University of BRISTOL

3D stress fields for beams with arbitrary cross-sections using the Unified Formulation with Serendipity Lagrange polynomial expansions

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MECHCOMP3

3rd International Conference on Mechanics of Composites

4-7 July 2017 (Bologna, Italy)

BRISTOL COMPOSITES INSTITUTE

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Outline

- Motivation & Objective
- Carrera Unified Formulation (CUF) for refined 1D models
- Various cross-section expansion models
  - *Taylor expansions (TE)*
  - *Lagrange expansions (LE)*
  - *Serendipity Lagrange expansions (SLE)*
- Numerical examples
- Node-dependent kinematics (NDK)
- Conclusions and future developments

Motivation

Classical to Refined Kinematic Models

Euler-Bernoulli beam theory (EBBT)

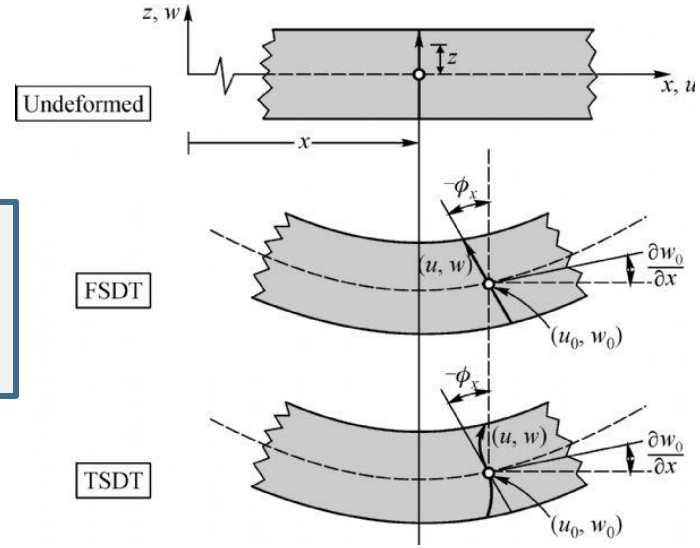
$$u(x,y,z) = u_0(y)$$
$$v(x,y,z) = v_0(y) - x\frac{\partial u_0}{\partial y} - z\frac{\partial w_0}{\partial y}$$
$$w(x,y,z) = w_0(y)$$

Timoshenko beam theory (TBT)

$$u(x,y,z) = u_0(y)$$
$$v(x,y,z) = v_0(y) + x\phi_z(y) - z\phi_x(y)$$
$$w(x,y,z) = w_0(y)$$

Higher order theories

Limited by the Saint-Venant Principle



General guideline[1]

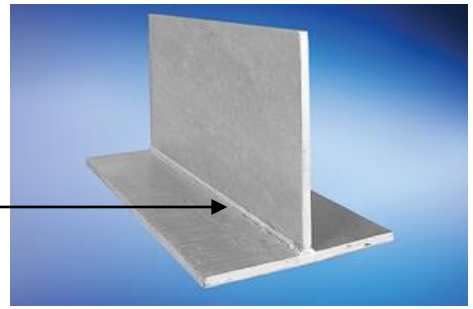
The richer the kinematic field, then the more accurate is the 1D model

[1] ***K. Washizu.*** *Variational Methods in Elasticity and Plasticity. Oxford, 1968.*

Reddy's third order theory for plates

$$u(x,y,z) = u_0(x,y) + z\phi_x(x,y) + z^3\left(\frac{-4}{3h^2}\right)\left(\phi_x + \frac{\partial w_0}{\partial y}\right)$$
$$v(x,y,z) = v_0(x,y) + z\phi_y(x,y) + z^3\left(\frac{-4}{3h^2}\right)\left(\phi_y + \frac{\partial w_0}{\partial y}\right)$$
$$w(x,y,z) = w_0(x,y)$$

What about this?



Stress concentration regions

Motivation

Classical to Refined Kinematic Models

3D finite element analysis

$$u(x,y,z) = N_i(x,y,z)\,u_i$$
$$v(x,y,z) = N_i(x,y,z)\,v_i$$
$$w(x,y,z) = N_i(x,y,z)\,w_i$$

N_i - 3D shape functions

Computationally Expensive!

Generalized beam theory (GBT)

$u(x,y,z) = \Psi_y(y)\,u(s)$ *(for prismatic thin-walled beams)*

$\gamma_{yz} = 0, \gamma_{xz} = 0, \varepsilon_{zz} = 0$ *(Thin-plate theory is adopted)*

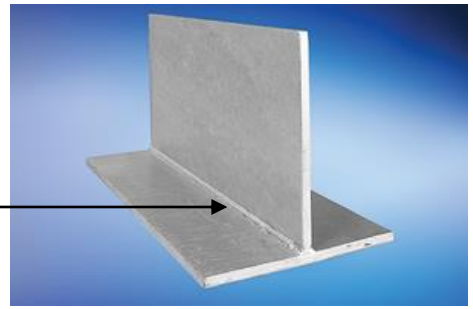
3D stress fields?

Proposed Solution

Carrera Unified Solution (CUF)

$$u(x,y,z) = N_i(y)\,F_\zeta(x,z)\,u_{\zeta i}$$
$$v(x,y,z) = N_i(y)\,F_\zeta(x,z)\,v_{\zeta i}$$
$$w(x,y,z) = N_i(y)\,F_\zeta(x,z)\,w_{\zeta i}$$

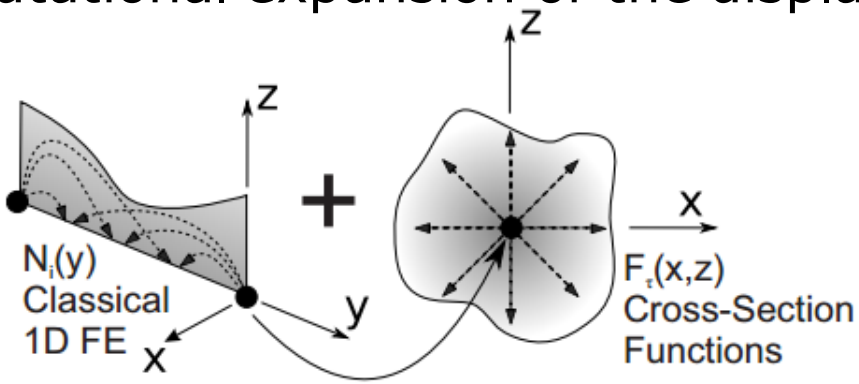
What about this?



