A thermal stress analysis of three-dimensional beams by one-dimensional hierarchical finite elements

G. De Pietro[1,2], Y. Hui[1,2,3], G. Giunta[1], S. Belouettar[1] and E. Carrera[2]

[1]Luxembourg Institute of Science and Technology, Luxembourg
[2]Politecnico di Torino, Italy
[3]School of Civil Engineering, Wuhan University, China

2nd International Conference on Mechanics of Composites
Porto, Portugal

July 11, 2016

Scope

Thermo-mechanical response investigation of three-dimensional composite beam structures through hierarchical one-dimensional finite elements based on Carrera unified formulation.

The intention behind this approach is two-fold:

- reduce the computational cost (when compared to full three-dimensional solutions).
- ensure accurate three-dimensional results via a one-dimensional approach.

Outline

The presentation is organised as follows:

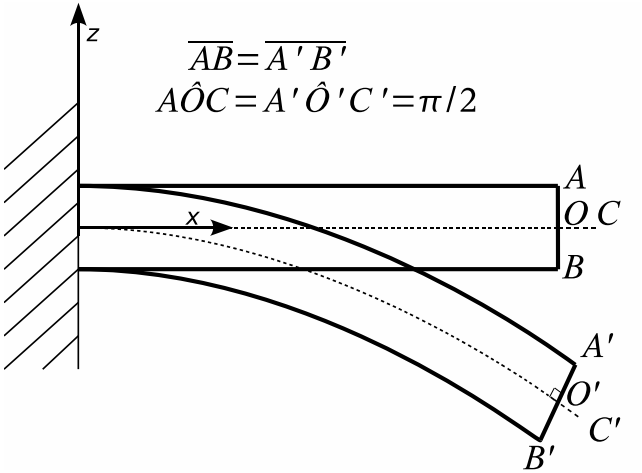
- Theoretical background
- Numerical results
- Conclusions

Classical Beam Theories

§ Euler-Bernoulli's theory:

$$
\begin{aligned}
u_x &= u_{x1} - u_{y1,x} y - u_{z1,x} z \\
u_y &= u_{y1} \\
u_z &= u_{z1}
\end{aligned}
$$

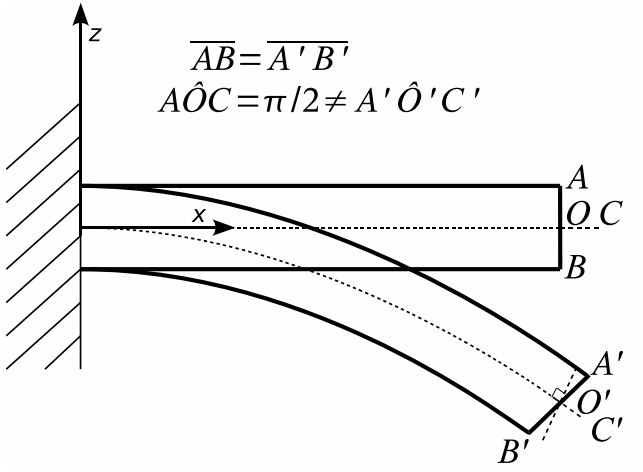
- Cross-section rigid on its plane.
- No shear stress (only axial stress).



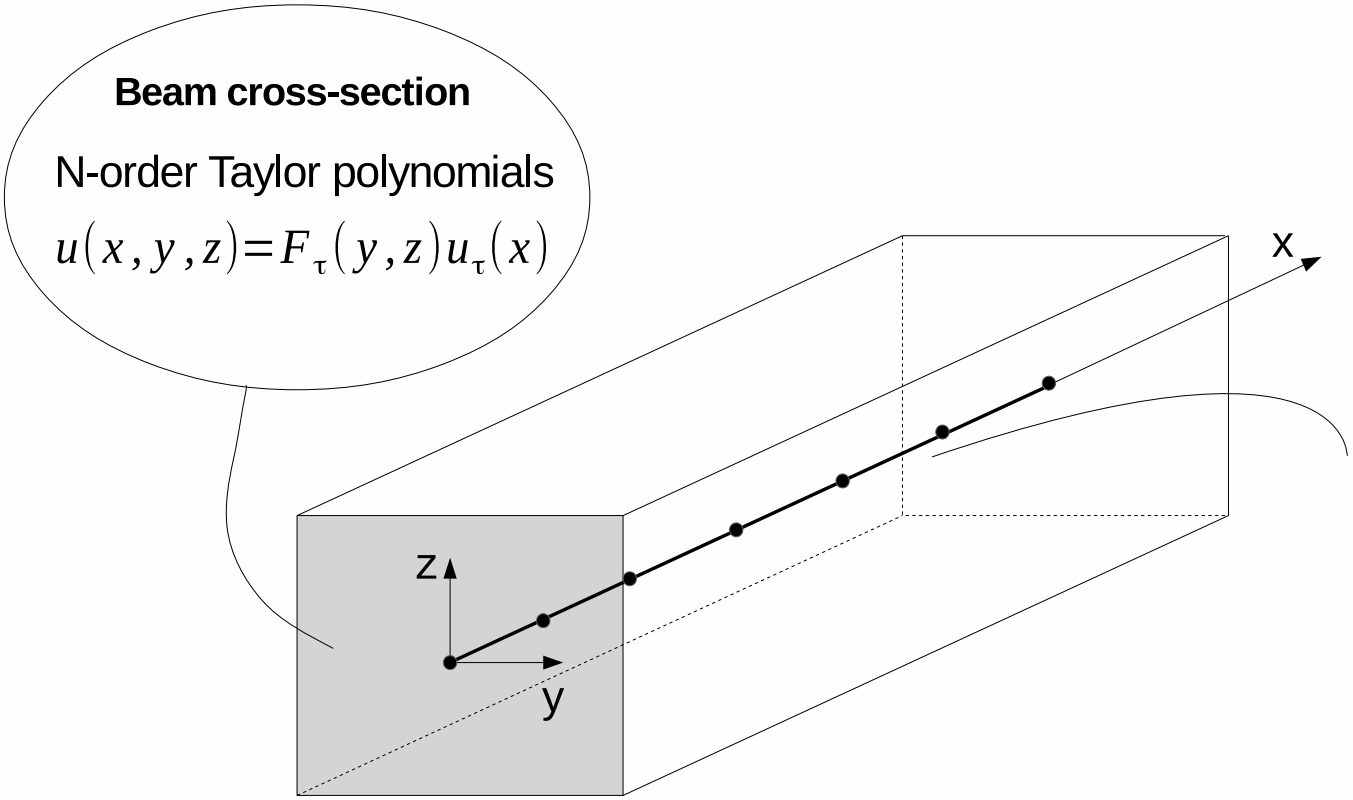
§ Timoshenko's theory:

$$
\begin{aligned}
u_x &= u_{x1} + u_{x2} y + u_{x3} z \\
u_y &= u_{y1} \\
u_z &= u_{z1}
\end{aligned}
$$

- Cross-section rigid on its plane.
- Shear stress (corrective factor).



One-dimensional Hierarchical Displacement Approximation



Beam cross section

$$\mathbf{u}\left(x,y,z\right) = F_\tau\left(y,z\right)\mathbf{u}_\tau\left(x\right) \quad \text{with} \quad \tau = 1,\ 2,\ \ldots,\ N_u$$

§ The compact expression is based on Einstein's notation: subscript τ indicates summation.

§ N_u is the number of accounted terms.

§ $F_\tau\left(y,z\right)$ are generic approximating functions.

§ Thanks to this compact notation, the element stiffness matrix can be derived in terms of '**fundamental nuclei**'.

§ Within this work, Taylor polynomials are chosen as expansion functions F_τ. Therefore, the generic N-order displacement field is:

$$\begin{aligned}
u_x &= u_{x1} + u_{x2}y + u_{x3}z + \cdots + u_{x\frac{\left(N^2+N+2\right)}{2}}y^N + \cdots + u_{x\frac{(N+1)(N+2)}{2}}z^N \\
u_y &= u_{y1} + u_{y2}y + u_{y3}z + \cdots + u_{y\frac{\left(N^2+N+2\right)}{2}}y^N + \cdots + u_{y\frac{(N+1)(N+2)}{2}}z^N \\
u_z &= u_{z1} + u_{z2}y + u_{z3}z + \cdots + u_{z\frac{\left(N^2+N+2\right)}{2}}y^N + \cdots + u_{z\frac{(N+1)(N+2)}{2}}z^N
\end{aligned}$$

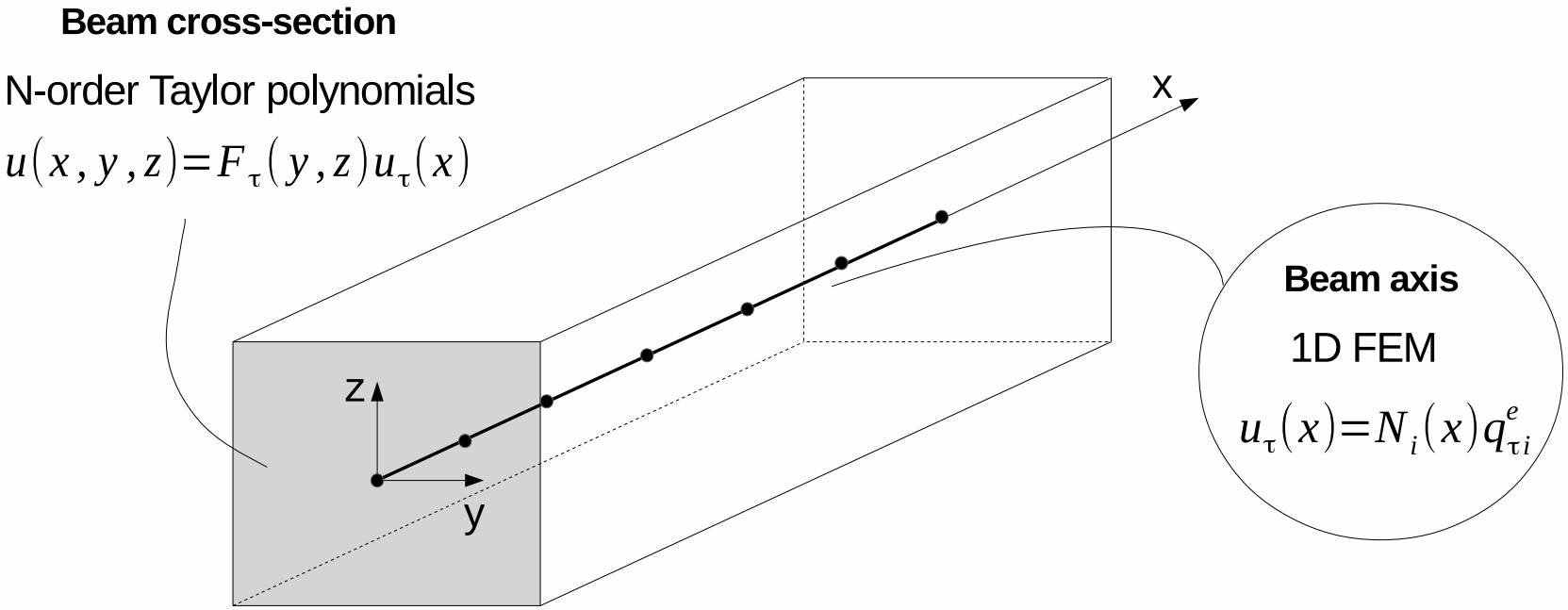
§ N_u and F_τ as functions of N can be obtained via Pascal's triangle as shown in the following Table:

N	N_u	F_τ
0	1	$F_1 = 1$
1	3	$F_2 = y \quad F_3 = z$
2	6	$F_4 = y^2 \quad F_5 = yz \quad F_6 = z^2$
...	...	...
N	$\frac{(N+1)(N+2)}{2}$	$F_{\frac{(N^2+N+2)}{2}} = y^N \quad F_{\frac{(N^2+N+4)}{2}} = y^{N-1}z \quad \ldots$
		$F_{\frac{N(N+3)}{2}} = yz^{N-1} \quad F_{\frac{(N+1)(N+2)}{2}} = z^N$

§ N is a free parameter of the formulation.

§ By properly choosing N, different beam theories accounting for higher-order effects such as shear deformations and cross-section in- and out-of plane warping can be straightforwardly obtained.

One-dimensional Hierarchical Displacement Approximation



Beam axis

The part of the displacement vector that depends upon the axial coordinate ($\mathbf{u}_\tau$) is approximated as follows:

$$\mathbf{u}_\tau(x) = N_i(x)\,\mathbf{q}_{\tau i} \quad \text{with} \quad \tau = 1,\ 2,\ \ldots,\ N_u \quad \text{and} \quad i = 1,\ 2,\ \ldots,\ N_n$$

§ $\mathbf{q}_{\tau i}$ are the nodal displacements unknowns typical of a finite element approximation.

§ $N_i(x)$ are the corresponding shape functions, which approximate the displacements along the beam axis in a C^0 sense up to an order $N_n - 1$ being N_n the number of nodes per element. This latter is a free parameter of the theoretical formulation.
Linear (B2), quadratic (B3) and cubic (B4) elements along the beam axis are considered.

Geometric Relations

A linear relation between strain and displacement vector is considered:

$$\boldsymbol{\varepsilon}_{tn} = \mathbf{D}_{np}\mathbf{u} + \mathbf{D}_{nx}\mathbf{u}$$
$$\boldsymbol{\varepsilon}_{tp} = \mathbf{D}_{p}\mathbf{u}$$

Total strain components have been grouped into vectors $\boldsymbol{\varepsilon}_{tn}$ with components orthogonal to the cross-section and $\boldsymbol{\varepsilon}_{tp}$ with components laying on Ω.

$\mathbf{D}_{np}$, $\mathbf{D}_{nx}$, and $\mathbf{D}_{p}$ are linear differential matrix operators.

Constitutive Relations

In the case of thermo-mechanical problems, Hooke's law reads:

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_e = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_\vartheta\right) = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\alpha}}T\right) = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\lambda}}T$$

where subscripts 'e' and 'ϑ' refer to the elastic and the thermal contributions, respectively. $\tilde{\mathbf{C}}$ is the material elastic stiffness, $\tilde{\boldsymbol{\alpha}}$ the vector of the thermal expansion coefficients, $\tilde{\boldsymbol{\lambda}}$ their product and T stands for temperature.

$$\boldsymbol{\sigma}_p = \tilde{\mathbf{C}}_{pp}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{pn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_p T$$
$$\boldsymbol{\sigma}_n = \tilde{\mathbf{C}}_{np}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{nn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_n T$$

Fourier's Heat Conduction Equation

Fourier's heat conduction equation for a multi-layered structure holds:

$$K_1^k \frac{\partial^2 T^k}{\partial x^2} + K_2^k \frac{\partial^2 T^k}{\partial y^2} + K_3^k \frac{\partial^2 T^k}{\partial z^2} = 0, \tag{1}$$

being K_i^k the thermal conductivity coefficients of the k-th layer. It has been solved via a Navier-type analytical solution, by assuming that the temperature does not depend upon the through-the-width co-ordinate y. The following temperature field:

$$T^k(x,z) = \Theta_\Omega(z)\,\Theta_n(x) = \left(\bar{T}_1^k e^{s_1^k z} + \bar{T}_2^k e^{s_2^k z}\right)\sin(\alpha x)$$

represents a solution of the considered heat conduction problem. [1]

[1] For more details, please see refer to Giunta et al.(2016), *A thermal stress finite element analysis of beam structures by hierarchical modelling*, Composites Part B: Engineering, 95, 179-195.

Principle of Virtual Displacements

The stiffness matrices are obtained in a nuclear form via the weak form of the Principle of Virtual Displacements:

$$\delta \mathscr{L}_{\text{int}} = 0$$

where:

- δ represents a virtual variation and
- $\mathscr{L}_{\text{int}}$ is the strain energy.

Stiffness Matrix

$$\delta \mathscr{L}_{\text{int}} = \int_{l^e} \int_{\Omega} \left(\delta \boldsymbol{\epsilon}_n^T \boldsymbol{\sigma}_n + \delta \boldsymbol{\epsilon}_p^T \boldsymbol{\sigma}_p \right) d\Omega dx$$

By substitution of the geometric relations, the material constitutive equations, the unified hierarchical approximation of the displacements it becomes:

Principle of Virtual Displacements

$$
\begin{aligned}
\delta L_{\text{int}} = \quad & \delta \mathbf{q}_{\tau i}^{T} \int_{l_e} \int_{\Omega} \Big\{ \left(\mathbf{D}_{nx} N_i\right)^T F_\tau \left[\mathbf{C}_{np} \left(\mathbf{D}_p F_s\right) N_j + \mathbf{C}_{nn} \left(\mathbf{D}_{np} F_s\right) N_j + \mathbf{C}_{nn} F_s \left(\mathbf{D}_{nx} N_j\right)\right] \\
& + \left(\mathbf{D}_{np} F_\tau\right)^T N_i \left[\mathbf{C}_{np} \left(\mathbf{D}_p F_s\right) N_j + \mathbf{C}_{nn} \left(\mathbf{D}_{np} F_s\right) N_j + \mathbf{C}_{nn} F_s \left(\mathbf{D}_{nx} N_j\right)\right] \\
& + \left(\mathbf{D}_p F_\tau\right)^T N_i \left[\mathbf{C}_{pp} \left(\mathbf{D}_p F_s\right) N_j + \mathbf{C}_{pn} \left(\mathbf{D}_{np} F_s\right) N_j + \mathbf{C}_{pn} F_s \left(\mathbf{D}_{nx} N_j\right)\right] \Big\} \, d\Omega \, dx \, \mathbf{q}_{sj} \\
& - \delta \mathbf{q}_{\tau i}^{T} \int_{l_e} \int_{\Omega} \left[\mathbf{D}_p^T F_\tau N_i \boldsymbol{\lambda}_{\mathbf{p}} + \left(\mathbf{D}_{nx}^T + \mathbf{D}_{np}^T\right) F_\tau N_i \boldsymbol{\lambda}_{\mathbf{n}}\right] \Theta_\Omega \Theta_n \, d\Omega \, dx
\end{aligned}
$$

This latter can be written in the following compact vector form:

$$
\delta L_{\text{int}} = \delta \mathbf{q}_{\tau i}^{T} \mathbf{K}^{\tau s i j} \mathbf{q}_{sj} - \delta \mathbf{q}_{\tau i}^{T} \mathbf{K}_{u\theta}^{\tau i}. \tag{2}
$$

§ The components of the element stiffness matrix fundamental nucleus $\mathbf{K}^{\tau s i j} \in \mathbb{R}^{3\times 3}$ are:

$$
\begin{aligned}
K_{xx}^{\tau s i j} &= I_{ij} \left(J_{\tau,y\, s,y}^{66} + J_{\tau,z\, s,z}^{55}\right) + I_{ij,x} J_{\tau,y\, s}^{16} + I_{i,xj} J_{\tau s,y}^{16} + I_{i,xj,x} J_{\tau s}^{11} \\
K_{yy}^{\tau s i j} &= I_{ij} \left(J_{\tau,y\, s,y}^{22} + J_{\tau,z\, s,z}^{44}\right) + I_{ij,x} J_{\tau,y\, s}^{26} + I_{i,xj} J_{\tau s,y}^{26} + I_{i,xj,x} J_{\tau s}^{66} \\
K_{zz}^{\tau s i j} &= I_{ij} \left(J_{\tau,y\, s,y}^{44} + J_{\tau,z\, s,z}^{33}\right) + I_{ij,x} J_{\tau,y\, s}^{45} + I_{i,xj} J_{\tau s,y}^{45} + I_{i,xj,x} J_{\tau s}^{55}
\end{aligned}
$$

$$
\begin{aligned}
K_{xy}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,y}^{26} + J_{\tau,z\,s,z}^{45}\right) + I_{ij,x}\,J_{\tau,y\,s}^{66} + I_{i,x\,j}\,J_{\tau s,y}^{12} + I_{i,x\,j,x}\,J_{\tau s}^{16} \\
K_{yx}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,y}^{26} + J_{\tau,z\,s,z}^{45}\right) + I_{ij,x}\,J_{\tau,y\,s}^{12} + I_{i,x\,j}\,J_{\tau s,y}^{66} + I_{i,x\,j,x}\,J_{\tau s}^{16} \\
K_{xz}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,z}^{36} + J_{\tau,z\,s,y}^{45}\right) + I_{ij,x}\,J_{\tau,z\,s}^{55} + I_{i,x\,j}\,J_{\tau s,z}^{13} + I_{i,x\,j,x}\,J_{\tau s}^{15} \\
K_{zx}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,z}^{45} + J_{\tau,z\,s,y}^{36}\right) + I_{ij,x}\,J_{\tau,z\,s}^{13} + I_{i,x\,j}\,J_{\tau s,z}^{55} \\
K_{yz}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,z}^{23} + J_{\tau,z\,s,y}^{44}\right) + I_{ij,x}\,J_{\tau,z\,s}^{45} + I_{i,x\,j}\,J_{\tau s,z}^{36} \\
K_{zy}^{\tau sij} &= I_{ij}\left(J_{\tau,y\,s,z}^{44} + J_{\tau,z\,s,y}^{23}\right) + I_{ij,x}\,J_{\tau,z\,s}^{36} + I_{i,x\,j}\,J_{\tau s,z}^{45}
\end{aligned}
$$

where:

$$
J_{\tau_{(,\eta)}\,s_{(,\xi)}}^{gh} = \int_{\Omega} \tilde{C}_{gh} F_{\tau_{(,\eta)}} F_{s_{(,\xi)}} \; d\Omega
$$

Weighted sum (in the continuum) of each elemental cross-section area where the weight functions account for the spatial distribution of geometry and material.

$$
I_{i_{(,x)}\,j_{(,x)}} = \int_{l^e} N_{i_{(,x)}} N_{j_{(,x)}} \; dx
$$

In order to avoid shear locking, reduced integration is used for the term I_{ij} in $K_{xx}^{\tau sij}$ since it is related to the shear deformations γ_{xy} and γ_{xz}.

