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THE EFFECTS OF SHEAR DEFORMATION AND CURVATURE ON BUCKLING AND VIBRATIONS OF CROSS-PLY LAMINATED COMPOSITE SHELLS†

E. CARRERA

DIAS Politecnico di Torino, 10129 *Torino, Italy*

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On the basis of Flügge's approximations and with account taken of the transverse shear deformation (SDT) and all rotatory inertias, the equations of in-plane buckling and free vibrations of multi-layered, anisotropic, doubly curved shells are presented. As a particular case, the equations relative to Love's approximations and Donnell's approximations and as well as of the corresponding classical theories (CLT) are derived. Analytical exact solutions are presented for buckling under axial compression and free vibrations of cross-ply laminated, simply supported, circular cylindrical and spherical composite shells. Depending on geometric parameters of the shell (length to radius ratio a/R, length to thickness ratio a/h and radius to thickness ratio R/h), numerical results (in tables and figures) are quoted to compare the different theories, and some trends are singled out about the effects of both transverse shear deformations and curvatures of the shell. Particular attention is given to the application of Flügge's approximations in which are presented solutions of the equilibrium equations containing terms of the same order in h/R. Finally, the effects of coupled out-of-plane stiffnesses on the theories is investigated.

1. INTRODUCTION

Many of the classical theories were developed originally for thin elastic isotropic shells, and are based on the Kirchoff–Love assumptions (or LFAT—Love First Approximation Theories): (1) the shell is thin; (2) the deflections of the shell are small; (3) the normal stresses perpendicular to the middle surface can be neglected in comparison with other stresses; (4) straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface. Although no precise definition of thinness is available, LFAT are expected to yield sufficiently accurate results when the ratio of the thickness to the radius of curvature of the reference surface can be neglected in comparison to unity. This particular approximation is known as Love's approximation. The assumption that the deflections of the shell are small permit one to refer all derivations and calculation to the original configuration of the shell, and ensures that the resulting theory will be a linear theory. It is expected that the third assumption will generally be valid except in the vicinity of highly concentrated loads. The fourth assumption implies that all of the transverse shear strain components in the direction of the normal to the reference surface vanish. The CLT (Classical Lamination Theory) is based on this assumption. A survey of various classical theories can be found in the work of Naghdi [1].

Subsequently, a second class of theories of thin elastic, isotropic moderately thick shells, which we shall denote as LSAT (Love Second Approximation Theories), has also been developed. To this grouping, we assign all shell theories in which one or more of Kirchoff–Love's original postulates are suspended. A theory of elastic shells in which the

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thinness assumption is delayed was independently derived by Flügge [2], Lur'e [3] and Byrne [4]. The effects of transverse shear deformations (SDT—Shear Deformation Theory) and normal stresses were considered by Hildebrand, Reissner and Thomas [5]. Non-linear theories were considered by Sanders [6].

Moreover, for shallow shell analysis other approximations on curvature terms have been introduced by Donnell [7] and Mushtari [8]. Numerous applications of these approximations, both of LFAT and LSAT, have been presented in the literature.

Many extensions of these early theories to anisotropic shells have been published (see the publications of Ambartsumyan [9, 10]). Some of these, based on CLT and useful for our purposes are included in the following. A study of error involved in Donnell-type theories, on free vibrations of orthotropic cylindrical shells, was carried out by Stavsky and Loewy [11]; extension of Flügge's theory to cylindrical anisotropic shells has been presented by Cheng and Ho [12] and Lei and Cheng [13]; a comparison of some shell theories (Love–Flügge–Sanders–Donnell), on vibrations of cross-ply unsymmetric circular cylindrical shells has been performed by Soldatos [14].

As is well known, laminated composite plates and shells exhibit much larger thickness shear effects than the corresponding structures made of a homogeneous isotropic material. In fact, CLT fails to predict accurately the static and dynamic responses when the structures in question are rather thick and/or exhibit high anisotropic ratios. SDT and HSDT (Higher order SDT) must be used in these cases. See the papers by Noor and Burton [15] and Kapania and Raciti [16] and related references for multi-layered plates; and those of Bert and Kumar [17], Bhimaraddi [18], Reddy and Liu [19], Soldatos [20], Di Sciuva [21], Librescu *et al.* [22] (in which the effect of normal stress is also considered), Di Sciuva and Carrera [23], and related references for multi-layered shells. Useful analysis for our purpose is quoted in reference [17], in which Love's, Donnell's and Sanders' equations were presented for free vibrations of cross-ply laminated cylindrical shells, and in reference [20] in which comparison between Donnell's and Love's approximations was carried out for angle-ply laminated shells. Finally, a good survey of the theories adopted in the dynamic analysis of composite laminated shells has been presented by Soldatos [24], in which governing equations and numerical results are quoted for Donnell's, Love's, Sanders' and Flügge's theories based on CLT approximations, and for Donnell's, Love's and Sanders' theories based on SDT approximations.

To the author's best knowledge, for the buckling and vibration analysis of laminated composite shells and shell panels, there is not as yet any shell theory available in the literature which, on the basis of Flügge's approximations, incorporates the effects of transverse shear deformation. That is, no numerical study is available in the literature in which the first and fourth Kirchoff–Love hypotheses are suspended, and a comparison between the effects of shear deformation and shell curvatures is developed. It would prove useful to add, in this comparison, a study of error given by the Donnell-type approximation. The focus of the current work is directed toward removing this absence.

On the basis of Flügge-type approximations and with account taken of the transverse shear deformation and all rotatory inertias, the equations of free vibrations and buckling under in-plane loading of composite multi-layered circular cylindrical and spherical shells are derived in what follows. Linear theory with normal (to the middle surface) displacement constant along the thickness is considered (transverse normal strain and stress are negligible). No HSDT will be considered. By using some tracers (δ_D, δ_L and δ_C), which have the form of Kronecker deltas, displacements models, strain–displacement relations and equations of motion corresponding to other theories will be obtained as special cases. For $\delta_D = \delta_L = \delta_C = 1$ one has Flügge–SDT equations, while for $\delta_D = 0$ and $\delta_L = 0$, Flügge-SDT equations give Donnell–SDT and Love–SDT equations respectively. Imposing $\delta_C = 0$, for all three theories, SDT equations give the corresponding CLT equations. Furthermore, a numerical investigation is performed to show the influence of various geometrical parameters (length to radius ratio a/R, length to thickness ratio a/h and radius to thickness ratio R/h) on the approximations given by several theories. Results are given for those geometries (circular cylinder and spherical shells), laminations (orthotropic cross-ply), boundary conditions (simply supported) and load conditions (buckling under axial compression and free vibrations) for which exact eigenvalue solutions of equilibrium differential equations exist. Particular attention is given to the application of Flügge's approximations in which solutions of the equilibrium equations containing terms of the same order in h/R are presented. Finally the effects of coupled out-of-plane stiffnesses on the theories are investigated.

2. PRELIMINARY

2.1. SHELL GEOMETRY AND NOTATION

The salient features of shell geometry are shown in Figure 1. α and β are the curvilinear orthogonal co-ordinates (coinciding with lines of principal curvature) on the shell reference surface Ω (middle surface of the shell). ζ denotes the rectilinear co-ordinate measured along the normal, $\vec{n}$, to Ω. The following relations hold in the given triorthogonal system of curvilinear co-ordinates: square of line element,

$$ds^2 = H_\alpha^2\, d\alpha^2 + H_\beta^2\, d\beta^2 + H_\zeta^2\, d\zeta^2; \tag{1}$$

area of an infinitesimal rectangle on Ω,

$$d\Omega = H_\alpha H_\beta\, d\alpha\, d\beta; \tag{2}$$

volume of an infinitesimal parallelepiped,

$$dV = H_\alpha H_\beta H_\zeta\, d\alpha\, d\beta\, d\zeta. \tag{3}$$

Here

$$H_\alpha = A(1-\zeta/R_\alpha), \qquad H_\beta = B(1-\zeta/R_\beta), \qquad H_\zeta = 1. \tag{4}$$

R_α and R_β are the radii of curvature in the directions of α and β. A and B are the coefficients of the first fundamental form of the shell reference surface. To be considered in what follows are circular cylindrical and spherical shells for which $A = B = 1$. C denotes the intersection curve between the reference surface Ω and the edge surface Ω_e. C_u denotes that part of C on which are prescribed (geometrical) boundary conditions. C_p denotes that part of C on which are natural (mechanical) boundary conditions. $\vec{s}$ is the unit vector tangent to C, and $\hat{l}$ and $\hat{m}$ are the direction cosines of $\vec{s}$ along the α- and β-directions, respectively. Θ denotes the fiber orientation with respect to the α-direction.

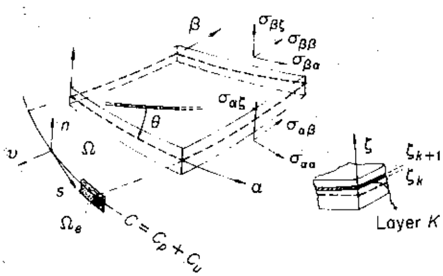

Figure 1. Shell geometry and notation.

2.2. STRESS-STRAIN RELATIONS OF AN ORTHOTROPIC LAMINA

Consider a shell of constant thickness h, consisting of a finite number ns of thin layers of orthotropic material and uniform thickness, perfectly bonded together. The principal axes of elasticity of any individual layer are assumed to be oriented parallel to the laminate axes. The material properties and the thickness of each layer may be entirely different. Upon neglecting the normal stresses $\sigma_{\zeta\zeta}$ the constitutive relations for any individual layer are [25]

$$\begin{pmatrix}\sigma_{\alpha\alpha}\\ \sigma_{\beta\beta}\\ \sigma_{\alpha\beta}\\ \sigma_{\alpha\zeta}\\ \sigma_{\beta\zeta}\end{pmatrix}=\begin{pmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{12} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{66} & 0 & 0\\ 0 & 0 & 0 & Q_{44} & 0\\ 0 & 0 & 0 & 0 & Q_{55}\end{pmatrix}\begin{pmatrix}\varepsilon_{\alpha\alpha}\\ \varepsilon_{\beta\beta}\\ \varepsilon_{\alpha\beta}\\ \varepsilon_{\alpha\zeta}\\ \varepsilon_{\beta\zeta}\end{pmatrix}, \tag{5}$$

where 1, 2 and 6 are the longitudinal L-direction of the fiber and the two orthogonal directions (T-directions) of the fiber axis L, respectively.

2.3. DISPLACEMENT FIELD

With the second and third Kirchoff–Love hypotheses taken into account, the displacement field is assumed to be of the form

$$u(\alpha,\beta,\zeta;t)=u^0(\alpha,\beta;t)+\zeta\Theta_\alpha,\qquad v(\alpha,\beta,\zeta;t)=v^0(\alpha,\beta;t)+\zeta\Theta_\beta,$$
$$w(\alpha,\beta,\zeta;t)=w^0(\alpha,\beta;t), \tag{6}$$

where $\Theta_\alpha=\delta_C\gamma^0_{\alpha\zeta}-w^0_{,\alpha}-u^0/R_\alpha$ and $\Theta_\beta=\delta_C\gamma^0_{\beta\zeta}-w^0_{,\beta}-v^0/R_\beta$, t denotes the time, u, v and w are the displacement components of a generic point of the shell along α, β and ζ respectively, u^0, v^0 and w^0 are the displacement components of a point on the shell reference surface Ω along α,β and ζ, respectively, and $\gamma^0_{\alpha\zeta}$ amd $\gamma^0_{\beta\zeta}$ denote the shear rotations in the α–ζ and β–ζ planes, respectively, the subscript notations ",α" and ",β" denote partial derivatives with respect to α and β, respectively. For $\delta_C=0$, one obtains the CLT approximations.

2.4. STRAIN-DISPLACEMENT RELATIONS

In non-linear theory †(in the von Kármán sense), the strain–displacement relations in general curvilinear co-ordinates take the following form [27]:

$$H_\alpha\varepsilon_{\alpha\alpha}=u_{,\alpha}-\frac{w}{R_\alpha}+\frac{1}{2}\frac{w^2_{,\alpha}}{H^2_\alpha},\qquad H_\alpha\varepsilon_{\alpha\zeta}=w_{,\alpha}-H_\alpha u_{,\zeta},$$
$$H_\beta\varepsilon_{\beta\beta}=v_{,\beta}-\frac{w}{R_\beta}+\frac{1}{2}\frac{w^2_{,\beta}}{H^2_\beta},\qquad H_\beta\varepsilon_{\beta\zeta}=w_{,\beta}-H_\beta v_{,\zeta},$$
$$H_\alpha H_\beta\varepsilon_{\alpha\beta}=H_\alpha u_{,\beta}+H_\beta v_{,\alpha}+w_{,\alpha}w_{,\beta}. \tag{7}$$

Upon substitution of equations (6) into (7) one obtains‡

$$H_\alpha\varepsilon_{\alpha\alpha}=\varepsilon^0_{\alpha\alpha}+\zeta K_{\alpha\alpha}+\tfrac{1}{2}(w^0_{,\alpha})^2,\qquad H_\alpha\varepsilon_{\alpha\zeta}=\delta_C\gamma^0_{\alpha\zeta},$$
$$H_\beta\varepsilon_{\beta\beta}=\varepsilon^0_{\beta\beta}+\zeta K_{\beta\beta}+\tfrac{1}{2}(w^0_{,\beta})^2,\qquad H_\beta\varepsilon_{\alpha\zeta}=\delta_C\gamma^0_{\beta\zeta},$$
$$H_\alpha H_\beta\varepsilon_{\alpha\beta}=\varepsilon^0_{\alpha\beta}+\zeta\tau+w^0_{,\alpha}w^0_{,\beta}, \tag{8}$$

† The non-linear components are considered because one is interested in buckling problems. The linearized buckling equations will be obtained via Euler's method [26].

