MITC9 Shell Finite Elements with Various Through-the-thickness Approximating Functions For the Analysis of Laminated Structures

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FULLCOMP ESR1 – Variable, mixed, linear and nonlinear kinematic shell formulations including thermal, hygrothermal, piezo and magnetic effects

- **Object:** (Laminated) Shell (formulations)
- **Methodology:** CUF

**2D FEM Formulation**

\[ \delta L_i = \delta u_{yj} k^{r_{ij}} u_{ri} \]

\[ k_{xx} = (\lambda + 2G) \int_{\Omega} N_{i,x} N_{j,x} d\Omega \int_{h} F_{r} F_{s} dz \]

\[ + G \int_{\Omega} N_{i,j} d\Omega \int_{h} F_{r,z} F_{s,c} dz + G \int_{\Omega} N_{i,y} N_{j,y} d\Omega \int_{h} F_{r} F_{s} dz; \]

\[ k_{xy} = \lambda \int_{\Omega} N_{i,y} N_{j,x} d\Omega \int_{h} F_{r,z} F_{s,c} dz + G \int_{\Omega} N_{i,x} N_{j,y} d\Omega \int_{h} F_{r} F_{s} dz \]

**Keywords:**

- **Variable kinematics**
  - Variable number of expansions: ESL-TYLN
  - Variable thickness function in different layers

- **Mixed kinematics**
  - Reissner’s Mixed Variational Theorem
  - Mixed ESL-LW kinematic description on different FEM nodes

- **Linear and nonlinear**
  - Geometry non-linearity
  - Material non-linearity

- **Multifield effects**
  - Thermal, Hygroscopic, Electric, Magnetic, · · ·
  - Thermo-electrical, Electro-magnetic, Thermo-magnetic, · · ·
Carrera Unified Formulation (CUF) and advanced 2D models.

Various thickness functions.

MITC9 element.

Numerical examples.

Conclusions.
Doubly-curved Shell Geometry*

Geometrical relations:

\[ \varepsilon_\alpha = \frac{1}{(1 + z/R_\alpha)} \left( \frac{1}{A} \frac{\partial u}{\partial \alpha} + \frac{v}{AB} \frac{\partial A}{\partial \beta} + \frac{w}{R_\alpha} \right) \]

\[ \varepsilon_\beta = \frac{1}{(u + z/R_\beta)} \left( u \frac{\partial B}{\partial \alpha} + \frac{1}{B} \frac{\partial v}{\partial \beta} + \frac{w}{R_\beta} \right) \]

\[ \varepsilon_z = \frac{\partial w}{\partial z} \]

\[ \gamma_{\alpha\beta} = \frac{A(1 + z/R_\alpha)}{B(1 + z/R_\beta)} \frac{\partial}{\partial \alpha} \left[ \frac{u}{A(1 + z/R_\alpha)} \right] + \frac{B(1 + z/R_\beta)}{A(1 + z/R_\alpha)} \frac{\partial}{\partial \alpha} \left[ \frac{v}{B(1 + z/R_\beta)} \right] \]

\[ \gamma_{\alpha z} = \frac{1}{A(1 + z/R_\alpha)} \frac{\partial w}{\partial \alpha} + A(1 + z/R_\alpha) \frac{\partial}{\partial z} \left[ \frac{u}{A(1 + z/R_\alpha)} \right] \]

\[ \gamma_{\beta z} = \frac{1}{B(1 + z/R_\beta)} \frac{\partial w}{\partial \beta} + B(1 + z/R_\beta) \frac{\partial}{\partial z} \left[ \frac{v}{B(1 + z/R_\beta)} \right] \]

\[ H_\alpha = A(1 + z/R_\alpha) \quad H_\beta = B(1 + z/R_\beta) \]

Note:

- Constant radii of curvature \( R_\alpha \) and \( R_\beta \) lead to Lamé parameters \( A = B = 1 \).
- Exact shell geometry (middle-surface) curvature is described.

An Example: A Higher-order Deformation Theory for Plate Written in CUF

- Displacement description

\[
\begin{align*}
\mathbf{u} &= u_0(x,y) + z \cdot u_1(x,y) + \cdots + z^N u_N(x,y) \\
\mathbf{v} &= v_0(x,y) + z \cdot v_1(x,y) + \cdots + z^N v_N(x,y) \\
\mathbf{w} &= w_0(x,y) + z \cdot w_1(x,y) + \cdots + z^N w_N(x,y)
\end{align*}
\]

\[
\mathbf{u}^T = \{u \quad v \quad w\}^T = \{F_s u_s \quad F_s v_s \quad F_s w_s\}^T = F_s \mathbf{u}_s^T
\]

\[
F_s(z) = z^{s-1} \quad s = 1, 2, \cdots, N + 1
\]

- FEM discretization

\[
\mathbf{u}(x,y,z) = N_i(x,y) \cdot \mathbf{u}_i(z) = N_i(x,y) \cdot F_s(z) \cdot \mathbf{U}_s
\]