Thermal load vector

The components of the thermal load vector fundamental nucleus $\overline{\mathbf{K}}_{u\theta}^{sj}$ are:

$$\overline{K}_{u\theta x}^{sj} = I_{\theta_n j,x} J_{\theta_\Omega s}^{1} + I_{\theta_n j} J_{\theta_\Omega s,y}^{6}$$
$$\overline{K}_{u\theta y}^{sj} = I_{\theta_n j} J_{\theta_\Omega s,y}^{2} + I_{\theta_n j,x} J_{\theta_\Omega s}^{6}$$
$$\overline{K}_{u\theta z}^{sj} = I_{\theta_n j} J_{\theta_\Omega s,z}^{3}$$

The generic term $J_{\tau_{(,\phi)}}^{g}$ is:

$$J_{\theta_\Omega s_{(,\phi)}}^{g} = \int_{\Omega} F_{s_{(,\phi)}} \, \overline{\lambda}_g \, \Theta_\Omega \, d\Omega,$$

whereas the term $I_{\theta_n j_{(,x)}}$ stands for:

$$I_{\theta_n j_{(,x)}} = \int_{l^e} \Theta_n N_{j_{(,x)}} \, dx,$$

where the temperature has been written as:

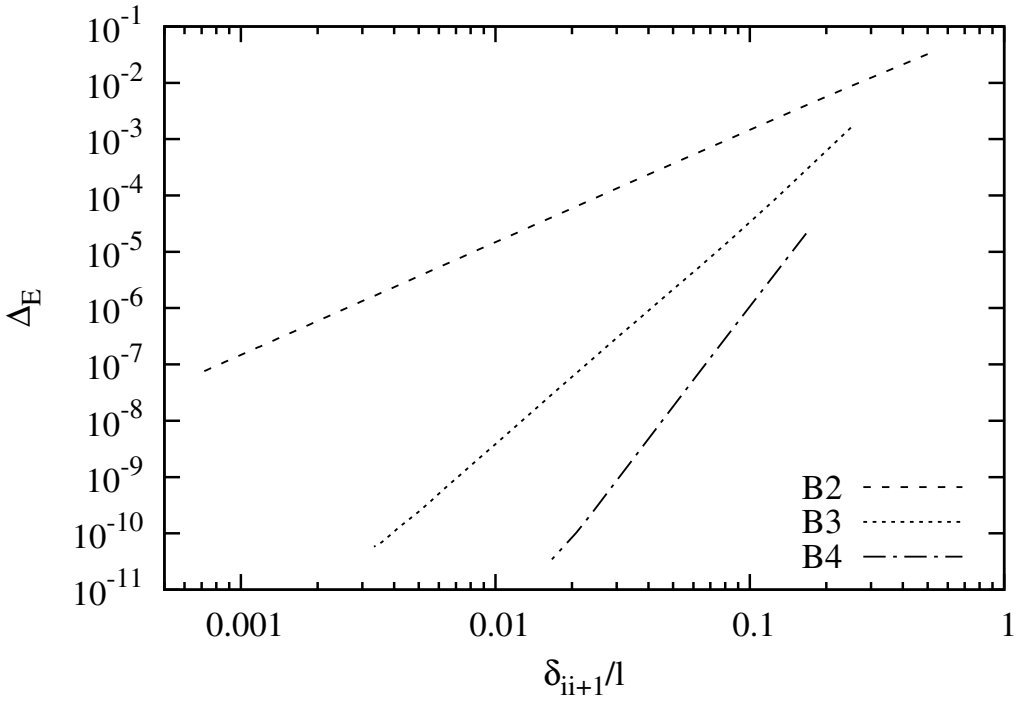
$$T(x,y,z) = \Theta_n(x) \, \Theta_\Omega(y,z)$$

Numerical Results

- The beam support is $[0, l] \times [-a/2, a/2] \times [-b/2, b/2]$. Square cross-section with $a = b = 1$ m are considered. Short beams are investigated ($l/b = 5$, $l/b = 3$.)
- Laminated and functionally graded beams are investigated.
- Different constraint configurations are considered.
- Three-dimensional FEM models are developed within the commercial code ANSYS and used for comparison.

Problem Convergence

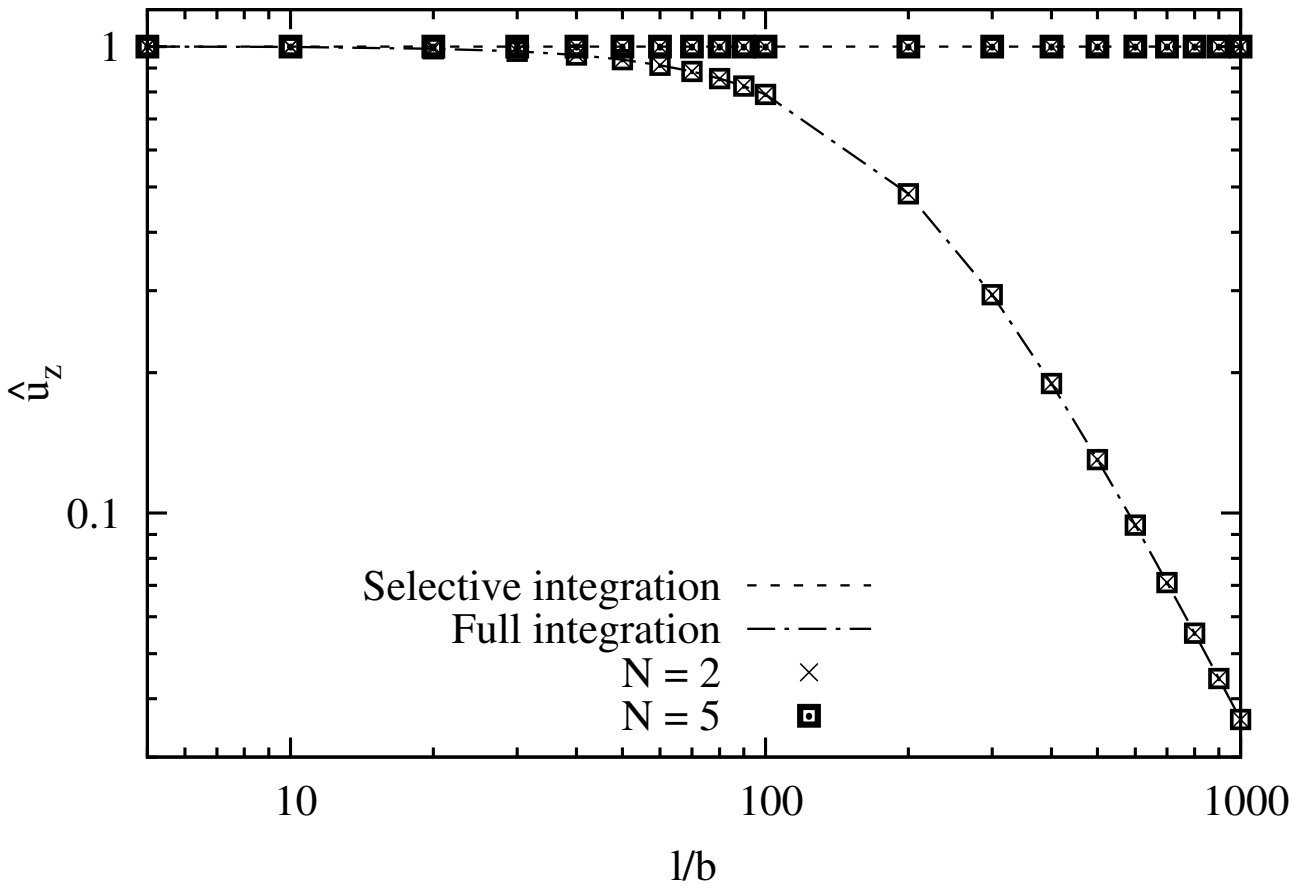
Strain energy relative error Δ_E versus the normalised distance δ_{ii+1}/l between two consecutive nodes, $l/a = 10$, isotropic beam, $N = 2$.



The error is computed by comparing the strain energy to a closed form Navier-type solution, which in the framework of a theory is an exact solution.

Shear Locking

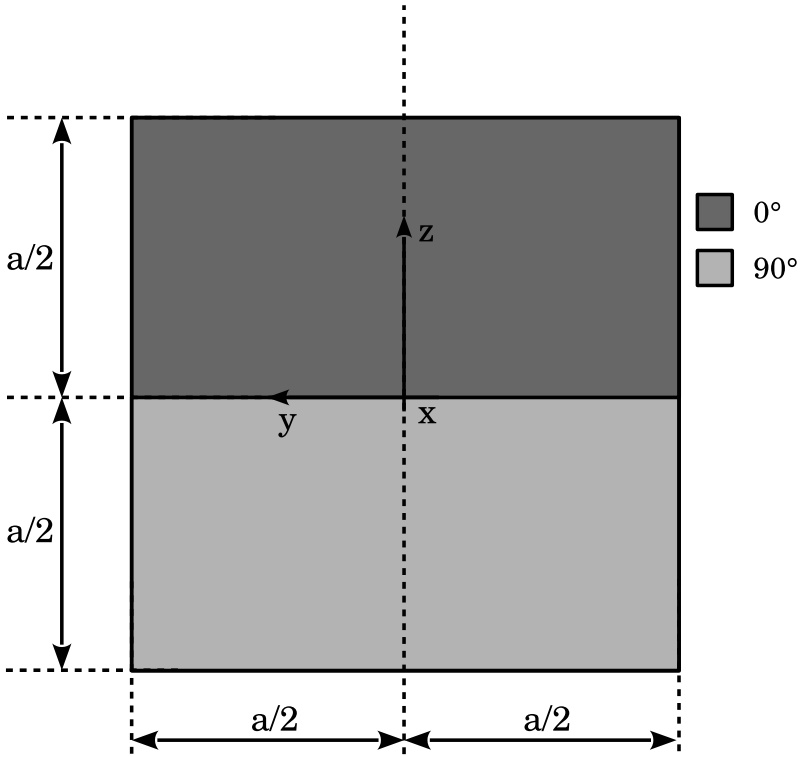
Transverse displacement ratio $\hat{u}_z = u_z\left(l/2,0,0\right)/u_z^{\mathrm{Nav}}\left(l/2,0,0\right)$ versus l/b for ***linear elements***, isotropic beam, $N = 2$ and 5.

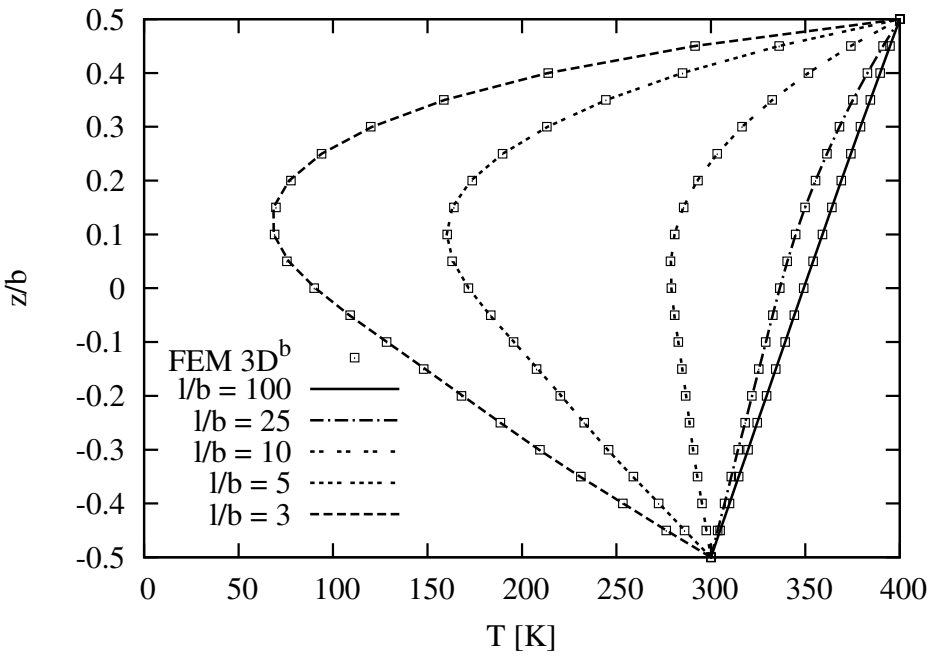


Laminated Beam

A [0/90] stacking sequences is investigated.

The material elastic and thermal properties (graphite-epoxy) are: $E_L = 172.72$ GPa, $E_T = 6.91$ GPa, $G_{LT} = 3.45$ GPa, $G_{TT} = 1.38$ GPa, $\nu_{LT} = \nu_{TT} = 0.25$, $K_L = 36.42$ W/mK, $K_T = 0.96$ W/mK, $\alpha_L = 0.57 \cdot 10^{-6}\mathrm{K}^{-1}$ and $\alpha_T = 35.60 \cdot 10^{-6}\mathrm{K}^{-1}$.





Displacement components [m] for a short laminated $[0/90]$ simply supported beam. $l/b = 3$.

	$-10^3 \times \tilde{u}_x$		$10^3 \times \tilde{u}_y$			$-10^3 \times \tilde{u}_z$	
FEM 3D-R[a]	6.5160		5.3068			8.7798	
FEM 3D-C[b]	6.5161		5.3067			8.7799	
TBT[c]	−0.0471		0.0000			−0.1627	
EBT[c]	−0.0471		0.0000			−0.1627	
	B2	B3, B4	B2	B3	B4	B2	B3,B4
$N = 14$	6.5109	6.5107	5.3072	5.3069	5.3069	8.8172	8.8172
$N = 11$	6.5077	6.5075	5.2930	5.2927	5.2927	8.8294	8.8294
$N = 9$	6.4999	6.4997	5.2944	5.2941	5.2941	8.8341	8.8341
$N = 7$	6.4977	6.4975	5.1728	5.1725	5.1725	8.8606	8.8606
$N = 5$	6.4557	6.4555	5.0304	5.0302	5.0302	8.7907	8.7906
$N = 3$	6.0633	6.0631	4.3927	4.3924	4.3925	8.6835	8.6834
$N = 2$	6.6233	6.6230	1.3257	1.3256	1.3256	8.7811	8.7810

a: Elements' number $40 \times 40 \times 40$. *b*: Elements' number $20 \times 20 \times 20$.
c: Navier-type solution.

Relative differences for $N \geq 9$ being 0.6% at worst

Stress components $\tilde{\sigma}_{xx}$, $\tilde{\sigma}_{xy}$ and $\tilde{\sigma}_{xz}$ [Pa] for a short laminated [0/90] simply supported beam. $l/b = 3$.

	$-10^{-8} \times \tilde{\sigma}_{xx}$			$10^{-6} \times \tilde{\sigma}_{xy}$			-$10^{-7} \times \tilde{\sigma}_{xz}$		
FEM 3D-R[a]		1.1519			7.6949			1.6506	
FEM 3D-C[b]		1.1473			7.7425			1.6555	
TBT[c]		0.1708			0.0000			0.0000	
EBT[c]		0.1708			–[d]			–[d]	
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N = 14$	1.1595	1.1599	1.1597	7.6536	7.6558	7.6540	1.6382	1.6389	1.6385
$N = 11$	1.1558	1.1561	1.1560	7.6922	7.6945	7.6926	1.6433	1.6439	1.6435
$N = 9$	1.1688	1.1691	1.1690	7.7875	7.7897	7.7879	1.7598	1.7605	1.7600
$N = 7$	1.1345	1.1348	1.1347	8.1668	8.1691	8.1672	1.8267	1.8274	1.8269
$N = 4$	1.2616	1.2619	1.2617	6.6134	6.6162	6.6141	1.4939	1.4946	1.4941
$N = 3$	1.0608	1.0610	1.0609	5.2571	5.2596	5.2578	0.8788	0.8794	0.8790
$N = 2$	0.9897	0.9899	0.9899	1.7501	1.7509	1.7503	0.5559	0.5564	0.5561

a: Elements' number $60 \times 60 \times 60$. *b*: Elements' number $20 \times 20 \times 20$.
c: Navier-type solution. *d*: Result not provided by the theory.

Stress components $\tilde{\sigma}_{yy}$, $\tilde{\sigma}_{zz}$ and $\tilde{\sigma}_{yz}$ [Pa] for a short laminated $[0/90]$ simply supported beam. $l/b = 3$.

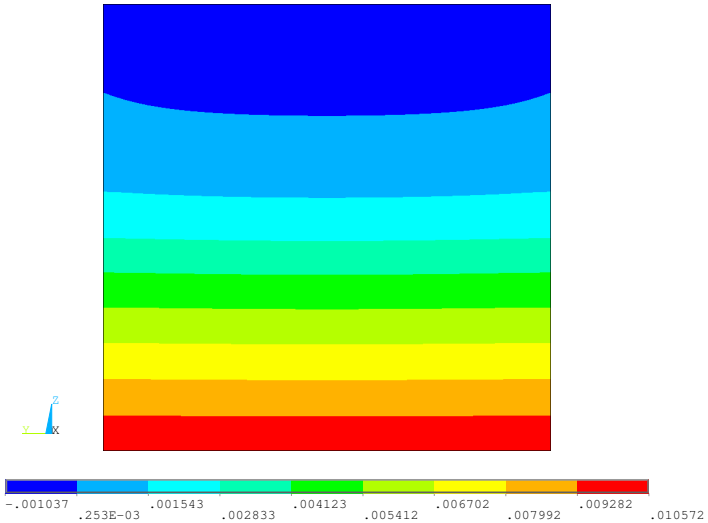
	$-10^{-7} \times \tilde{\sigma}_{yy}$			$10^{-6} \times \tilde{\sigma}_{zz}$			$-10^{-6} \times \tilde{\sigma}_{yz}$	
FEM 3D-R[a]		4.0438			5.2706			3.0341
FEM 3D-C[b]		3.9743			5.2944			3.0583
	B2	B3	B4	B2	B3	B4	B2	B3, B4
$N = 14$	4.0576	4.0581	4.0581	5.3601	5.3581	5.3581	3.0746	3.0744
$N = 11$	4.0384	4.0390	4.0389	5.4208	5.4188	5.4187	2.8618	2.8616
$N = 9$	4.2086	4.2091	4.2091	4.8464	4.8445	4.8444	3.1198	3.1197
$N = 7$	3.9584	3.9590	3.9589	4.9673	4.9653	4.9653	3.3943	3.3941
$N = 4$	3.0461	3.0467	3.0466	10.190	10.188	10.188	2.0006	2.0004
$N = 3$	4.6004	4.6009	4.6009	19.038	19.036	19.036	1.6835	1.6834
$N = 2$	10.188	10.189	10.189	20.056	20.053	20.053	0.0198	0.0198

a: Elements' number $60 \times 60 \times 60$. b: Elements' number $20 \times 20 \times 20$.

