A thermal stress finite element analysis of beam structures by hierarchical modelling

G. Giunta[1], G. De Pietro[1,2], S. Belouettar[1] and E. Carrera[2]

[1]Luxembourg Institute of Science and Technology, Luxembourg
[2]Politecnico di Torino, Italy

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Scope

Within this talk, the hierarchical one-dimensional finite element modelling based on Carrera's unified formulation is used to investigate the **thermal-stress in isotropic and composite three-dimensional beams**.
The intention behind this approach is two-fold:

- reduce the computational cost (when compared to full three-dimensional solutions).
- ensure accurate three-dimensional results via a one-dimensional approach.

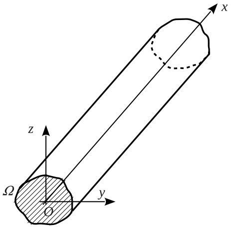
Outline

The presentation is organised as follows:

- Theoretical background
- Numerical results
- Conclusions

Beam Structures and Modelling

A beam is a structure whose axial extension (l) is predominant if compared to any other dimension orthogonal to it.
The cross-section (Ω) is identified by intersecting the beam with planes that are orthogonal to its axis.
A Cartesian reference system is adopted: y- and z-axis are two orthogonal directions laying on Ω. The x coordinate is coincident to the axis of the beam.



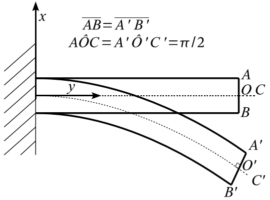
One-dimensional modelling: the beam is seen as a line that connects the centroids of each cross-section.

Classical Theories

§ Euler-Bernoulli's theory:

$$
\begin{aligned}
u_x &= u_{x1} - u_{y1,x}y - u_{z1,x}z \\
u_y &= u_{y1} \\
u_z &= u_{z1}
\end{aligned}
$$

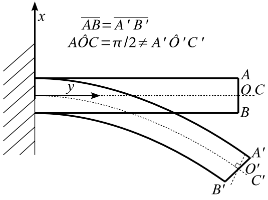
- Cross-section rigid on its plane.
- No shear stress (only axial stress).



§ Timoshenko's theory:

$$
\begin{aligned}
u_x &= u_{x1} + u_{x2}y + u_{x3}z \\
u_y &= u_{y1} \\
u_z &= u_{z1}
\end{aligned}
$$

- Cross-section rigid on its plane.
- Shear stress (corrective factor).



One-dimensional Hierarchical Approximation

Several displacement-based theories can be formulated on the basis of the following **generic kinematic field**:

$$\mathbf{u}\,(x,y,z) = F_\tau\,(y,z)\,\mathbf{u}_\tau\,(x) \quad \text{with} \quad \tau = 1,\ 2,\ \ldots,\ N_u$$

§ N_u stands for the number of unknowns over the cross-section. It depends on the approximation order N that is a free parameter of the formulation.

§ The compact expression is based on Einstein's notation: subscript τ indicates summation. Thanks to this notation, problem's governing differential equations and boundary conditions can be derived in terms of a single '**fundamental nucleo**'.

§ The complexity related to higher than classical approximation terms is tackled and the theoretical formulation is valid for the generic approximation order and approximating functions $F_\tau\,(y,z)$.
Within this work, $F_\tau\,(y,z)$ are assumed to be Mac Laurin's polynomials.

§ According to the previous choice of polynomial functions, the generic, N-order displacement field is:

$$u_x = u_{x1} + u_{x2}y + u_{x3}z + \cdots + u_{x\frac{(N^2+N+2)}{2}}\, y^N + \cdots + u_{x\frac{(N+1)(N+2)}{2}}\, z^N$$
$$u_y = u_{y1} + u_{y2}y + u_{y3}z + \cdots + u_{y\frac{(N^2+N+2)}{2}}\, y^N + \cdots + u_{y\frac{(N+1)(N+2)}{2}}\, z^N$$
$$u_z = u_{z1} + u_{z2}y + u_{z3}z + \cdots + u_{z\frac{(N^2+N+2)}{2}}\, y^N + \cdots + u_{z\frac{(N+1)(N+2)}{2}}\, z^N$$

§ N_u and F_τ as functions of N can be obtained via Pascal's triangle as shown in the following Table:

N	N_u	F_τ
0	1	$F_1 = 1$
1	3	$F_2 = y \quad F_3 = z$
2	6	$F_4 = y^2 \quad F_5 = yz \quad F_6 = z^2$
$\dots$	$\dots$	$\dots$
N	$\frac{(N+1)(N+2)}{2}$	$F_{\frac{(N^2+N+2)}{2}} = y^N \quad F_{\frac{(N^2+N+4)}{2}} = y^{N-1}z \quad \dots$
		$F_{\frac{N(N+3)}{2}} = yz^{N-1} \quad F_{\frac{(N+1)(N+2)}{2}} = z^N$

Finite Element Approximation

The part of the displacement vector that depends upon the axial coordinated ($\mathbf{u}_\tau$) is approximated as follows:

$$\mathbf{u}_\tau\left(x\right) = N_i\left(x\right)\mathbf{q}^e_{\tau i} \quad \text{with} \quad \tau = 1,\ 2,\ \ldots,\ N \quad \text{and} \quad i = 1,\ 2,\ \ldots,\ N_n$$

§ $\mathbf{q}_{\tau i}$ are the nodal displacements unknowns typical of a finite element approximation.

§ $N_i\left(x\right)$ are the corresponding shape functions, which approximate the displacements along the beam axis in a C^0 sense up to an order $N_n - 1$ being N_n the number of nodes per element. This latter is a free parameter of the theoretical formulation.
Linear (B2), quadratic (B3) and cubic (B4) elements along the beam axis are considered.

Geometric Equations

In the case of small displacements with respect to a characteristic dimension of Ω, linear relations between strain and displacement components hold:

$$\boldsymbol{\varepsilon}_n = \mathbf{D}_{np}\mathbf{u} + \mathbf{D}_{nx}\mathbf{u}$$
$$\boldsymbol{\varepsilon}_p = \mathbf{D}_p\mathbf{u}$$

Strain components have been grouped into vectors $\boldsymbol{\varepsilon}_n$ that lay on the cross-section and $\boldsymbol{\varepsilon}_p$ laying on planes orthogonal to Ω.