Stress concentration regions

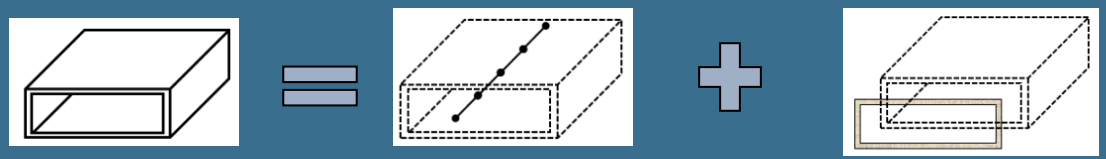
The Unified Formulation - Overview

Computational expansion of the displacement field



Ref: ***E, Carrera.*** Finite Element Analysis of Structures through Unified Formulation. *Wiley Publications, 2014.*

3D structure | 1D FE model | Cross-section expansion



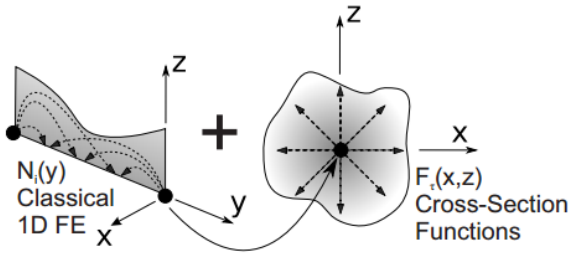
$$u(x,y,z) = N_i(y)\, F_\zeta(x,z)\, u_{\zeta i}$$
$$v(x,y,z) = N_i(y)\, F_\zeta(x,z)\, v_{\zeta i}$$
$$w(x,y,z) = N_i(y)\, F_\zeta(x,z)\, w_{\zeta i}$$

$N_i(y)$ - 1D shape functions (Linear, quadratic, cubic)

- Taylor-like polynomial expansions (TE)
- Lagrange polynomial expansions (LE)
- Serendipity Lagrange expansions (SLE)

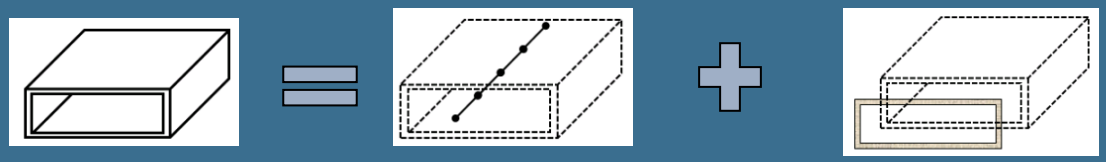
The Unified Formulation - Overview

Computational expansion of the displacement field



$$
\begin{aligned}
u(x,y,z) &= u_1(y) + xu_2(y) + zu_2(y) + x^2 u_4(y) + xzu_5(y) + z^2 u_6(y) + \cdots \\
v(x,y,z) &= u_1(y) + xv_2(y) + zv_2(y) + x^2 v_4(y) + xzv_5(y) + z^2 v_6(y) + \cdots \\
w(x,y,z) &= \underbrace{w_1(y)}_{\substack{p=0 \\ \zeta=1}} + \underbrace{xw_2(y) + zw_2(y)}_{\substack{p=1 \\ \zeta=2,\ \zeta=3}} + \underbrace{x^2 w_4(y) + xzw_5(y) + z^2 w_6(y)}_{\substack{p=2 \\ \zeta=4,\ \zeta=5,\ \zeta=6}} + \cdots
\end{aligned}
$$

3D structure | 1D FE model | Cross-section expansion



$$
\begin{aligned}
u(x,y,z) &= N_i(y)\, F_\zeta(x,z)\, u_{\zeta i} \\
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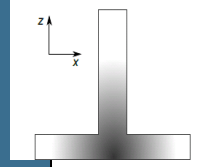
w(x,y,z) &= N_i(y)\, F_\zeta(x,z)\, w_{\zeta i}
\end{aligned}
$$

$N_i(y)$ - 1D shape functions (Linear, quadratic, cubic)

Taylor-like polynomial expansions (TE)

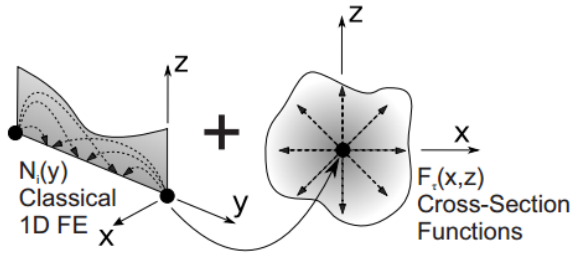


Lagrange polynomial expansions (LE)

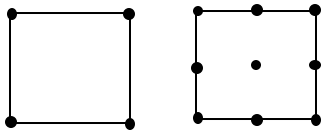
Serendipity Lagrange expansions (SLE)

The Unified Formulation - Overview

Computational expansion of the displacement field



$$u(x,y,z) = L_1 u_1(y) + L_2 u_2(y) + L_3 u_3(y) + L_4 u_4(y) + \cdots$$
$$v(x,y,z) = L_1 v_1(y) + L_2 v_2(y) + L_3 v_3(y) + L_4 v_4(y) + \cdots$$
$$w(x,y,z) = L_1 w_1(y) + L_2 w_2(y) + L_3 w_3(y) + L_4 w_4(y) + \cdots$$

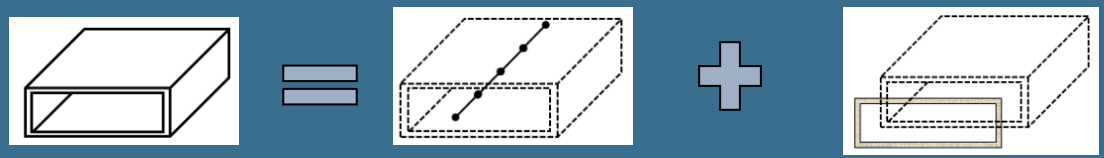


L4 L9

$L4$ element (2D linear shape functions)

$\zeta = 1, \quad \zeta = 2, \quad \zeta = 3, \quad \zeta = 4$

3D structure | 1D FE model | Cross-section expansion



$$u(x,y,z) = N_i(y)\, F_\zeta(x,z)\, u_{\zeta i}$$
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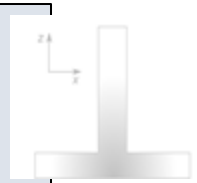
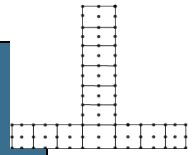
$N_i(y)$ - 1D shape functions (Linear, quadratic, cubic)

Taylor-like polynomial expansions (TE)



Lagrange polynomial expansions (LE)



Serendipity Lagrange expansions (SLE)

The Unified Formulation - Overview

Advanced Numerical Techniques 1 (7th July Morning session – 9:40)
-Sergio Minera

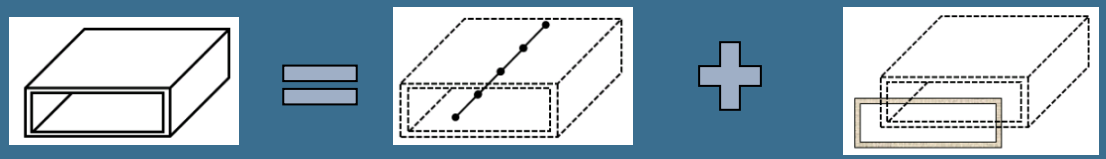
Computational expansion of the displacement field

$$u(x,y,z) = \underbrace{\sum_{k=1}^{4} L_k u_k(y)}_{p=1} \quad \underbrace{+L_5 u_5(y) + L_6 u_6(y) + L_7 u_7(y) + L_8 u_8(y) + \cdots}_{p=2}$$

$$v(x,y,z) = \sum_{k=1}^{4} L_k v_k(y) \quad +L_5 v_5(y) + L_6 v_6(y) + L_7 v_7(y) + L_8 v_8(y) + \cdots$$

$$w(x,y,z) = \sum_{k=1}^{4} L_k w_k(y) \quad +L_5 w_5(y) + L_6 w_6(y) + L_7 w_7(y) + L_8 w_8(y) + \cdots$$