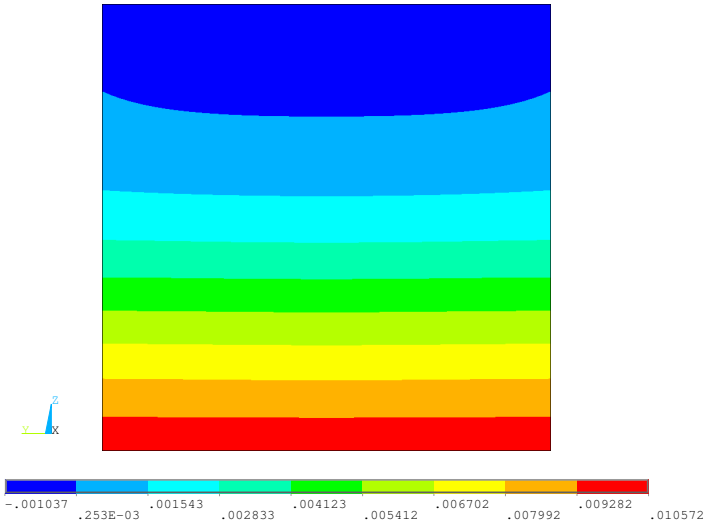
Relative differences for $N = 14$ being 1.7% at worst

Displacement cross-section variation

Axial displacement u_x [m] over the cross-section at $x = l$ for $l/b = 3$, laminated cantilever beam.



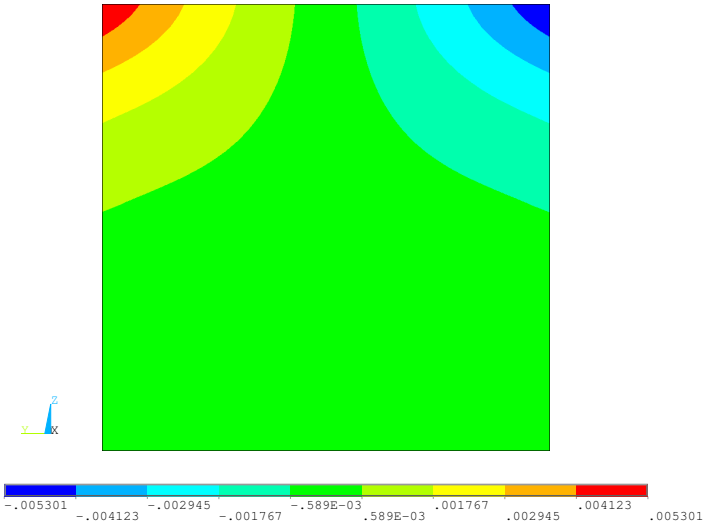
(a) FEM 3D-R



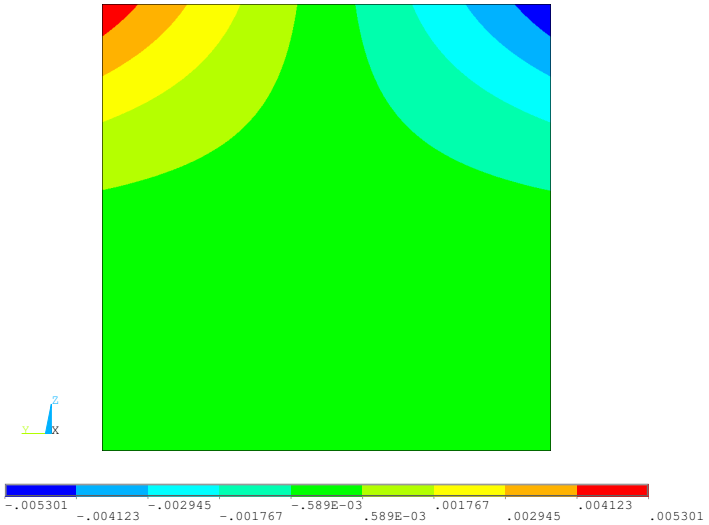
(b) N=14

Displacement cross-section variation

Through-the-width displacement u_y [m] over the cross-section at $x = l/2$ for $l/b = 3$, laminated cantilever beam.



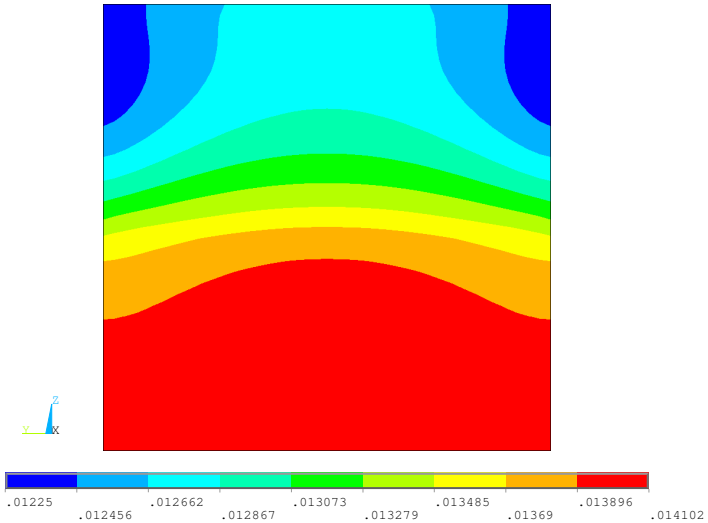
(a) FEM 3D-R



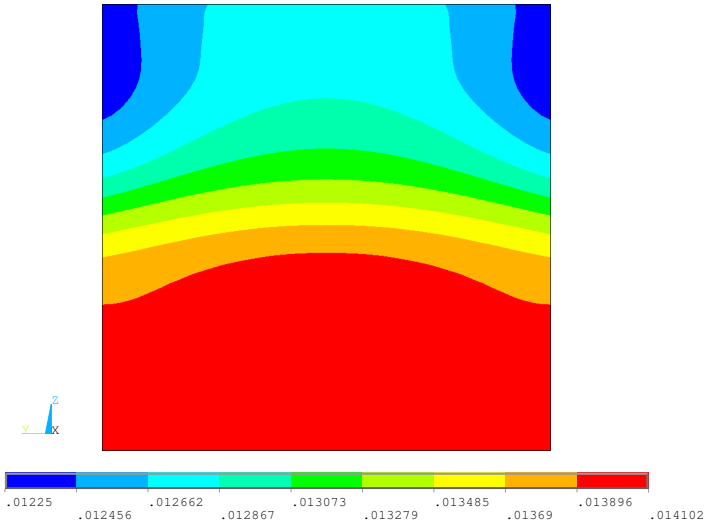
(b) N=14

Displacement cross-section variation

Through-the-thickness displacement u_z [m] over the cross-section at $x = l$ for $l/b = 3$, laminated cantilever beam.



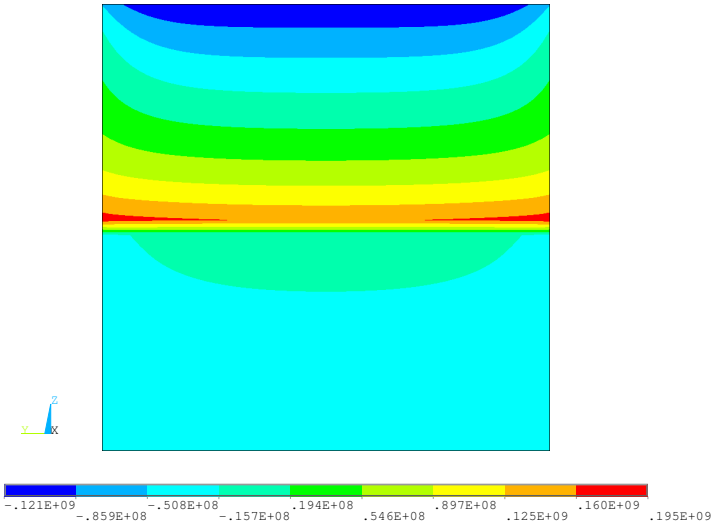
(a) FEM 3D-R



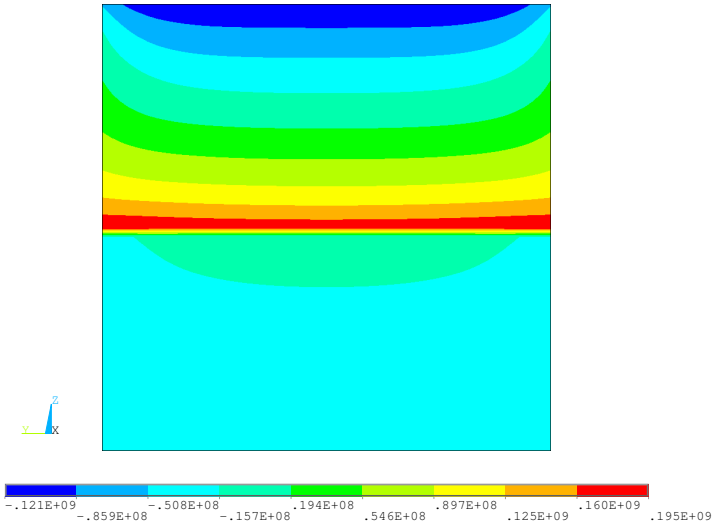
(b) N=14

Stress cross-section variation

Axial stress σ_{xx} [Pa] over the cross-section at $x = l/2$ for $l/b = 3$, laminated cantilever beam.



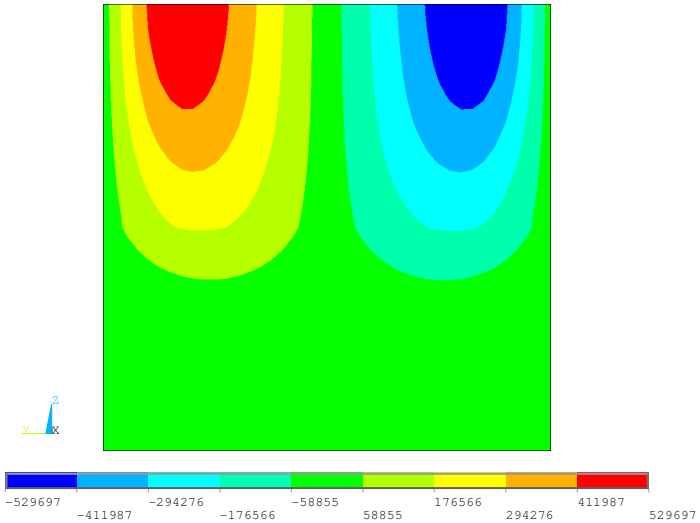
(a) FEM 3D-R



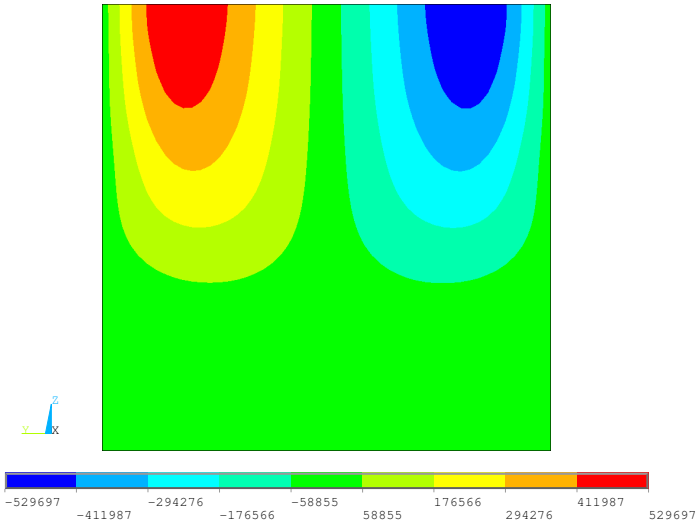
(b) N=14

Stress cross-section variation

Shear stress σ_{xy} [Pa] over the cross-section at $x/l = 2$ for $l/b = 3$, laminated cantilever beam.



(a) FEM 3D-R

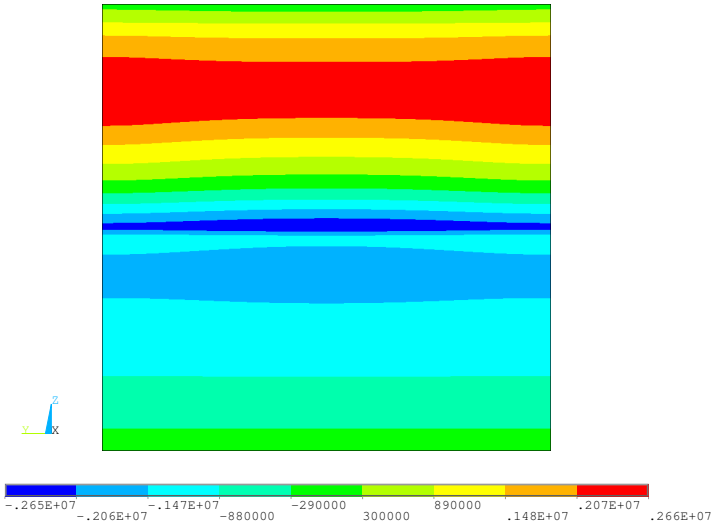


(b) N=14

Stress cross-section variation

Shear stress σ_{xz} [Pa] over the cross-section at $x/l = 2$ for $l/b = 3$, laminated cantilever beam.

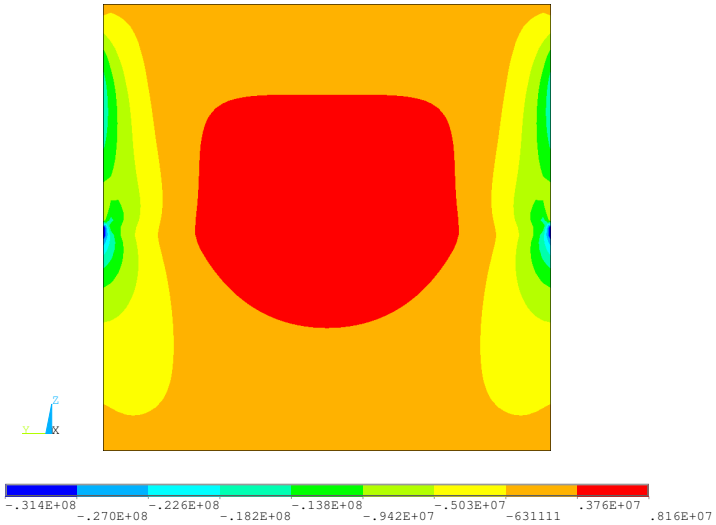
(a) FEM 3D-R



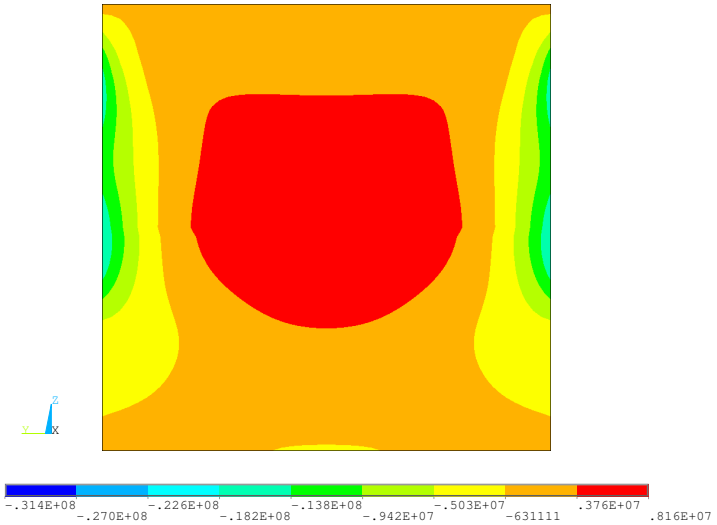
(b) N=14

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] over the cross-section at $x = l/2$ for $l/b = 3$, laminated cantilever beam.



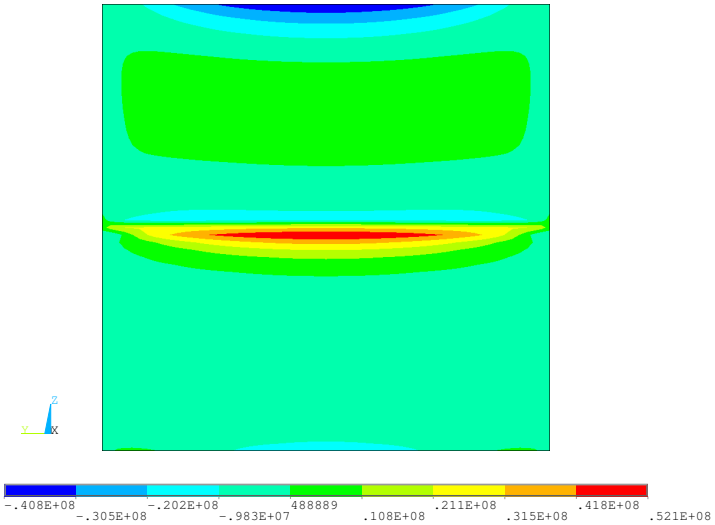
(a) FEM 3D-R



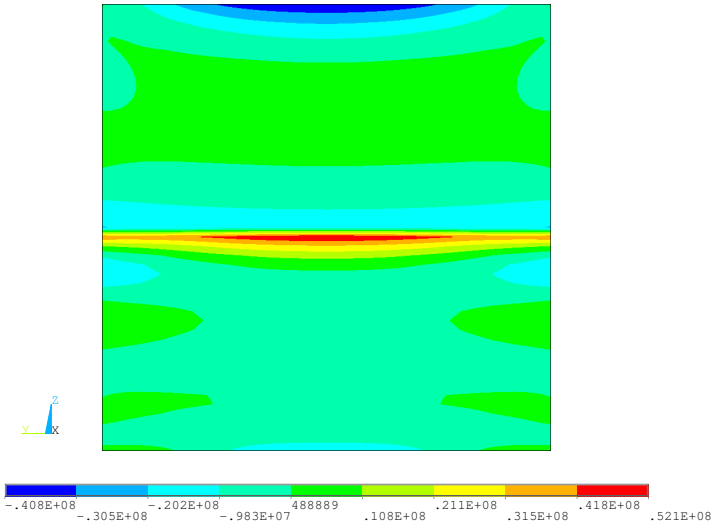
(b) N=14

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] over the cross-section at $x = l/2$ for $l/b = 3$, laminated cantilever beam.