$\mathbf{D}_{np}$, $\mathbf{D}_{nx}$, and $\mathbf{D}_p$ are the following differential matrix operators:

$$\mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial}{\partial y} & 0 & 0 \\ \dfrac{\partial}{\partial z} & 0 & 0 \end{bmatrix} \quad \mathbf{D}_{nx} = \mathbf{I}\frac{\partial}{\partial x} \quad \mathbf{D}_p = \begin{bmatrix} 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix}$$

$\mathbf{I}$ is the unit matrix.

Constitutive Relations

In the case of thermo-mechanical problems, Hooke's law reads:

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_e = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_\vartheta\right) = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\alpha}}T\right) = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\lambda}}T$$

where subscripts 'e' and 'ϑ' refer to the elastic and the thermal contributions, respectively.

- $\tilde{\mathbf{C}}$ is the material elastic stiffness,
- $\tilde{\boldsymbol{\alpha}}$ the vector of the thermal expansion coefficients,
- $\tilde{\boldsymbol{\lambda}}$ their product and
- T stands for temperature.

According to the stress and strain vectors splitting, the previous equation becomes:

$$\boldsymbol{\sigma}_p = \tilde{\mathbf{C}}_{pp}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{pn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_p T$$
$$\boldsymbol{\sigma}_n = \tilde{\mathbf{C}}_{np}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{nn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_n T$$

Matrices $\tilde{\mathbf{C}}_{pp}$, $\tilde{\mathbf{C}}_{pn}$, $\tilde{\mathbf{C}}_{np}$ and $\tilde{\mathbf{C}}_{nn}$ are:

$$\tilde{\mathbf{C}}_{pp} = \begin{bmatrix} \tilde{C}_{22} & \tilde{C}_{23} & 0 \\ \tilde{C}_{23} & \tilde{C}_{33} & 0 \\ 0 & 0 & \tilde{C}_{44} \end{bmatrix} \quad \tilde{\mathbf{C}}_{pn} = \tilde{\mathbf{C}}_{np}^T = \begin{bmatrix} \tilde{C}_{12} & \tilde{C}_{26} & 0 \\ \tilde{C}_{13} & \tilde{C}_{36} & 0 \\ 0 & 0 & \tilde{C}_{45} \end{bmatrix}$$

$$\tilde{\mathbf{C}}_{nn} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{16} & 0 \\ \tilde{C}_{16} & \tilde{C}_{66} & 0 \\ 0 & 0 & \tilde{C}_{55} \end{bmatrix}$$

The coefficients $\tilde{\boldsymbol{\lambda}}_n$ and $\tilde{\boldsymbol{\lambda}}_p$:

$$\tilde{\boldsymbol{\lambda}}_n^T = \left\{ \begin{array}{ccc} \tilde{\lambda}_1 & \tilde{\lambda}_6 & 0 \end{array} \right\} \quad \tilde{\boldsymbol{\lambda}}_p^T = \left\{ \begin{array}{ccc} \tilde{\lambda}_2 & \tilde{\lambda}_3 & 0 \end{array} \right\}$$

are related to the thermal expansion coefficients $\tilde{\boldsymbol{\alpha}}_n$ and $\tilde{\boldsymbol{\alpha}}_p$:

$$\tilde{\boldsymbol{\alpha}}_n^T = \left\{ \begin{array}{ccc} \tilde{\alpha}_1 & 0 & 0 \end{array} \right\} \quad \tilde{\boldsymbol{\alpha}}_p^T = \left\{ \begin{array}{ccc} \tilde{\alpha}_2 & \tilde{\alpha}_3 & 0 \end{array} \right\}$$

through the following equations:

$$\begin{aligned} \tilde{\boldsymbol{\lambda}}_p &= \tilde{\mathbf{C}}_{pp}\tilde{\boldsymbol{\alpha}}_p + \tilde{\mathbf{C}}_{pn}\tilde{\boldsymbol{\alpha}}_n \\ \tilde{\boldsymbol{\lambda}}_n &= \tilde{\mathbf{C}}_{np}\tilde{\boldsymbol{\alpha}}_p + \tilde{\mathbf{C}}_{nn}\tilde{\boldsymbol{\alpha}}_n \end{aligned}$$

Principle of Virtual Displacements

The stiffness matrices are obtained in a nuclear form via the weak form of the Principle of Virtual Displacements:

$$\delta \mathscr{L}^e_{\text{int}} = 0$$

where:

- δ represents a virtual variation and
- $\mathscr{L}^e_{\text{int}}$ is the strain energy.

Stiffness Matrix

$$\delta \mathscr{L}^e_{\text{int}} = \int_l \int_\Omega \left(\delta \boldsymbol{\epsilon}_n^T \boldsymbol{\sigma}_n + \delta \boldsymbol{\epsilon}_p^T \boldsymbol{\sigma}_p \right) d\Omega dx$$

By substitution of the geometrical relations, the material constitutive equations, the unified hierarchical approximation of the displacements it becomes:

Principle of Virtual Displacements

$$\delta \mathscr{L}^e_{\text{int}} =$$

$$\delta \mathbf{q}^T_{\tau i} \int_l \int_\Omega \Big\{ (\mathbf{D}_{nx} N_i)^T F_\tau \left[\tilde{\mathbf{C}}_{np} (\mathbf{D}_p F_s) N_j + \tilde{\mathbf{C}}_{nn} (\mathbf{D}_{np} F_s) N_j + \tilde{\mathbf{C}}_{nn} F_s (\mathbf{D}_{nx} N_j) \right]$$

$$+ (\mathbf{D}_{np} F_\tau)^T N_i \left[\tilde{\mathbf{C}}_{np} (\mathbf{D}_p F_s) N_j + \tilde{\mathbf{C}}_{nn} (\mathbf{D}_{np} F_s) N_j + \tilde{\mathbf{C}}_{nn} F_s (\mathbf{D}_{nx} N_j) \right]$$

$$+ (\mathbf{D}_p F_\tau)^T N_i \left[\tilde{\mathbf{C}}_{pp} (\mathbf{D}_p F_s) N_j + \tilde{\mathbf{C}}_{pn} (\mathbf{D}_{np} F_s) N_j + \tilde{\mathbf{C}}_{pn} F_s (\mathbf{D}_{nx} N_j) \right] \Big\} \, d\Omega \, dx \, \mathbf{q}_{sj}$$