3D structure | 1D FE model | Cross-section expansion



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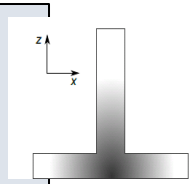
Taylor-like polynomial expansions (TE)



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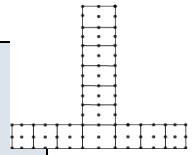
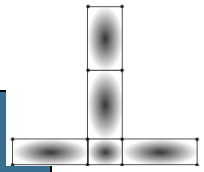
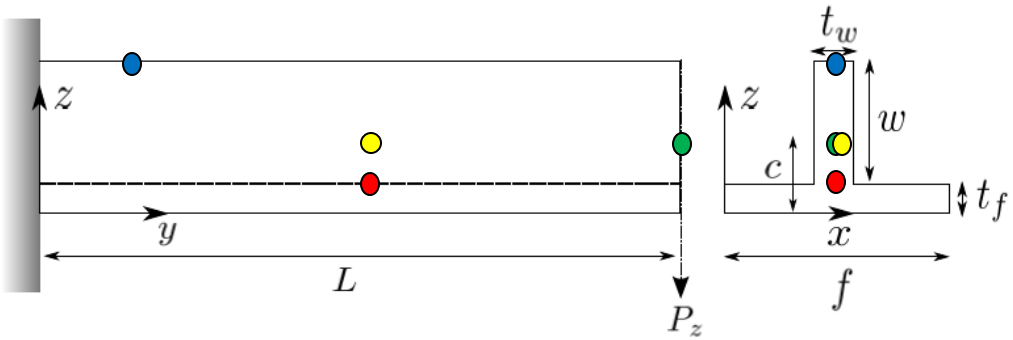


Serendipity Lagrange expansions (SLE)