‡ In the following it will be supposed that $H^2_\alpha=H^2_\beta\approx 1$ in the non-linear components of w.

where

$$\varepsilon^0_{\alpha\alpha} = u^0_{,\alpha} - w^0/R_\alpha, \qquad \varepsilon^0_{\beta\beta} = v^0_{,\beta} - w^0/R_\beta,$$
$$\varepsilon^0_{\alpha\beta} = H_\alpha u^0_{,\beta} + H_\beta v^0_{,\alpha}, \qquad \tau = H_\alpha K_{\alpha\beta} + H_\beta K_{\beta\alpha},$$
$$K_{\alpha\alpha} = \Theta_{\alpha,\alpha} = \delta_C \gamma^0_{\alpha\zeta,\alpha} - w^0_{,\alpha\alpha} - \delta_D u^0_{,\alpha}/R_\alpha, \qquad K_{\beta\beta} = \Theta_{\beta,\beta} = \delta_C \gamma^0_{\beta\zeta,\beta} - w^0_{,\beta\beta} - \delta_D v^0_{,\beta}/R_\beta,$$
$$K_{\alpha\beta} = \Theta_{\alpha,\beta} = \delta_C \gamma^0_{\alpha\zeta,\beta} - w^0_{,\alpha\beta} - \delta_D u^0_{,\beta}/R_\alpha, \qquad K_{\beta\alpha} = \Theta_{\beta,\alpha} = \delta_C \gamma^0_{\beta\zeta,\alpha} - w^0_{,\beta\alpha} - \delta_D v^0_{,\alpha}/R_\beta. \tag{9}$$

$K_{\alpha\alpha}$ and $K_{\beta\beta}$ are the changes of curvatures along the α- and β-directions on Ω; τ is the twist of Ω. For $\delta_D = 0$, one obtains Donnell's changes of curvatures and twist.

3. SURVEY OF THE THEORIES CONSIDERED

It may be useful to recall some details of the theories considered in this paper.

3.1. LOVE'S THEORY

The following approximation (first Kirchhoff–Love hypothesis) is associated [28] with this theory:

$$H_\alpha = 1 - \zeta/R_\alpha \approx 1, \qquad H_\beta = 1 - \zeta/R_\beta \approx 1. \tag{10}$$

The approximations (10) are made in the geometrical as well as in the strain–displacements relations. Sometimes, the approximations $\gamma^0_{\alpha\zeta} = \gamma^0_{\beta\zeta} = 0$ are also associated with this name. In the present paper these two approximations will be examined separately.

3.2. THEORY OF FLÜGGE–LUR'E–BYRNE

This theory was independently derived by Flügge [2], Lur'e [3] and Byrne [4] (in the following it will be called simply Flügge's theory). The thinness assumption (first Kirchhoff–Love hypothesis) is delayed and terms such as $1/H_\alpha$ and $1/H_\beta$ are expressed as expansions in powers to ζ/R:†

$$\frac{1}{H_\alpha} = 1 + \delta_L \left\{ \frac{\zeta}{R_\alpha} + \left(\frac{\zeta}{R_\alpha}\right)^2 + \cdots + \left(\frac{\zeta}{R_\alpha}\right)^{NT} + O\left(\frac{\zeta}{R_\alpha}\right)^{NT+1} \right\},$$
$$\frac{1}{H_\beta} = 1 + \delta_L \left\{ \frac{\zeta}{R_\beta} + \left(\frac{\zeta}{R_\beta}\right)^2 + \cdots + \left(\frac{\zeta}{R_\beta}\right)^{NT} + O\left(\frac{\zeta}{R_\beta}\right)^{NT+1} \right\}. \tag{11}$$

Here the series have been truncated to NT order. For $\delta_L = 0$ (or $NT = 0$) one has Love's theory.

3.3. DONNELL'S THEORY

This theory is universally adopted for shallow shell analysis (for the definition of shallow shells, see references [29, 30]). It was independently introduced by Donnell [7] and Mushtari [8]. In this theory one neglects the terms $u_{,\alpha}/R_\alpha$ and $v_{,\beta}/R_\beta$ in the changes of curvatures (9), and the components of transverse shear force along the shell co-ordinates α and β in equilibrium equations along the two same co-ordinates, respectively (see equations 15(a, b) of section 4).

† R without a suffix is understood to denote R_β for cylindrical shells and R_β, or R_α, for a spherical shell.

3.4. CLT—CLASSICAL LAMINATION THEORY

The fourth Kirchoff-Love hypothesis is associated with this theory. The transverse shear deformability of the shell is neglected:

$$\gamma^0_{\alpha\zeta}=\gamma^0_{\beta\zeta}=0. \tag{12}$$

It is the extension of Kirchoff's universally adopted in the analysis of anisotropic thin plates.

3.5. SDT—SHEAR DEFORMATION THEORY

The fourth Kirchhoff-Love hypothesis is not invoked. It is the extension to moderately thick shells of Mindlin's theory universally adopted in the analysis of moderately thick plates. In recent papers [15, 19, 20] on composite structures, this theory is called FSDT (First-Shear Deformation Theory) to distinguish it from other numerous forms of HSDT (Higher order Shear Deformation Theory).

4. GOVERNING EQUATIONS OF VIBRATIONS AND BUCKLING

Consider the motion of an elastic body under prescribed boundary conditions. If the body is assumed to execute an arbitrary set of infinitesimal virtual displacements δu, δv and δu, from the actual configuration, the following variational equation of motion holds [26]:

$$\delta\Phi+\sum_{s=1}^{ns}\int_{V_s}\mu_s(\ddot{u}\delta u+\ddot{v}\delta v+\ddot{w}\delta w)_s\,\mathrm{d}V_s=0. \tag{13}$$

$\delta\Phi$ is the virtual variation of strain energy, u_s is the material mass density of the sth layer, and the overdot indicates differentiation with respect to time; the integral gives the virtual variation of inertial forces. V_s represents the volume of the sth layer. The variation of the total strain energy in the global original volume V of the laminate is given by

$$\delta\Phi=\sum_{s=1}^{ns}\int_{V_s}[\sigma_{\alpha\alpha}\delta\varepsilon_{\alpha\alpha}+\sigma_{\beta\beta}\delta\varepsilon_{\beta\beta}+\sigma_{\alpha\beta}\delta\varepsilon_{\beta\alpha}+\{\sigma_{\alpha\zeta}\delta\varepsilon_{\alpha\zeta}+\sigma_{\beta\zeta}\delta\varepsilon_{\beta\zeta}\}\delta_C]_s\,\mathrm{d}V_s. \tag{14}$$

Substitution of the strain of expression (9), after necessary integrations, yields the following non-linear differential equations of motion and boundary conditions in terms of force and moment stress resultants (see references [21] and [31] for more details): equations of motion,

$$N_{\alpha\alpha,\alpha}+N_{\beta\alpha,\beta}-\delta_D\frac{\tilde{Q}_\alpha}{R_\alpha}=\left(M-2\frac{P}{R_\alpha}+\frac{J}{R_\alpha^2}\right)\ddot{u}^0+\left(P-\frac{J}{R_\alpha}\right)\ddot{\gamma}^0_{\alpha\zeta}\delta_C-\left(P-\frac{J}{R_\alpha}\right)\ddot{w}^0_{,\alpha}, \tag{15a}$$

$$N_{\alpha\beta,\alpha}+N_{\beta\beta,\beta}-\delta_D\frac{\tilde{Q}_\beta}{R_\beta}=\left(M-2\frac{P}{R_\beta}+\frac{J}{R_\beta^2}\right)\ddot{v}^0+\left(P-\frac{J}{R_\beta}\right)\ddot{\gamma}^0_{\beta\zeta}\delta_C-\left(P-\frac{J}{R_\beta}\right)\ddot{w}_{,\beta}, \tag{15b}$$

$$\tilde{Q}_{\alpha,\alpha}+\tilde{Q}_{\beta,\beta}+\frac{N_{\alpha\alpha}}{R_\alpha}+\frac{N_{\beta\beta}}{R_\beta}+(N_{\alpha\alpha}w^0_{,\alpha}+N_{\alpha\beta}w^0_{,\beta})_{,\alpha}+(N_{\beta\beta}w^0_{,\beta}+N_{\beta\alpha}w^0_{,\alpha})_{,\beta}$$
$$=M\ddot{w}^0+\left[\left(P-\frac{J}{R_\alpha}\right)\ddot{u}^0+J\ddot{\gamma}^0_{\alpha\zeta}\delta_C-J\ddot{w}^0_{,\alpha}\right]_{,\alpha}+\left[\left(P-\frac{J}{R_\beta}\right)\ddot{v}^0+J\ddot{\gamma}^0_{\beta\zeta}\delta_C-J\ddot{w}^0_{,\beta}\right]_{,\beta}, \tag{15c}$$

$$\{\tilde{Q}_\alpha-Q_\alpha\}\delta_C=\left\{\left[P-\frac{J}{R_\alpha}\right]\ddot{u}^0+J\ddot{\gamma}^0_{\alpha\zeta}-J\ddot{w}^0_{,\alpha}\right\}\delta_C, \tag{15d}$$

$$\{\tilde{Q}_\beta-Q_\beta\}\delta_C=\left\{\left[P-\frac{J}{R_\beta}\right]\ddot{v}^0+J\ddot{\gamma}^0_{\beta\zeta}-J\ddot{w}^0_{,\beta}\right\}\delta_C; \tag{15e}$$

boundary conditions,

Mechanical on C_p — Geometrical on C_u

$$N_{\alpha\upsilon}-M_{\alpha\upsilon}/R_\alpha=\bar{N}_{\alpha\nu}-\bar{M}_{\alpha\upsilon}/R_\alpha, \qquad u^0=\bar{u}^0,$$

$$N_{\beta\upsilon}-M_{\beta\upsilon}/R_\beta=\bar{N}_{\beta\upsilon}-\bar{M}_{\beta\upsilon}/R_\beta, \qquad v^0=\bar{v}^0,$$

$$\begin{aligned}V_n-M_{\nu t,t}&+N_{\alpha\alpha}w^0_{,\alpha}+N_{\alpha\beta}w^0_{,\beta}+N_{\beta\beta}w^0_{,\beta}+N_{\beta\alpha}w^0_{,\alpha}\\&=\bar{V}_n-\bar{M}_{\nu t,t}+\overline{N_{\alpha\alpha}w^0_{,\alpha}}+\overline{N_{\alpha\beta}w^0_{,\beta}}+\overline{N_{\beta\beta}w^0_{,\beta}}+\overline{N_{\beta\alpha}w^0_{,\alpha}}\\&+\left[\left(P-\frac{J}{R_\alpha}\right)\ddot{u}^0+\delta_CJ\ddot{\gamma}^0_{\alpha\zeta}-J\ddot{w}^0_{,\alpha}\right]\hat{l}\\&+\left[\left(P-\frac{J}{R_\beta}\right)\ddot{v}^0+\delta_CJ\ddot{\gamma}^0_{\beta\zeta}-J\ddot{w}^0_{,\beta}\right]\hat{m}, \qquad w^0=\bar{w}^0,\end{aligned}$$

$$M_\upsilon=\bar{M}_\upsilon, \qquad w^0_{,\nu}=\bar{w}^0_{,\nu},$$
$$\{M_{\alpha\upsilon}\}\delta_C=\{\bar{M}_{\alpha\upsilon}\}\delta_C, \qquad \{|^0_{\alpha\zeta}\}\delta_C=\{\bar{\gamma}^0_{\alpha\zeta}\}\delta_C,$$
$$\{M_{\beta\nu}\}\delta_C=\{\bar{M}_{\beta\upsilon}\}\delta_C, \qquad \{\delta^0_{\beta\zeta}\}\delta_C=\{\bar{\gamma}^0_{\beta\zeta}\}\delta_C.$$

Here the following symbols and notation have been introduced:

$$N_{\alpha\upsilon}=N_{\alpha\alpha}\hat{l}+N_{\beta\alpha}\hat{m}, \qquad N_{\beta\upsilon}=N_{\alpha\beta}\hat{l}+N_{\beta\beta}\hat{m}, \qquad M_{\alpha\upsilon}=M_{\alpha\alpha}\hat{l}+M_{\beta\alpha}\hat{m},$$
$$M_{\beta\upsilon}=M_{\alpha\beta}\hat{l}+M_{\beta\beta}\hat{m}, \qquad M_{\alpha\upsilon}=M_{\alpha\upsilon}\hat{l}+M_{\beta\upsilon}\hat{m}, \qquad M_{\beta\upsilon}=M_{\beta\upsilon}\hat{l}-M_{\alpha\upsilon}\hat{m},$$
$$\tilde{Q}_\alpha=M_{\alpha\alpha,\alpha}+M_{\beta\alpha,\beta}, \qquad \tilde{Q}_\beta=M_{\beta\beta,\beta}+M_{\alpha\beta,\alpha},$$
$$V_n=\tilde{Q}_\alpha\hat{l}+\tilde{Q}_\beta\hat{m}\langle(\cdots)\rangle=\sum_{s=1}^{NS}\int_{\zeta_{s-1}}^{\zeta_s}(\cdots)\,\mathrm{d}\zeta; \tag{16}$$

$$(M,P,J)=\left\langle(1,\zeta,\zeta^2)\left[\mu_s\left(1+\delta_L\left\{-\frac{R_\alpha+R_\beta}{R_\alpha R_\beta}\zeta+\frac{\zeta^2}{R_\alpha R_\beta}\right\}\right)\right]\right\rangle,$$
$$(N_{\alpha\alpha},N_{\alpha\beta},Q_\alpha,M_{\alpha\alpha},M_{\alpha\beta})=\langle(\sigma_{\alpha\alpha},\sigma_{\alpha\beta},\sigma_{\alpha\zeta},[\sigma_{\alpha\alpha},\sigma_{\alpha\beta}]\zeta)H_\beta\rangle,$$
$$(N_{\beta\beta},N_{\beta\alpha},Q_\beta,M_{\beta\beta},M_{\beta\alpha})=\langle(\sigma_{\beta\beta},\sigma_{\beta\alpha},\sigma_{\beta\zeta},[\sigma_{\beta\beta},\sigma_{\beta\alpha}]\zeta)H_\alpha\rangle. \tag{17}$$

These equations are complete in all curvature terms and rotatory inertias; thus they can give, as a particular case, all theories considered in this paper. Notice that for CLT one has a reduction both of the number of equilibrium equations and of the number of boundary conditions.

Neglecting all non-linear terms one obtains the linear equilibrium equations of motion, which will be used to determine the free vibration frequencies of multi-layered shells. The buckling equations can be obtained via Euler's method of adjacent states of equilibrium [26]. If the pre-buckling deformation can be neglected then, via Euler's method, one obtains equations that are formally the same as equations (15), in which the stress resultants multiplied by displacements w (present in equation 15(c)) are considered to be prescribed external forces, constant in the shell, and it is assumed that they vary neither in magnitude nor in direction during buckling; that is,

$$(N_{\alpha\alpha}w^0_{,\alpha}+N_{\alpha\beta}w^0_{,\beta})_{,\alpha}+(N_{\beta\beta}w^0_{,\beta}+N_{\beta\alpha}w^0_{,\alpha})_{,\beta}\to\bar{N}_{\alpha\alpha}w^0_{,\alpha\alpha}+\bar{N}_{\alpha\beta}w^0_{,\alpha\beta}+\bar{N}_{\beta\alpha}w^0_{,\beta\alpha}+\bar{N}_{\beta\beta}w^0_{,\beta\beta}. \tag{18}$$

Notice that the pre-buckling strain can be neglected when no structural coupling phenomena are present. There is structural coupling when geometrical imperfections are present, or the shell is curved, or the multi-layered shell is unsymmetrically laminated. This question has been well discussed in references [32] and [33]. Finally, the calculation of the buckling load as a bifurcation point (eigenvalue problem) fails when the shell exhibits snap-through phenomena [34]. For the aims of this paper just buckling-bifurcation problems are considered; structural coupling and snap-through phenomena, which would necessarily require non-linear analysis, are neglected.