In a compact vectorial form:

$$\delta L_{int} = \delta \mathbf{q}^T_{\tau i} \mathbf{K}^{\tau s i j} \mathbf{q}_{sj}$$

§ The components of the element stiffness matrix $\mathbf{K}^{\tau s i j} \in \mathbb{R}^{3\times 3}$ fundamental nucleo are:

$$\begin{aligned}
K^{\tau s i j}_{xx} &= I_{ij} \left(J^{66}_{\tau,_y s,_y} + J^{55}_{\tau,_z s,_z} \right) + I_{ij,_x} J^{16}_{\tau,_y s} + I_{i,_x j} J^{16}_{\tau s,_y} + I_{i,_x j,_x} J^{11}_{\tau s} \\
K^{\tau s i j}_{yy} &= I_{ij} \left(J^{22}_{\tau,_y s,_y} + J^{44}_{\tau,_z s,_z} \right) + I_{ij,_x} J^{26}_{\tau,_y s} + I_{i,_x j} J^{26}_{\tau s,_y} + I_{i,_x j,_x} J^{66}_{\tau s} \\
K^{\tau s i j}_{zz} &= I_{ij} \left(J^{44}_{\tau,_y s,_y} + J^{33}_{\tau,_z s,_z} \right) + I_{ij,_x} J^{45}_{\tau,_y s} + I_{i,_x j} J^{45}_{\tau s,_y} + I_{i,_x j,_x} J^{55}_{\tau s}
\end{aligned}$$

$$
\begin{aligned}
K_{xy}^{\tau sij} &= \quad I_{ij}\left(J^{26}_{\tau,_y s,_y} + J^{45}_{\tau,_z s,_z}\right) + I_{ij,_x} J^{66}_{\tau,_y s} + I_{i,_x j} J^{12}_{\tau s,_y} + I_{i,_x j,_x} J^{16}_{\tau s} \\
K_{yx}^{\tau sij} &= \quad I_{ij}\left(J^{26}_{\tau,_y s,_y} + J^{45}_{\tau,_z s,_z}\right) + I_{ij,_x} J^{12}_{\tau,_y s} + I_{i,_x j} J^{66}_{\tau s,_y} + I_{i,_x j,_x} J^{16}_{\tau s} \\
K_{xz}^{\tau sij} &= \quad I_{ij}\left(J^{36}_{\tau,_y s,_z} + J^{45}_{\tau,_z s,_y}\right) + I_{ij,_x} J^{55}_{\tau,_z s} + I_{i,_x j} J^{13}_{\tau s,_z} + I_{i,_x j,_x} J^{15}_{\tau s} \\
K_{zx}^{\tau sij} &= \quad I_{ij}\left(J^{45}_{\tau,_y s,_z} + J^{36}_{\tau,_z s,_y}\right) + I_{ij,_x} J^{13}_{\tau,_z s} + I_{i,_x j} J^{55}_{\tau s,_z} \\
K_{yz}^{\tau sij} &= \quad I_{ij}\left(J^{23}_{\tau,_y s,_z} + J^{44}_{\tau,_z s,_y}\right) + I_{ij,_x} J^{45}_{\tau,_z s} + I_{i,_x j} J^{36}_{\tau s,_z} \\
K_{zy}^{\tau sij} &= \quad I_{ij}\left(J^{44}_{\tau,_y s,_z} + J^{23}_{\tau,_z s,_y}\right) + I_{ij,_x} J^{36}_{\tau,_z s} + I_{i,_x j} J^{45}_{\tau s,_z}
\end{aligned}
$$

where:

$$
J^{gh}_{\tau_{(,\eta)} s_{(,\xi)}} = \int_{\Omega} \tilde{C}_{gh} F_{\tau_{(,\eta)}} F_{s_{(,\xi)}} \; d\Omega
$$

Weighted sum (in the continuum) of each elemental cross-section area where the weight functions account for the spatial distribution of geometry and material.

$$
I_{i_{(,x)} j_{(,x)}} = \int_{l} N_{i_{(,x)}} N_{j_{(,x)}} \; dx
$$

In order to avoid shear locking, reduced integration is used for the term I_{ij} in $K_{exx}^{\tau sij}$ since it is related to the shear deformations γ_{xy} and γ_{xz}.

Thermo-mechanical coupling vector

The components of the thermo-mechanical coupling vector $\overline{\mathbf{K}}^{sj}_{u\theta}$ are:

$$\overline{K}^{sj}_{u\theta x} = I_{\theta_n j_{,x}} J^1_{\theta_\Omega s} + I_{\theta_n j} J^6_{\theta_\Omega s_{,y}}$$
$$\overline{K}^{sj}_{u\theta y} = I_{\theta_n j} J^2_{\theta_\Omega s_{,y}} + I_{\theta_n j_{,x}} J^6_{\theta_\Omega s}$$
$$\overline{K}^{sj}_{u\theta z} = I_{\theta_n j} J^3_{\theta_\Omega s_{,z}}$$