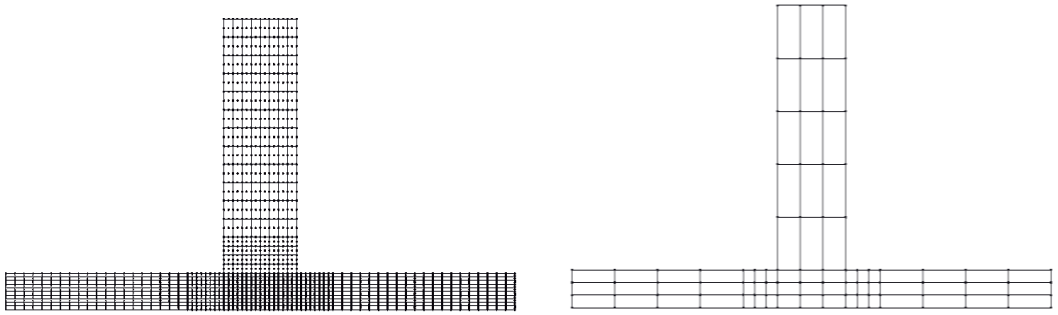


Numerical Results

T-section beam structure



Cross-section mesh

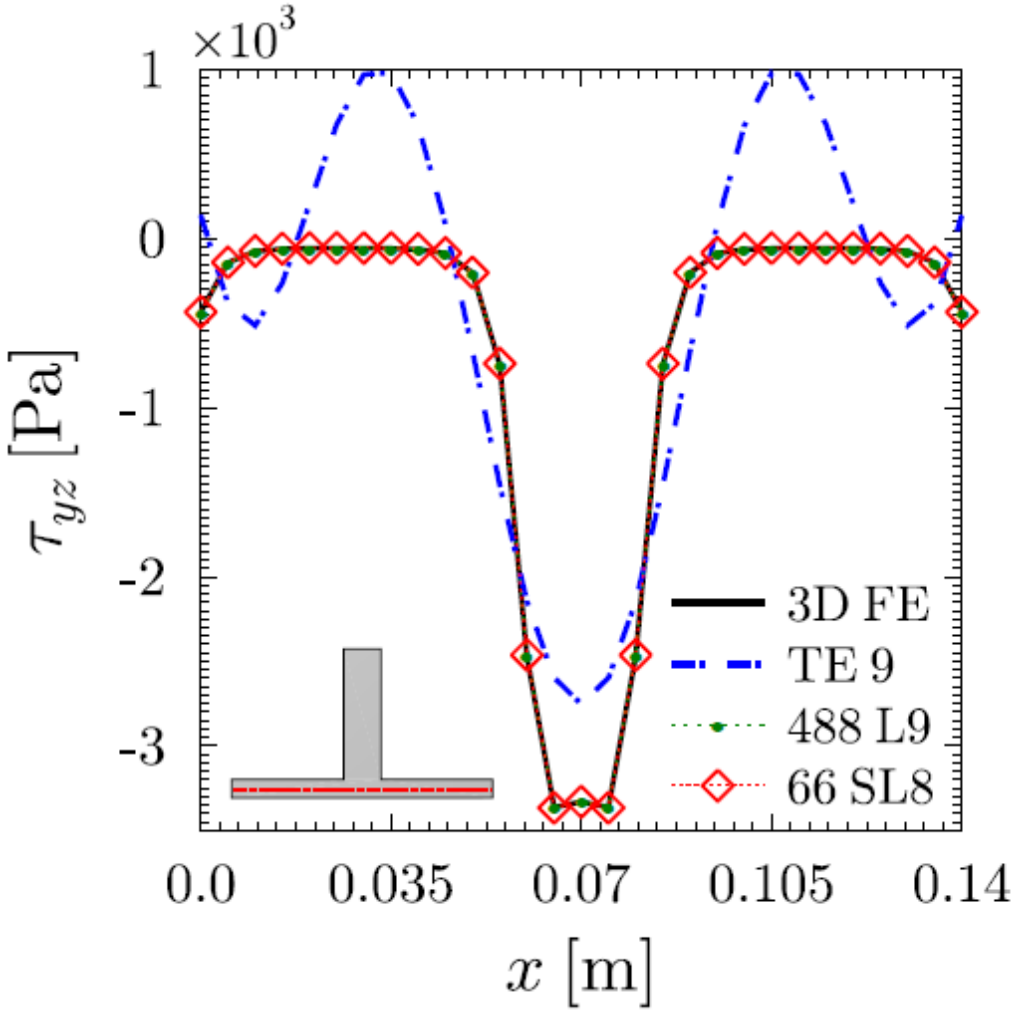


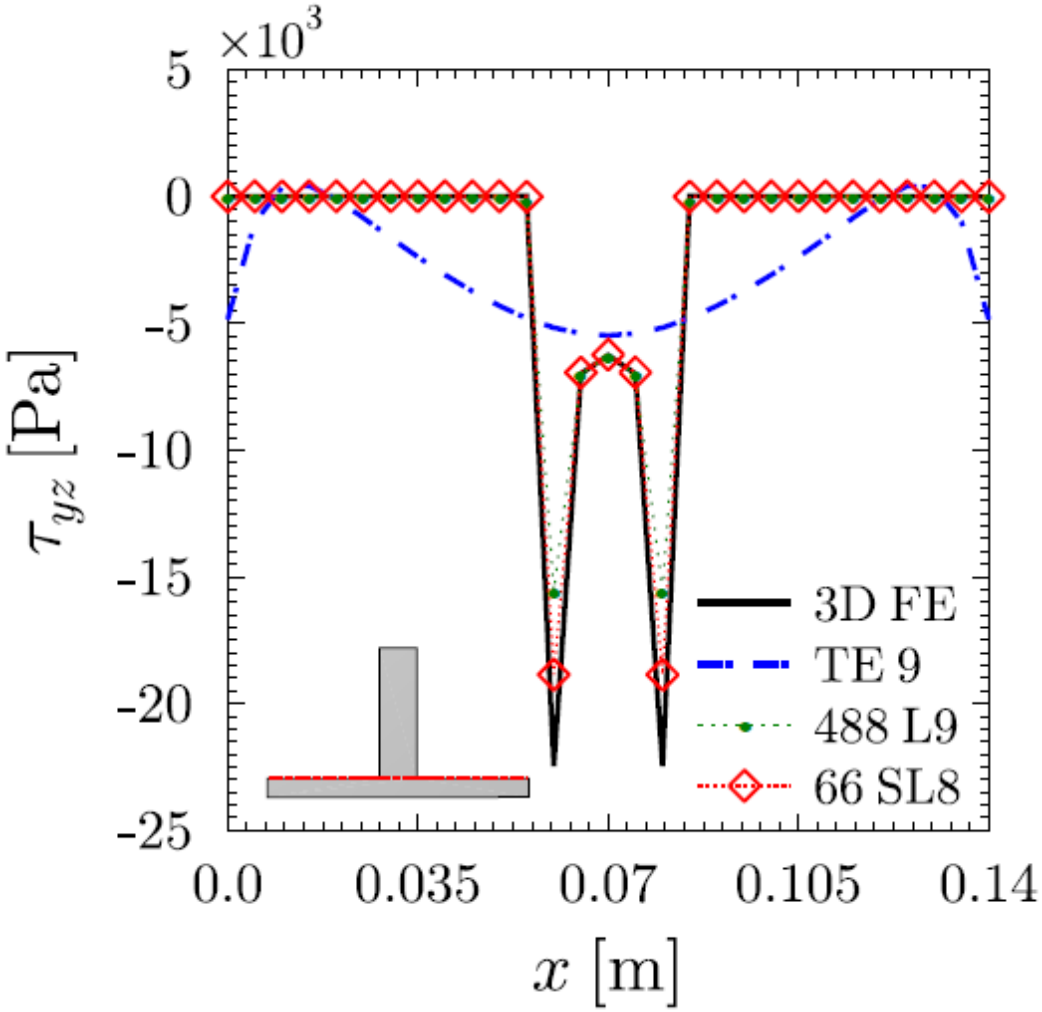
488 L9 66 SL8

DOFs ~550k

Model	$u_z \times 10^{-5}$ (m) $[f/2, L, 0.025]$	σ_{yy} (Pa) $[f/2, L/5, w]$	τ_{yz} (Pa) $[f/2, L/2, 0.025]$	τ_{yz} (Pa) $[f/2, L/2, 0.01]$
Solid	-2.6304	258410	-8830.4	-6266.3
Taylor Expansion				
T5	-2.6248	258323	-8520.6	-44031.3
T6	-2.6268	258321	-8999.9	-4667.6
T7	-2.6274	258327	-9080.2	-5061.7
T8	-2.6280	258326	-8897.8	-5159.9
T9	-2.6284	258324	-8802.7	-5485.1
Lagrange Expansion				
40 L9	-2.6298	258326	-8973.3	-44031.3
126 L9	-2.6301	258327	-8816.4	-6544.9
184 L9	-2.6303	258327	-8843.8	-6247.3
336 L9	-2.6303	258327	-8844.4	-6309.4
432 L9	-2.6304	258327	-8844.9	-6304.8
488 L9	-2.6304	258327	-8845.6	-6277.7
Serendipity Lagrange Expansion				
66 SL5	-2.6304	258327	-8826.2	-5990.7
66 SL6	-2.6305	258327	-8815.7	-6135.0
66 SL7	-2.6305	258327	-8838.1	-6207.0
66 SL8	-2.6305	258327	-8824.7	-6243.1
66 SL9	-2.6305	258327	-8826.6	-6259.8

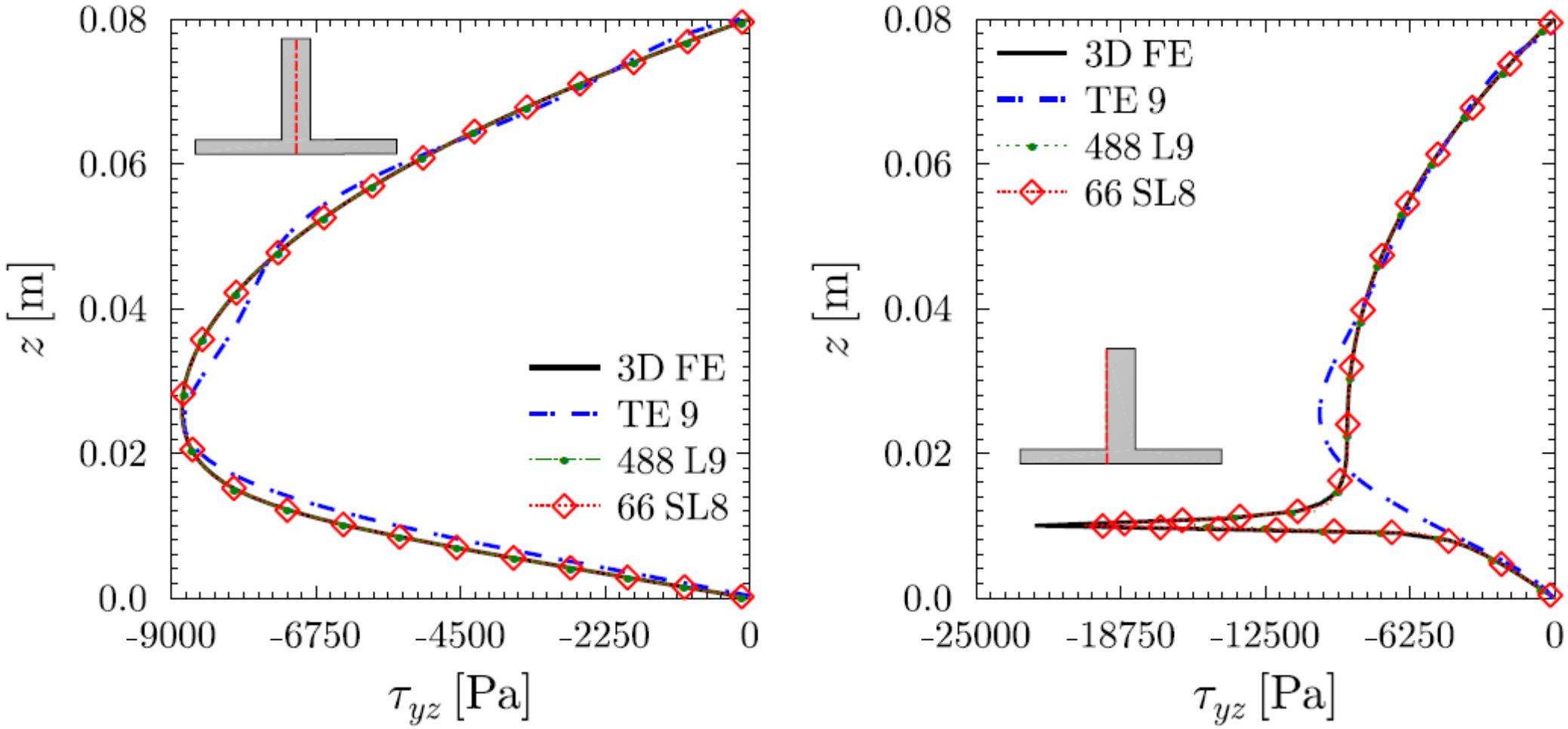
Numerical Results *(T-section beam structure)*