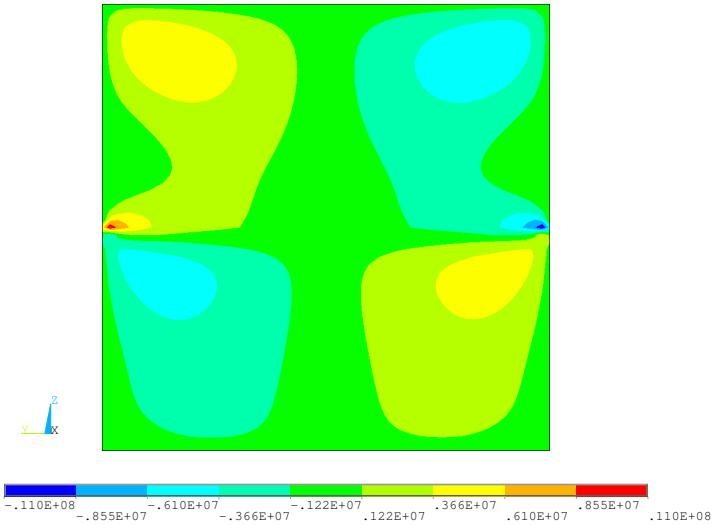
(a) FEM 3D-R



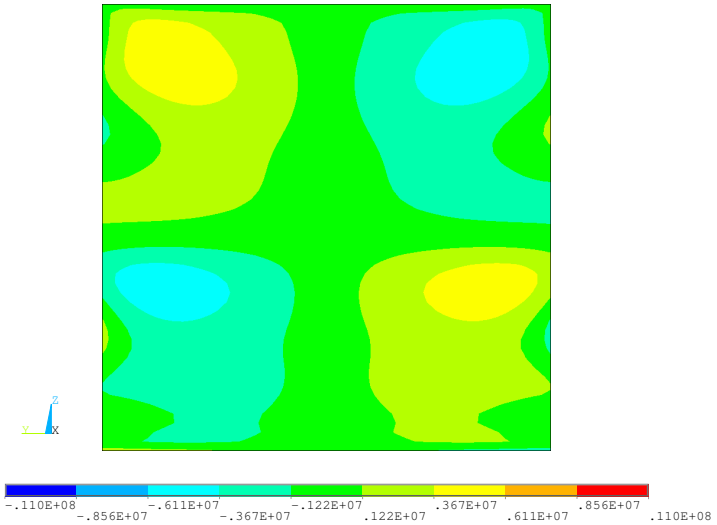
(b) N=14

Stress cross-section variation

Shear stress σ_{yz} [Pa] over the cross-section at $x = l/2$ for $l/b = 3$, laminated cantilever beam.



(a) FEM 3D-R



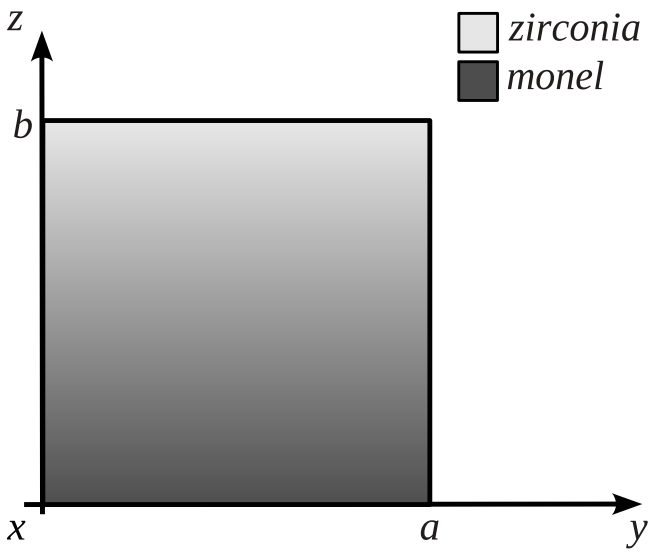
(b) N=14

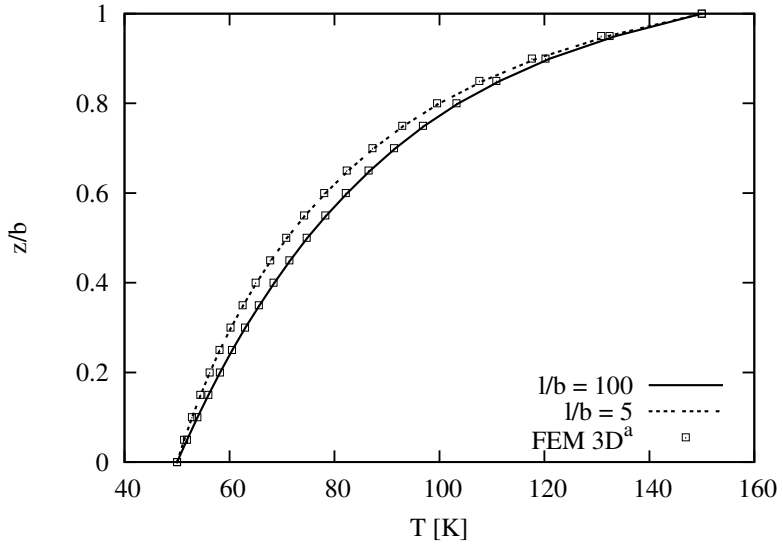
Functionally Graded (FG) Beam

$$f = (f_1 - f_2)\left(\frac{z}{b}\right)^{n_z} + f_2 \tag{3}$$

The elastic and thermal properties of the constituent materials are:

	E [GPa]	ν	K [W/mK]	α [10^{-6} K^{-1}]
Zirconia	151.01	0.300	2.09	10.
Monel	179.40	0.368	25.00	15.





Displacement components [m] for a short FG simply supported beam, $l/b = 5$, $n_z = 1$.

	$10^3 \times \overline{u}_z$		$-10^3 \times \overline{u}_x$		$10^4 \times \overline{u}_y$		
FEM 3D-R[a]	1.4917		1.9976		6.9765		
FEM 3D-C[b]	1.5089		2.0064		7.0247		
TBT[c]	1.4956		1.9809		0.0000		
EBT[c]	1.4957		1.9808		0.0000		
	B2	B3, B4	B2	B3, B4	B2	B3	B4
$N = 13$	1.5160	1.5161	2.0107	2.0107	7.0261	7.0255	7.0256
$N = 10$	1.5160	1.5161	2.0106	2.0107	7.0203	7.0197	7.0197
$N = 7$	1.5160	1.5161	2.0111	2.0112	6.9947	6.9942	6.9942
$N = 4$	1.5158	1.5159	2.0066	2.0066	6.8768	6.8763	6.8763
$N = 2$	1.4553	1.4554	2.0030	2.0030	6.1463	6.1458	6.1458

a: Elements' number $60 \times 60 \times 60$. b: Elements' number $20 \times 20 \times 20$.
c: Navier-type solution.

Relative differences for $N \geq 4$ being 1.6% at worst

Stress components $\tilde{\sigma}_{xx}$, $\tilde{\sigma}_{xy}$ and $\tilde{\sigma}_{xz}$ [Pa] for a FG simply supported beam, $l/b = 5$, $n_z = 1$.

	$10^{-6} \times \overline{\sigma}_{xx}$			$10^{-5} \times \overline{\sigma}_{xy}$			-$10^{-6} \times \overline{\sigma}_{xz}$		
FEM 3D-R[a]		8.5121			9.6899			2.9668	
FEM 3D-C[b]		8.7013			9.8166			2.9902	
TBT[c]		10.245			0.0000			0.0000	
EBT[c]		10.233			–[d]			–[d]	
	B2	B3	B4	B2	B3	B4	B2	B3	B4
$N = 13$	8.7748	8.8411	8.7890	9.8564	9.8253	9.8426	3.0193	2.9964	3.0093
$N = 11$	8.8734	8.9397	8.8876	9.7936	9.7625	9.7798	3.0138	2.9909	3.0038
$N = 9$	8.8863	8.9524	8.9005	9.8572	9.8262	9.8435	2.9891	2.9663	2.9791
$N = 7$	8.8519	8.9186	8.8662	10.057	10.026	10.043	3.0873	3.0645	3.0773
$N = 4$	6.3049	6.3717	6.3193	1.8705	1.8403	1.8576	2.8007	2.7779	2.7907
$N = 2$	22.250	22.316	22.264	25.875	25.844	25.860	2.8220	2.8002	2.8125

a: Elements' number $60 \times 60 \times 60$. *b*: Elements' number $20 \times 20 \times 20$.
c: Navier-type solution. *d*: Result not provided by the theory.

Stress components $\overline{\sigma}_{yy}$, $\overline{\sigma}_{zz}$ and $\overline{\sigma}_{yz}$ [Pa] for a FG simply supported beam, $l/b = 5$, $n_z = 1$.

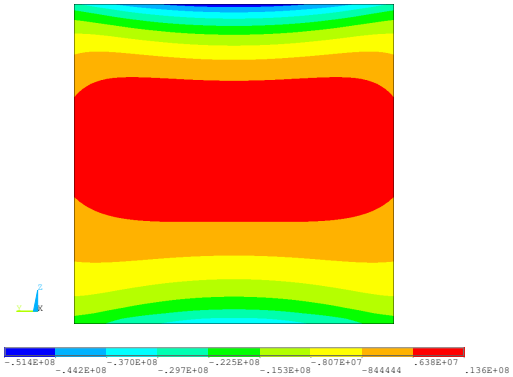
	$10^{-6} \times \overline{\sigma}_{yy}$			$10^{-6} \times \overline{\sigma}_{zz}$			$-10^{-6} \times \overline{\sigma}_{yz}$	
FEM 3D-R[a]		4.7891			6.3622			4.0031
FEM 3D-C[b]		4.8664			6.4933			4.1017
	B2	B3	B4	B2	B3	B4	B2	B3, B4
$N = 13$	4.9353	4.9490	4.9229	6.4851	6.4987	6.4726	4.0826	4.0824
$N = 11$	5.0380	5.0518	5.0256	6.6771	6.6909	6.6646	4.0817	4.0815
$N = 9$	5.0549	5.0684	5.0426	6.6987	6.7121	6.6863	4.0294	4.0292
$N = 7$	4.9814	4.9957	4.9690	6.6642	6.6784	6.6518	3.9214	3.9212
$N = 4$	1.9721	1.9866	1.9599	1.9702	1.9846	1.9580	1.7367	1.7366
$N = 2$	26.398	26.411	26.385	26.845	26.857	26.832	0.0552	0.0552

a: Elements' number $60 \times 60 \times 60$. b: Elements' number $20 \times 20 \times 20$.

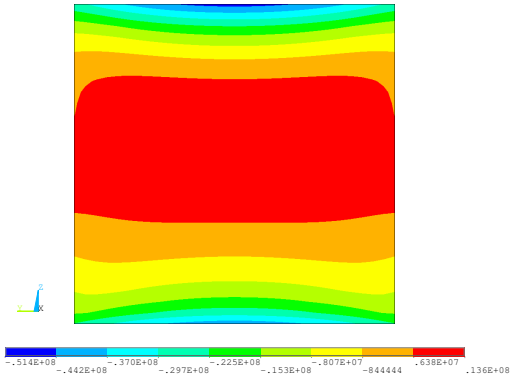
Relative differences for $N = 13$, $B4$ being 3.3% at worst

Stress cross-section variation

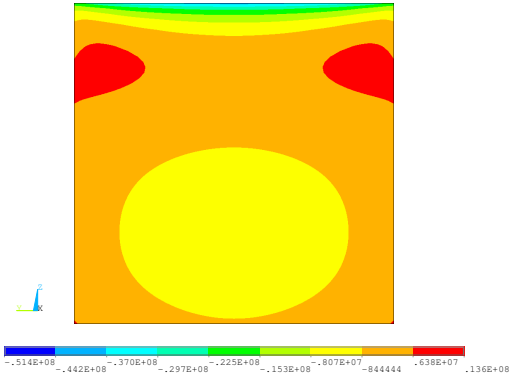
Axial stress σ_{xx} [Pa] at $x/l = 1/2$, cantilever FG beam, $l/b = 5$.



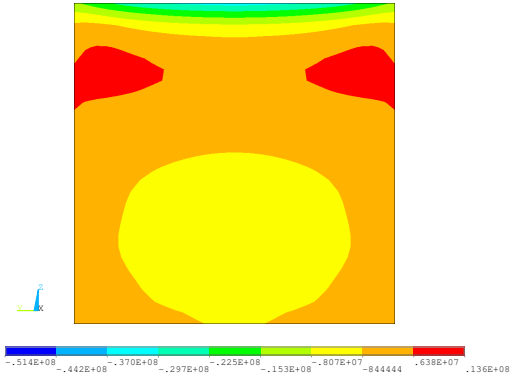
(a) FEM 3D^a, $n_z = 0.5$



(b) N=13, $n_z = 0.5$



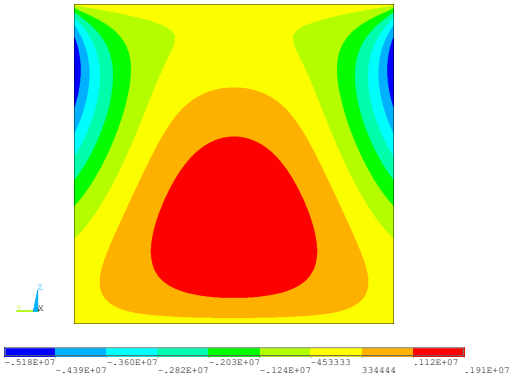
(c) FEM 3D^a, $n_z = 2$



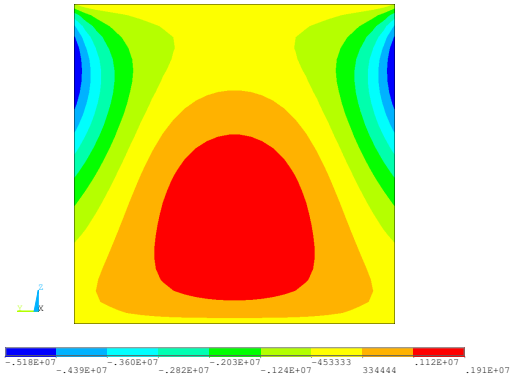
(d) N=13, $n_z = 2$

Stress cross-section variation

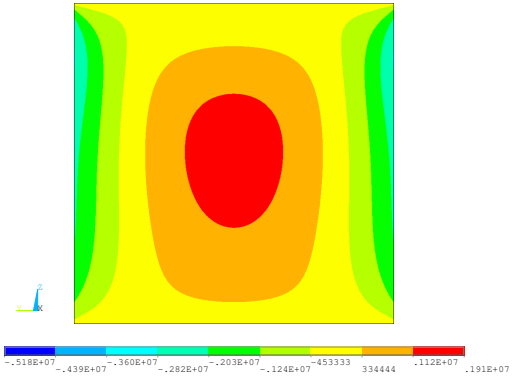
Shear stress σ_{xz} [Pa] at $x/l = 1/4$, cantilever FG beam, $l/b = 5$.



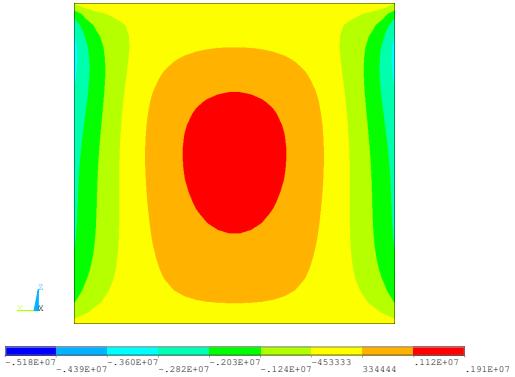
(a) FEM 3D^a, $n_z = 0.5$



(b) N=13, $n_z = 0.5$



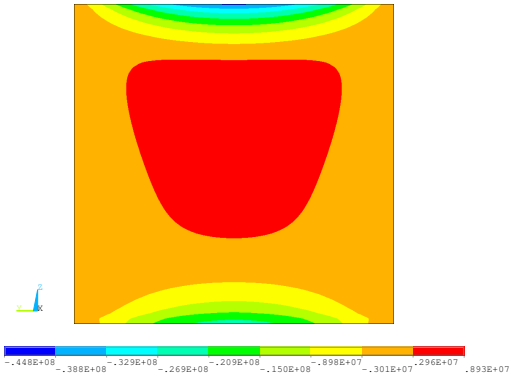
(c) FEM 3D^a, $n_z = 2$



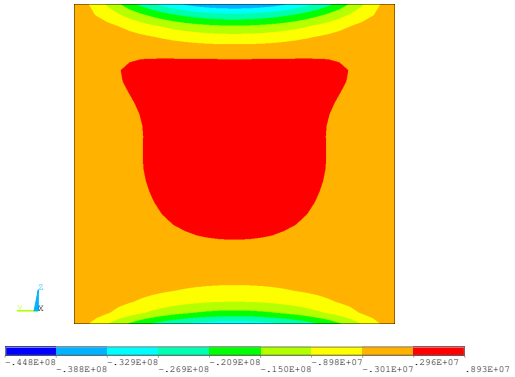
(d) N=13, $n_z = 2$

Stress cross-section variation

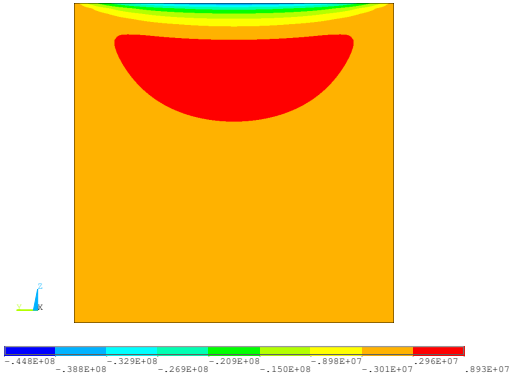
Through-the-width normal stress σ_{yy} [Pa] at $x/l = 1/2$, cantilever FG beam, $l/b = 5$.



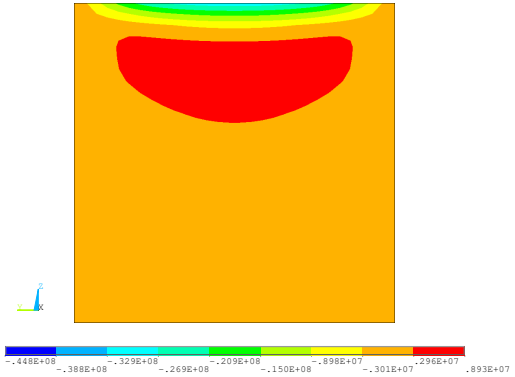
(a) FEM 3D^a, $n_z = 0.5$



(b) N=13, $n_z = 0.5$



(c) FEM 3D^a, $n_z = 2$



(d) N=13, $n_z = 2$

Conclusions

	Laminated		FG	
	Simply supported	Cantilever	Simply supported	Cantilever
u	0.6% ($N \geq 9$)	0.4% ($N \geq 9$, $B4$)	1.6% ($N \geq 4$)	0.4% ($N \geq 3$)
σ	1.7% ($N = 14$)	1.9% ($N = 14$)	3.3% ($N = 13$, $B4$)	3.3% ($N = 13$, $B4$)

Model	DOFs
FEM-$3D^a$	$2.7 * 10^6$
FEM-$3D^b$	$1.1 * 10^5$
$N = 14$	$4.4 * 10^4$
$N = 13$	$3.8 * 10^4$
$N = 9$	$2.0 * 10^4$
$N = 7$	$1.3 * 10^4$
$N = 4$	$5.4 * 10^3$
$N = 3$	$3.6 * 10^3$

Hierarchical one-dimensional finite elements computational cost:

$$DOFs = 3 \cdot \frac{(N+1)(N+2)}{2} \cdot N_n$$

with N the order of the beam theory and N_n the number of total nodes.

- A unified formulation for one-dimensional beam finite elements has been presented for the thermal stress analysis.
- The numerical investigation and validation showed that the proposed formulation allows obtaining accurate results for all the considered cases with reduced the computational costs when compared to three-dimensional FEM solutions.

Acknowledgements
This work has been carried out within the FULLCOMP project funded by the European Union's Horizon's 2020 research and innovation programme under grant agreement No 642121.

Many thanks for your kind attention!

Laminated beam. Simply supported case.