The generic term $J^g_{\tau_{(,\phi)}}$ is:

$$J^g_{\theta_\Omega s_{(,\phi)}} = \int_\Omega F_{s_{(,\phi)}} \; \overline{\lambda}_g \; \Theta_\Omega \; d\Omega.$$

whereas the term $I_{\theta_n j_{(,x)}}$ stands for:

$$I_{\theta_n j_{(,x)}} = \int_l \Theta_n N_{j_{(,x)}} \; dx.$$

The temperature has been written as:

$$T(x,y,z) = \Theta_n(x)\, \Theta_\Omega(y,z)$$

Fourier's Heat Conduction Equation

The beam models are derived considering the temperature as an external loading resulting from the internal thermal stresses. This requires the temperature profile to be known over the whole beam domain. Fourier's heat conduction equation for a multi-layered beam:

$$\frac{\partial}{\partial x}\left(\tilde{K}_1^k \frac{\partial T^k}{\partial x}\right) + \frac{\partial}{\partial y}\left(\tilde{K}_2^k \frac{\partial T^k}{\partial y}\right) + \frac{\partial}{\partial z}\left(\tilde{K}_3^k \frac{\partial T^k}{\partial z}\right) = 0$$

is solved via a Navier-type solution by ideally dividing the cross-section Ω into N_{Ω^k} non-overlapping sub-domains (or layers) along the through-the-thickness direction z:

$$\Omega = \bigcup_{k=1}^{N_{\Omega^k}} \Omega^k$$

For a k^{th} layer, the Fourier differential equation becomes:

$$\tilde{K}_1^k \frac{\partial^2 T^k}{\partial x^2} + \tilde{K}_2^k \frac{\partial^2 T^k}{\partial y^2} + \tilde{K}_3^k \frac{\partial^2 T^k}{\partial z^2} = 0$$

where $\tilde{K}_i^k$ are the thermal conductivity coefficients. In order to obtain a closed form analytical solution, it is further assumed that the temperature does not depend upon the through-the-width co-ordinate y. The continuity of the temperature and the through-the-thickness heat flux q_z hold at each interface between two consecutive sub-domains:

$$T_{\top}^{k} = T_{\perp}^{k+1}$$

$$q_{z\top}^{k} = q_{z\perp}^{k+1} \quad \text{with} \quad q_z^k = \tilde{K}_3^k \frac{\partial T^k}{\partial z}$$

Subscripts '$\top$' and '$\perp$' stand for sub-domain's top and bottom, respectively. The following temperatures are also imposed at cross-section through-the-thickness top and bottom:

$$T^{N_{\Omega^k}} = T_{\top}^{N_{\Omega^k}} \sin(\alpha x)$$
$$T^1 = T_{\perp}^1 \sin(\alpha x)$$

where $T_{\top}^{N_{\Omega^k}}$ and $T_{\perp}^1$ are maximal amplitudes and α is:

$$\alpha = \frac{m\pi}{l}$$

with $m \in \mathbf{N}^+$ representing the half-wave number along the beam axis.

The following temperature field:

$$T^k(x,z) = \Theta_\Omega^k(z) \sin \Theta_n(x) = \bar{T}^k e^{s^k z} \sin(\alpha x)$$

represents a solution of the considered heat conduction problem. $\bar{T}^k$ is an unknown constant obtained by imposing the boundary conditions, whereas s is:

$$s_{1,2}^k = \pm \sqrt{\frac{K_1^k}{K_3^k}}\ \alpha$$

$\Theta_\Omega^k(z)$, therefore, becomes:

$$\Theta_\Omega^k(z) = \bar{T}_1^k e^{s_1 z} + \bar{T}_2^k e^{s_2 z}$$

or, equivalently:

$$\Theta_\Omega^k(z) = C_1^k \cosh\left(\sqrt{\frac{K_1^k}{K_3^k}}\ z\right) + C_2^k \sinh\left(\sqrt{\frac{K_1^k}{K_3^k}}\ z\right)$$

For a cross-section division into N_{Ω^k} sub-domains, $2 \cdot N_{\Omega^k}$ unknowns C_j^k are present. The problem is mathematically well posed since the boundary conditions yield a linear algebraic system of $2 \cdot N_{\Omega^k}$ equations in C_j^k.

Numerical Results

- The beam support is $[0, l] \times [-a/2, a/2] \times [-b/2, b/2]$. Square cross-section with $a = b = 1$ m are considered. The length-to-side ratio l/b is equal to 100 and 10.
- The thermal boundary conditions are: $T_{\top} = 400$ K and $T_{\perp} = 300$ K. A half-wave is considered for the temperature variation along the beam axis.
- Simply supported beams are considered for which a closed-form Navier-type analytical solution is present.
- Three-dimensional FEM models are also developed within the commercial code ANSYS.
- The degrees of freedom of the three-dimensional FEM mechanical models are about $8 \cdot 10^5$. The number of DOFs for the most expensive one-dimensional model ($N = 14$ with 121 nodes) are about $4 \cdot 10^4$.

Isotropic Beam

Isotropic beams made of an aluminium alloy are first considered. The mechanical properties are: $E = 72$ GPa, $\nu = 0.3$, $K = 121$ W/mK, $\alpha = 23 \cdot 10^{-6}$ K^{-1}.

Unless differently stated, the following displacements and stresses considered:

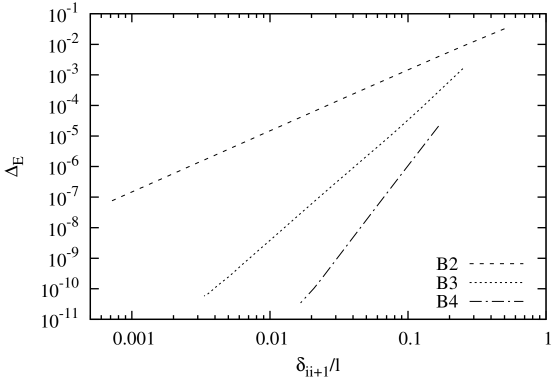
$$\tilde{u}_x = u_x\left(0, -\frac{a}{2}, \frac{b}{2}\right) \quad \tilde{u}_y = u_y\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) \quad \tilde{u}_z = u_z\left(\frac{l}{2}, 0, \frac{b}{2}\right)$$

$$\tilde{\sigma}_{xx} = \sigma_{xx}\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) \quad \tilde{\sigma}_{xz} = \sigma_{xz}\left(0, -\frac{a}{2}, 0\right) \quad \tilde{\sigma}_{xy} = \sigma_{xy}\left(0, \frac{a}{4}, \frac{b}{2}\right)$$

$$\tilde{\sigma}_{zz} = \sigma_{zz}\left(\frac{l}{2}, 0, 0\right) \quad \tilde{\sigma}_{yy} = \sigma_{yy}\left(\frac{l}{2}, 0, \frac{b}{2}\right) \quad \tilde{\sigma}_{yz} = \sigma_{yz}\left(\frac{l}{2}, \frac{a}{4}, \frac{b}{4}\right)$$