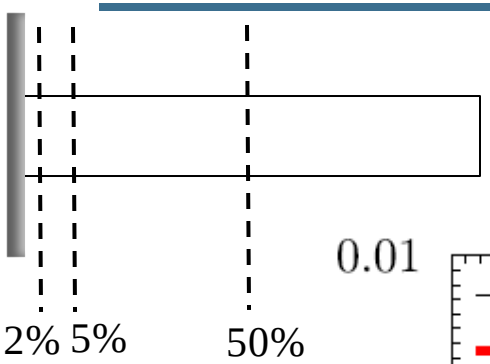
Shear stress along the flange at mid-span

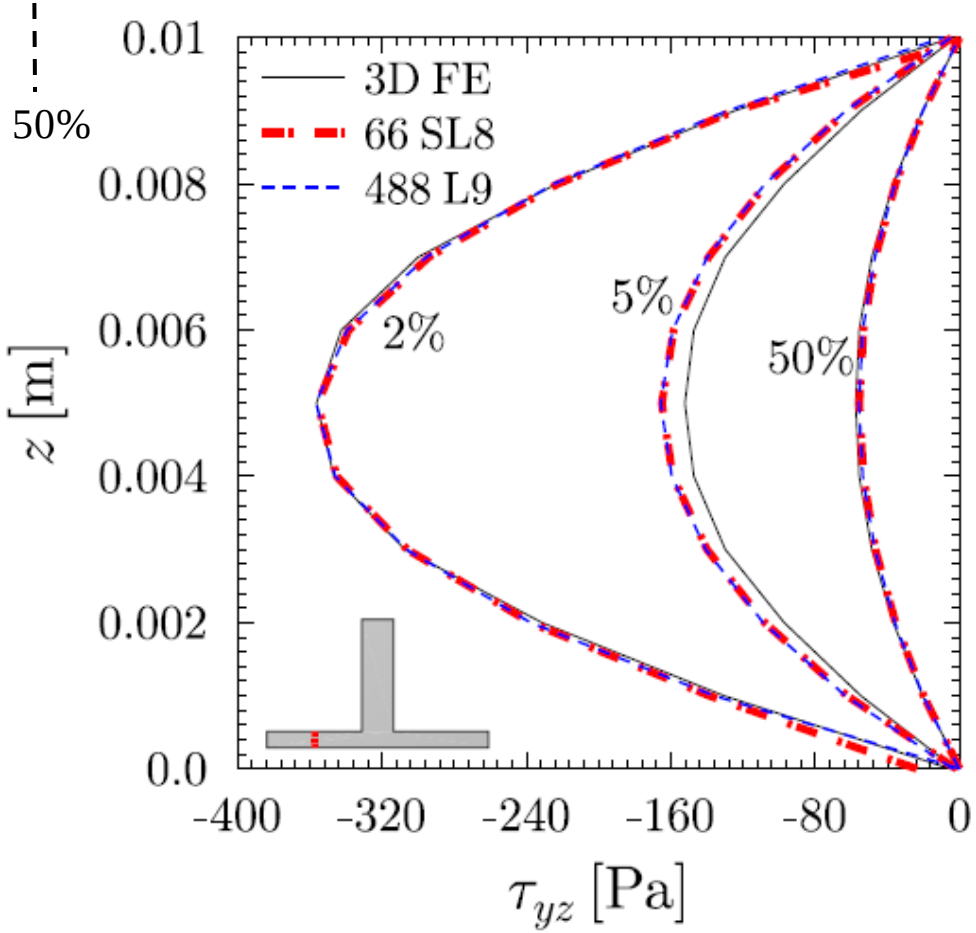
Numerical Results *(T-section beam structure)*

