1), should therefore be replaced by a more accurate approximation if its form is simple enough for easy use. Such alternatives for N/A are the subject of the present Note.

Consider a case of tension without bending: In Fig. 1, let $M_0 = -PR$, and examine the cross-section at $\theta = \pi/2$. Exact theory (Timoshenko and Goodier, 1970) and finite element solutions show that strain ϵ_θ on the inner portion of this cross-section varies approximately as c/r, where c is a constant. If it is also assumed that stress σ_θ is uniaxial, as is customary, then $\sigma_\theta = E\epsilon_\theta$, and

$$N = \int \sigma_\theta dA = \int \frac{Ec}{r}\,dA = Ec\,\frac{A}{r_n}. \tag{3}$$

Hence, $c = Nr_n/AE$. The direct stress term becomes Nr_n/Ar, and Eq. (1) is replaced by

$$\sigma_\theta = \frac{M(r_n - r)}{Aer} + \frac{Nr_n}{Ar}. \tag{4}$$

Unfortunately Eq. (4) places the centroid of the direct stress distribution at $r = r_n$ rather than at $r = R$, and moment equilibrium is not satisfied. A remedy for this inconsistency is to augment the direct stress by a term that varies linearly with distance from the centroid. Thus, the direct stress becomes $[(r_n/r) + \alpha(r-R)]N/A$, where constant α can be determined by equilibrium of moments about point Q in Fig. 1(b) with $\theta = \pi/2$.

$$\int \sigma_\theta (r - R)\,dA + M_o + PR = 0. \tag{5}$$

Hence, with $I_G = \int (R-r)^2 dA$ the centroidal moment of inertia of A, we obtain $\alpha = Ae/I_G$, and Eq. (4) is replaced by

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