Problem Convergence

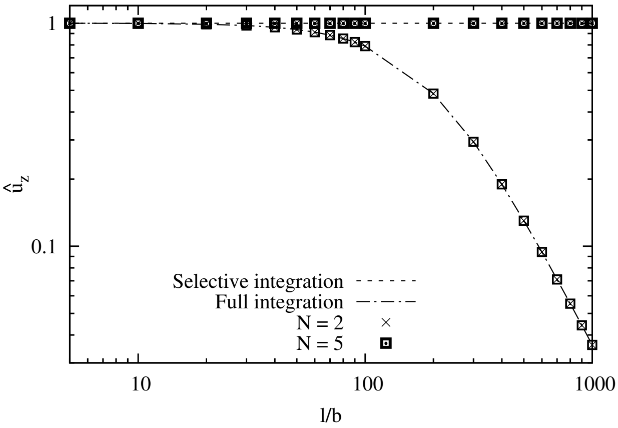
Strain energy relative error Δ_E versus the normalised distance δ_{ii+1}/l between two consecutive nodes for ***linear element***, $l/a = 10$ and $N = 2$.



The error is computed by comparing the strain energy to a closed form Navier-type solution, which in the framework of a theory is an exact solution.

Shear Locking

Transverse displacement ratio $\hat{u}_z = u_z\,(l/2,0,0)\,/u_z^{\mathrm{Nav}}\,(l/2,0,0)$ versus l/a via linear elements, $N = 2$ and 5.



Displacement components [m] for a slender and short isotropic beams

	$-10 \times \tilde{u}_x$		$10^3 \times \tilde{u}_y$			$\tilde{u}_z$	
FEM 3D[a]	2.9287		4.6118			2.3347	
FEM 3D[b]	2.9287		4.5977			2.3347	
	B2	B3, B4	B2	B3	B4	B2	B3, B4
$N \geq 3$	2.9286	2.9287	4.6003	4.5997	4.5999	2.3345	2.3347
$N = 2$	2.9286	2.9287	4.6000	4.5994	4.5996	2.3345	2.3347

a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

	$-10^2 \times \tilde{u}_x$		$10^3 \times \tilde{u}_y$			$10^2 \times \tilde{u}_z$	
FEM 3D[a]	2.9511		4.5903			2.7603	
FEM 3D[b]	2.9511		4.5903			2.7603	
	B2	B3, B4	B2	B3	B4	B2	B3, B4
$N = 9, 10$	2.9510	2.9511	4.5906	4.5902	4.5903	2.7601	2.7603
$N = 7, 8$	2.9511	2.9511	4.5906	4.5902	4.5902	2.7601	2.7603
$N = 6$	2.9510	2.9511	4.5901	4.5897	4.5897	2.7601	2.7603
$N = 5$	2.9510	2.9511	4.5899	4.5895	4.5895	2.7601	2.7603
$N = 4$	2.9515	2.9515	4.5893	4.5889	4.5889	2.7605	2.7607
$N = 3$	2.9515	2.9515	4.5892	4.5888	4.5888	2.7606	2.7607
$N = 2$	2.9492	2.9493	4.5645	4.5641	4.5641	2.7574	2.7575

a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

Stress components [MPa] for a short isotropic beam

	$\tilde{\sigma}_{xx}$			$-\tilde{\sigma}_{xz}$			$\tilde{\sigma}_{xy}$		
FEM 3D[a]		5.1254			3.1353			2.1063	
FEM 3D[b]		5.1330			3.1338			2.1106	
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N = 14$	5.0794	5.2868	5.1369	3.1711	3.0906	3.1363	2.0971	2.1047	2.1002
$N = 10$	5.1094	5.3131	5.1670	3.1532	3.0727	3.1184	2.1189	2.1266	2.1220
$N = 9$	5.1111	5.3149	5.1686	3.1532	3.0727	3.1184	2.1204	2.1280	2.1235
$N = 7$	5.1431	5.3505	5.2007	3.1631	3.0826	3.1283	2.1435	2.1512	2.1467
$N = 5$	5.0699	5.2926	5.1275	3.3697	3.2892	3.3349	1.9983	2.0060	2.0014
$N = 4$	4.2994	4.5448	4.3570	2.7925	2.7120	2.7578	1.0515	1.0593	1.0548
$N = 3$	4.2549	4.5027	4.3125	2.7926	2.7121	2.7578	1.0096	1.0174	1.0128
$N = 2$	0.5261	0.7870	0.5837	2.5234	2.4431	2.4888	2.5845	2.5921	2.5874

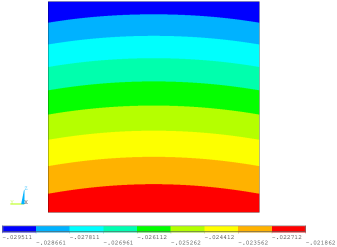
a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

	$10 \times \tilde{\sigma}_{zz}$			$-\tilde{\sigma}_{yy}$			$10 \times \tilde{\sigma}_{yz}$		
FEM 3D[a]		7.6369			2.6898			5.4339	
FEM 3D[b]		7.6828			2.6836			5.4878	
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N = 14$	8.0610	8.3715	7.6197	2.6398	2.6412	2.6909	5.4214	5.4212	5.4213
$N = 10$	8.0524	8.3609	7.6112	2.6529	2.6533	2.7040	5.4237	5.4234	5.4237
$N = 9$	8.0526	8.3610	7.6113	2.6569	2.6569	2.7080	5.4247	5.4244	5.4246
$N = 7$	8.0812	8.4009	7.6399	2.6541	2.6496	2.7052	5.4001	5.4003	5.4000
$N = 5$	8.1837	8.4681	7.7424	2.3692	2.3567	2.4202	4.6323	4.6315	4.6322
$N = 4$	7.4561	7.8019	7.0148	1.7941	1.7644	1.8451	2.9494	2.9498	2.9493
$N = 3$	7.4333	7.7792	6.9920	1.9537	1.9231	2.0048	2.5478	2.5479	2.5478
$N = 2$	32.196	32.486	31.755	11.028	10.994	11.079	0.2602	0.2599	0.2601