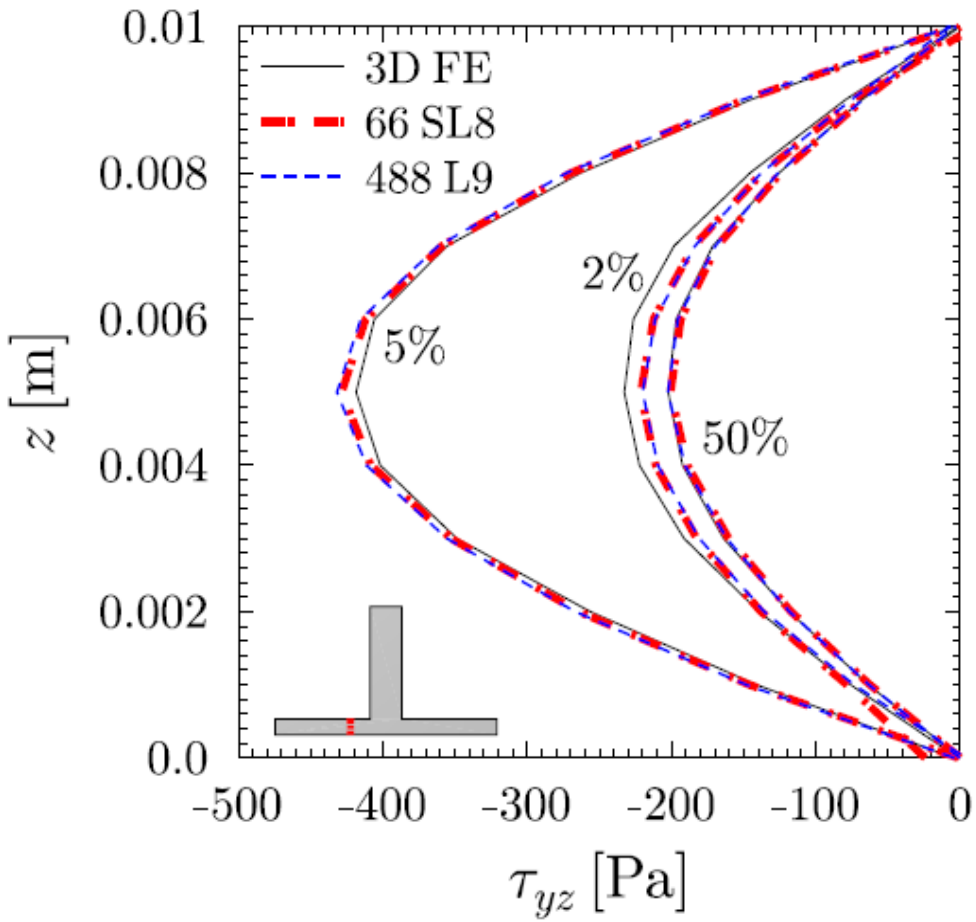


Through-thickness shear stress in the web at mid-span

Numerical Results *(T-section beam structure)*

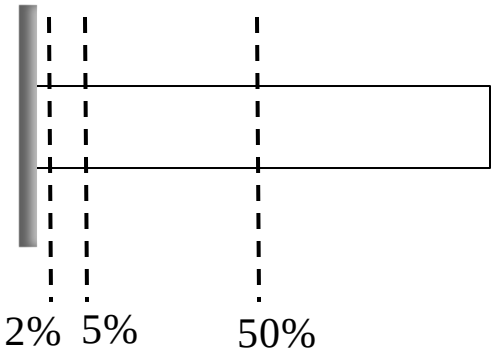


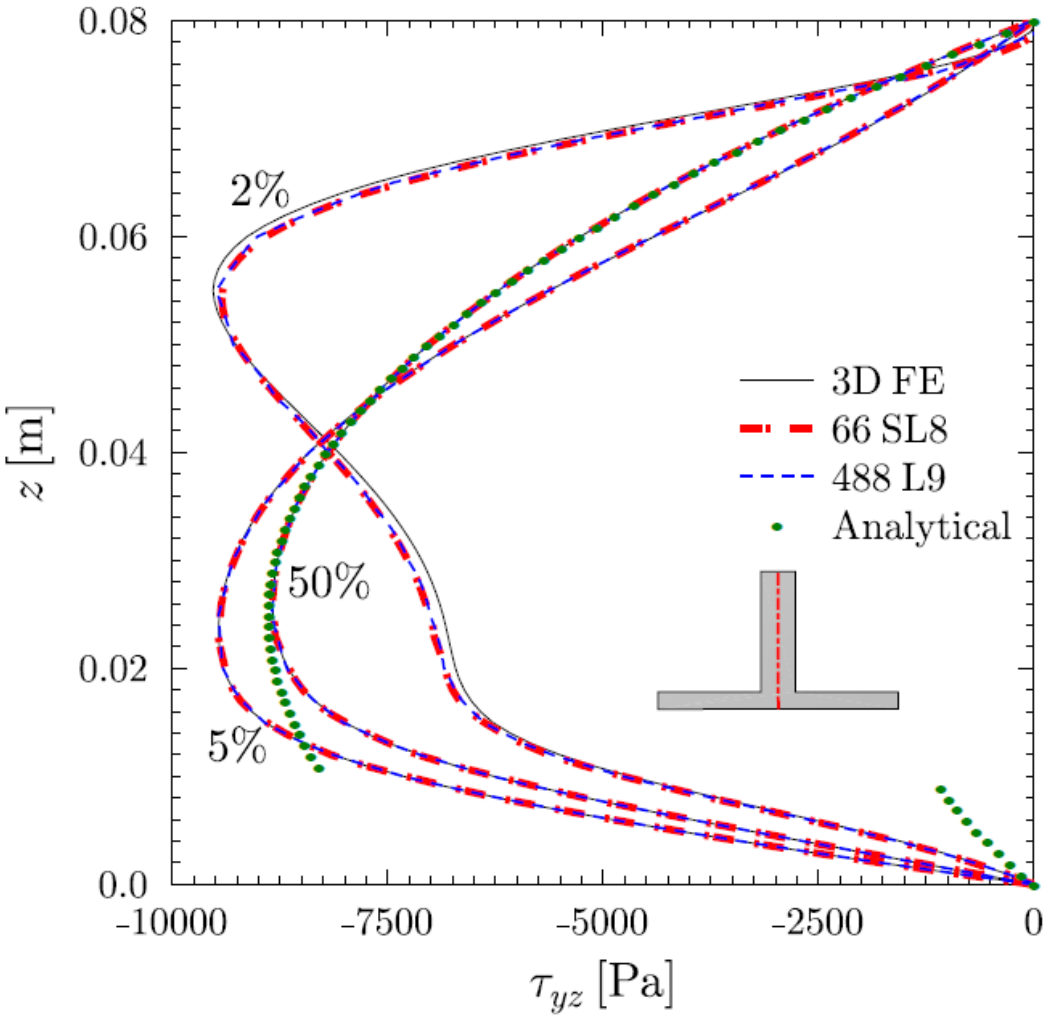




Through-thickness shear stress in the flange at various locations along the beam

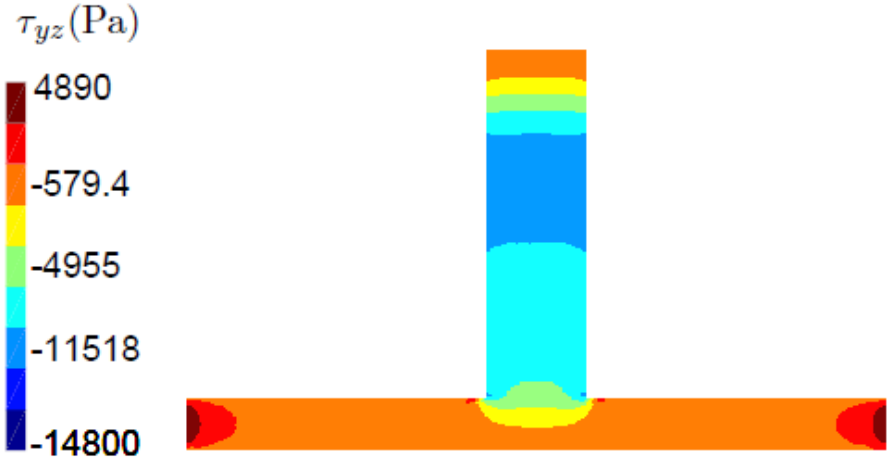
Numerical Results *(T-section beam structure)*