4.1. EQUATIONS OF BUCKLING AND VIBRATIONS IN TERMS OF DISPLACEMENTS

Upon substitution of the relations (5), and after linearizing expression (8), the stress resultants can be written in the form

$$R_{ij} = \sum_{k=1}^{5} L_k^{R_{ij}} s_k \quad \text{with} \quad (s_1, s_2, s_3, s_4, s_5) = (u^0, v^0, w^0, \gamma^0_{\alpha\zeta}, \gamma^0_{\beta\zeta}),$$

$$R = N, M, \quad i, j = \alpha, \beta. \tag{19}$$

For the R_{ij} corresponding to $N_{\alpha\alpha}$, the $L_k^{R_{ij}}$ $(k = 1, \ldots, 5)$ differential operators are as follows:

$$L_1^{N_{\alpha\alpha}} = \left\langle Q_{11} \frac{H_\beta}{H_\alpha}\left(1 - \delta_D \frac{\zeta}{R_\alpha}\right)\right\rangle_{,\alpha} + \left\langle Q_{16}\left(1 - \delta_D \frac{\zeta}{R_\alpha}\right)\right\rangle_{,\beta},$$

$$L_2^{N_{\alpha\alpha}} = \left\langle Q_{12}\left(1 - \delta_D \frac{\zeta}{R_\beta}\right)\right\rangle_{,\beta} + \left\langle Q_{16} \frac{H_\beta}{H_\alpha}\left(1 - \delta_D \frac{\zeta}{R_\beta}\right)\right\rangle_{,\alpha},$$

$$L_3^{N_{\alpha\alpha}} = -\left\langle Q_{11} \frac{1}{R_\alpha}\frac{H_\beta}{H_\alpha} + Q_{12}\frac{1}{R_\beta}\right\rangle - \left\langle Q_{11} \frac{H_\beta}{H_\alpha}\right\rangle_{,\alpha\alpha} - \langle Q_{12}\zeta\rangle_{,\beta\beta} - \left\langle Q_{16}\left(1 + \frac{H_\beta}{H_\alpha}\right)\zeta\right\rangle_{,\alpha\beta},$$

$$L_4^{N_{\alpha\alpha}} = \left\langle Q_{11}\zeta \frac{H_\beta}{H_\alpha}\right\rangle_{,\alpha} + \langle Q_{16}\zeta\rangle_{,\beta}, \qquad L_5^{N_{\alpha\alpha}} = \langle Q_{12}\zeta\rangle_{,\beta} + \left\langle Q_{16}\zeta \frac{H_\beta}{H_\alpha}\right\rangle_{,\alpha}. \tag{20}$$

The operators for $M_{\alpha\alpha}$ ($L_i^{M_{\alpha\alpha}}$, $i = 1, \ldots, 5$) are directly obtained from those of $N_{\alpha\alpha}$, with Q_{ij} replaced by ζQ_{ij}; the operators for $N_{\alpha\beta}$ and $M_{\alpha\beta}$ ($L_i^{N_{\alpha\beta}}$ and $L_i^{M_{\alpha\beta}}$, $i = 1, \ldots, 5$) are obtained directly from those of $N_{\alpha\alpha}$ and $M_{\alpha\alpha}$, with Q_{11} replaced by Q_{21}, Q_{12} by Q_{22}, and Q_{16} by Q_{26}. The operators for $N_{\beta\beta}$, $M_{\beta\beta}$, $N_{\beta\alpha}$ and $M_{\beta\alpha}$ can be obtained directly from those for $N_{\alpha\alpha}$, $M_{\alpha\alpha}$, $N_{\alpha\beta}$ and $M_{\alpha\beta}$, by making the following changes: $1 \rightleftharpoons 2$, $\alpha \rightleftharpoons \beta$ and $4 \rightleftharpoons 5$, wherever they compare. This tedious shifting is very advantageous in the implementation of a computer program. In fact, by means of an appropriate computer code it is possible to express the generic stress resultants (19) in terms of displacements in a complete and automatic way. This automation has been used in the work reported here and it has been extensively presented by the author [35].

The shear resultants expressed in the form (19), give the following differential operators:

$$L_1^{Q_\alpha} = L_2^{Q_\alpha} = L_3^{Q_\alpha} = 0, \qquad L_1^{Q_\beta} = L_2^{Q_\beta} = L_3^{Q_\beta} = 0,$$

$$L_4^{Q_\alpha} = \langle Q_{44} H_\beta / H_\alpha\rangle, \qquad L_5^{Q_\alpha} = \langle Q_{45}\rangle, \qquad L_4^{Q_\beta} = \langle Q_{54}\rangle, \qquad L_5^{Q_\beta} = \langle Q_{55} H_\alpha / H_\beta\rangle. \tag{21}$$

In the expression for the stress resultants the following integrals are present:

$$(I_{ij}^{0\alpha}, I_{ij}^{1\alpha}, I_{ij}^{2\alpha}) = \langle (1, \zeta, \zeta^2) Q_{ij} H_\beta / H_\alpha\rangle,$$

$$(I_{ij}^{0\beta}, I_{ij}^{1\beta}, I_{ij}^{2\beta}) = \langle (1, \zeta, \zeta^2) Q_{ij} H_\alpha / H_\beta\rangle, \qquad i, j = 1, 2, 6, 4, 5. \tag{22}$$

If one makes an expansion in series of powers of ζ/R, following Flügge's approximations (11), the expressions (22) for H_β/H_α and H_α/H_β give respectively:†

$$\frac{H_\beta}{H_\alpha}=1+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}\zeta+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}\zeta^2+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}\zeta^3+O\left(\frac{\zeta}{R_\alpha}\right)^4\right\}\delta_L, \qquad NT=3,$$

$$\frac{H_\alpha}{H_\beta}=1+\left\{\frac{R_\alpha-R_\beta}{R_\alpha R_\beta}\zeta+\frac{R_\alpha-R_\beta}{R_\alpha R_\beta^2}\zeta^2+\frac{R_\alpha-R_\beta}{R_\alpha R_\beta^3}\zeta^3+O\left(\frac{\zeta}{R_\beta}\right)^4\right\}\delta_L, \qquad NT=3. \tag{23}$$

Upon introducing the laminate stiffnesses

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij})=\langle(1,\zeta,\zeta^2,\zeta^3,\zeta^4,\zeta^5,\zeta^6)Q_{ij}\rangle, \qquad i,j=1,2,6,4,5, \tag{24}$$

the expressions (23) ($NT=3$ is assumed) give

$$\begin{aligned}(I_{ij}^{0\alpha}, I_{ij}^{1\alpha}, I_{ij}^{2\alpha})&=(A_{ij}, B_{ij}, D_{ij})\\&+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}(B_{ij}, D_{ij}, E_{ij})+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}(D_{ij}, E_{ij}, F_{ij})\right.\\&\left.+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}(E_{ij}, F_{ij}, G_{ij})+O\left(\frac{F_{ij}, G_{ij}, H_{ij}}{R_\alpha^4}\right)\right\}\delta_L,\\(I_{ij}^{0\beta}, I_{ij}^{1\beta}, I_{ij}^{2\beta})&=(A_{ij}, B_{ij}, D_{ij})\\&+\left\{\frac{R_\alpha-R_\beta}{R_\alpha R_\beta}(B_{ij}, D_{ij}, E_{ij})+\frac{R_\alpha-R_\beta}{R_\alpha R_\beta^2}(D_{ij}, E_{ij}, F_{ij})\right.\\&\left.+\frac{R_\alpha-R_\beta}{R_\alpha R_\beta^3}(E_{ij}, F_{ij}, G_{ij})+O\left(\frac{F_{ij}, G_{ij}, H_{ij}}{R_\beta^4}\right)\right\}\delta_L,\end{aligned} \tag{25}$$

and expressions (20) and (21) give

$$\begin{aligned}L_1^{N_{\alpha\alpha}}&=\left(I_{11}^{0\alpha}-\delta_D\frac{I_{11}^{1\alpha}}{R_\alpha}\right)_{,\alpha}+\left(A_{16}-\delta_D\frac{B_{16}}{R_\alpha}\right)_{,\beta},\\L_2^{N_{\alpha\alpha}}&=\left(A_{12}-\delta_D\frac{\beta_{12}}{R_\beta}\right)_{,\beta}+\left(I_{16}^{0\alpha}-\delta_D\frac{I_{16}^{1\alpha}}{R_\beta}\right)_{,\alpha},\\L_3^{N_{\alpha\alpha}}&=-\left(\frac{I_{11}^{0\alpha}}{R_\alpha}+A_{12}\frac{1}{R_\beta}\right)-(I_{11}^{0\alpha})_{,\alpha\alpha}-(B_{12})_{,\beta\beta}-(B_{16}+I_{16}^{1\alpha})_{,\alpha\beta},\end{aligned} \tag{26}$$

$$L_4^{N_{\alpha\alpha}}=(I_{11}^{1\alpha})_{,\alpha}+(B_{16})_{,\beta}, \qquad L_5^{N_{\alpha\alpha}}=(B_{12})_{,\beta}+(I_{16}^{1\alpha})_{,\alpha},$$

$$L_4^{Q_\alpha}=I_{44}^{0\alpha}, L_5^{Q_\alpha}=A_{45}, \qquad L_4^{Q_\beta}=A_{54}, \qquad L_5^{Q_\beta}-I_{55}^{0\beta}. \tag{27}$$

Upon substitution of the previous notations in equations (15), one has the following linear equations of free vibrations and buckling (stability equations) of circular cylinder and spherical multi-layered shells, expressed in terms of displacements:

$$\sum_{k=1}^{5}\left[L_{k,\alpha}^{N_{\alpha\alpha}}+L_{k,\beta}^{N_{\beta\alpha}}-\delta_D\frac{L_k^{\tilde Q_\alpha}}{R_\alpha}\right]s_k=RH_{eq15\alpha}, \qquad \sum_{k=1}^{5}\left[L_{k,\alpha}^{N_{\alpha\beta}}+L_{k,\beta}^{N_{\beta\beta}}-\delta_D\frac{L_k^{Q_\beta}}{R_\beta}\right]s_k=RH_{eq15b},$$

$$\sum_{k=1}^{5}\left[L_{k,\alpha}^{\tilde Q_\alpha}+L_{k,\beta}^{\hat Q_\beta}\frac{L_k^{N_{\alpha\alpha}}}{R_\alpha}\frac{L_k^{N_{\beta\beta}}}{R_\beta}\right]s_k+\bar N_{\alpha\alpha}w^0_{,\alpha\alpha}+\bar N_{\alpha\beta}w^0_{,\alpha\beta}+\bar N_{\beta\alpha}w^0_{,\beta\alpha}+\bar N_{\beta\beta}w^0_{,\beta\beta}=RH_{eq15c},$$

$$\sum_{k=1}^{5}[L_k^{\hat Q_\alpha}-L_k^{Q_\alpha}]s_k=RH_{eq15d}, \qquad \sum_{k=1}^{5}[L_k^{\hat Q_\beta}-L_k^{Q_\beta}]s_k=RH_{eq15e}, \tag{28}$$

† Cylindrical shells can be obtained by imposing $R_\alpha=\infty$ or $R_\beta=\infty$.

when $RH_{eq...}$ denotes the right sides in equations (15). The boundary conditions are the same as those of equations (15). If one does not consider the dynamic terms, equations (28) coincide with the static equations of buckling.

The explicit version of system (28) is given in Appendix A.

4.2. NOTE ON THE APPLICATIONS OF FLÜGGE'S THEORY

In the literature [12, 13], Flügge's theory is developed by direct substitution of expression (25) in the equilibrium equations. In the present paper a comparison of the following methods of application of Flügge's theory is developed: (f1) direct substitution of expansion (25) into the equilibrium equations; (f2) substitution of expansion (25) into the equilibrium equations and reducing the NT order by one when terms of type I_{ij}/R are compared in the stress resultants; (f3) as f2, but further reduction of the NT order by one unit when the stress resultants are compared when divided by a radius of curvature in the equilibrium equations.

Note that in the f3 method components of the same order in h/R in the same equation are neglected.

To clarify the distinctions among these three methods, consider the first term in the first equation in Appendix A; one has

$$\{[I_{11}^{0\alpha}-\delta_D I_{11}^{1\alpha}/R_\alpha]-(\delta_D/R_\alpha)[I_{11}^{1\alpha}-\delta_D I_{11}^{2\alpha}/R_\alpha]\}u^0_{,\alpha\alpha}. \tag{29}$$

The terms in the first square brackets derives from $N_{\alpha\alpha}$, while the remaining terms derive from $-M_{\alpha\alpha,\alpha}/R_\alpha$. If one supposes $NT=3$, substituting expression (25) in expression (29) and following f1, one has

$$\begin{aligned}\Bigg\{\Bigg[&A_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}B_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}E_{11}\right\}\delta_L\\ &-\frac{\delta_D}{R_\alpha}\left(B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}F_{11}\right\}\delta_L\right)\Bigg]\\ &-\frac{\delta_D}{R_\alpha}\Bigg[B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}F_{11}\right\}\delta_L\\ &-\frac{\delta_D}{R_\alpha}\left(D_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}E_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}F_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}G_{11}\right\}\delta_L\right)\Bigg]\Bigg\}u^0_{,\alpha\alpha},\end{aligned} \tag{30}$$

in which the power series expansion is limited to order $NT=3$ for all integrals in expression (29).

Following f2, one has

$$\begin{aligned}\Bigg\{\Bigg[&A_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}B_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}E_{11}\right\}\delta_L\\ &-\frac{\delta_D}{R_\alpha}\left(B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}\right\}\delta_L\right)\Bigg]\\ &-\frac{\delta_D}{R_\alpha}\Bigg[B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}F_{11}\right\}\delta_L\Bigg]\\ &-\frac{\delta_D}{R_\alpha}(D_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}E_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}F_{11}\right\}\delta_L\Big)\Bigg]\Bigg\}u^0_{,\alpha\alpha},\end{aligned} \tag{31}$$

in which the power series expansion has been limited to the following: (a) order $NT=3$ for $I_{11}^{0\alpha}$ and for $I_{11}^{1\alpha}$ quoted inside the second square brackets; (b) order $NT=2$ for $I_{11}^{1\alpha}$ quoted inside the first square brackets and for $I_{11}^{2\alpha}$.