As far as displacements and stresses are concerned, they have been evaluated at the following points:

$$\begin{aligned}
\tilde{u}_x &= u_x\left(0, -\frac{a}{2}, -\frac{b}{2}\right) & \tilde{u}_y &= u_y\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) & \tilde{u}_z &= u_z\left(\frac{l}{2}, 0, -\frac{b}{2}\right) \\
\tilde{\sigma}_{xx} &= \sigma_{xx}\left(\frac{l}{2}, 0, \frac{b}{2}\right) & \tilde{\sigma}_{xz} &= \sigma_{xz}\left(0, -\frac{a}{2}, \frac{b}{4}\right) & \tilde{\sigma}_{xy} &= \sigma_{xy}\left(0, \frac{a}{4}, \frac{b}{2}\right) \\
\tilde{\sigma}_{zz} &= \sigma_{zz}\left(\frac{l}{2}, 0, \frac{b}{4}\right) & \tilde{\sigma}_{yy} &= \sigma_{yy}\left(\frac{l}{2}, 0, \frac{b}{2}\right) & \tilde{\sigma}_{yz} &= \sigma_{yz}\left(\frac{l}{2}, -\frac{a}{4}, \frac{b}{4}\right)
\end{aligned} \tag{4}$$

FGM beam. Simply supported case.

As far as displacements and stresses are concerned, they have been evaluated at the following points:

$$\begin{aligned}
\tilde{u}_x &= u_x\left(0, \frac{a}{2}, b\right) & \tilde{u}_y &= u_y\left(\frac{l}{2}, a, b\right) & \tilde{u}_z &= u_z\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) \\
\tilde{\sigma}_{xx} &= \sigma_{xx}\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) & \tilde{\sigma}_{xy} &= \sigma_{zz}\left(0, \frac{a}{4}, 0\right) & \tilde{\sigma}_{xz} &= \sigma_{xz}\left(0, 0, \frac{b}{2}\right) \\
\tilde{\sigma}_{yy} &= \sigma_{yy}\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) & \tilde{\sigma}_{zz} &= \sigma_{zz}\left(\frac{l}{2}, \frac{a}{2}, \frac{b}{2}\right) & \tilde{\sigma}_{yz} &= \sigma_{yz}\left(\frac{l}{2}, \frac{a}{4}, \frac{3}{4}b\right)
\end{aligned} \tag{5}$$

Geometric Relations

In the case of small displacements with respect to a characteristic dimension of Ω, linear relations between strain and displacement components hold:

$$\boldsymbol{\varepsilon}_n = \mathbf{D}_{np}\mathbf{u} + \mathbf{D}_{nx}\mathbf{u}$$
$$\boldsymbol{\varepsilon}_p = \mathbf{D}_p\mathbf{u}$$

Strain components have been grouped into vectors $\boldsymbol{\varepsilon}_n$ that lay on the cross-section and $\boldsymbol{\varepsilon}_p$ laying on planes orthogonal to Ω.

$\mathbf{D}_{np}$, $\mathbf{D}_{nx}$, and $\mathbf{D}_p$ are the following differential matrix operators:

$$\mathbf{D}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{\partial}{\partial y} & 0 & 0 \\ \dfrac{\partial}{\partial z} & 0 & 0 \end{bmatrix} \quad \mathbf{D}_{nx} = \mathbf{I}\frac{\partial}{\partial x} \quad \mathbf{D}_p = \begin{bmatrix} 0 & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \end{bmatrix}$$

$\mathbf{I}$ is the unit matrix.

Constitutive Relations

In the case of thermo-mechanical problems, Hooke's law reads:

$$\boldsymbol{\sigma} = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_e = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}_\vartheta\right) = \tilde{\mathbf{C}}\left(\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\alpha}}T\right) = \tilde{\mathbf{C}}\boldsymbol{\varepsilon}_t - \tilde{\boldsymbol{\lambda}}T$$

where subscripts 'e' and 'ϑ' refer to the elastic and the thermal contributions, respectively.

- $\tilde{\mathbf{C}}$ is the material elastic stiffness,
- $\tilde{\boldsymbol{\alpha}}$ the vector of the thermal expansion coefficients,
- $\tilde{\boldsymbol{\lambda}}$ their product and
- T stands for temperature.

According to the stress and strain vectors splitting, the previous equation becomes:

$$\begin{aligned}\boldsymbol{\sigma}_p &= \tilde{\mathbf{C}}_{pp}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{pn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_p T \\ \boldsymbol{\sigma}_n &= \tilde{\mathbf{C}}_{np}\boldsymbol{\varepsilon}_{tp} + \tilde{\mathbf{C}}_{nn}\boldsymbol{\varepsilon}_{tn} - \tilde{\boldsymbol{\lambda}}_n T\end{aligned}$$

Matrices $\tilde{\mathbf{C}}_{pp}$, $\tilde{\mathbf{C}}_{pn}$, $\tilde{\mathbf{C}}_{np}$ and $\tilde{\mathbf{C}}_{nn}$ are:

$$\tilde{\mathbf{C}}_{pp} = \begin{bmatrix} \tilde{C}_{22} & \tilde{C}_{23} & 0 \\ \tilde{C}_{23} & \tilde{C}_{33} & 0 \\ 0 & 0 & \tilde{C}_{44} \end{bmatrix} \quad \tilde{\mathbf{C}}_{pn} = \tilde{\mathbf{C}}_{np}^T = \begin{bmatrix} \tilde{C}_{12} & \tilde{C}_{26} & 0 \\ \tilde{C}_{13} & \tilde{C}_{36} & 0 \\ 0 & 0 & \tilde{C}_{45} \end{bmatrix}$$

$$\tilde{\mathbf{C}}_{nn} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{16} & 0 \\ \tilde{C}_{16} & \tilde{C}_{66} & 0 \\ 0 & 0 & \tilde{C}_{55} \end{bmatrix}$$

The coefficients $\tilde{\boldsymbol{\lambda}}_n$ and $\tilde{\boldsymbol{\lambda}}_p$:

$$\tilde{\boldsymbol{\lambda}}_n^T = \left\{ \begin{array}{ccc} \tilde{\lambda}_1 & \tilde{\lambda}_6 & 0 \end{array} \right\} \quad \tilde{\boldsymbol{\lambda}}_p^T = \left\{ \begin{array}{ccc} \tilde{\lambda}_2 & \tilde{\lambda}_3 & 0 \end{array} \right\}$$

are related to the thermal expansion coefficients $\tilde{\boldsymbol{\alpha}}_n$ and $\tilde{\boldsymbol{\alpha}}_p$:

$$\tilde{\boldsymbol{\alpha}}_n^T = \left\{ \begin{array}{ccc} \tilde{\alpha}_1 & 0 & 0 \end{array} \right\} \quad \tilde{\boldsymbol{\alpha}}_p^T = \left\{ \begin{array}{ccc} \tilde{\alpha}_2 & \tilde{\alpha}_3 & 0 \end{array} \right\}$$

through the following equations:

$$\begin{aligned} \tilde{\boldsymbol{\lambda}}_p &= \tilde{\mathbf{C}}_{pp}\tilde{\boldsymbol{\alpha}}_p + \tilde{\mathbf{C}}_{pn}\tilde{\boldsymbol{\alpha}}_n \\ \tilde{\boldsymbol{\lambda}}_n &= \tilde{\mathbf{C}}_{np}\tilde{\boldsymbol{\alpha}}_p + \tilde{\mathbf{C}}_{nn}\tilde{\boldsymbol{\alpha}}_n \end{aligned}$$

Fourier's Heat Conduction Equation

Fourier's heat conduction equation for the k-th layer of the beam holds:

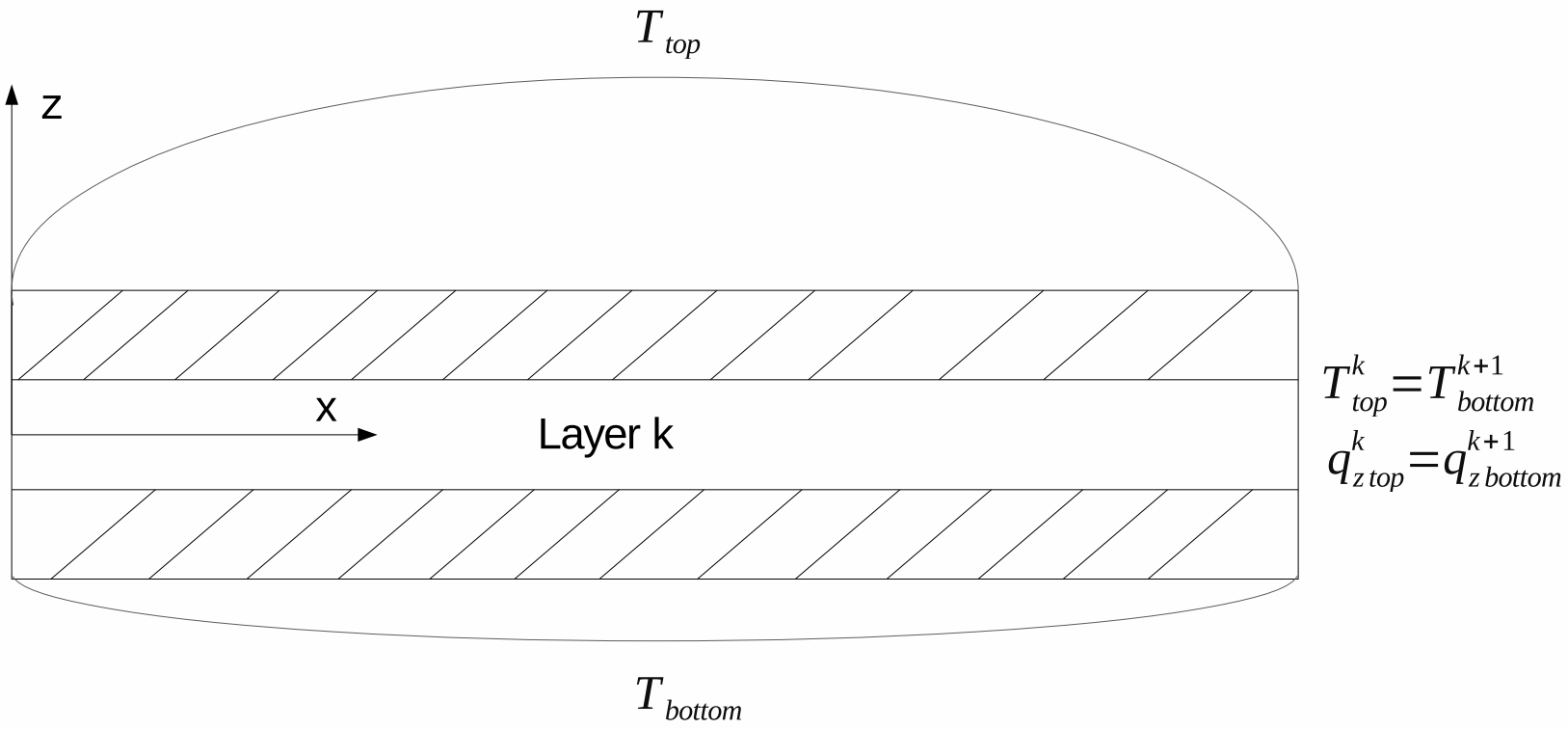
$$\tilde{K}_1^k \frac{\partial^2 T^k}{\partial x^2} + \tilde{K}_2^k \frac{\partial^2 T^k}{\partial y^2} + \tilde{K}_3^k \frac{\partial^2 T^k}{\partial z^2} = 0$$

where $\tilde{K}_i^k$ are the thermal conductivity coefficients.
In order to obtain a closed form analytical solution, it is further assumed that the temperature does not depend upon the through-the-width co-ordinate y. The following temperature field:

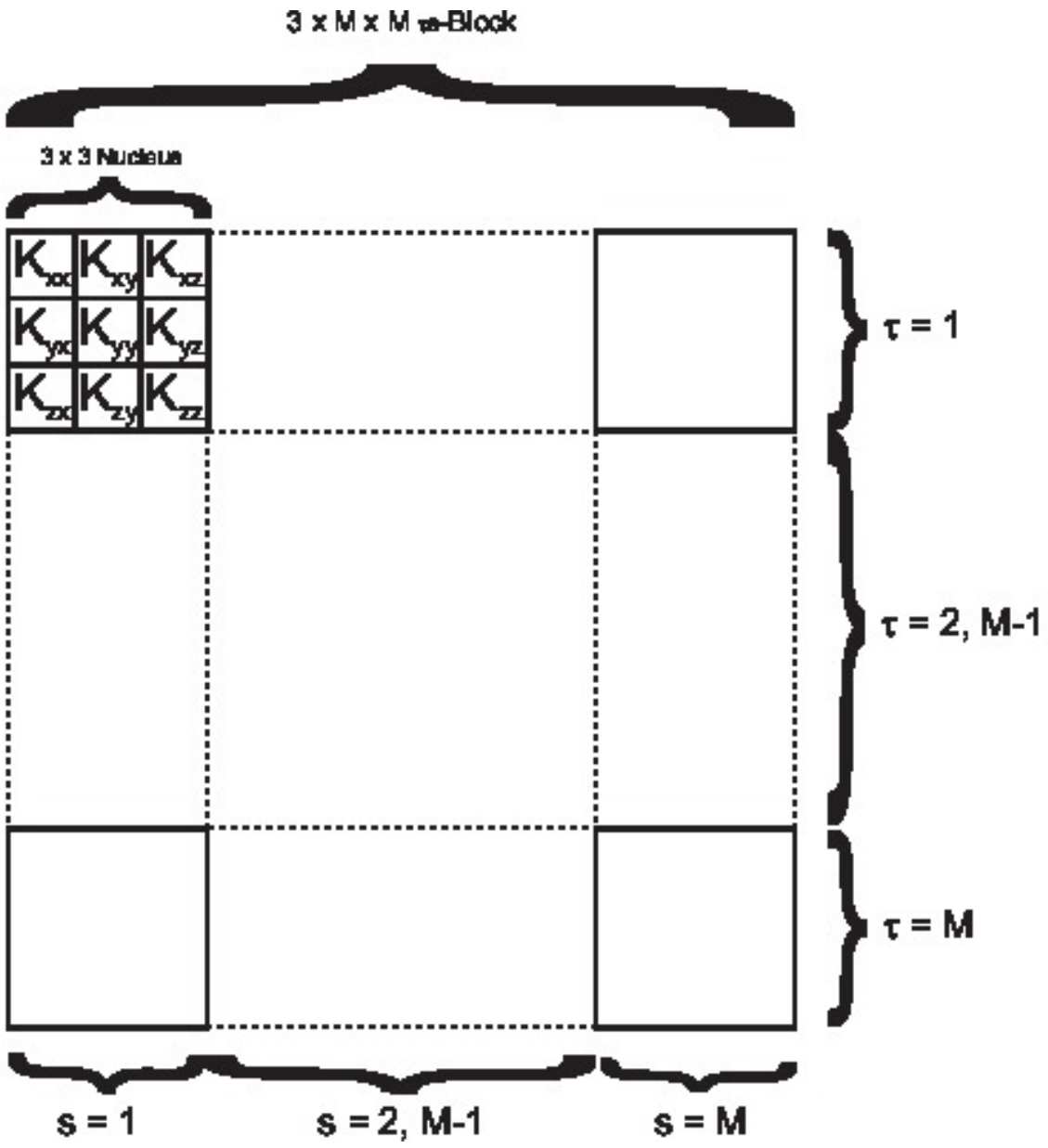
$$T^k(x,z) = \Theta_\Omega^k(z)\,\Theta_n(x) = \left(\bar{T}_1^k e^{s_1^k z} + \bar{T}_2^k e^{s_2^k z}\right)\sin(\alpha x)$$

represents a solution of the considered heat conduction problem. $\bar{T}^k$ are unknown constants, whereas s is:

$$s_{1,2}^k = \pm\sqrt{\frac{K_1^k}{K_3^k}}\,\alpha$$

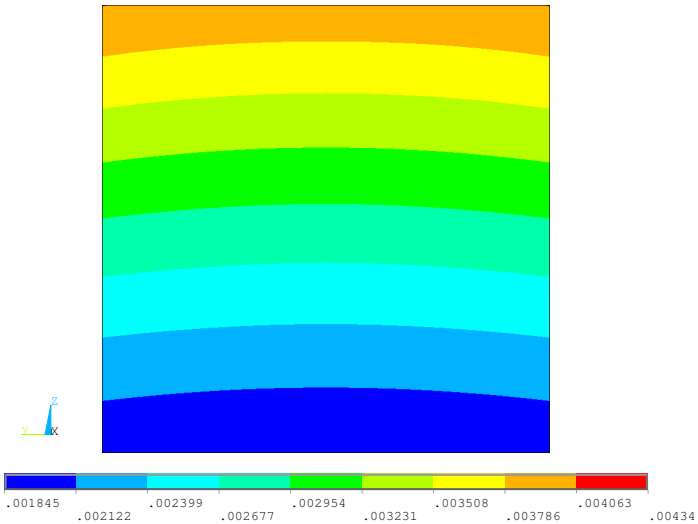


For a cross-section division into N_{Ω^k} sub-domains, $2 \cdot N_{\Omega^k}$ unknowns $\bar{T}^k_j$ are present. The problem is mathematically well posed since the boundary conditions yield a linear algebraic system of $2 \cdot N_{\Omega^k}$ equations in $\bar{T}^k_j$.

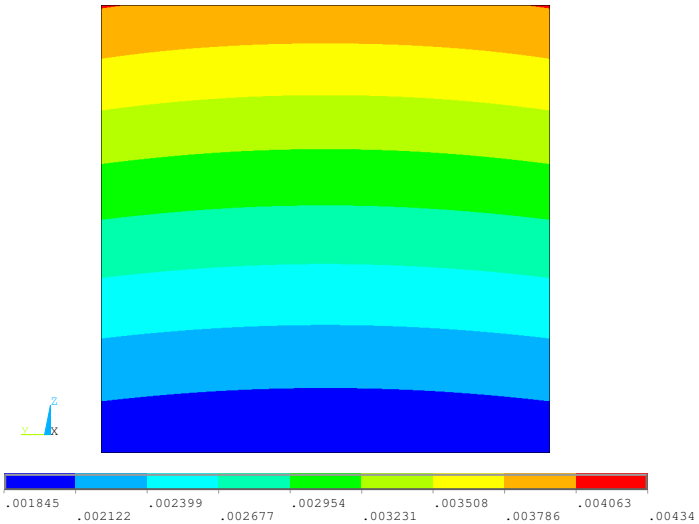

3 x M x M τs-Block
3 x 3 Nucleus
Kxx Kxy Kxz
Kyx Kyy Kyz
Kzx Kzy Kzz
τ = 1
τ = 2, M-1
τ = M
s = 1
s = 2, M-1
s = M

Displacement cross-section variation

Axial displacement u_x [m] at $x/l = 1$, cantilever FGM beam, $l/b = 5$.