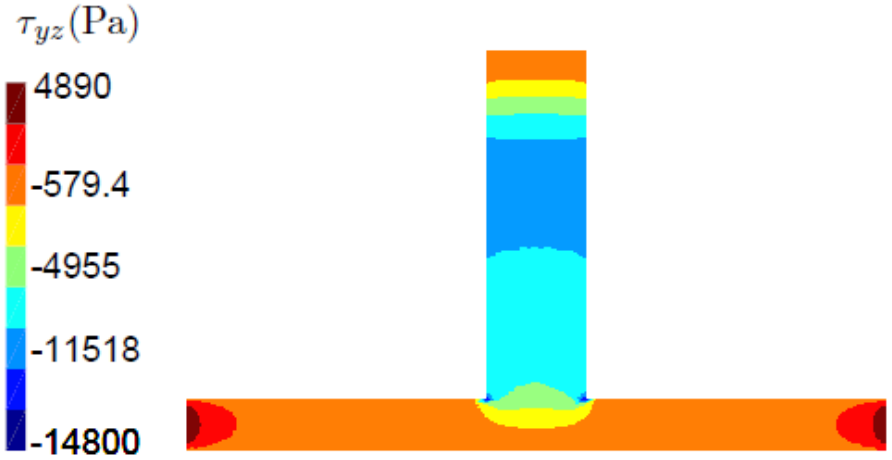


Through-thickness shear stress in the web at various locations along the beam

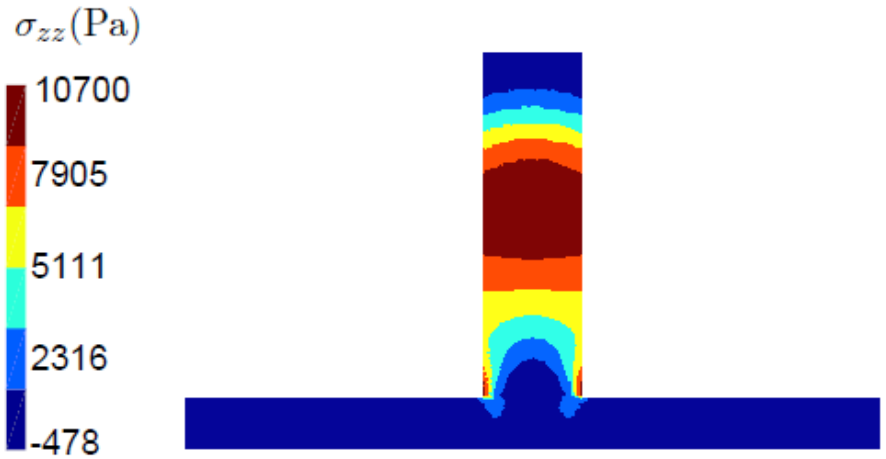
Numerical Results *(T-section beam structure)*



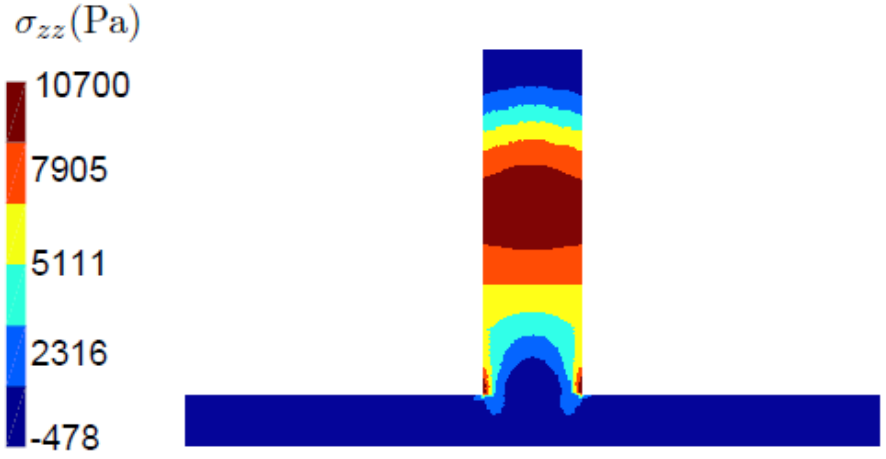
(a) ANSYS



(b) SL



(a) ANSYS

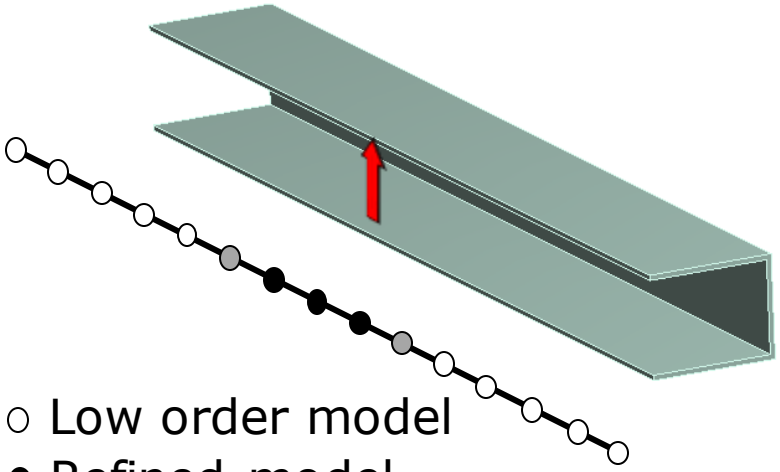


(b) SL

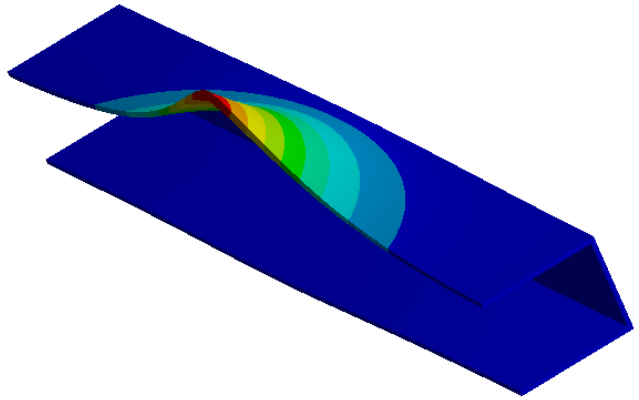
Distribution of transverse shear and normal stress in the cross-section at 2% of the beam length from the clamped end

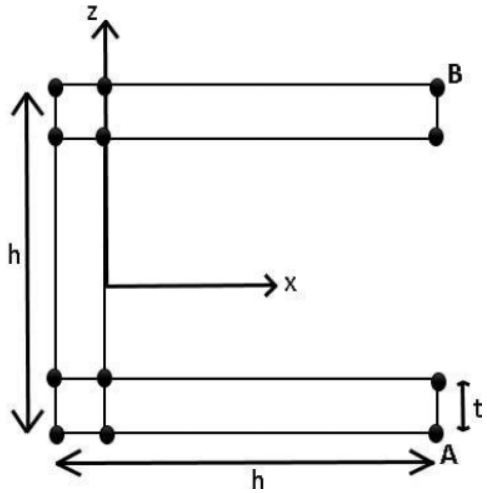
Node-Dependent Kinematics (NDK)