Following f3, one has

$$\left\{\left[A_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}B_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^3}E_{11}\right\}\delta_L\right.$$
$$-\frac{\delta_D}{R_\alpha}\left(B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}\right\}\delta_L\right)\Bigg]$$
$$-\frac{\delta_D}{R_\alpha}\left[B_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}D_{11}+\frac{R_\beta-R_\alpha}{R_\beta R_\alpha^2}E_{11}\right\}\delta_L\right.$$
$$\left.\left.-\frac{\delta_D}{R_\alpha}\left(D_{11}+\left\{\frac{R_\beta-R_\alpha}{R_\beta R_\alpha}E_{11}\right\}\delta_L\right)\right]\right\}u^0_{,\alpha\alpha}, \tag{32}$$

in which the power series expansion has been limited to the following: (a) order $NT=3$ for $I_{11}^{0\alpha}$; (b) and $NT=2$ for the two integrals $I_{11}^{1\alpha}$; (c) order $NT=1$ for $I_{11}^{2\alpha}$. The minimum value of NT is of course, zero, which corresponds to Love's approximation. As shown in the numerical investigations, the f3 method gives the most acceptable results.

It is noted at last that in the expressions (26) and (27) for the stress resultants only the terms H_β/H_α and H_α/H_β are present, and that for spherical shells (for which $R_\alpha=R_\beta$) expressions (23) gives

$$H_\beta/H_\alpha=1, \qquad H_\alpha/H_\beta=1. \tag{33}$$

Therefore, in buckling analyses of spherical shells Love's and Flügge's approximations coincide and do not introduce approximations. In this sense they give the exact result. This conclusion is not true if one is interested in the calculation of stresses, strains or displacements, for which one also has the terms $1/H_\alpha$ and $1/H_\beta$. In free vibration analyses Love's theory coincides with Flügge's theory if Flügge's contributions are neglected in the evaluation of the dynamic properties M, P and J (expression (17)).

4.3. THE NAVIER SOLUTION FOR CROSS-PLY LAMINATED SHELLS

Exact solution of the partial differential equations (28) on an arbitrary domain and for general boundary conditions is not possible. However, for simply supported and axially compressed shells the projection of which in the α–β plane is a rectangle one can solve the equation exactly, provided that the lamination scheme is of cross-ply type (90°/0°/90°/0°/90° ...). The Navier solutions exist if the following stiffnesses are zero (orthotropic behavior of multi-layered shell):

$$A_{i6}=B_{i6}=D_{i6}=E_{i6}=F_{i6}=G_{i6}=0, \quad i=1,2,$$
$$A_{45}=B_{45}=D_{45}=E_{45}=F_{45}=G_{45}=0. \tag{34}$$

The boundary conditions are assumed to be of the form (simply supported)

$$\alpha=0,a, \qquad \bar v^0=\bar w^0=\bar\gamma^0_{\alpha\zeta}=\bar N_{\alpha\alpha}=\bar M_{\alpha\alpha}=0,$$
$$\beta=0,b, \qquad \bar u^0=\bar w^0=\bar\gamma^0_{\beta\zeta}=\bar N_{\beta\beta}=\bar M_{\beta\beta}=0. \tag{35}$$

Following the Navier solution procedure, one assumes the following solution form that satisfies the boundary conditions (34):

$$(u^0,\gamma^0_{\alpha\zeta})=\sum_{mn}(A_u^{mn},A_{\gamma\alpha\zeta}^{mn})\cos(p\alpha)\sin(q\beta)\,\mathrm{e}^{\mathrm{i}\omega t}, \qquad w^0=\sum_{mn}A_w^{mn}\sin(p\alpha)\sin(q\beta)\,\mathrm{e}^{\mathrm{i}\omega t},$$
$$(v^0,\gamma^0_{\beta\zeta})=\sum_{mn}(A_v^{mn},A_{\gamma\beta\zeta}^{mn})\sin(p\alpha)\cos(q\beta)\,\mathrm{e}^{\mathrm{i}\omega t},$$
$$\text{with}\quad \mathrm{i}=\sqrt{-1}, \quad p=m\pi/a, \quad q=n\pi/b. \tag{36}$$

Here a and b are the lengths along the α- and β-directions, respectively, m and n denote the half-wave numbers into the a- and β-directions, respectively, and ω is the frequency of natural vibrations of the shell. Substituting equations (36) into equations (28) gives the eigenvalue problem

$$([K]-\lambda[G])[A]=(0), \quad [A]^{\mathrm{T}}=(A_u^{mn},\ldots,A_{\gamma\beta\zeta}^{mn}), \qquad \text{for any } m,n. \tag{37}$$

$[K]$ is the stiffness matrix of the shell. When $[G]$ is the geometric stiffness matrix of the shell $[B]$, the eigenvalues correspond to the buckling loads ($\lambda=-N_{\alpha\alpha}$). When $[G]$ is the dynamic matrix of the shell $[M]$, the eigenvalues coincide with the squares of the natural frequencies ($\lambda=\omega^2$). Explicit versions of the matrices $[K]$, $[B]$ and $[M]$ are given in Appendix B.

5. NUMERICAL RESULTS

Cross-ply laminated circular cylindrical and spherical shells are to be considered. Two lamination schemes are considered: 90°/0° (unsymmetric cross-ply) and 0°/90°/0° (symmetric cross-ply). Where comparisons are made with results available in the literature, the mechanical and geometrical properties and the values of shear corrector factor of the multi-layered shells are the same as those quoted in the references; in these numerical studies, the following properties of the lamina are used: MAT1: $E_l/E_t=40$, $G_{lt}/E_t=0{\cdot}5$, $E_{tt}/E_t=0{\cdot}2$, $\nu=0{\cdot}25$, $\mu=1{\cdot}0$, $\chi=1{\cdot}0$; MAT2: $E_l/E_t=25$, $G_{lt}/E_t=0{\cdot}5$, $E_{tt}/E_t=0{\cdot}2$, $\nu=0{\cdot}25$, $\mu=1{\cdot}0$, $\chi=1{\cdot}0$. E_l and E_t are the Young's moduli along and orthogonal to the fiber direction, respectively. G_{lt} and G_{tt} are the thickness shear moduli. ν is the Poisson ratio, and χ denotes the shear correction factor. All results for the frequencies here have been obtained with account taken of all rotatory and tangential inertias, with the exception of results in Table 1 in which the values obtained with rotatory inertias are indicated with R.I. at the beginning of the line. When completely closed circular cylindrical shells, only the even numbers of half-waves have been computed.

For convenience, the following five error indicators are introduced:

$$E_{sD}=\frac{SDT_D-CLT_D}{SDT_D}, \qquad E_{sL}=\frac{SDT_L-CLT_L}{SDT_L}, \qquad E_{sF}=\frac{SDT_F-CLT_F}{SDT_F},$$

$$E_{cD}=\frac{SDT_D-SDT_F}{SDT_F}, \qquad E_{cL}=\frac{SDT_L-SDT_F}{SDT_F}.$$

Here D, L and F denote Donnell's, Love's and Flügge's theories, respectively. E_{sI} ($I=D, L, F$) denotes the errors introduced by neglecting transverse shear deformations for the different theories. E_{cL} denotes the error given by Love's theory when terms of type (ζ/R) in the equilibrium equations are neglected. E_{cD} denotes the error given by Donnell's approximations on curvature terms (shallow shell theory).

5.1. CIRCULAR CYLINDRICAL PANELS AND COMPLETE CYLINDRICAL SHELLS

Two well known effects of transverse shear deformation are summarized in Figures 2 and 3. In Figure 2 the effect of thickness shear elastic moduli magnitude (or anisotropic ratio) of the lamina is considered, for the buckling load parameter $\bar{N}_{\alpha\alpha}=N_{\alpha\alpha}/(E_t h^3)$ of a symmetric cross-ply laminated cylindrical panel. Love's theory values have been plotted. The effect of shear deformation increases very strongly when the anisotropic ratio increases ($E_{sL}=5{\cdot}6\%$ when $E_l/E_t=1$, $G_{lt}/E_l=0{\cdot}5$ and $E_{sL}=72\%$ when $E_l/E_t=40$, $G_{lt}/E_l=1/80$). In Figure 3 the effects of the half-wave numbers n and m on the frequency parameter $\bar{\omega}_R=\omega a^2\sqrt{M/(E_t h^3)}$ are shown. A complete circular cylinder and two values of a/R_β

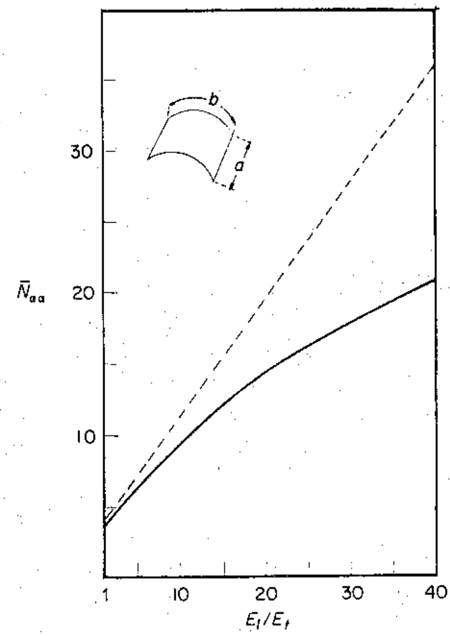

Figure 2. Effect of shear deformation on buckling loads parameter versus fiber anisotropic ratio. Love's values. $R_\beta/a=5$, $a=b$, $a/h=10$, $m=n=1$, symmetric cross-ply 0°/90°/0°. - - -, CLT; ——, SDT.

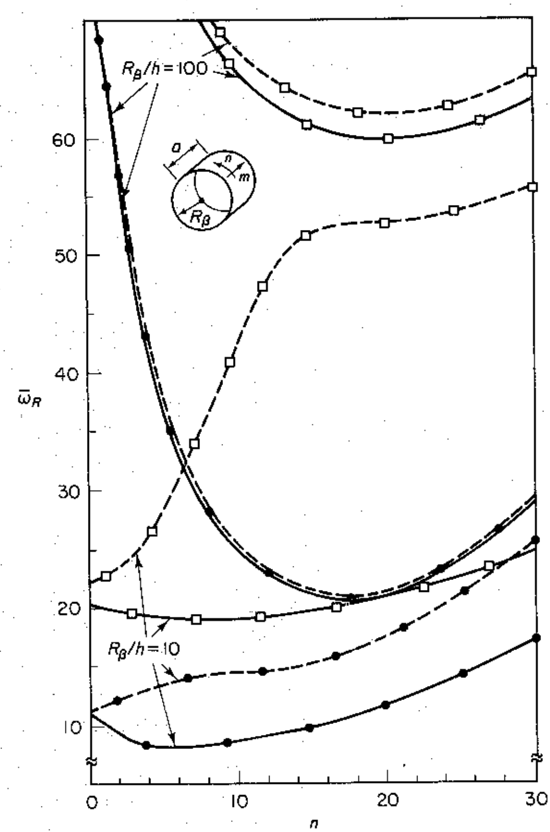

Figure 3. Effect of shear deformation on frequencies parameter versus circumferential and longitudinal half-wave numbers. Love's values. $a/R_\beta=0{\cdot}5$; symmetric cross-ply 0°/90/0°, MAT 2. - - -, CLT; ——, SDT; ●, $m=1$; □, $m=2$.

(length to radius ratio) are considered. E_{sL} increases very strongly when m and n increase (when $n=0$ one has a pure membrane natural mode in the circumferential direction), and CLT may lead to an incorrect characteristic mode which is associated with the fundamental frequencies of free vibrations (see references [11, 14, 20]).

In Figure 4 the typical effects of curvature terms are considered [11, 14]. A comparison of Donnell's, Love's and Flügge's theories for a complete cylinder is presented. The fundamental frequency parameters $\bar{\omega}$ versus the length to radius ratio are compared. SDT values are plotted. The half-wave number of the characteristic mode decreases when a/R_β increases. In respect to the change of the characteristic mode, the differences in the values given by the different theories increases. Notice that E_{cD} and E_{cL} increase when a/R_β increases. The values relative to CLT are not plotted because they may be confused with those of SDT: that is, the errors E_{sI} $(I=D, L, F)$ are negligible in comparison to E_{cL} and E_{cD} in the geometries investigated. In Table 1 a comparison is shown with the results given in reference [14] for the frequency parameter $\bar{\omega}=\omega b^2\sqrt{M/A_{11}}$. Complete cylinders are considered. In reference [14] no results are given for SDT theories and rotatory inertia terms are neglected . There is a very good agreement with the present analysis for both Donnell's and Love's theories. Some differences from Flügge's theory are present, especially when $R_\beta/h=20$ (R_β/h is the radius to thickness ratio). In the development of Flügge's theory in reference [14], the effects of the stiffnesses E_{ij}, F_{ij} and G_{ij} have been neglected, and Love's approximation has been used in the evaluation of M, P and J. It is noted that E_{sL} increases when n and h/R_β increase, while E_{cD} increases when a/R_β increases and R_β/h decreases. Rotatory inertia terms decrease the value of $\bar{\omega}$. In respect to the lowest frequency parameters one has $E_{cJ}>E_{sI}$ $(J=D, L$ and $I=D, L, F)$. It is important to notice that a value for Love–CLT can be smaller than a value for Donnell-SDT ($a/R_\beta=20$, $R_\beta/h=100$, $n=4$: $E_{sL}=0$; $E_{cD}=5{\cdot}2\%$), and a value for Flügge–CLT

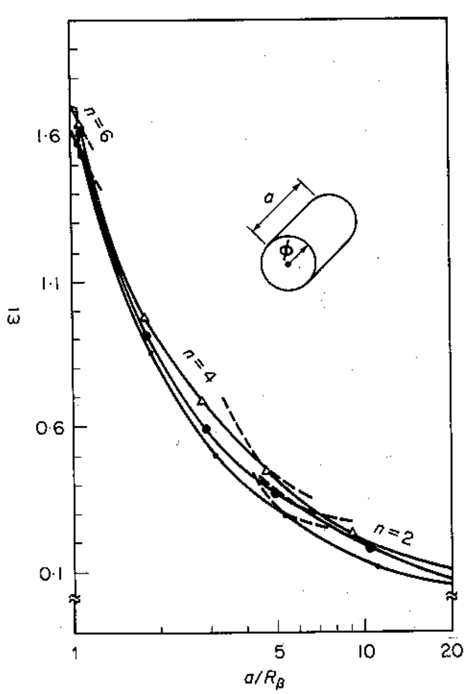

Figure 4. Comparison for frequencies $\bar{w}$ parameter among different theories, versus length-to-radius ratio. SDT values $R_\beta/h=10$, $m=1$; antisymmetric cross-ply 90°/0°, MAT 1. △, Donnell; ○, Love; ●, Flügge (f3).