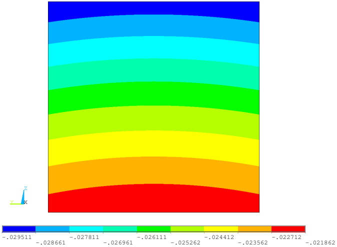
a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

Displacement cross-section variation

Axial displacement u_x [m] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, isotropic beam.



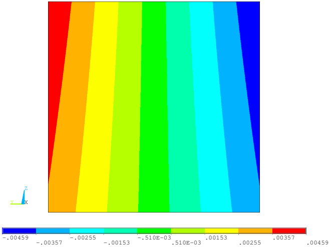
(a) FEM 3D-R



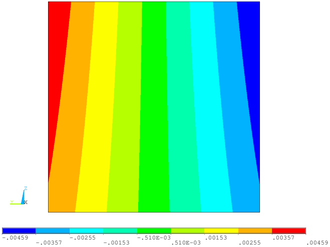
(b) $N = 2$

Displacement cross-section variation

Through-the-width displacement u_y [m] over the cross-section at $x = l/2$, B4 for $l/b = 10$, isotropic beam.



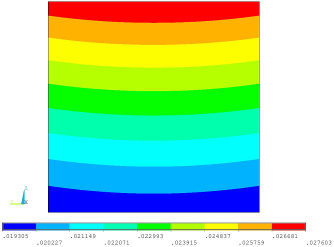
(a) FEM 3D-R



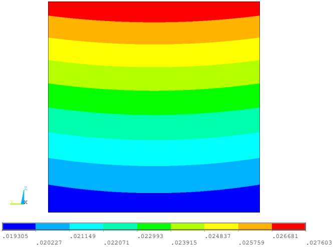
(b) $N = 2$

Displacement cross-section variation

Through-the-thickness displacement u_z [m] over the cross-section at $x = l/2$, B4 for $l/b = 10$, isotropic beam.



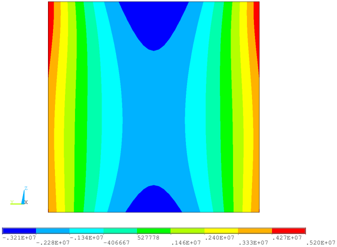
(a) FEM 3D-R



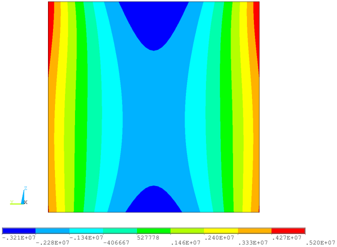
(b) $N = 2$

Stress cross-section variation

Axial stress σ_{xx} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, isotropic beam.



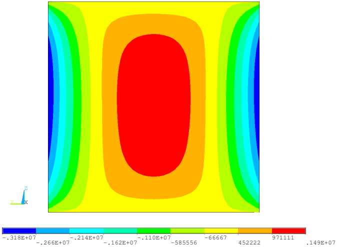
(a) FEM 3D-R



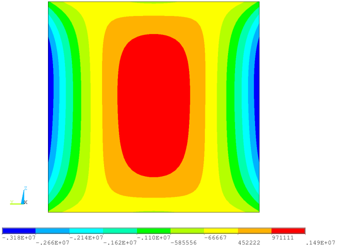
(b) $N = 7$

Stress cross-section variation

Shear stress σ_{xz} [Pa] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, isotropic beam.



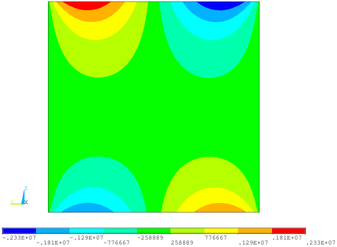
(a) FEM 3D-R



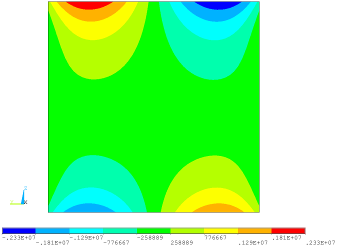
(b) $N = 7$

Stress cross-section variation

Shear stress σ_{xy} [Pa] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, isotropic beam.



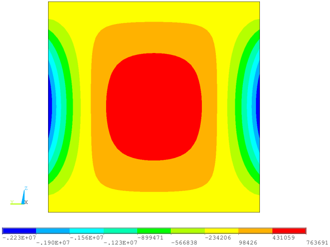
(a) FEM 3D-R



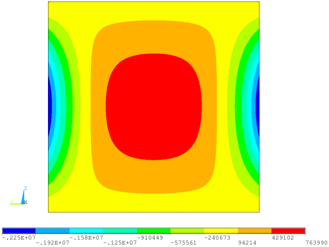
(b) $N = 7$

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, isotropic beam.



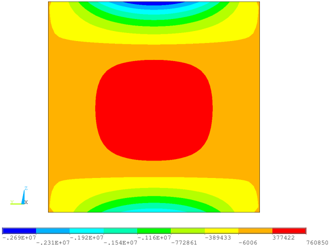
(a) FEM 3D-R



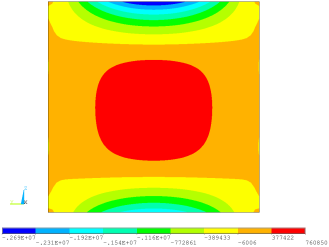
(b) $N = 7$

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] over the cross-section, B4 for $l/b = 10$, isotropic beam.