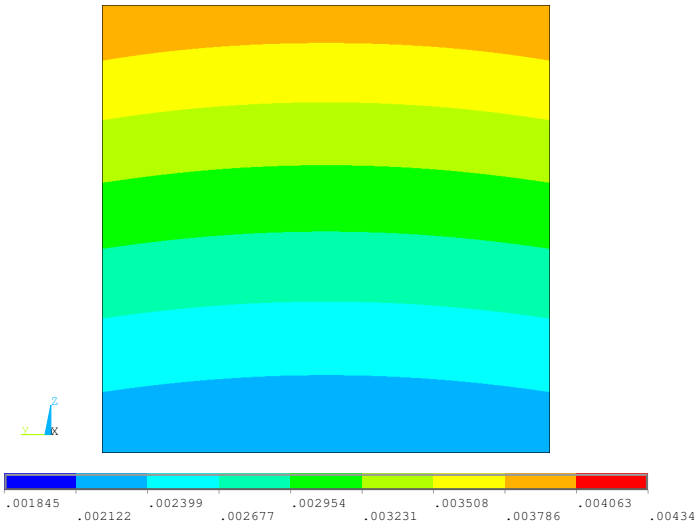
(a) FEM 3D^a, $n_z = 0.5$



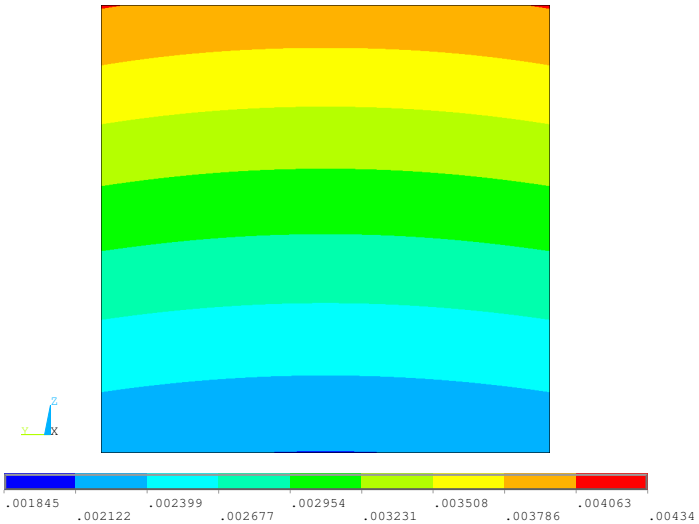
(b) N=13, $n_z = 0.5$

Displacement cross-section variation

Axial displacement u_x [m] at $x/l = 1$, cantilever FGM beam, $l/b = 5$.



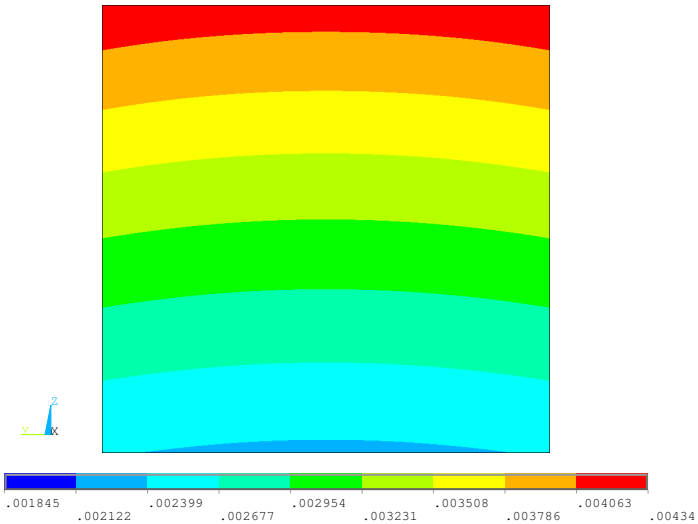
(a) FEM 3D^a, $n_z = 1$



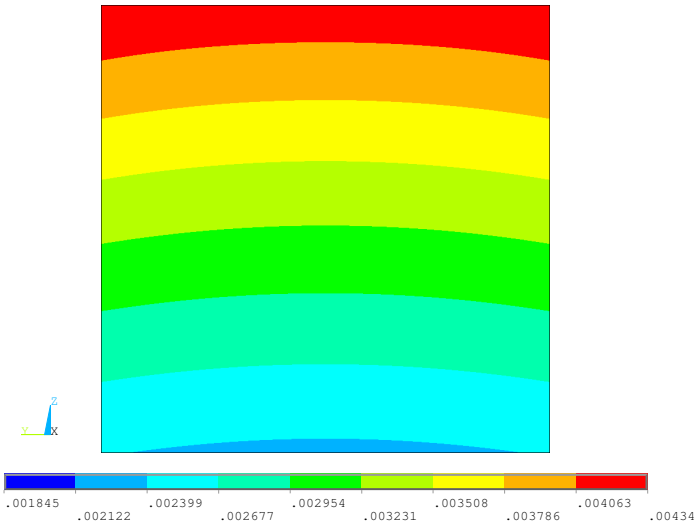
(b) N=13, $n_z = 1$

Displacement cross-section variation

Axial displacement u_x [m] at $x/l = 1$, cantilever FGM beam, $l/b = 5$.



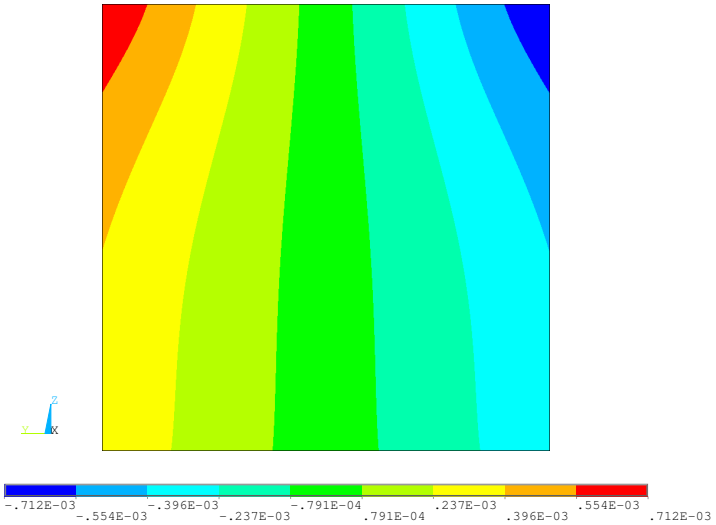
(a) FEM 3D^a, $n_z = 2$



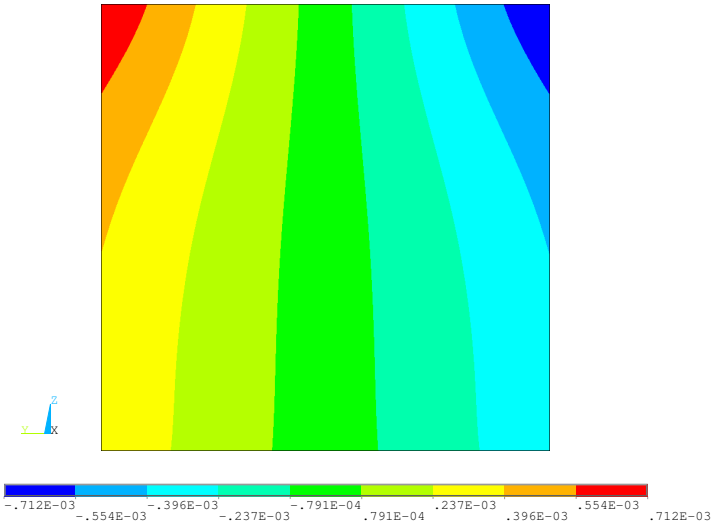
(b) N=13, $n_z = 2$

Displacement cross-section variation

Through-the-width displacement u_y [m] at $x/l = 1/2$, cantilever FGM beam. $l/b = 5$.



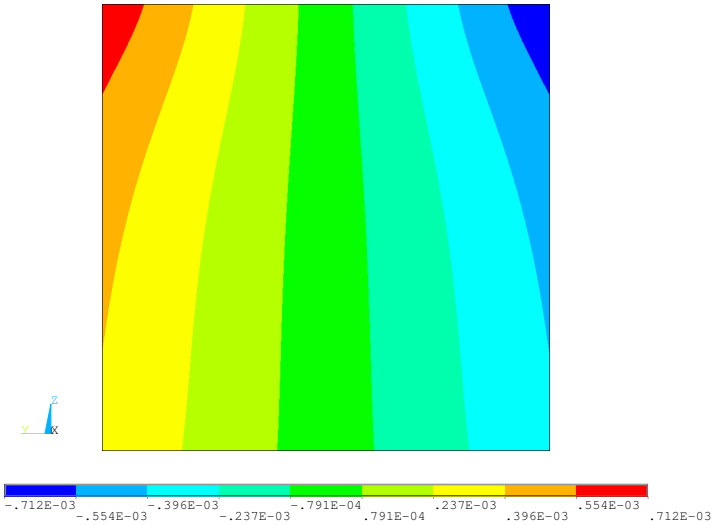
(a) FEM 3D^a, $n_z = 0.5$



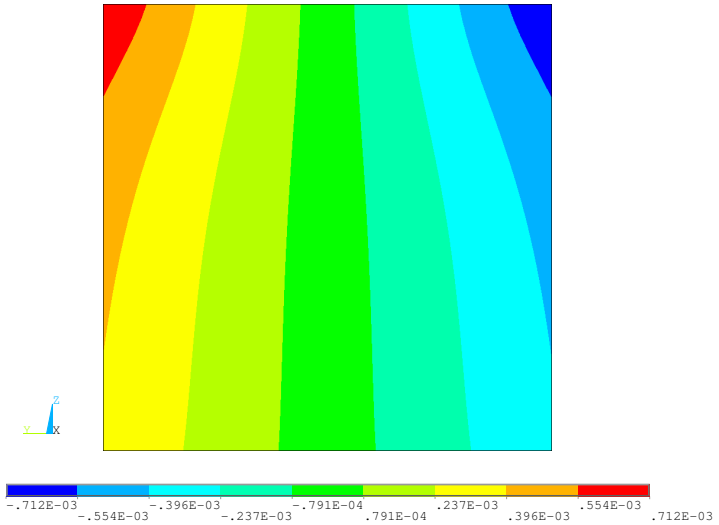
(b) N=13, $n_z = 0.5$

Displacement cross-section variation

Through-the-width displacement u_y [m] at $x/l = 1/2$, cantilever FGM beam. $l/b = 5$.



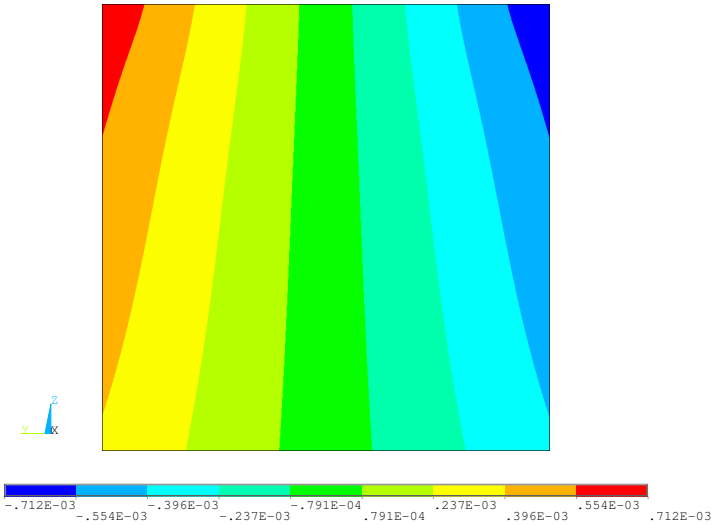
(a) FEM 3D^a, $n_z = 1$



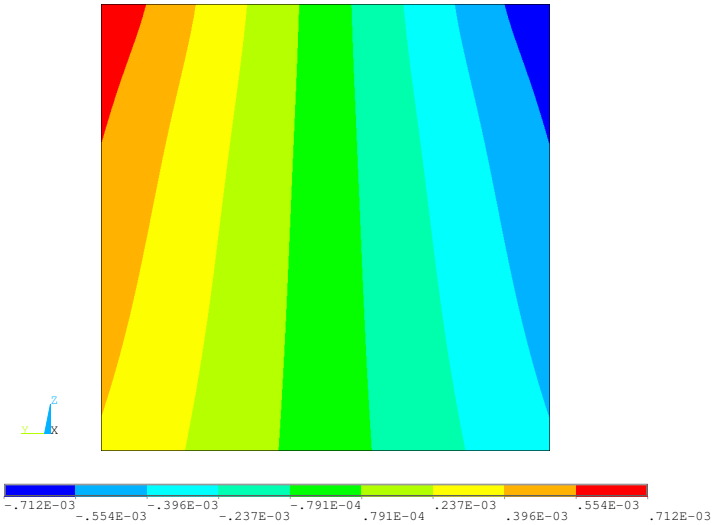
(b) N=13, $n_z = 1$

Displacement cross-section variation

Through-the-width displacement u_y [m] at $x/l = 1/2$, cantilever FGM beam. $l/b = 5$.



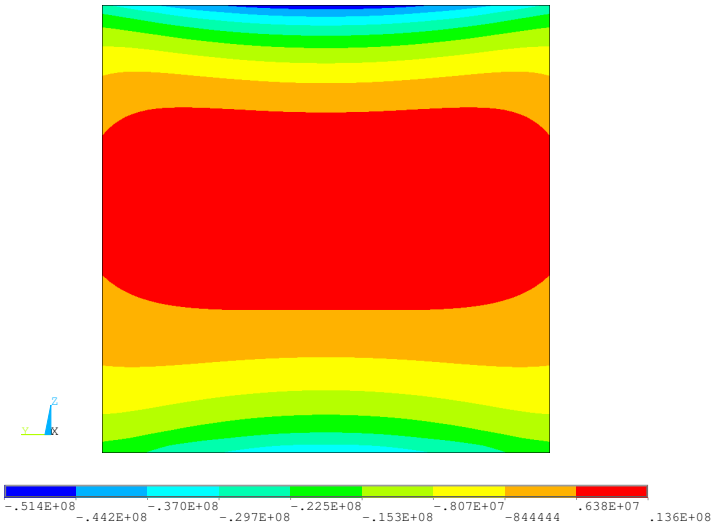
(a) FEM 3D^a, $n_z = 2$



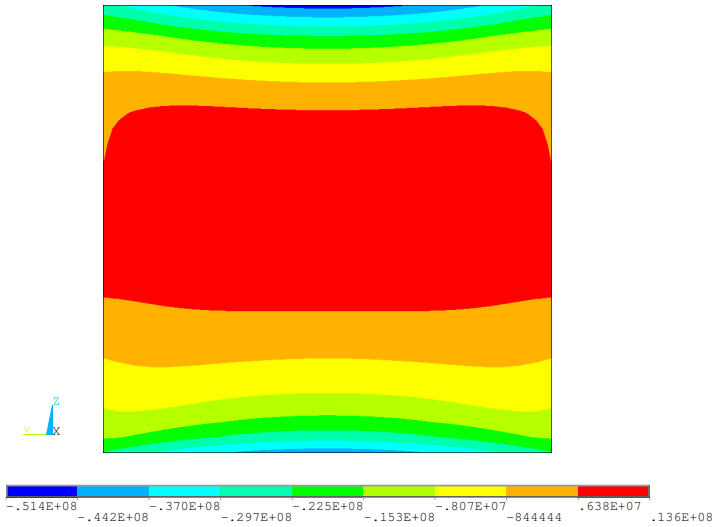
(b) N=13, $n_z = 2$

Stress cross-section variation

Axial stress σ_{xx} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



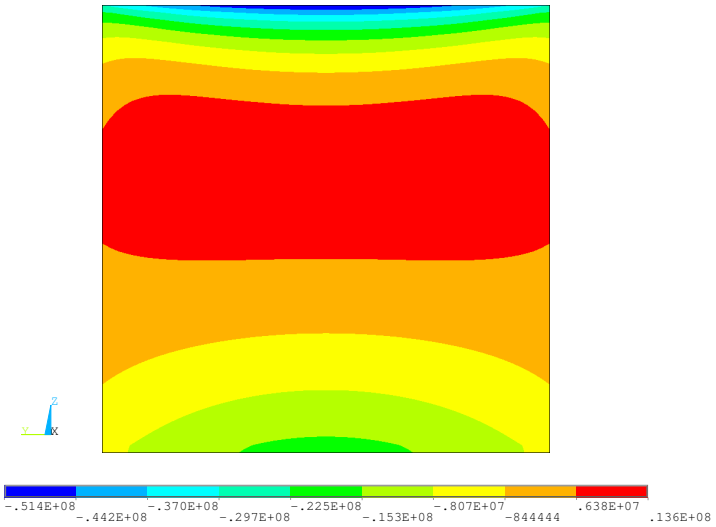
(a) FEM 3D^a, $n_z = 0.5$



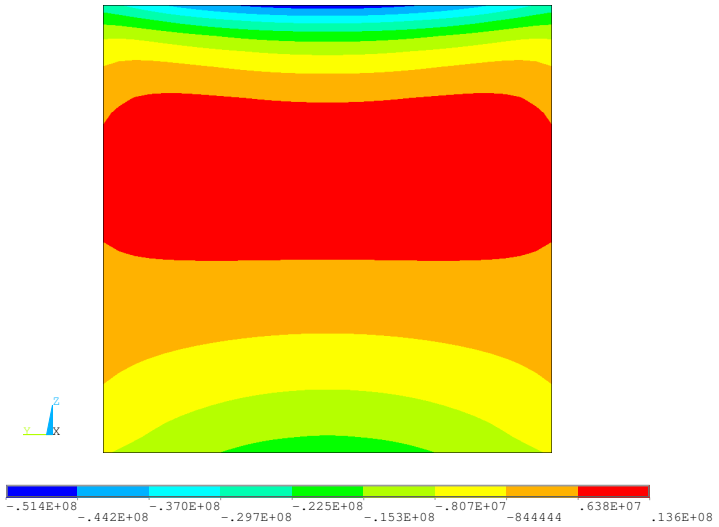
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Axial stress σ_{xx} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