TABLE 1

Comparison for $\bar{\omega}$ with reference [14]; $R_\alpha = \infty$, $m = 1$, *cross-ply* 90%; R.I. *indicates results including rotatory inertia*

		Ref [14] (CLT)			Present analysis					
					Donnell		Love		Flügge (f1, 3)	
a/R_β	n	Donnell	Love	Flügge	CLT	SDT	CLT	SDT	CLT	SDT
1	2	2·109	2·106	2·106	2·109	2·109	2·106	2·106	2·106	2·106
	4	1·347	1·344	1·344	1·347	0·1347	1·344	1·344	1·344	1·344
	6	0·9619	0·9589	0·9588	0·9619	0·9618	0·9589	0·9588	0·9588	0·9586
	8	0·7534	0·7495	0·7496	0·7534	0·7533	0·7459	0·7494	0·7494	0·7490
	10	0·6478	0·6423	0·6426	0·6478	0·6476	0·6423	0·6421	0·6421	0·6412
	12	0·6213	0·6138	0·6147	0·6213	0·6207	0·6138	0·6133	0·6135	0·6117
R.I.	12	—	—	—	0·6212	0·6206	0·6137	0·6131	0·6134	0·6116
20	2	0·7730	0·7696	0·7691	0·0773	0·0773	0·0769	0·0769	0·0769	0·0769
	4	0·0476	0·0410	0·0411	0·0476	0·0476	0·0410	0·0410	0·0410	0·0409
R.I.	4	—	—	—	0·0476	0·0476	0·0410	0·0410	0·0410	0·0409
	6	0·0887	0·0790	0·0794	0·0887	0·0887	0·0790	0·0790	0·0789	0·0785
	8	0·1594	0·1491	0·1498	0·1594	0·2592	0·1491	0·1490	0·1490	0·1481
	10	0·2515	0·2409	0·2421	0·2515	0·2511	0·2409	0·2405	0·2407	0·2391
	12	0·3642	0·3533	0·3550	0·3643	0·3635	0·3533	0·3526	0·3531	0·3504
$R_\beta/h = 20$										
1	2	2·132	2·119	2·121	2·133	2·133	2·119	2·119	2·119	2·117
	4	1·429	1·411	1·413	1·430	1·427	1·411	1·409	1·409	1·403
	6	1·171	1·141	1·148	1·172	1·165	1·142	1·136	1·139	1·125
R.I.	6	—	—	—	1·170	1·163	1·140	1·134	1·137	1·123
	8	1·224	1·179	1·193	1·224	1·206	1·179	1·163	1·175	1·142
	10	1·521	1·463	1·489	1·521	1·479	1·464	1·425	1·459	1·391
	12	1·997	1·930	1·970	1·998	1·914	1·931	1·852	1·924	1·804
20	2	0·0844	0·0762	0·0760	0·0844	0·0844	0·0762	0·0762	0·0760	0·0760
R.I.	2	—	—	—	0·0844	0·0844	0·0762	0·0762	0·0760	0·0760
	4	0·1843	0·1384	0·1415	0·1844	0·1834	0·1384	0·1377	0·1379	0·1336
	6	0·4342	0·3820	0·3913	0·4343	0·4292	0·3821	0·3777	0·3809	0·3661
	8	0·7884	0·7311	0·7490	0·7886	0·7723	0·7313	0·7165	0·7291	0·6949
	10	1·244	1·181	1·210	1·244	1·205	1·181	1·145	1·178	1·111
	12	1·801	1·731	1·773	1·801	1·721	1·731	1·656	1·726	1·609

can be smaller than a value for Love-SDT ($a/R_\beta = 20$, $R_\beta/h = 20$, $n = 2$: $E_{sL} = 0$; $E_{cD} = 10{\cdot}8\%$; $E_{cL} = 0{\cdot}3\%$).

A better comparison of the different theories is shown in Figures 5 and 6, in which a square panel, initially flat, is considered as its curvature increases until it becomes a complete cylinder. The buckling load parameter $\bar{N}_{\alpha\alpha}$ versus $\varphi = b/R_\beta$ is plotted (when $\varphi = 0$ one has the flat plate, and when $\varphi = 2\pi$ one has the complete cylinder). The smaller value of $\bar{N}_{\alpha\alpha}$ for $m = 1$ has been computed; it is not always coincident with the fundamental value of the buckling load (see Table 2). In Figure 5 $a/h = 10$. Some interesting observations can be made: (1) the difference between CLT and SDT values is different for several theories ($E_{sD} \neq E_{sL} \neq E_{sF}$); (2) these errors increase when n increases and they decrease when φ increases; (3) in respect to the change of the characteristic mode one has larger differences among the different theories; (4) depending on the angle φ a value for

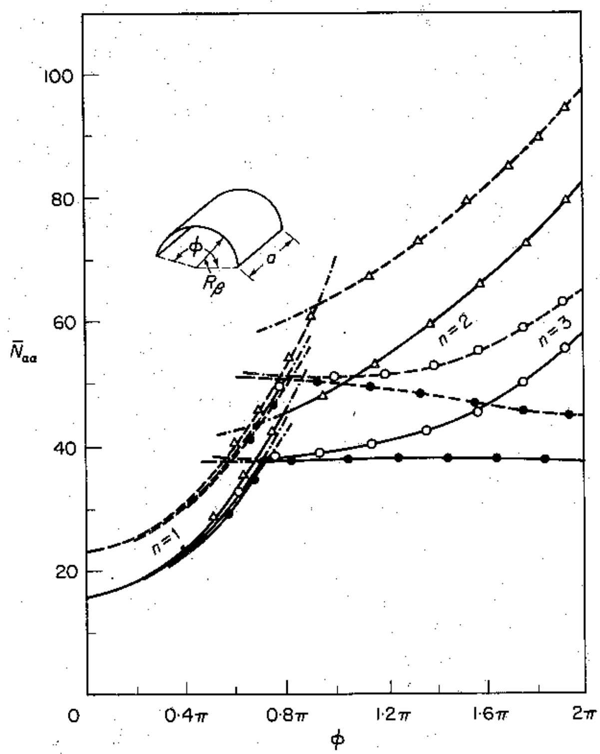

Figure 5. Comparison for buckling load parameter versus φ-angle among different theories (Love–Donnell–Flügge). CLT (- - -) and SDT (——) values. $a/h=10$, $a=b$, $m=1$; symmetric cross-ply 0°/90°/0°, MAT 2. Key as Figure 4.

Love–CLT can be smaller than that for Donnell–SDT ($E_{cD}>E_{sL}$), and a value for Flügge–CLT can be smaller than a value for Love–SDT ($E_{sL}>E_{cL}$). The same comparison is shown in Figure 6 for $a/h=20$, and analogous observations can be made. Note that in Figure 6 values of R_β/h greater than those of Figure 5 are considered, and values of E_{cD} are greater than those of Figure 5.

Error analysis results for several geometrical parameters are illustrated in Table 2 for square cylindrical symmetrically laminated panels, in respect to the fundamental buckling load parameter $\bar{N}_{\alpha\alpha}$. It is known that for flat plates (with fixed mechanical properties of the lamina) the error given when neglecting the shear deformations depends on only the thickness parameter a/h; this is not true for cylindrical panels. From previous investigations one can establish the following trends: the error indicators E_{sI} ($I=D, L, F$) increase when both a/h and a/R_β decrease; the error indicators E_{cI} ($I=D, L$) increase when a/R_β increases and R_β/h decreases. Thus in Table 2 the following additional geometrical parameters have been introduced: $-a^2/(Rh)=(a/h)*(a/R)$ for E_{sI}, $I=D, L, F$; $-R^2/(ah)=(R/h)/(a/R)$ for E_{cI}, $I=D, L$. Although there is some improvement, it is shown in Table 2 that E_{sI} and E_{cI} are not directly dependent on $a^2/(Rh)$ and $R^2/(ah)$, respectively. Notice that when the cylindrical panel tends towards a ring ($a/R_\beta \rightsquigarrow 0$), then E_{sI} ($I=D, L, F$) depend on a/h only (flat plate behaviour).

5.2. SPHERICAL PANELS

In Table 3 a comparison with results of reference [19] is presented for spherical symmetrically laminated panels. The value $\chi=1$ has been used in the results of the present

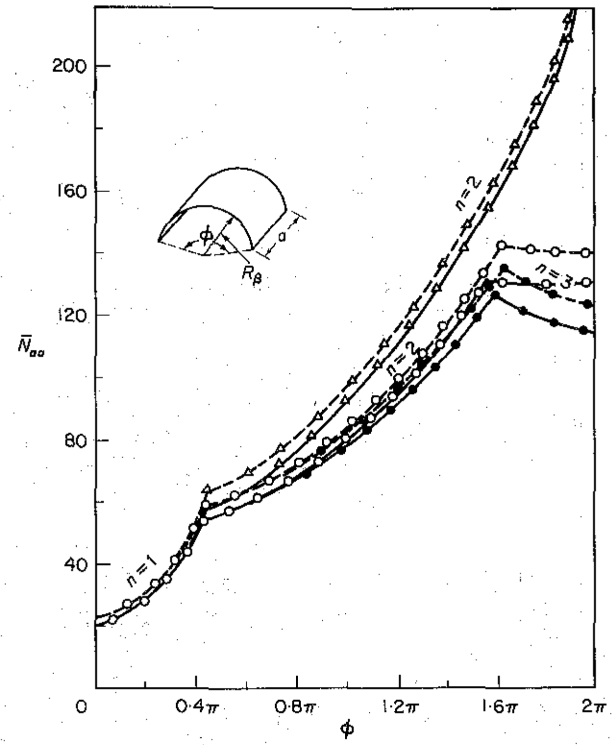

Figure 6. As Figure 5 but for $a/h = 20$.

analysis, while in reference [19] $\chi = 5/6$. Values of the fundamental frequency parameter $\bar{\omega}_R$ for several geometrical parameters have been computed. Analogous conclusions can be reached for cylindrical shells. For the geometries considered, one has E_{sL} larger than E_{cL}. Note that E_{cL} is negligible; in fact it depends only on the different values of M, P and J computed by Flügge's and Love's theories respectively.

In section 4.2 it has been demonstrated that in the buckling analysis of spherical shells, Flügge theory coincides with Love's theory; thus Figure 7 gives a comparison between Love's and Donnell's theories. The buckling load parameter versus R/h is plotted for symmetrically laminated spherical shells. Both CLT and SDT values have been computed. The following trend is noted. When $R/h \rightsquigarrow \infty$ (flat plate) Donnell's values coincide with Love's values, while with R/h decreasing CLT values coincide with SDT values (to emphasize that the shell is thick enough and $E_l/E_t = 25$, $G_{lt}/E_l = 1/50$). It is concluded that to decrease R/h it can be more exact to consider all curvature terms than to take into account shear deformations of the shell, while for R/h increasing the trend is reversed. The same analysis is performed in Figure 8 for the fundamental frequency parameter $\bar{\omega}$, with the mechanical properties of MAT2. The same conclusions as for Figure 7 can be drawn. The values for Flügge's theory are not plotted because they may be confused with those of Love's theory: that is, E_{cL} is negligible in comparison with E_{cD} and E_{sI} ($I = D, L$).

5.3. COMPARISON AMONG THE f1, f2 AND f3 FLÜGGE'S THEORIES AND THE EFFECTS OF THE COUPLING STIFFNESSES B_{ij}, E_{ij} AND G_{ij} ($i, j = 1, 2, 6, 4, 5$)

In Table 4 a comparison with reference [18] is shown for $\bar{\omega}_R$. SDT values and coupled and uncoupled †solutions are presented. In reference [18] Donnell's theory was adopted,

† In the uncoupled solution one has $B_{ij} = E_{ij} = G_{ij} = 0$, for $i, j = 1, 2, 6, 4, 5$.

TABLE 2

Comparison of the different theories for $N_{\alpha\alpha}$ versus R_β/h and a/h; $R_\alpha=\infty$, $a=b$ superscripts denote m and n, respectively; cross-ply 0°/90°0°; MAT 2

a/R_β	R_β/h	Love CLT	Love SDT	Donnell CLT	Donnell SDT	Flügge (f3, 3) CLT	Flügge (f3, 3) SDT	Errors E_{sL}	Errors E_{cD}	Errors E_{cL}	a^2/Rh	R^2/ah
$a/h=10$												
1	10	27·69^{1,1}	20·33^{1,1}	28·17^{1,1}	20·74^{1,1}	27·67^{1,1}	20·30^{1,1}	36·20	2·17	0·15	10	10
0.5	20	24·54^{1,1}	17·13^{1,1}	24·66^{1,1}	17·24^{1,1}	24·54^{1,1}	17·12^{1,1}	43·26	0·70	0·06	5	40
0·2	50	23·66^{1,1}	16·24^{1,1}	23·68^{1,1}	16·26^{1,1}	23·68^{1,1}	16·24^{1,1}	45·69	0·10	0·00	2	250
0·1	100	23·54^{1,1}	16·11^{1,1}	23·54^{1,1}	16·12^{1,1}	23·54^{1,1}	16·11^{1,1}	46·12	0·06	0·00	1	1000
$a/h=20$												
2	10	65·42^{1,2}	60·97^{1,2}	70·97^{1,2}	66·17^{1,2}	65·25^{1,1}	60·79^{1,1}	7·30	8·85	0·29	40	5
1	20	41·72^{1,1}	39·27^{1,1}	42·20^{1,1}	39·73^{1,1}	41·72^{1,1}	39·26^{1,1}	6·23	1·20	0·03	20	20
0·4	50	26·41^{1,1}	23·94^{1,1}	26·49^{1,1}	24·01^{1,1}	26·41^{1,1}	23·94^{1,1}	10·32	0·29	0·00	8	125
0·2	100	24·22^{1,1}	21·75^{1,1}	24·22^{1,1}	21·77^{1,1}	24·22^{1,1}	21·75^{1,1}	11·36	0·09	0·00	4	500
$a/h=50$												
5	10	353·5^{2,4}	342.1^{2,4}	387·3^{2,4}	374·5^{2,4}	351·4^{2,4}	340·0^{2,4}	3·33	9·47	0·60	250	2
2·5	20	204·9^{2,3}	196·8^{2,3}	210·3^{2,3}	202·1^{2,3}	204·8^{2,3}	196·7^{2,3}	4·12	2·75	0·05	125	8
1	50	79·62^{1,2}	78·80^{1,1}	81·06^{1,1}	80·22^{1,1}	79·61^{1,2}	78·78^{1,2}	1·04	1·83	0·03	50	50
0·5	100	52·62^{1,1}	52·18^{1,1}	52·74^{1,1}	52·30^{1,1}	52·62^{1,1}	52·18^{1,1}	0·84	0·23	0·00	25	200
$a/h=100$												
5	20	748·2^{2,5}	742·4^{2,5}	788·7^{3,6}	772·1^{3,6}	745·7^{2,5}	740·0^{2,5}	0·80	4·3	0·32	500	4
2	50	318·5^{2,4}	315·2^{2,4}	324·2^{2,4}	320·9^{2,4}	318·4^{2,4}	315·2^{2,4}	0·95	1·59	0·00	200	25
1	100	163·7^{1,2}	163·5^{1,2}	165·2^{1,2}	164·9^{1,2}	163·7^{1,2}	163·5^{1,2}	0·12	0·86	0·00	100	100