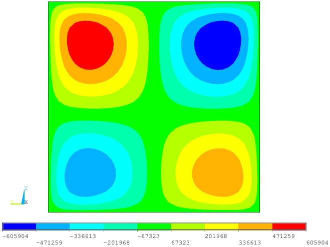
(a) FEM 3D-R



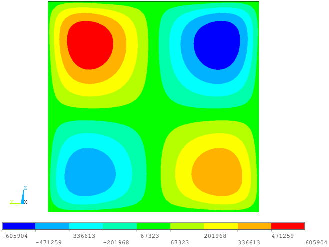
(b) $N = 13$

Stress cross-section variation

Shear stress σ_{yz} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, isotropic beam.



(a) FEM 3D-R

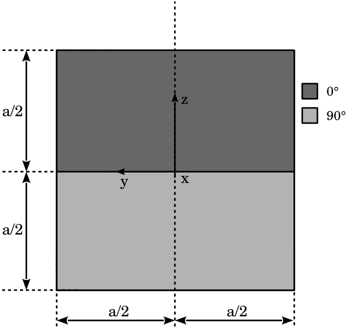


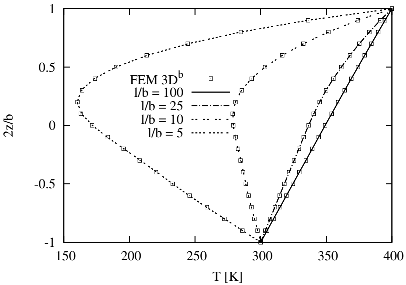
(b) $N = 9$

Laminated Beam

A [0/90] stacking sequences is investigated.

The material elastic and thermal properties are: $E_L = 172.72$ GPa, $E_T = 6.91$ GPa, $G_{LT} = 3.45$ GPa, $G_{TT} = 1.38$ GPa, $\nu_{LT} = \nu_{TT} = 0.25$, $K_L = 36.42$ W/mK, $K_T = 0.96$ W/mK, $\alpha_L = 0.57 \cdot 10^{-6}\mathrm{K}^{-1}$ and $\alpha_T = 35.60 \cdot 10^{-6}\mathrm{K}^{-1}$.





Displacement components [m] for a short laminated $[0/90]$ beam

	$10^3 \times \tilde{u}_x$			$10^3 \times \tilde{u}_y$			$-10^2 \times \tilde{u}_z$		
FEM 3D-R[a]	2.9118			7.2636			5.9587		
FEM 3D-C[b]	2.9118			7.2633			5.9589		
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N = 14$	2.9191	2.9192	2.9192	7.2813	7.2809	7.2809	5.9734	5.9738	5.9738
$N = 11$	2.9207	2.9208	2.9208	7.2554	7.2550	7.2550	5.9781	5.9785	5.9785
$N = 10$	2.9213	2.9214	2.9214	7.3252	7.3248	7.3248	5.9820	5.9824	5.9824
$N = 9$	2.9230	2.9230	2.9230	7.2652	7.2648	7.2648	5.9841	5.9845	5.9845
$N = 8$	2.9275	2.9275	2.9275	7.1785	7.1781	7.1781	5.9855	5.9859	5.9859
$N = 7$	2.9203	2.9204	2.9204	7.2285	7.2281	7.2281	5.9859	5.9863	5.9863
$N = 6$	2.9155	2.9156	2.9156	7.3180	7.3176	7.3176	5.9989	5.9993	5.9993
$N = 5$	2.9250	2.9251	2.9251	7.3445	7.3441	7.3441	6.0036	6.0040	6.0040
$N = 4$	2.8792	2.8793	2.8793	8.2410	8.2406	8.2406	5.9276	5.9280	5.9280
$N = 3$	2.8122	2.8123	2.8123	8.0597	8.0592	8.0592	5.8456	5.8460	5.8460
$N = 2$	2.7856	2.7857	2.7857	2.8734	2.8732	2.8732	5.7710	5.7714	5.7714

a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

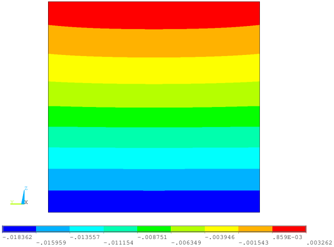
Stress components $\tilde{\sigma}_{xx}$, $\tilde{\sigma}_{xz}$, $\tilde{\sigma}_{xy}$ [MPa] for a short laminated [0/90] beam

	$-\tilde{\sigma}_{xx}$			$\tilde{\sigma}_{xz}$			$\tilde{\sigma}_{xy}$		
FEM 3D-R[a]		197.44			2.7923			2.1428	
FEM 3D-C[b]		197.69			2.7941			2.1479	
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N=14$	198.26	198.32	198.28	2.7571	2.7447	2.7516	2.1430	2.1443	2.1434
$N=11$	198.23	198.30	198.26	2.5753	2.5630	2.5698	2.1658	2.1671	2.1662
$N=10$	197.94	198.00	197.97	2.5452	2.5329	2.5398	2.1742	2.1755	2.1746
$N=9$	198.89	198.95	198.91	2.8818	2.8694	2.8763	2.1754	2.1767	2.1758
$N=8$	199.88	199.95	199.91	2.8468	2.8345	2.8414	2.2105	2.2118	2.2109
$N=7$	198.81	198.87	198.84	3.3072	3.2948	3.3017	2.2748	2.2761	2.2752
$N=6$	194.03	194.09	194.05	3.5028	3.4903	3.4972	2.3185	2.3198	2.3189
$N=5$	191.96	192.02	191.99	3.3370	3.3245	3.3314	1.9073	1.9086	1.9077
$N=4$	196.96	197.02	196.99	3.9790	3.9665	3.9734	1.7738	1.7753	1.7743
$N=3$	196.77	196.83	196.80	1.7305	1.7184	1.7252	1.7871	1.7886	1.7877
$N=2$	212.50	212.56	212.52	2.1559	2.1439	2.1506	0.9702	0.9707	0.9704

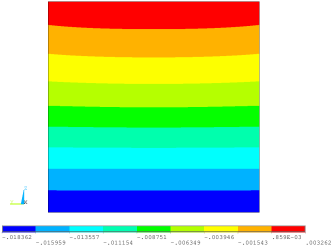
a: Elements' number $40 \times 40 \times 40$. b: Elements' number $20 \times 20 \times 20$.