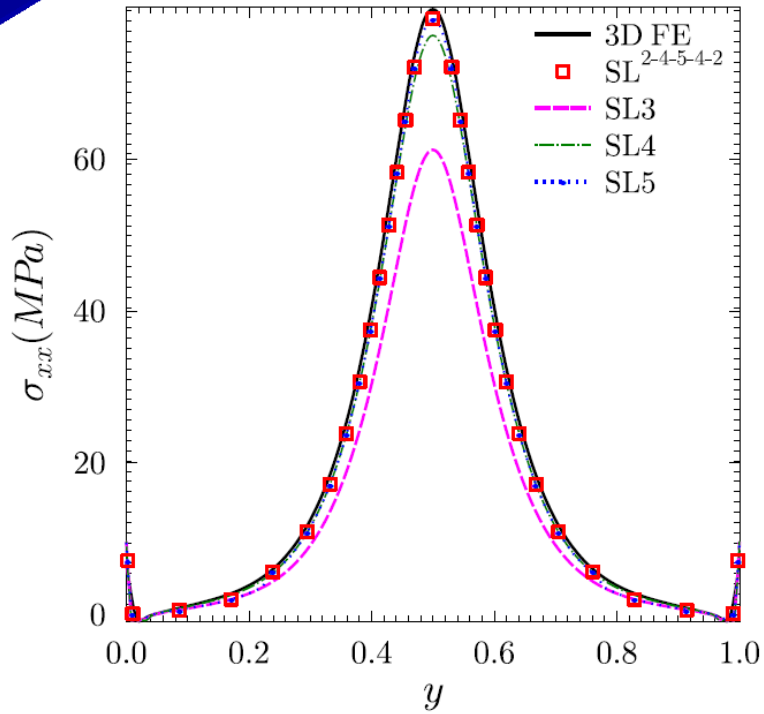
Clamped at both ends
Point load at B



○ Low order model
● Refined model







- ❑ Need for local kinematic refinement
- ❑ CUF can be used to derive NDK models without any 'ad hoc' assumptions.
- ❑ Less model building effort when used with SLE model
- ❑ Increases accuracy
- ❑ Reduces computational cost

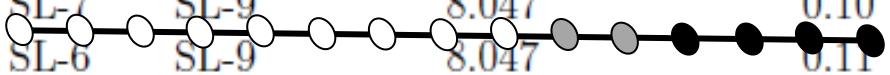
Node-Dependent Kinematics (NDK)

Cantilever C-section beam
Load applied at free end
Vertical deflection at Point A

Model	u_z (mm)	Error (%)	DOFs
ANSYS	8.055	-	66471
SL-9	8.054	0.01	18375
SL-8	8.040	0.18	14925
SL-7	8.011	0.55	11850
SL-6	7.990	0.80	9150
SL-5	7.947	1.34	6825
SL-4	7.923	1.64	4875

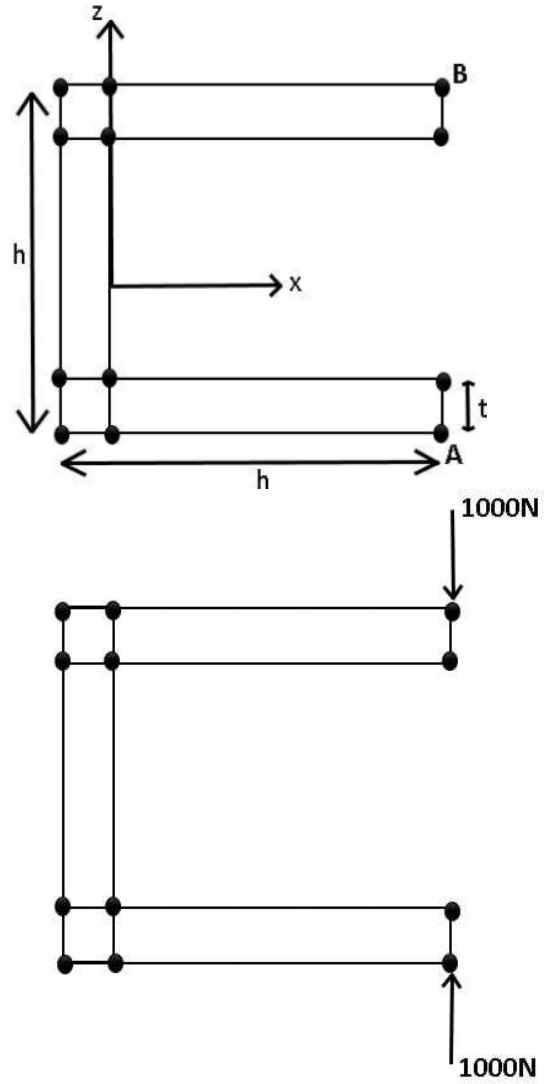
Expansion order for each beam element

E1-6	E7	E8			
SL-2	SL-7	SL-9	8.047	0.10	5874
SL-2	SL-6	SL-9	8.047	0.11	5550
SL-2	SL-4	SL-9	[illegible]	0.15	5037
SL-2	SL-3	SL-9	8.039	0.20	4848



○ Low order model

● Refined model

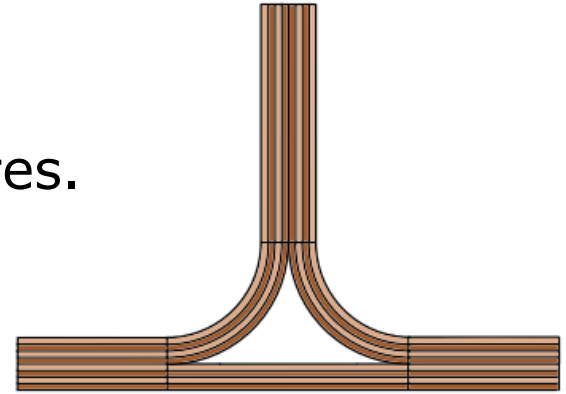


Conclusions

- The *Serendipity Lagrange expansion (SLE) model* is introduced within the framework of Carrera Unified Formulation.
- Able to capture *full 3D stress fields* and accuracy is *not limited to Saint-Venant's Principle*.
- The SLE model retains benefits of both the Lagrange model (*cross-section discretization*) and the Taylor model (*order of expansion*), without retaining any of their disadvantages.
- The SLE model discretises cross-sections locally, and therefore, is *able to capture localised stress fields* unlike the Taylor expansion model.
- *Local refinement* can be achieved using *Node-dependent kinematics* technique with SLE model.

Ongoing Developments & Future Extensions

- CUF-SLE model for laminated composite structures.
- Extension of CUF to Tow-Steered Composite structures.
- Introduction of CUF-SLE model to analyse curved cross-section beams.



Acknowledgement

This research has been developed in the framework of the FULLCOMP project. The H2020 Marie Sklodowska-Curie European Training Network is gratefully acknowledged.

For more details visit: www.fullcomp.net

THANK YOU FOR YOUR ATTENTION!

ANY QUESTIONS?

University of BRISTOL

3D stress fields for beams with arbitrary cross-sections using the Unified Formulation with Serendipity Lagrange polynomial expansions

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