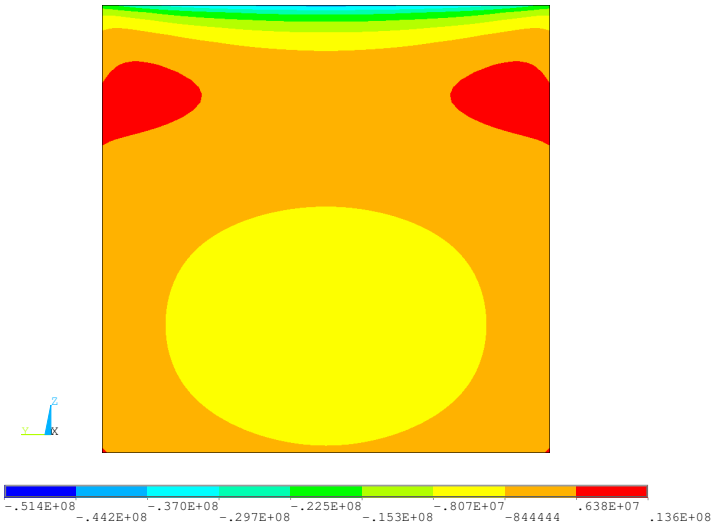
(a) FEM 3D^a, $n_z = 1$



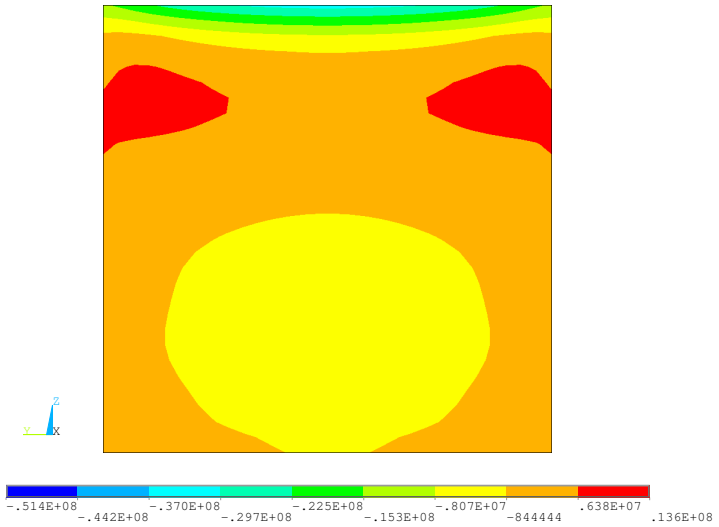
(b) N=13, $n_z = 1$

Stress cross-section variation

Axial stress σ_{xx} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



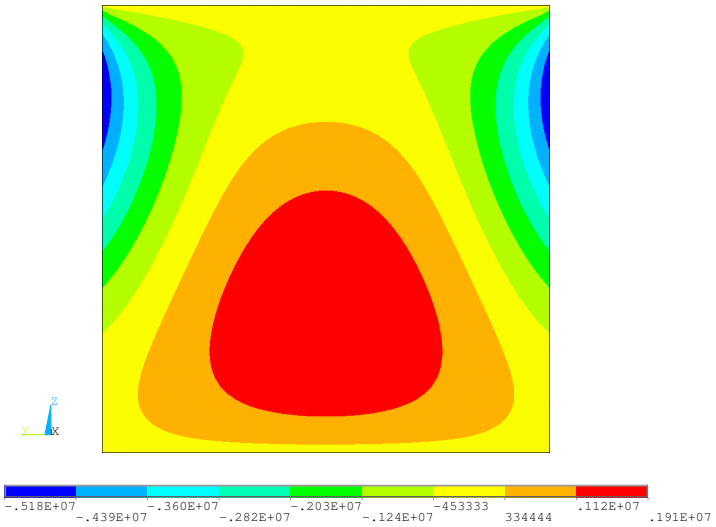
(a) FEM 3D^a, $n_z = 2$



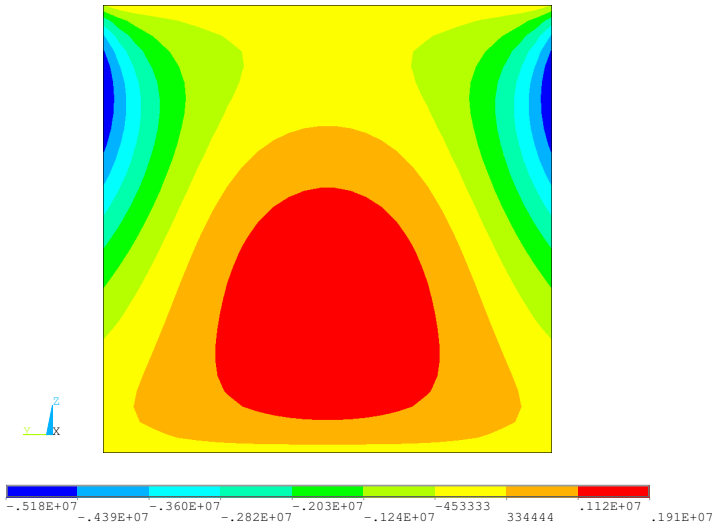
(b) N=13, $n_z = 2$

Stress cross-section variation

Shear stress σ_{xz} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



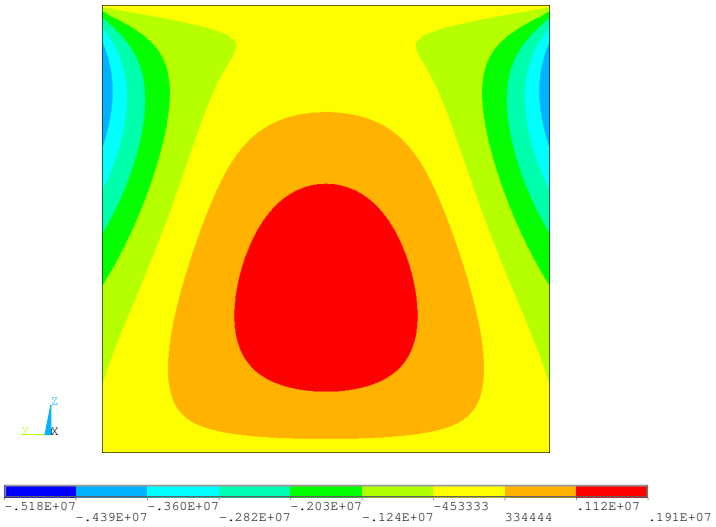
(a) FEM 3D^a, $n_z = 0.5$



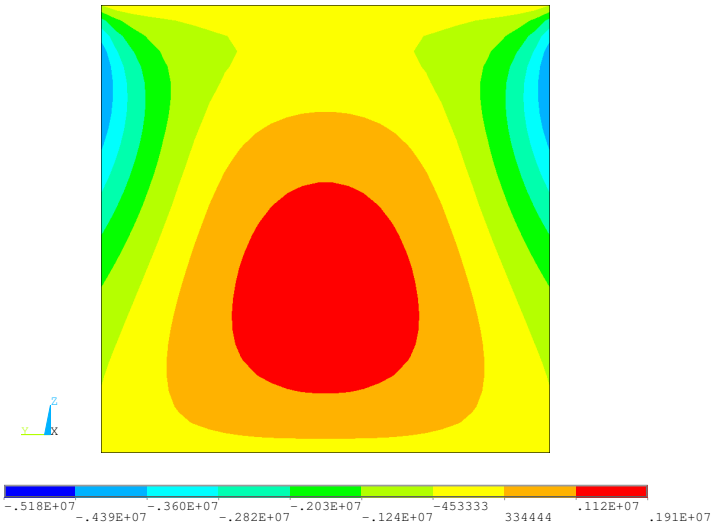
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Shear stress σ_{xz} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



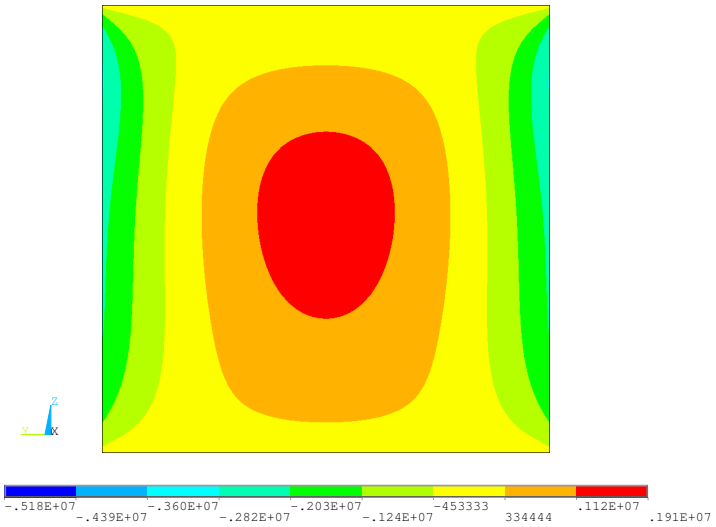
(a) FEM 3D^a, $n_z = 1$



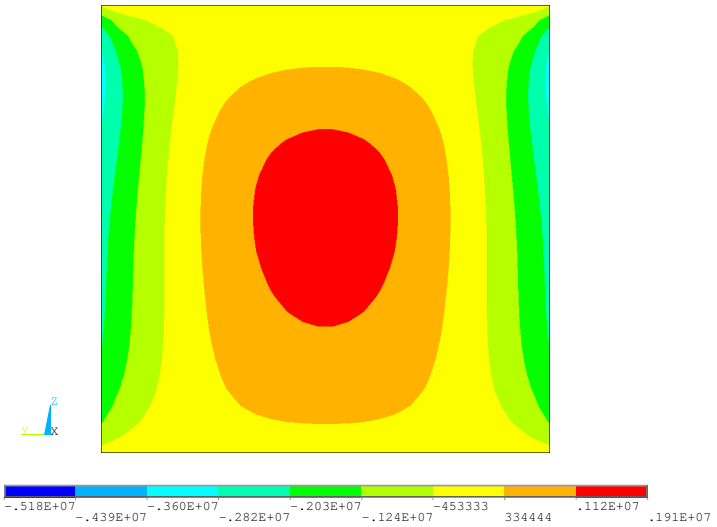
(b) N=13, $n_z = 1$

Stress cross-section variation

Shear stress σ_{xz} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



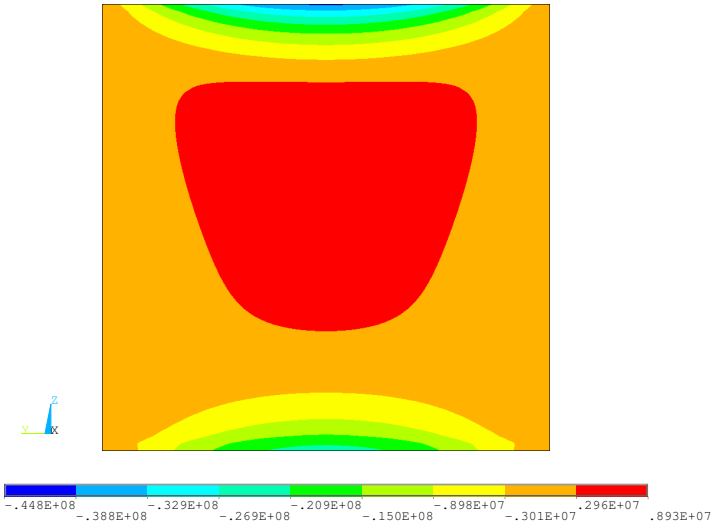
(a) FEM 3D^a, $n_z = 2$



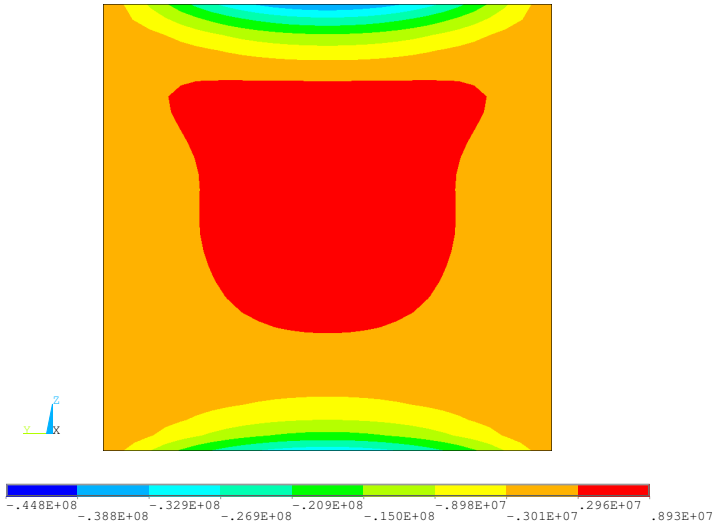
(b) N=13, $n_z = 2$

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



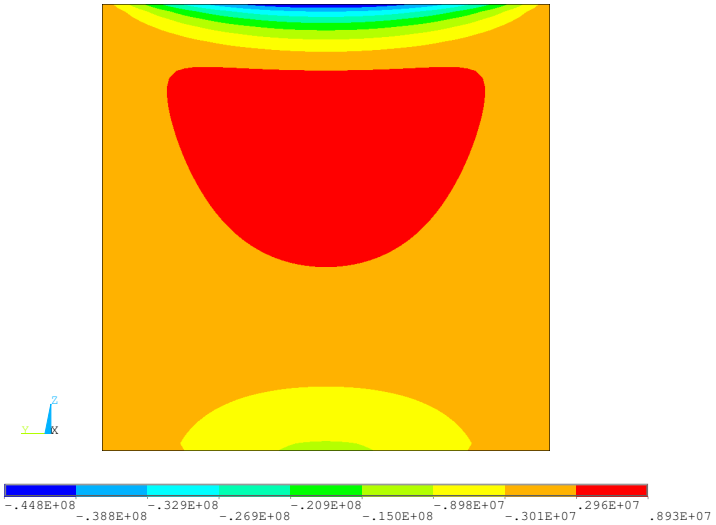
(a) FEM 3D^a, $n_z = 0.5$



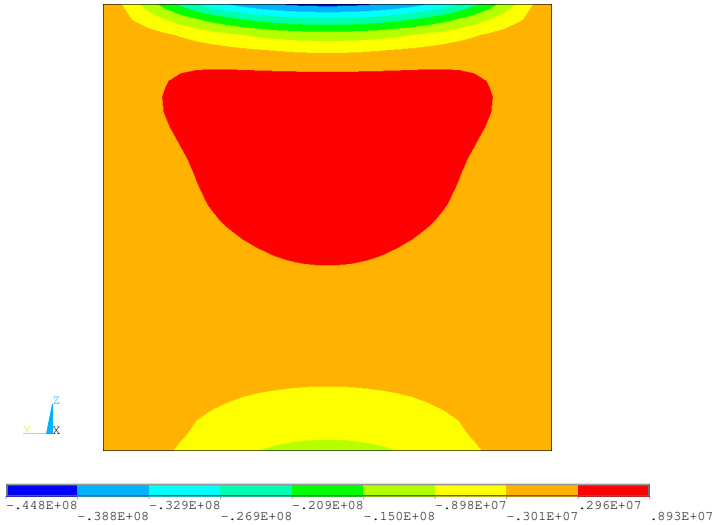
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



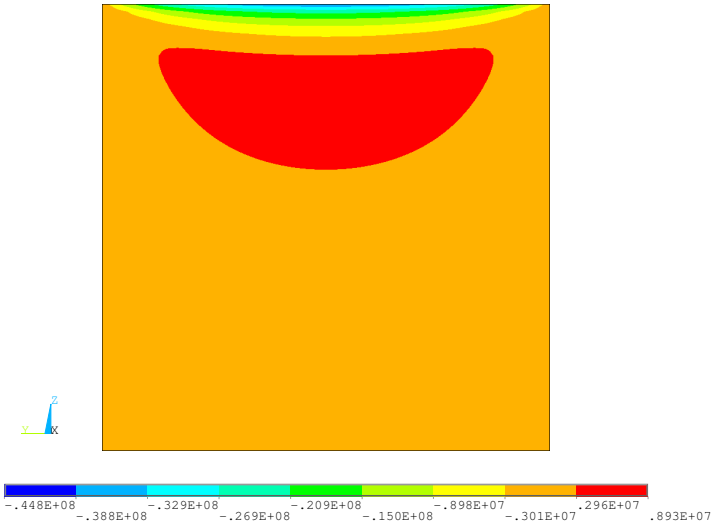
(a) FEM 3D^a, $n_z = 1$



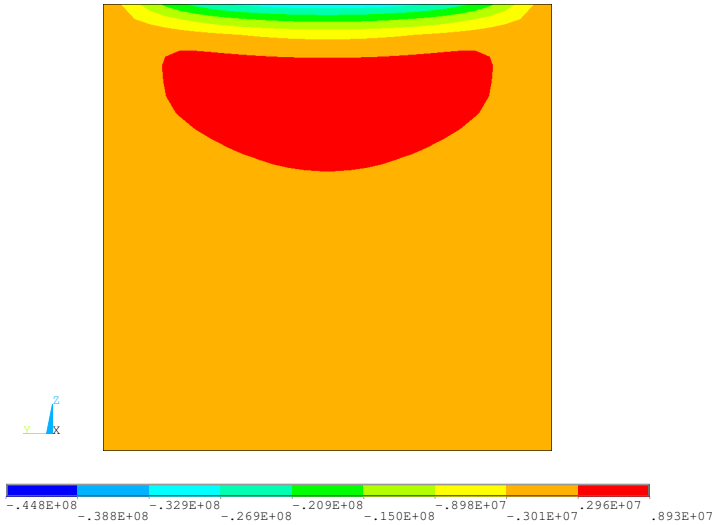
(b) N=13, $n_z = 1$

Stress cross-section variation

Through-the-width normal stress σ_{yy} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



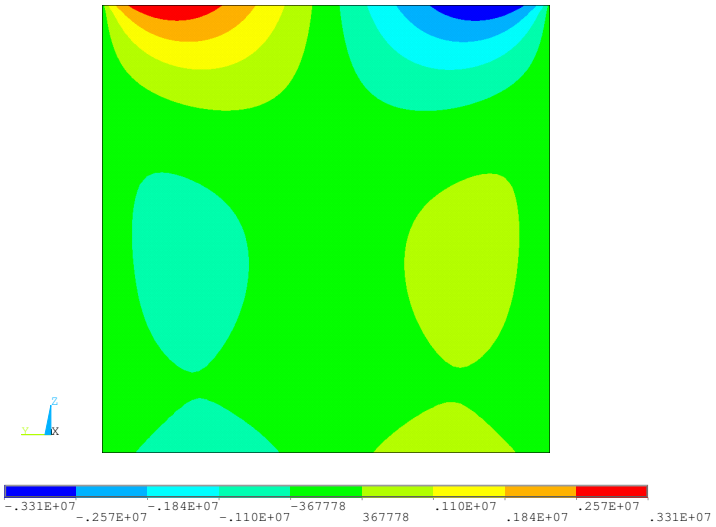
(a) FEM 3D^a, $n_z = 2$



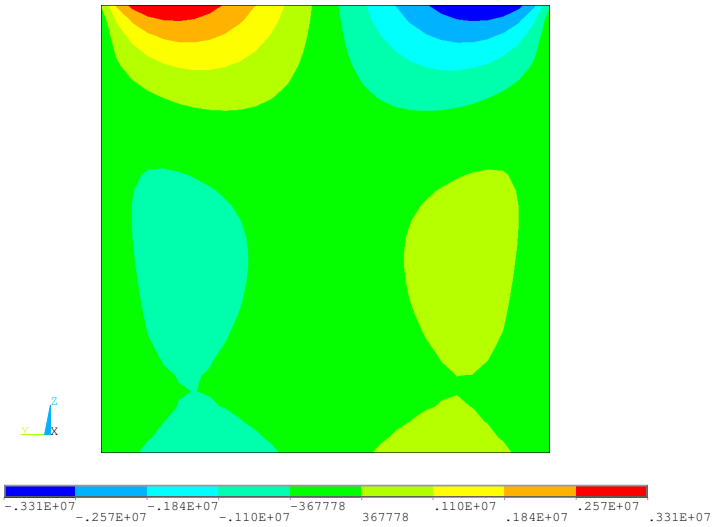
(b) N=13, $n_z = 2$

Stress cross-section variation

Shear stress σ_{xy} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



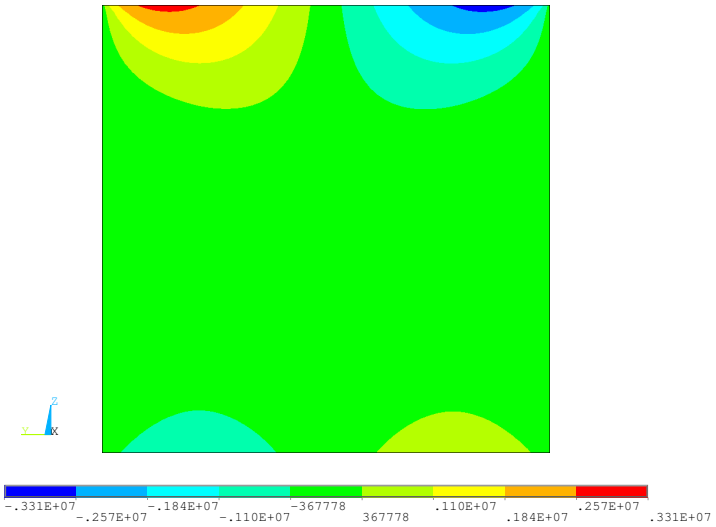
(a) FEM 3D^a, $n_z = 0.5$



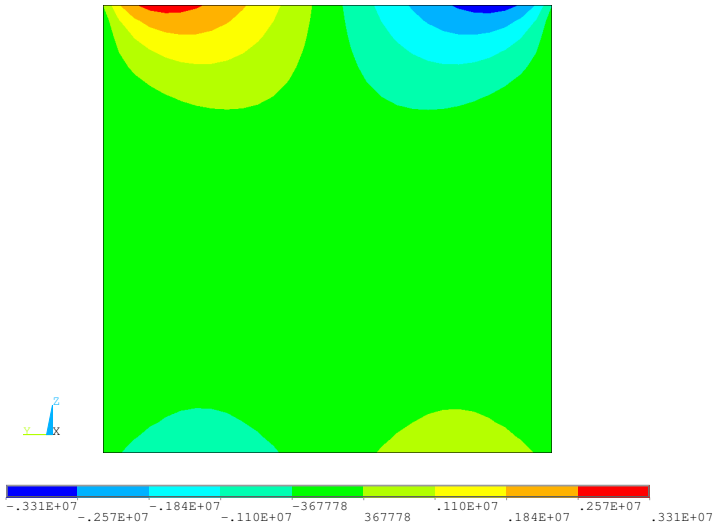
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Shear stress σ_{xy} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



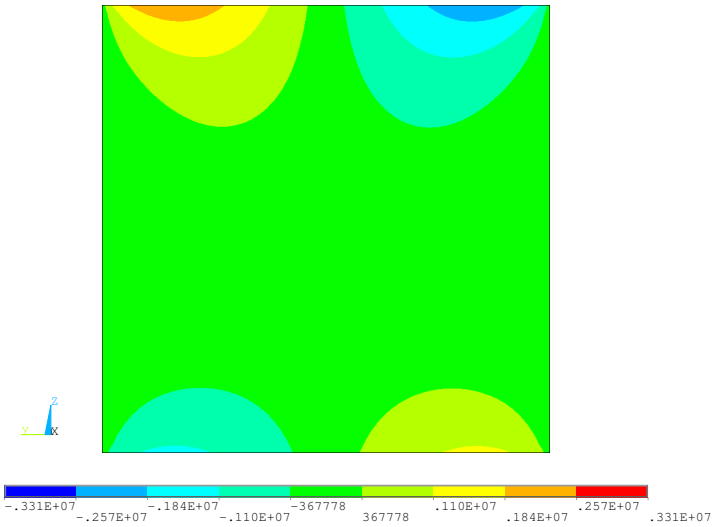
(a) FEM 3D^a, $n_z = 1$



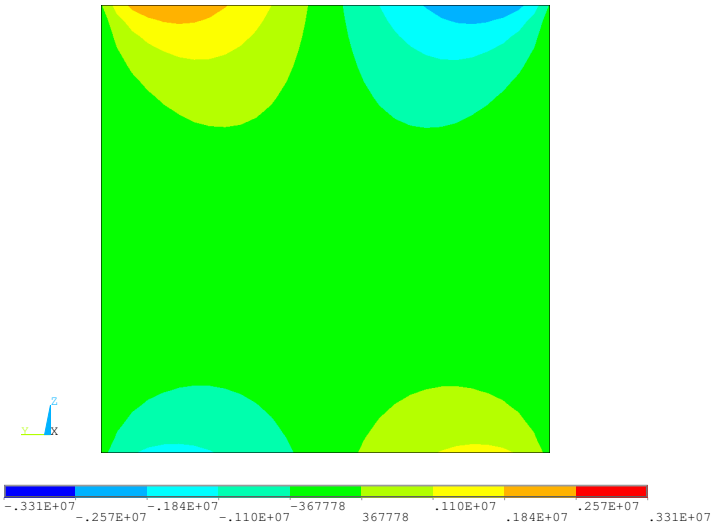
(b) N=13, $n_z = 1$

Stress cross-section variation

Shear stress σ_{xy} [Pa] at $x/l = 1/4$, cantilever FGM beam, $l/b = 5$.