Displacement cross-section variation

Axial displacement u_x [m] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



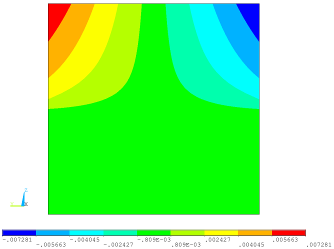
(a) FEM 3D-R



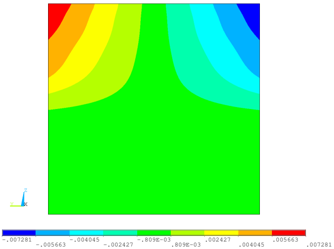
(b) $N = 14$

Displacement cross-section variation

Through-the-width displacement u_y [m] over the cross-section at $x = l/2$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



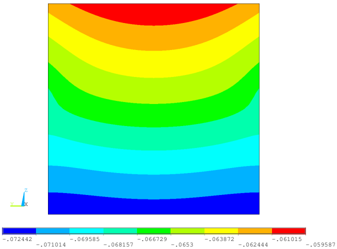
(a) FEM 3D-R



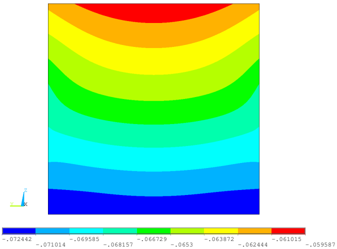
(b) $N = 14$

Displacement cross-section variation

Through-the-thickness displacement u_z [m] over the cross-section at $x = l/2$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



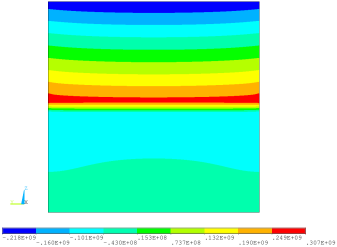
(a) FEM 3D-R



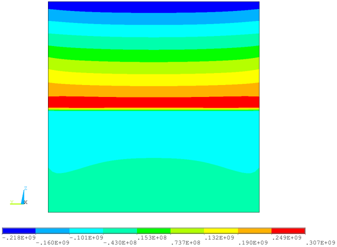
(b) $N = 14$

Stress cross-section variation

Axial stress σ_{xx} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



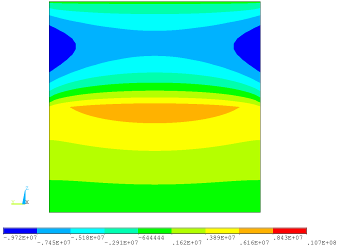
(a) FEM 3D-R



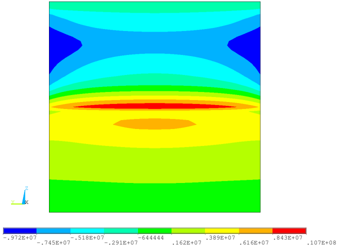
(b) $N = 14$

Stress cross-section variation

Shear stress σ_{xz} [Pa] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



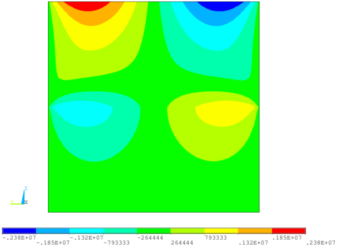
(a) FEM 3D-R



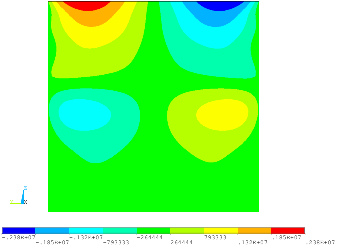
(b) $N = 14$

Stress cross-section variation

Shear stress σ_{xy} [Pa] over the cross-section at $x/l = 0$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



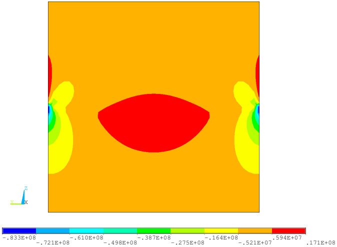
(a) FEM 3D-R



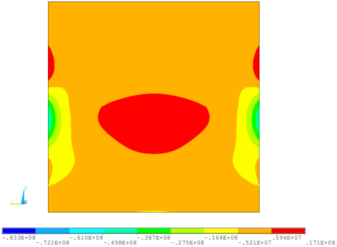
(b) $N = 14$

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



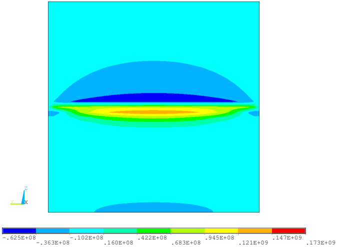
(a) FEM 3D-R



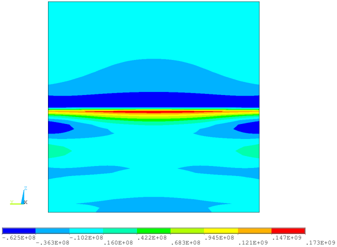
(b) $N = 14$

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] over the cross-section at $x = l/2$, B4 for $l/b = 10$, laminated $[0, 90]$ beam.



(a) FEM 3D-R



(b) $N = 14$

Conclusions

- A unified formulation for one-dimensional beam finite elements has been presented for the thermal stress analysis.
- Higher-order models that account for shear deformations and in- and out-of-plane warping can be formulated straightforwardly.
- The numerical investigation and validation showed that the proposed formulation allows obtaining accurate results reducing the computational costs when compared to three-dimensional FEM solutions.

Acknowledgement

This work has been carried out within the FULLCOMP project funded by the European Union's Horizon's 2020 research and innovation programme under grant agreement No 642121.

Many thanks for your kind attention!