TABLE 3

Comparison for $\bar{\omega}_R$ with reference [19]; $R_\alpha = R_\beta$; $m = n = 1$, $a/b = 1$, *cross-ply* 0°/90°/0°

		Reference [19]	Present analysis								
			Donnell		Love		Flügge (f3, 3)		Errors (%)		
R_β/h	R_β/a	SDT	SDT	CLT	SDT	CLT	SDT	CLT	E_{sL}	E_{cD}	E_{cL}
$a/h = 10$											
5	0·5	—	23·830	24·744	22·482	22·839	22·442	22·798	1·59	6·00	0·18
10	1	—	17·124	18·823	16·377	17·828	16·369	17·820	8·86	4·56	0·05
20	2	—	13·942	16·208	13·681	15·882	13·680	15·878	16·09	1·92	0·03
50	5	12·372	12·772	15·291	12·725	15·234	12·724	15·233	19·72	0·38	0·01
100	10	12·215	12·585	15·151	12·573	15·136	12·573	15·136	20·38	0·10	0·00
200	20	12·176	12·538	15·116	12·535	15·112	12·535	15·112	20·56	0·02	0·00
500	50	12·165	12·529	15·106	12·528	15·105	12·528	15·106	20·57	0·01	0·00
1000	100	12·163	12·523	15·104	12·523	15·104	12·523	15·104	20·61	0·00	0·00
∞	Plate	12·162	12·523	15·104	12·523	15·104	12·523	15·104	20·61	0·00	0·00
$a/h = 100$											
50	0·5	—	208·74	208·74	208·54	208·54	208·54	208·54	0·00	0·10	0·00
100	1	—	126·00	126·01	125·86	125·86	125·86	125·87	0·00	0·10	0·00
200	2	—	68·036	68·044	67·966	67·966	67·966	67·965	0·00	0·10	0·00
500	5	30·923	31·023	31·041	30·995	31·013	30·995	31·013	0·06	0·09	0·00
1000	10	20·347	20·345	20·382	20·334	20·361	20·334	20·361	0·18	0·05	0·00
2000	20	16·627	16·631	16·665	16·626	16·661	16·626	16·661	0·21	0·03	0·00
5000	50	15·424	15·431	15·466	15·430	15·460	15·430	15·460	0·19	0·00	0·00
1000	100	15·244	15·250	15·287	15·250	15·287	15·250	15·287	0·24	0·00	0·00
∞	Plate	15·183	15·191	15·227	15·191	15·227	15·191	15·227	0·24	0·00	0·00

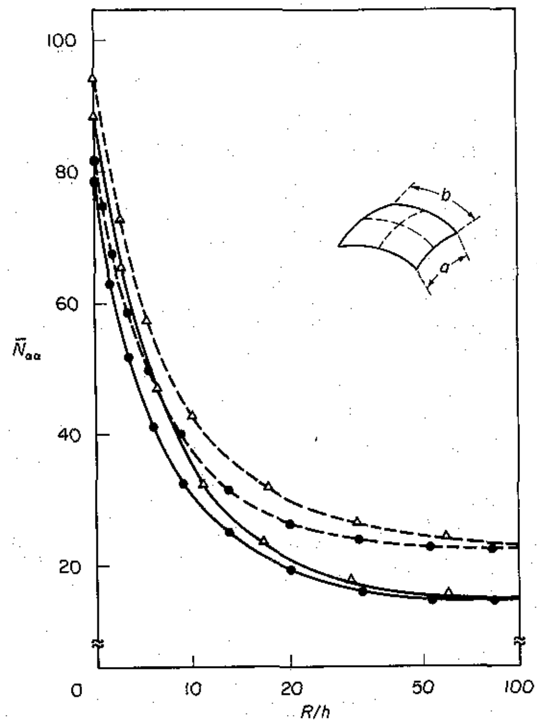

Figure 7. Comparison for buckling load parameter $\bar{N}_{\alpha\alpha}$ versus radius thickness ratio between Donnell's (△) and Love's (●) theories. CLT (- - -) and SDT (——) values. $a/h=10$, $a=b$, $R_\alpha=R_\beta$, $m=n=1$; symmetric cross-ply 0°/90°/0°, MAT 2.

and rotatory and tangential inertia terms were neglected. In the same work it was said that not always (depending on m, n and geometrical parameters) is it conservative for $\bar{\omega}_R$ to neglect inertia terms; this is confirmed by the present analysis. Notice that the values computed by using Flügge's theory can be smaller or larger than Love's theory values.

In Table 5 a comparison of the f1, f2 and f3 Flügge's theories for $\bar{\omega}$ versus NT and R_β/h, for a square cylindrical panel, is presented. Both CLT and SDT values are quoted. When R_β/h increases the differences between f1, f2 and f3 and $NT=1$, $NT=2$ and $NT=3$ decrease. The difference between f1 and f2 is smaller than the difference between f3 and f1, or between f3 and f2. Notice the different behavior between even and odd numbers of NT. Moreover, it is shown in Table 5 that it is not always possible to forecast if an increase in order of NT, or considering several ways of applying Flügge theory, will produce an increase or a decrease in $\bar{\omega}$.

To explain these phenomena, the effect of coupling stiffnesses among the different theories is shown in Table 6. Values of $\bar{\omega}$ versus n are quoted for a square cylindrical panel. The previous trend is confirmed. In fact, the differences between the f1 and f3 theories and between $NT=2$ and $NT=3$ orders in uncoupled solutions are less important than in coupled solutions. It can be concluded that the behavior of terms of type ζ/R_β is not predictable *a priori*. An explanation of this can be found from the following consideration. To increase the NT order by one unit (or to change the method of development of Flügge theories) produces in the stress resultants (19) some changes of the type

$$A_{ij} \to (A_{ij}+B_{ij}), \quad B_{ij} \to (B_{ij}+C_{ij}), \ldots, \text{etc.}, \qquad i,j=1,2,6,4,5. \tag{38}$$

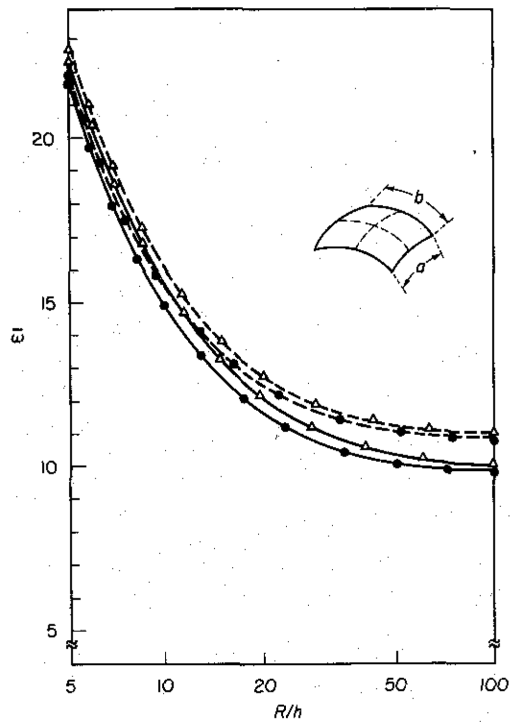

Figure 8. Comparison for fundamental frequency parameter $\bar{\omega}$ versus radius thickness ratio between Donnell's (△) and for Love's (●) theories. CLT (- - -) and SDT (——) values. $a/h=10$, $a=b$, $R_\alpha=R_\beta$, $m=n=1$; antisymmetric cross-ply 90°/0°, MAT 1.

TABLE 4

Comparison for $\bar{\omega}_R$ with reference [18]; $R_\alpha=\infty$, $a/b=1$, *cross-ply* 90%

a/h	R_β/h	m	n	Donnell	Love	Flügge (f3, 3)	Reference [18]
Coupled solutions—SDT							
50	312·5	1	1	14·08	14·08	14·09	14·08
50	312·5	1	2	32·35	32·35	42·40	—
50	312·5	2	1	35·14	35·14	35·19	—
50	312·5	2	2	47·65	47·65	47·72	—
10	31·25	1	1	11·63	11·62	11·76	11·64
10	31·25	1	2	28·01	28·09	28·48	—
10	31·25	2	1	28·36	28·35	28·72	—
10	31·25	2	2	39·50	39·57	40·04	—
Uncoupled solutions—SDT							
50	312·5	1	1	20·57	20·57	20·55	20·59
50	312·5	1	2	53·72	53·72	53·68	53·58
50	312·5	2	1	55·50	55·50	55·59	55·40
50	312·5	2	2	75·71	75·71	75·68	75·51
10	31·25	1	1	17·29	17·21	17·21	17·11
10	31·25	1	2	39·71	39·61	39·62	38·21
10	31·25	2	1	40·01	39·97	39·97	38·63
10	31·25	2	2	53·96	53·89	53·89	51·77

TABLE 5

Comparison of f1, f2 *and* f3 *Flügge's theories for* $\bar{w}$; $R_\alpha=\infty$, $a/h=10$, $a=b$, $m=n=1$, *cross-ply* 90°/0°, MAT 1

	NT = 1		NT = 2		NT = 3		
	CLT	SDT	CLT	SDT	CLT	SDT	R_β/h
f1	0·2966	0·2834	0·2878	0·2749	0·2885	0·2771	
f2	0·2966	0·2834	0·2877	0·2749	0·2884	0·2771	5
f3	0·2965	0·2834	0·3030	0·2991	0·2738	0·2631	
f1	0·2540	0·2370	0·2530	0·2362	0·2532	0·2368	
f2	0·2540	0·2369	0·2530	0·2361	0·2531	0·2366	10
f3	0·2890	0·2750	0·2886	0·2740	0·2518	0·2354	
f1	0·2446	0·2282	0·2434	0·2270	0·2433	0·2271	
f2	0·2446	0·2282	0·2434	0·2270	0·2435	0·2273	20
f3	0·2446	0·2282	0·2488	0·2328	0·2432	0·2270	
f1	0·2423	0·2281	0·2422	0·2280	0·2422	0·2280	
f2	0·2423	0·2281	0·2422	0·2280	0·2422	0·2280	100
f3	0·2423	0·2281	0·2423	0·2281	0·2422	0·2280	

In the present numerical investigations it has been shown that it is not possible to forecast if values of $N_{\alpha\alpha}$ or ω increase or decrease when the *NT* order increases, or when f1, f2 or f3 theories are considered. In this context it appears reasonable that in the coupled solution, in which all stiffnesses are not zero, this trend is more evident than in the uncoupled solution.

6. CONCLUDING REMARKS

On the basis of Flügge's approximations and with account taken of the shear deformation, the equations of buckling and free vibrations of doubly curved multi-layered shells have been derived. Results for buckling and vibrations of cross-ply laminated circular cylindrical and spherical shells have been given. A comparison of some two-dimensional theories (Donnel–Love–Flügge, based on both SDT and CLT) has been presented for various values of the geometrical parameters.

The following results from the open literature have been confirmed.

1. In the geometrical and mechanical configurations investigated it has been confirmed that Love's approximation, with respect to Flügge's approximation, introduces an error of less than 5% when $R/h \geqslant 20$ (this is the definition of thin shell quoted in reference [28]). The same statement cannot be made for Donnell's theory.

2. The effects of curvature terms (terms of type ζ/R_β and Donnell-type approximations) increase when a/R increases and R/h decreases.

3. The effects of transverse shear deformations of the shell increase when a/R and a/h decrease.

4. The transverse shear deformations of the shell depend only on a/h when its behavior is the same as that of a ring.

5. In respect to the change of the characteristic mode one finds larger differences among the results of the different theories.

The following new conclusions can be drawn.

6. It is impossible to attribute to only one shell geometrical parameter the effects of curvature terms or the effects of transverse shear deformation.

TABLE 6

Effects of coupled stiffnesses (B_{ij}, E_{ij} and G_{ij}, $i, j = 1, 2, 3, 4, 5$) on $\bar{\omega}$; comparison of different theories; $R_\beta = 10$, $a/R_\beta = 1$, $a/h = 10$, $a = b$, $m = 1$, cross-ply 90°/0°, MAT 1

	Love		Donnell		f1 (NT=2)		f1 (NT=3)		f3 (NT=2)		f3 (NT3=3)	
n	CLT	SDT	CLT	SDT	CLT	SDT	CLT	SDT	CLT	SDT	CLT	SDT
Coupled solutions												
1	0·2549	0·2458	0·2676	0·2574	0·2530	0·2362	0·2532	0·2368	0·2886	0·2740	0·2518	0·2354
2	0·6227	0·5363	0·6502	0·5565	0·6187	0·5117	0·6189	0·5126	0·6335	0·5299	0·6184	0·5120
3	1·317	0·9998	1·360	1·023	1·310	0·9643	1·316	0·9726	1·210	0·9641	1·310	0·9641
4	2·237	1·513	2·299	1·537	2·225	1·471	2·228	1·476	2·226	1·471	2·226	1·471
Uncoupled solutions												
1	0·4009	0·3531	0·4190	0·3683	0·4009	0·3532	0·4009	0·3532	0·4000	0·3521	0·4002	0·3521
2	1·128	0·7634	1·157	0·7812	1·130	0·7643	1·130	0·7643	1·130	0·7638	1·130	0·7638
3	2·455	1·281	2·485	1·295	2·460	1·282	2·459	1·282	2·459	1·282	2·459	1·282
4	3·701	1·802	3·701	1·814	3·702	1·804	3·702	1·804	3·702	1·803	3·702	1·803

7. With increasing a/R and decreasing R/h, values of Love–SDT or Donnell–SDT can be larger than values of Flügge–CLT or Love–CLT, respectively. Analogous inconsistencies are found when taking into account in a consistent manner the terms of the same order in h/R in the same equilibrium equation in the development of Flügge's theory (values for f3-CLT can be smaller than values for f1-SDT).