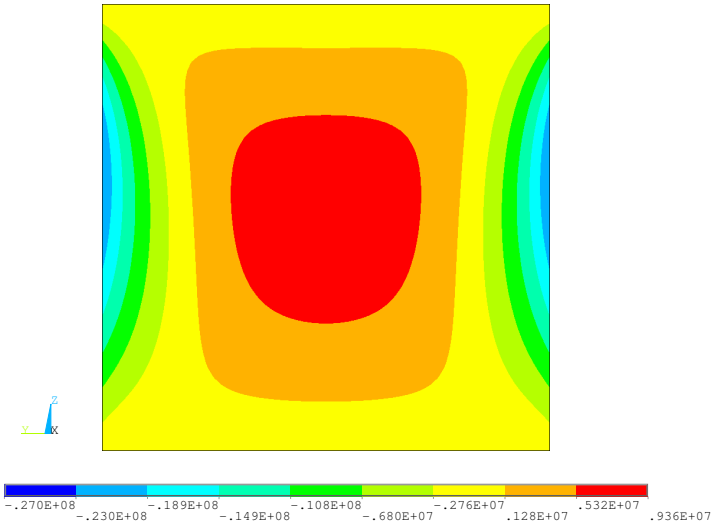
(a) FEM 3D^a, $n_z = 2$



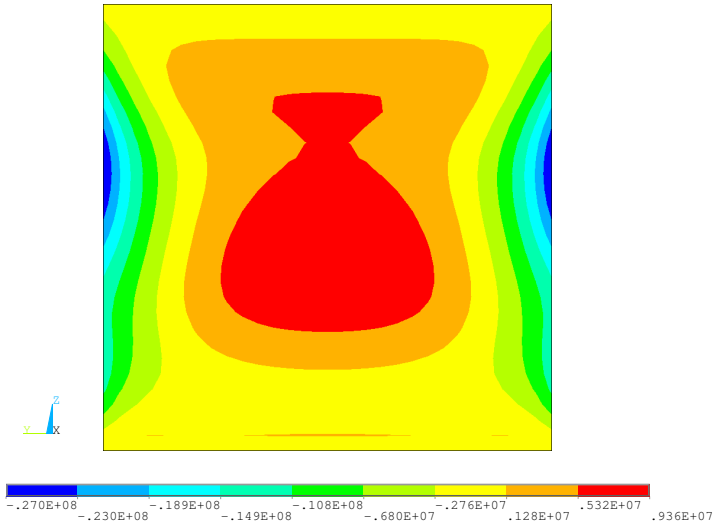
(b) N=13, $n_z = 2$

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



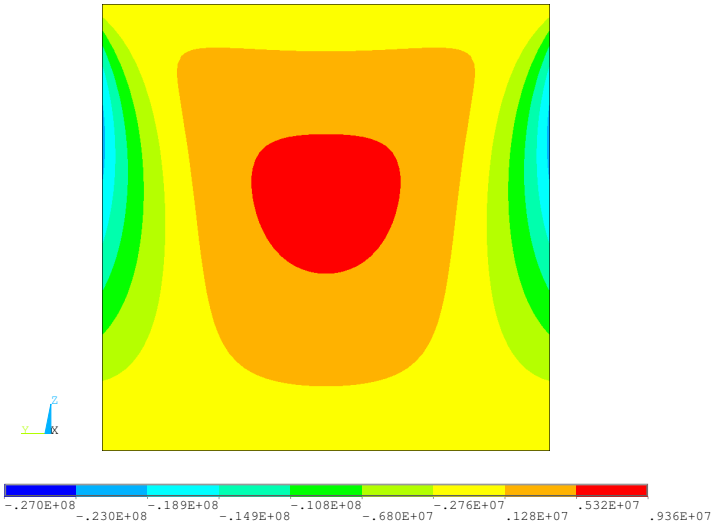
(a) FEM 3D^a, $n_z = 0.5$



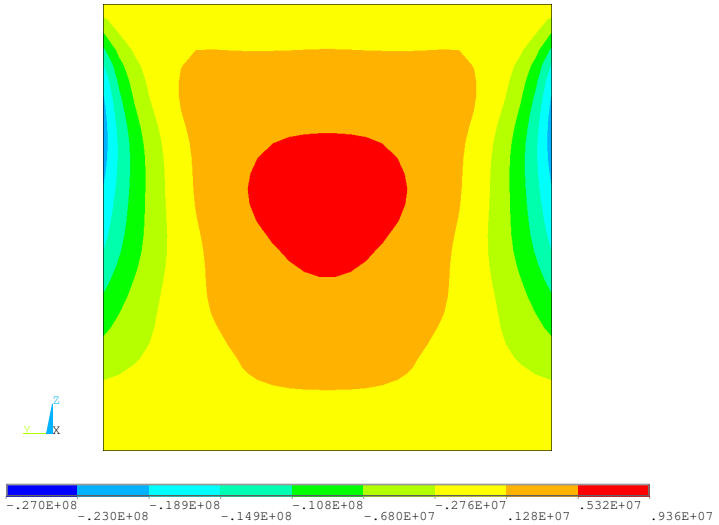
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



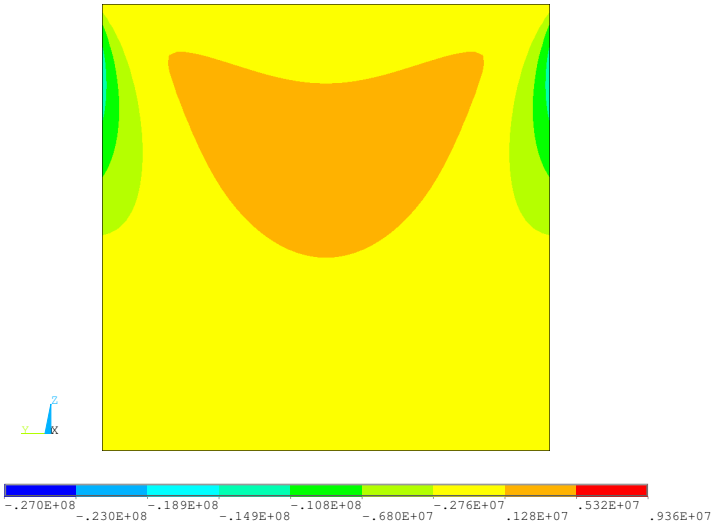
(a) FEM 3D^a, $n_z = 1$



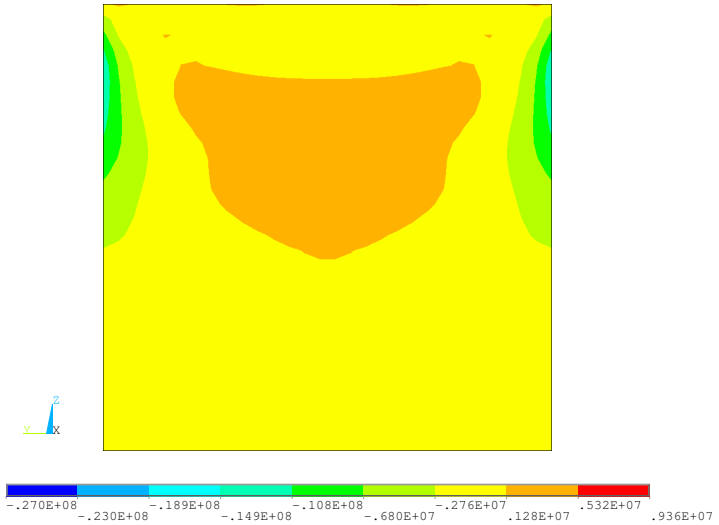
(b) N=13, $n_z = 1$

Stress cross-section variation

Through-the-thickness normal stress σ_{zz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



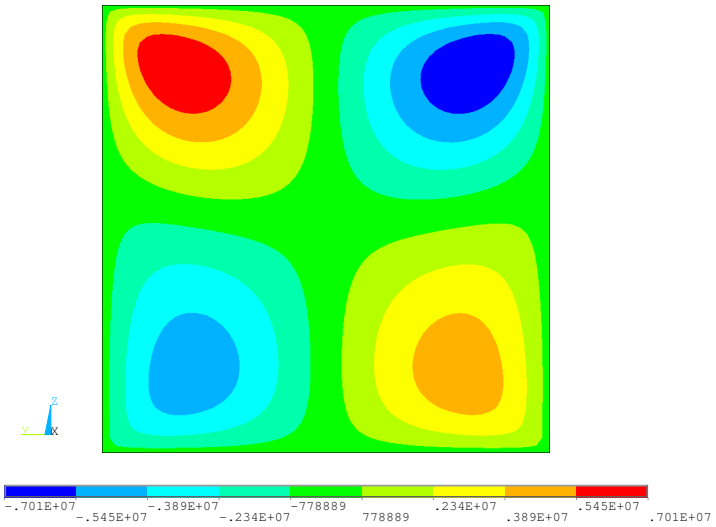
(a) FEM 3D^a, $n_z = 2$



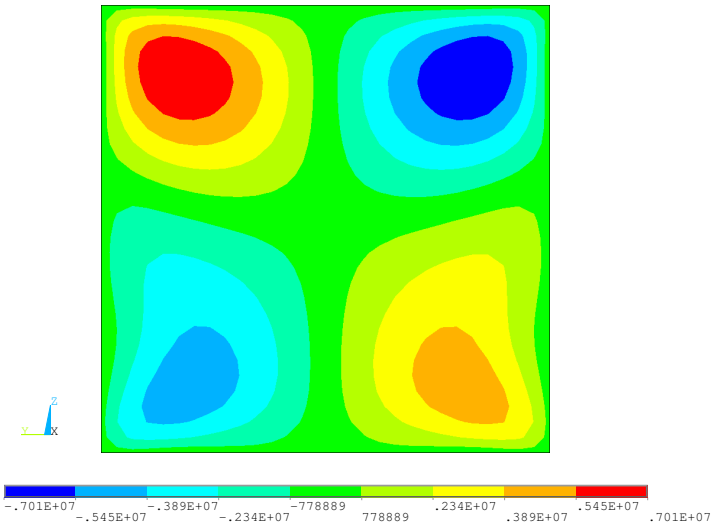
(b) N=13, $n_z = 2$

Stress cross-section variation

Shear stress σ_{yz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



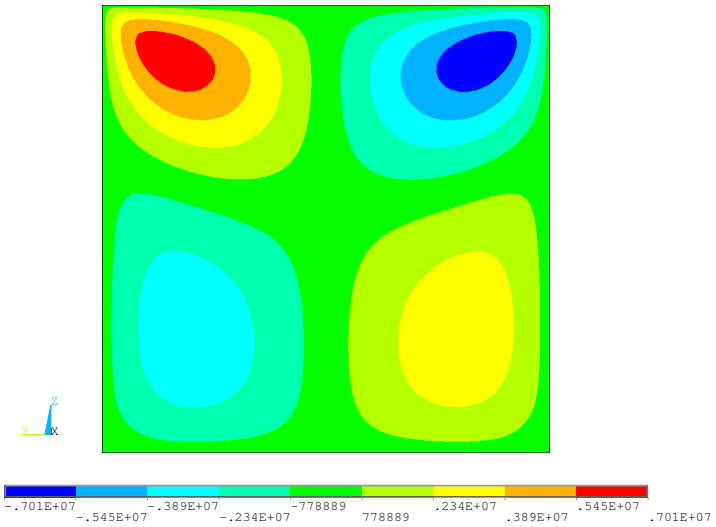
(a) FEM 3D^a, $n_z = 0.5$



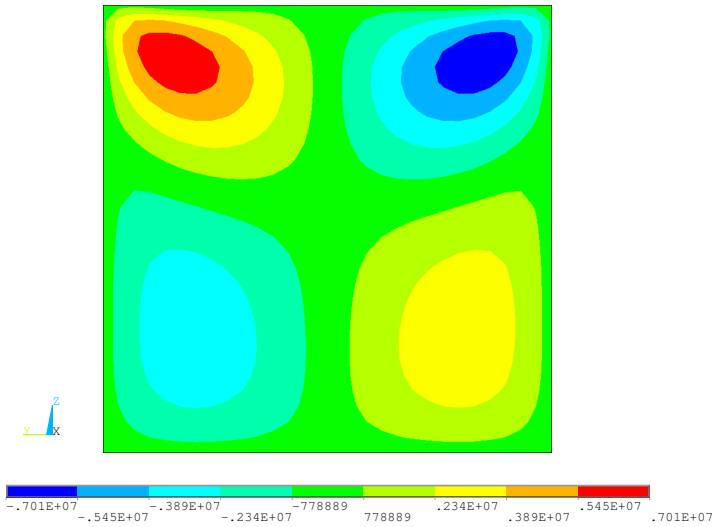
(b) N=13, $n_z = 0.5$

Stress cross-section variation

Shear stress σ_{yz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



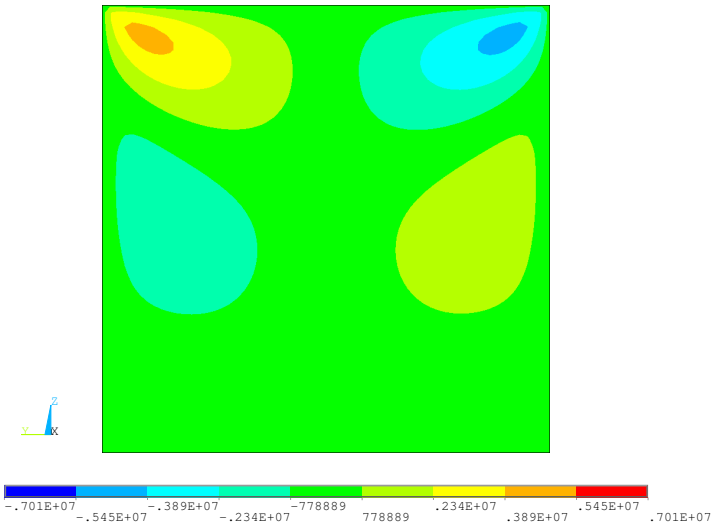
(a) FEM 3D^a, $n_z = 1$



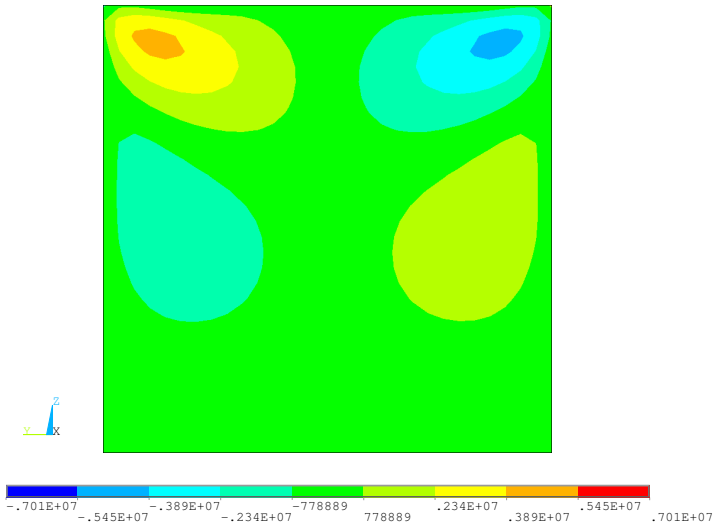
(b) N=13, $n_z = 1$

Stress cross-section variation

Shear stress σ_{yz} [Pa] at $x/l = 1/2$, cantilever FGM beam, $l/b = 5$.



(a) FEM 3D^a, $n_z = 2$



(b) N=13, $n_z = 2$

Results overview: displacements in slender beams.

Isotropic		
	Simply supported	Cantilever
u	0.3% ($N \geq 3$)	1.3% ($N \geq 3$)

Laminated		
	Simply supported	Cantilever
u	1.1% ($N \geq 9$)	1.0% ($N \geq 9$)

FGM		
	Simply supported	Cantilever
u	1.1% ($N \geq 7$)	1.1% on u_x, u_z ($N \geq 3$) 7.0% on u_y

Results overview: displacements and stresses in short beams.

Isotropic		
	Simply supported	Cantilever
u	0.4% ($N \geq 3$)	0.6% ($N \geq 3$)
σ	0.3% ($N = 14$, $B4$)	0.7% ($N = 14$)

Laminated		
	Simply supported	Cantilever
u	0.6% ($N \geq 9$)	0.4% ($N \geq 9$, $B4$)
σ	1.7% ($N = 14$)	1.9% ($N = 14$)

FGM		
	Simply supported	Cantilever
u	1.6% ($N \geq 4$)	0.4% ($N \geq 3$)
σ	3.3% ($N = 13$, $B4$)	3.3% ($N = 13$, $B4$)

Computational costs in terms of degrees of freedom (DOFs)

Hierarchical one-dimensional finite elements computational cost:

$$DOFs = 3\frac{(N+1)(N+2)}{2}N_n$$

with N the order of the beam theory and N_n the number of total nodes.

Model	DOFs
FEM-3D^a	$2.7 * 10^6$
FEM-3D^b	$1.1 * 10^5$
$N = 14$	$4.4 * 10^4$
$N = 13$	$3.8 * 10^4$
$N = 9$	$2.0 * 10^4$
$N = 7$	$1.3 * 10^4$
$N = 4$	$5.4 * 10^3$
$N = 3$	$3.6 * 10^3$