8. It is not predictable *a priori* whether an increasing of NT order in the development of Flügge theory will produce an increase or a decrease in ω, and this trend is more evident in the coupled solutions.

9. In buckling analysis of spherical shells, Love's and Flügge's theories coincide with the exact result, while in free vibration analysis Love's theory introduces approximations only in the estimates of the inertia properties of the shell, M, P and J.

Additional investigations directed towards linking the two effects directly to some mixed geometrical–mechanical parameters of shell are probably necessary, but success for such research cannot be guaranteed. In the present paper results for global stability parameters (buckling loads and free vibrations) have been presented. It is the author's opinion that larger differences among different theories could be found in the computation of local parameters such as stresses, strains or displacements. Moreover, it would be interesting to extend the present study to HSDT. This could be a subject for future investigation.

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APPENDIX A: EXPLICIT EXPRESSIONS FOR SYSTEM (28)

For equation (28a):

$$\left\{I_{11}^{0\alpha}-\delta_D\frac{I_{11}^{1\alpha}}{R_\alpha}-\frac{\delta_D}{R_\alpha}\left[I_{11}^{1\alpha}-\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right]\right\}u^0_{,\alpha\alpha}+2\left\{A_{16}-\delta_D\frac{B_{16}}{R_\alpha}-\frac{\delta_D}{R_\alpha}\left[B_{16}-\delta_D\frac{D_{16}}{R_\alpha}\right]\right\}U^0_{,\alpha\beta}$$

$$+\left\{I_{66}^{0\beta}-\delta_D\frac{I_{66}^{1\beta}}{R_\alpha}-\frac{\delta_D}{R_\alpha}\left[I_{66}^{1\beta}-\delta_D\frac{I_{66}^{2\beta}}{R_\alpha}\right]\right\}u^0_{,\beta\beta}+\left\{I_{16}^{0\beta}-\delta_D\frac{I_{16}^{1\alpha}}{R_\alpha}-\frac{\delta_D}{R_\beta}\left[I_{16}^{1\alpha}-\delta_D\frac{I_{16}^{2\alpha}}{R_\alpha}\right]\right\}v^0_{,\alpha\alpha}$$

$$+\left\{A_{12}-\delta_D\frac{B_{12}}{R_\alpha}-\frac{\delta_D}{R_\beta}\left[B_{12}-\delta_D\frac{D_{12}}{R_\alpha}\right]+A_{66}-\delta_D\frac{B_{66}}{R_\alpha}-\frac{\delta_D}{R_\beta}\left[B_{66}-\delta_D\frac{D_{66}}{R_\alpha}\right]\right\}v^0_{,\alpha\beta}$$

$$+\left\{I_{26}^{0\beta}-\delta_D\frac{I_{26}^{1\beta}}{R_\alpha}-\frac{\delta_D}{R_\beta}\left[I_{26}^{1\beta}-\delta_D\frac{I_{26}^{2\beta}}{R_\alpha}\right]\right\}v^0_{,\beta\beta}+\left\{-\frac{I_{11}^{0\alpha}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{I_{11}^{1\alpha}}{R_\alpha}-\frac{A_{12}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{B_{12}}{R_\alpha}\right\}w^0_{,\alpha}$$

$$+\left\{-\frac{I_{26}^{0\beta}}{R_\beta}+\frac{\delta_D}{R_\alpha}\frac{I_{26}^{1\beta}}{R_\beta}-\frac{A_{16}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{B_{16}}{R_\alpha}\right\}w^0_{,\beta}+\left\{-I_{11}^{1\alpha}+\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}w^0_{,\alpha\alpha\alpha}$$

$$+\left\{-2B_{16}-I_{16}^{1\alpha}-\frac{\delta_D}{R_\alpha}\left[2D_{16}-I_{16}^{2\alpha}\right]\right\}w^0_{,\alpha\alpha\beta}$$

$$+\left\{-B_{12}+\frac{\delta_D}{R_\alpha}D_{12}-B_{66}-I_{66}^{1\beta}+\frac{\delta_D}{R_\alpha}[D_{66}+I_{66}^{2\beta}]\right\}w^0_{,\alpha\beta\beta}+\left\{-I_{26}^{1\beta}+\delta_D\frac{I_{26}^{2\beta}}{R_\alpha}\right\}w^0_{,\beta\beta\beta}$$

$$+\left\{\left\{I_{11}^{1\alpha}-\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}\gamma^0_{\alpha\zeta,\alpha\alpha}+2\left\{B_{16}-\delta_D\frac{D_{16}}{R_\beta}\right\}\gamma^0_{\alpha\zeta,\alpha\beta}+\left\{I_{66}^{1\beta}-\delta_D\frac{I_{66}^{2\beta}}{R_\alpha}\right\}\gamma^0_{\alpha\zeta,\beta\beta}\right.$$

$$+\left\{I_{16}^{1\alpha}-\delta_D\frac{I_{16}^{2\alpha}}{R_\alpha}\right\}\gamma^0_{\beta\zeta,\alpha\alpha}+\left\{B_{12}+B_{66}-\frac{\delta_D}{R_\beta}[D_{12}+D_{66}]\right\}\gamma^0_{\beta\zeta,\alpha\beta}$$

$$\left.+\left\{I_{26}^{1\beta}-\frac{\delta_D}{R_\alpha}\frac{I_{26}^{2\beta}}{R_\alpha}\right\}\gamma^0_{\beta\zeta,\beta\beta}\right\}\delta_C=RH_{eq.15a}.$$

For equation (28b),

$$+\left\{I_{26}^{0\alpha}-\delta_D\frac{I_{26}^{1\beta}}{R_\beta}-\frac{\delta_D}{R_\alpha}\left[I_{26}^{1\beta}-\delta_D\frac{I_{26}^{2\beta}}{R_\beta}\right]\right\}u^0_{,\beta\beta}+\left\{A_{12}-\delta_D\frac{B_{12}}{R_\beta}-\frac{\delta_D}{R_\alpha}\left[B_{12}-\delta_D\frac{D_{12}}{R_\beta}\right]\right.$$

$$\left.+A_{66}-\delta_D\frac{B_{66}}{R_\beta}-\frac{\delta_D}{R_\alpha}\left[B_{66}-\delta_D\frac{D_{66}}{R_\beta}\right]\right\}u^0_{,\beta\alpha}+\left\{I_{16}^{0\alpha}-\delta_D\frac{I_{16}^{1\alpha}}{R_\beta}-\frac{\delta_D}{R_\alpha}\left[I_{16}^{1\alpha}-\delta_D\frac{I_{16}^{2\alpha}}{R_\beta}\right]\right\}u^0_{,\alpha\alpha}$$

$$+\left\{I_{22}^{0\beta}-\delta_D\frac{I_{22}^{1\beta}}{R_\beta}-\frac{\delta_D}{R_\beta}\left[I_{22}^{1\beta}-\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right]\right\}v^0_{,\beta\beta}+2\left\{A_{26}-\delta_D\frac{B_{26}}{R_\beta}-\frac{\delta_D}{R_\beta}\left[B_{26}-\delta_D\frac{D_{26}}{R_\beta}\right]\right\}v^0_{,\beta\alpha}$$

$$+\left\{I_{66}^{0\alpha}-\delta_D\frac{I_{66}^{1\alpha}}{R_\beta}-\frac{\delta_D}{R_\beta}\left[I_{66}^{1\alpha}-\delta_D\frac{I_{66}^{2\alpha}}{R_\beta}\right]\right\}v^0_{,\alpha\alpha}+\left\{-\frac{I_{22}^{0\beta}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{I_{22}^{1\beta}}{R_\beta}-\frac{A_{12}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{B_{12}}{R_\beta}\right\}w^0_{,\beta}$$

$$+\left\{-\frac{I_{16}^{0\alpha}}{R_\alpha}+\frac{\delta_D}{R_\beta}\frac{I_{16}^{1\alpha}}{R_\alpha}-\frac{A_{26}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{B_{26}}{R_\beta}\right\}w^0_{,\alpha}+\left\{-I_{22}^{1\beta}+\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}w^0_{,\beta\beta\beta}$$

$$+\left\{-2B_{26}-I_{26}^{1\beta}-\frac{\delta_D}{R_\beta}[2D_{26}-I_{26}^{2\beta}]\right\}w^0_{,\beta\beta\alpha}$$

$$+\left\{-B_{12}+\frac{\delta_D}{R_\beta}D_{12}-B_{66}-I_{66}^{1\alpha}+\frac{\delta_D}{R_\beta}[D_{66}+I_{66}^{2\alpha}]\right\}w^0_{,\beta\alpha\alpha}$$

$$+\left\{-I_{16}^{1\alpha}+\delta_D\frac{I_{16}^{2\alpha}}{R_\beta}\right\}w^0_{,\alpha\alpha\alpha}+\left\{\left\{I_{26}^{1\beta}-\delta_D\frac{I_{26}^{2\beta}}{R_\beta}\right\}\gamma^0_{\alpha\zeta,\beta\beta}\right.$$

$$+\left\{B_{12}+B_{66}-\frac{\delta_D}{R_\alpha}[D_{12}+D_{66}]\right\}\gamma^0_{\alpha\zeta,\beta\alpha}$$

$$+\left\{I_{16}^{1\alpha}-\frac{\delta_D}{R_\beta}\frac{I_{16}^{2\alpha}}{R_\beta}\right\}\gamma^0_{\alpha\zeta,\alpha\alpha}+\left\{I_{22}^{1\beta}-\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}\gamma^0_{\beta\zeta,\beta\beta}$$

$$\left.+2\left\{B_{26}-\delta_D\frac{D_{26}}{R_\alpha}\right\}\gamma^0_{\beta\zeta,\beta\alpha}+\left\{I_{66}^{1\alpha}-\delta_D\frac{I_{66}^{2\alpha}}{R_\beta}\right\}\gamma^0_{\beta\zeta,\alpha\alpha}\right\}\delta_C=RH_{eq.15b}.$$

For equation (28c),

$$-\left\{-\frac{I_{11}^{0\alpha}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{I_{11}^{1\alpha}}{R_\alpha}-\frac{A_{12}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{B_{12}}{R_\alpha}\right\}u^0_{,\alpha}-\left\{-\frac{I_{26}^{0\beta}}{R_\beta}+\frac{\delta_D}{R_\alpha}\frac{I_{26}^{1\beta}}{R_\beta}-\frac{A_{16}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{B_{16}}{R_\alpha}\right\}u^0_{,\beta}$$

$$-\left\{-I_{11}^{1\alpha}+\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}u^0_{,\alpha\alpha\alpha}-\left\{-2B_{16}-I_{16}^{1\alpha}-\frac{\delta_D}{R_\alpha}[2D_{16}-I_{16}^{2\alpha}]\right\}u^0_{,\alpha\alpha\beta}$$

$$-\left\{-B_{12}+\frac{\delta_D}{R_\alpha}D_{12}-B_{66}-I_{66}^{1\beta}+\frac{\delta_D}{R_\alpha}[D_{66}+I_{66}^{2\beta}]\right\}u^0_{,\alpha\beta\beta}+\left\{-I_{26}^{1\beta}+\delta_D\frac{I_{26}^{2\beta}}{R_\alpha}\right\}u^0_{,\beta\beta\beta}$$

$$-\left\{-\frac{I_{22}^{0\beta}}{R_\beta}-\frac{\delta_D}{R_\beta}\frac{I_{22}^{1\beta}}{R_\beta}-\frac{A_{12}}{R_\beta}+\frac{\delta_D}{R_\alpha}\frac{B_{12}}{R_\beta}\right\}v^0_{,\beta}-\left\{-\frac{I_{16}^{0\alpha}}{R_\alpha}+\frac{\delta_D}{R_\beta}\frac{I_{16}^{1\alpha}}{R_\alpha}-\frac{A_{26}}{R_\alpha}+\frac{\delta_D}{R_\beta}\frac{B_{26}}{R_\beta}\right\}v^0_{,\alpha}$$
$$-\left\{-I_{22}^{1\beta}+\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}v^0_{,\beta\beta\beta}-\left\{-2B_{26}-I_{26}^{1\beta}-\frac{\delta_D}{R_\beta}[2D_{26}-I_{26}^{2\beta}]\right\}v^0_{,\beta\beta\alpha}$$
$$-\left\{-B_{12}+\frac{\delta_D}{R_\beta}D_{12}-B_{66}-I_{66}^{1\alpha}+\frac{\delta_D}{R_\beta}[D_{66}+I_{66}^{2\alpha}]\right\}v^0_{,\beta\alpha\alpha}-\left\{-I_{16}^{1\alpha}+\delta_D\frac{I_{16}^{2\alpha}}{R_\beta}\right\}v^0_{,\alpha\alpha\alpha}$$
$$-\left\{\frac{1}{R_\alpha}\left[\frac{I_{11}^{0\alpha}}{R_\alpha}+\frac{A_{12}}{R_\beta}\right]+\frac{1}{R_\beta}\left[\frac{I_{22}^{0\beta}}{R_\beta}+\frac{A_{12}}{R_\alpha}\right]\right\}w^0-\left\{2\left[\frac{I_{11}^{1\alpha}}{R_\alpha}+\frac{B_{12}}{R_\beta}\right]\right\}w^0_{,\alpha\alpha}$$
$$-\left\{\frac{2}{R_\alpha}[I_{16}^{1\alpha}+B_{16}]+\frac{2}{R_\beta}[I_{26}^{1\beta}+B_{26}]\right\}w^0_{,\alpha\beta}-\left\{2\left[\frac{I_{22}^{1\beta}}{R_\beta}+\frac{B_{12}}{R_\alpha}\right]\right\}w^0_{,\beta\beta}-\{I_{11}^{2\alpha}\}w^0_{,\alpha\alpha\alpha\alpha}$$
$$-\{2I_{16}^{2\alpha}+2D_{16}\}w^0_{,\alpha\alpha\alpha\beta}-\{I_{66}^{2\alpha}+I_{66}^{2\beta}+2D_{12}+2D_{66}\}w^0_{,\alpha\alpha\beta\beta}$$
$$-\{2I_{26}^{2\beta}+2D_{26}\}w^0_{,\beta\beta\beta\alpha}-\{I_{22}^{2\beta}\}w^0_{,\beta\beta\beta\beta}$$
$$+\left\{\frac{I_{11}^{1\alpha}}{R_\alpha}+\frac{B_{12}}{R_\beta}\right\}\gamma^0_{\alpha\zeta,\alpha}+\left\{\frac{I_{26}^{1\beta}}{R_\beta}+\frac{B_{16}}{R_\alpha}\right\}\gamma^0_{\alpha\zeta,\beta}+\{I_{11}^{2\alpha}\}\gamma^0_{\alpha\zeta,\alpha\alpha\alpha}+\{I_{16}^{2\alpha}+2D_{16}\}\gamma^0_{\alpha\zeta,\alpha\alpha\beta}$$
$$+\{I_{66}^{2\beta}+D_{12}+D_{66}\}\gamma^0_{\alpha\zeta,\alpha\beta\beta}+\{I_{26}^{2\beta}\}\gamma^0_{\alpha\zeta,\beta\beta\beta}\left\{\frac{I_{16}^{1\alpha}}{R_\alpha}+\frac{B_{26}}{R_\beta}\right\}\gamma^0_{\beta\zeta,\alpha}$$
$$+\left\{\frac{I_{22}^{1\beta}}{R_\beta}+\frac{B_{12}}{R_\alpha}\right\}\gamma^0_{\beta\zeta,\beta}+\{I_{16}^{2\alpha}\}\gamma^0_{\beta\zeta,\alpha\alpha\alpha}$$
$$+\{I_{66}^{2\alpha}+D_{12}+D_{66}\}\gamma^0_{\beta\zeta,\alpha\alpha\beta}+\{I_{26}^{2\beta}+2D_{26}\}\gamma^0_{\beta\zeta,\alpha\beta\beta}+\{I_{22}^{2\beta}\}\gamma^0_{\beta\zeta,\beta\beta\beta}\}\delta_C$$
$$+\bar N_{\alpha\alpha}w^0_{,\alpha\alpha}+\bar N_{\alpha\beta}w^0_{,\alpha\beta}+\bar N_{\beta\alpha}w^0_{,\alpha\beta}+\bar N_{\beta\beta}w^0_{,\beta\beta}=RH_{eq15c}.$$

For equation (28d),

$$\left\{\left\{I_{11}^{1\alpha}-\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}u^0_{,\alpha\alpha}+2\left\{B_{16}-\delta_D\frac{D_{16}}{R_\beta}\right\}u^0_{,\alpha\beta}+\left\{I_{66}^{1\beta}-\delta_D\frac{I_{66}^{2\beta}}{R_\alpha}\right\}u^0_{,\beta\beta}\right.$$
$$+\left\{I_{16}^{1\alpha}-\delta_D\frac{I_{16}^{2\alpha}}{R_\alpha}\right\}v^0_{,\alpha\alpha}+\left\{B_{12}+B_{66}-\frac{\delta_D}{R_\beta}[D_{12}+D_{66}]\right\}v^0_{,\alpha\beta}+\left\{I_{26}^{1\beta}-\frac{\delta_D}{R_\alpha}\frac{I_{26}^{2\beta}}{R_\alpha}\right\}v^0_{,\beta\beta}$$
$$-\left\{\frac{I_{11}^{1\alpha}}{R_\alpha}+\frac{B_{12}}{R_\beta}\right\}w^0_{,\alpha}-\left\{\frac{I_{26}^{1\beta}}{R_\beta}+\frac{B_{16}}{R_\alpha}\right\}w^0_{,\beta}-\{I_{11}^{2\alpha}\}w^0_{,\alpha\alpha\alpha}-\{I_{16}^{2\alpha}+2D_{16}\}w^0_{,\alpha\alpha\beta}$$
$$-\{I_{66}^{2\beta}+D_{12}+D_{66}\}w^0_{,\alpha\beta\beta}-\{I_{26}^{2\beta}\}w^0_{,\beta\beta\beta}+\{-I_{44}^{0\alpha}\}\gamma^0_{\alpha\zeta}$$
$$+\{I_{11}^{2\alpha}\}\gamma^0_{\alpha\zeta,\alpha\alpha}+\{2D_{16}\}\gamma^0_{\alpha\zeta,\alpha\alpha}+\{I_{66}^{2\beta}\}\gamma^0_{\alpha\zeta,\beta\beta}$$
$$+\{-A_{45}\}\gamma^0_{\beta\zeta}+\{I_{16}^{2\alpha}\}\gamma^0_{\beta\zeta,\alpha\alpha}+\{D_{12}+D_{66}\}\gamma^0_{\beta\zeta,\alpha\alpha}+\{I_{26}^{2\beta}\}\gamma^0_{\beta\zeta,\beta\beta}\}\delta_C=RH_{eq.15d}.$$

For equation (28e),

$$\left\{\left\{I_{16}^{1\alpha}-\delta_D\frac{I_{16}^{2\alpha}}{R_\alpha}\right\}u^0_{,\alpha\alpha}+\left\{B_{12}+B_{66}-\frac{\delta_D}{R_\beta}[D_{12}+D_{66}]\right\}u^0_{,\alpha\beta}+\left\{I_{26}^{1\beta}-\frac{\delta_D}{R_\alpha}\frac{I_{26}^{2\beta}}{R_\alpha}\right\}u^0_{,\beta\beta}\right.$$
$$+\left\{I_{22}^{1\beta}-\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}v^0_{,\beta\beta}+2\left\{B_{26}-\delta_D\frac{D_{26}}{R_\alpha}\right\}v^0_{,\beta\alpha}+\left\{I_{66}^{1\alpha}-\delta_D\frac{I_{66}^{2\alpha}}{R_\beta}\right\}v^0_{,\alpha\alpha}$$
$$-\left\{\frac{I_{16}^{1\alpha}}{R_\alpha}+\frac{B_{26}}{R_\beta}\right\}w^0_{,\alpha}-\left\{\frac{I_{22}^{1\beta}}{R_\beta}+\frac{B_{12}}{R_\alpha}\right\}w^0_{,\beta}-\{I_{16}^{2\alpha})w^0_{,\alpha\alpha\alpha}-\{I_{66}^{2\alpha}+D_{12}+D_{66}\}w^0_{,\alpha\alpha\beta}$$
$$-\{I_{26}^{2\beta}+2D_{26}\}w^0_{,\alpha\beta\beta}-\{I_{22}^{2\beta}\}w^0_{,\beta\beta\beta}\{-A_{45}\}\gamma^0_{\alpha\zeta}$$
$$+\{I_{16}^{2\alpha}\}\gamma^0_{\alpha\zeta,\alpha\alpha}+\{D_{12}+D_{66}\}\gamma^0_{\alpha\zeta,\alpha\alpha}+\{I_{26}^{2\beta}\}\gamma^0_{\alpha\zeta,\beta\beta}$$
$$+\{-I_{55}^{0\beta}\}\gamma^0_{\beta\zeta}+\{I_{66}^{2\alpha}\}\gamma^0_{\beta\zeta,\alpha\alpha}+\{2D_{26}\}\gamma^0_{\beta\zeta,\alpha\alpha}+\{I_{22}^{2\beta}\}\gamma^0_{\beta\zeta,\beta\beta}\}\delta_C=RH_{eq.15e}.$$

These equations give as particular cases the equations quoted in references [23] and [33] for buckling and vibration analysis of anisotropic cylindrical shells respectively, and those in reference [36] for buckling and vibration analysis of anisotropic flat plates.

APPENDIX B: EXPLICIT VERSION OF MATRICES IN EQUATION (37)

For the stiffness matrix $[K]$,

$$k_{11}=\left\{I_{11}^{0\alpha}-\delta_D\frac{I_{11}^{1\alpha}}{R_\alpha}-\frac{\delta_D}{R_\alpha}\left[I_{11}^{1\alpha}-\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right]\right\}(-p^2)+\left\{I_{66}^{0\beta}-\delta_D\frac{I_{66}^{1\beta}}{R_\alpha}\frac{\delta_D}{R_\alpha}\left[I_{66}^{1\beta}-\delta_D\frac{I_{66}^{2\beta}}{R_\alpha}\right]\right\}(-q^2),$$

$$k_{12}=k_{21}=\left\{A_{12}-\delta_D\frac{B_{12}}{R_\alpha}-\frac{\delta_D}{R_\alpha}\left[B_{12}-\delta_D\frac{D_{12}}{R_\alpha}\right]+A_{66}-\delta_D\frac{B_{66}}{R_\alpha}-\frac{\delta_D}{R_\beta}\left[B_{66}-\delta_D\frac{D_{66}}{R_\alpha}\right]\right\}(-pq),$$

$$k_{13}=-k_{31}=\left\{-\frac{I_{11}^{0\alpha}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{I_{11}^{1\alpha}}{R_\alpha}-\frac{A_{12}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{B_{12}}{R_\alpha}\right\}(p)+\left\{-I_{11}^{1\alpha}+\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}(-p^3)$$
$$+\left\{-B_{12}+\frac{\delta_D}{R_\alpha}D_{12}-B_{66}-I_{66}^{1\beta}+\frac{\delta_D}{R_\alpha}[D_{66}+I_{66}^{2\beta}]\right\}(-pq^2),$$

$$k_{14}-k_{41}=\left\{\left\{I_{11}^{1\alpha}-\delta_D\frac{I_{11}^{2\alpha}}{R_\alpha}\right\}(-p^2)+\left\{I_{66}^{1\beta}-\delta_D\frac{I_{66}^{2\beta}}{R_\alpha}\right\}(-q2)\right\}\delta_C,$$

$$k_{15}=k_{51}=\left\{\left\{B_{12}+B_{66}-\delta_D\left[\frac{D_{12}+D_{66}}{R\beta}\right]\right\}(-pq)\right\}\delta_C,$$

$$k_{22}=\left\{I_{22}^{0\beta}-\delta_D\frac{I_{22}^{1\beta}}{R_\beta}-\frac{\delta_D}{R_\beta}\left[I_{22}^{1\beta}-\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right]\right\}(-p^2)+\left\{I_{66}^{0\alpha}-\delta_D\frac{I_{66}^{1\alpha}}{R_\beta}-\frac{\delta_D}{R_\beta}\left[I_{66}^{1\alpha}-\delta_D\frac{I_{66}^{2\alpha}}{R_\beta}\right]\right\}(-q^2),$$

$$k_{23}=-k_{32}=\left\{-\frac{I_{22}^{0\beta}}{R_\beta}+\frac{\delta_D}{R_\beta}\frac{I_{22}^{1\beta}}{R_\beta}-\frac{A_{12}}{R_\alpha}+\frac{\delta_D}{R_\alpha}\frac{B_{12}}{R_\beta}\right\}(q)+\left\{-I_{22}^{1\beta}+\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}(-q^3)$$
$$+\left\{-B_{12}+\frac{\delta_D}{R_\beta}D_{12}-B_{66}-I_{66}^{1\alpha}+\frac{\delta_D}{R_\beta}[D_{66}+I_{66}^{2\alpha}]\right\}(-p^2q),$$

$$k_{24}=k_{42}=\left\{\left\{B_{12}+B_{66}-\delta_D\left[\frac{D_{12}+D_{66}}{R_\alpha}\right]\right\}(-q^2)\right\}\delta_C,$$

$$k_{25}=k_{52}=\left\{\left\{I_{22}^{1\beta}-\delta_D\frac{I_{22}^{2\beta}}{R_\beta}\right\}(-q^2)+\left\{I_{66}^{1\alpha}-\delta_D\frac{I_{66}^{2\alpha}}{R_\beta}\right\}(-p^2)\right\}\delta_C,$$

$$k_{33}=\left\{\frac{1}{R_\beta}\left[\frac{I_{11}^{0\alpha}}{R_\alpha}+\frac{A_{12}}{R_\beta}\right]+\frac{1}{R_\alpha}\left[\frac{I_{22}^{0\beta}}{R_\beta}+\frac{A_{12}}{R_\alpha}\right]\right\}(-p^2)-\left\{2\left[\frac{I_{11}^{1\alpha}}{R_\alpha}+\frac{B_{12}}{R_\beta}\right]\right\}(-p^2)$$
$$-\left\{2\left[\frac{I_{22}^{1\beta}}{R_\beta}+\frac{B_{12}}{R_\alpha}\right]\right\}(-q^2)-\{I_{11}^{2\alpha}\}(p^4)$$
$$-\{I_{66}^{2\alpha}+I_{66}^{2\beta}-2D_{12}+2D_{66}\}(p^2q^2)-\{I_{22}^{2\beta}\}(q^4),$$

$$k_{34}=-k_{43}=\left\{\left\{\frac{I_{11}^{1\alpha}}{R_\alpha}+\frac{B_{12}}{R_\beta}(-p)+\{I_{11}^{2\alpha}\}(p^3)+\{I_{66}^{2\beta}+D_{12}+D_{66}\}(pq^2)\right\}\delta_C,\right.$$

$$k_{35}=-k_{53}=\left\{\left\{\frac{I_{22}^{1\beta}}{R_\beta}+\frac{B_{12}}{R_\alpha}\right\}(-q)+\{I_{66}^{2\alpha}+D_{12}+D_{66}\}(p^2q)+\{I_{22}^{2\beta}\}(q^3)\right\}\delta_C,$$

$$k_{44}=\{\{-I_{44}^{0\alpha}\}+\{I_{11}^{2\alpha}\}(-p^2)+\{I_{66}^{2\beta}\}(-q^2)\}\delta_C,$$

$$k_{45}=k_{54}=\{\{-A_{45}\}+\{D_{12}+D_{66}\}(-p^2)\}\delta_C,$$

$$k_{55}=\{\{-I_{55}^{0\beta}\}+\{I_{66}^{2\alpha}\}(-p^2)+\{I_{22}^{2\beta}\}(-q^2)\}\delta_C.$$

For the dynamic matrix $[M]$,

$$m_{11}=M-2\frac{P}{R_\alpha}+\frac{1}{R_\alpha^2},\qquad m_{12}=m_{21}=0,$$

$$m_{13}=-m_{31}=\left(P-\frac{J}{R_\alpha}\right)(p),\qquad m_{14}=m_{41}=\left(P-\frac{J}{R_\alpha}\right)\delta_C,$$

$$m_{15}=m_{51}=0,\qquad m_{22}=M-2\frac{P}{R_\beta}+\frac{1}{R_\beta^2},$$

$$m_{23}=-m_{32}=-\left(P-\frac{J}{R_\beta}\right)(q),\qquad m_{24}=m_{42}=\left(P-\frac{J}{R_\beta}\right)\delta_C,$$

$$m_{25}=m_{52}=0,\qquad m_{33}=M+Jp^2+Jq^2,\qquad m_{34}=-m_{43}=Jp\delta_C,$$

$$m_{35}=-m_{53}=-Jq\delta_C,\qquad m_{44}=J\delta_C,\qquad m_{45}=m_{54}=0,\qquad m_{55}=J\delta_C.$$

For the geometric stiffnesses matrix $[B]$ only the term b_{33} is different from zero and for buckling under axial compression (in the α-direction) one has $b_{33}=-p^2$.