- PVD

\[
\delta L_{int} = \int_V \delta \mathbf{e}^T \sigma dV = \int_V \delta \mathbf{u}^T \mathbf{b}^T \mathbf{C} \delta \mathbf{u} dV
\]

\[
= \int_V \delta \mathbf{U}^T N^T b^T \mathbf{C} \mathbf{N} dV = \int_V \delta \mathbf{U}^T N^T F^T b^T Cb \mathbf{F} \mathbf{N} dV
\]

\[
= \delta \mathbf{U}^T \cdot \int_V \delta N^T F^T b^T Cb \mathbf{F} \mathbf{N} dV \cdot \mathbf{U} = \delta \mathbf{U}^T \cdot \mathbf{K} \cdot \mathbf{U}
\]

\[
\delta L_{ext} = \delta \mathbf{u}^T \mathbf{P} = \delta \mathbf{U}^T N^T F^T \mathbf{P}
\]

\[
K_{ij} = \frac{1}{V} \int_V N_j F_s b^T Cb F_{s} F_{ij} N_i dV
\]

Fundamental Nucleus

3 X M X M

K_{j i}

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CARRERA UNIFIED FORMULATION - 2D MODELS - POLITECNICO DI TORINO (ITALY) - WWW.MUL2.COM
CUF for Two Major Frameworks of Refined 2D Models

Equivalent-Single-Layer models (ESL)

Layer-Wise models (LW)

\[ u = F_0 u_0 + F_1 u_1 + \cdots + F_N u_N \]

Note:
- \( F_\tau \) defined on the whole through-thickness domain \( z \).

\[ u^k = F_t u^k_t + F_b u^k_b + F_r u^k_r \]

Note:
- \( F_\tau \) defined on each layer thickness domain \( z^k \);
- Continuity constraints at layer interfaces: \( u^k_t = u^{k+1}_b \).
CUF 2D Models Based on Series Expansion – Variable ESL Kinematics

- **Taylor Series (ESL-TYL)**: \( F_\tau = z^n \).
  
  \[
  \begin{align*}
  u & = u_1 + z u_2 + z^2 u_3 + \ldots + z^N u_{N+1} \\
  v & = u_1 + z v_2 + z^2 v_3 + \ldots + z^N v_{N+1} \\
  w & = w_1 + z w_2 + z^2 w_3 + \ldots + z^N w_{N+1} \\
  n & = 0 \quad n = 1 \quad n = 2 \quad \ldots \quad n = N \\
  \tau & = 1 \quad \tau = 2 \quad \tau = 3 \quad \ldots \quad \tau = N + 1
  \end{align*}
  \]

- **Exponential Series (ESL-EPN)**: \( F_\tau = e^{nz} \).
  
  \[
  \begin{align*}
  u & = u_1 + e^z u_2 + e^{2z} u_3 + \ldots + e^{Nz} u_{N+1} \\
  v & = u_1 + e^z v_2 + e^{2z} v_3 + \ldots + e^{Nz} v_{N+1} \\
  w & = w_1 + e^z w_2 + e^{2z} w_3 + \ldots + e^{Nz} w_{N+1} \\
  n & = 0 \quad n = 1 \quad n = 2 \quad \ldots \quad n = N \\
  \tau & = 1 \quad \tau = 2 \quad \tau = 3 \quad \ldots \quad \tau = N + 1
  \end{align*}
  \]
CUF 2D Models Based on Series Expansion – Variable ESL Kinematics

Hyperbolic Series (ESL-HPB\(N\)): \(\tau = 2k, F_\tau = sinh(nz); \tau = 2k + 1, F_\tau = cosh(nz)\).

\[
\begin{align*}
\tau = 1 & : u = u_1 + \sinh(z) u_2 + \cosh(z) u_3 + \sinh(2z) u_4 + \ldots \\
\tau = 2 & : v = u_1 + \sinh(z) v_2 + \cosh(z) v_3 + \sinh(2z) v_4 + \ldots \\
\tau = 2 & : w = w_1 + \sinh(z) w_2 + \cosh(z) w_3 + \sinh(2z) w_4 + \ldots \\
\end{align*}
\]

\(n = 0\) \(\tau = 1\) \(n = 1\) \(\tau = 2\) \(n = 2\) \(\tau = 3\) \(n = 3\) \(\tau = 4\)

Trigonometric Series (ESL-TRG\(N\)): \(\tau = 2k, F_\tau = sin(nz); \tau = 2k + 1, F_\tau = cos(nz)\).

\[
\begin{align*}
\tau = 1 & : u = u_1 + \sin(z) u_2 + \cos(z) u_3 + \sin(2z) u_4 + \ldots \\
\tau = 2 & : v = u_1 + \sin(z) v_2 + \cos(z) v_3 + \sin(2z) v_4 + \ldots \\
\tau = 2 & : w = w_1 + \sin(z) w_2 + \cos(z) w_3 + \sin(2z) w_4 + \ldots \\
\end{align*}
\]

\(n = 0\) \(\tau = 1\) \(n = 1\) \(\tau = 2\) \(n = 2\) \(\tau = 3\) \(n = 3\) \(\tau = 4\)

Zig-Zag model (ESL-XXX\(NZ\)): \(F_{N+2} = (-1)^k \zeta_k u_Z\) (Murakami’s function)

\[
u = F_1 \, u_1 + \ldots + F_{N+1} \, u_{N+1} + (-1)^k \zeta_k u_Z.
\]
CUF 2D Models Based on Interpolation Polynomials – Variable LW Kinematics

- **Kinematics of Layer-Wise Models (LWN):**

  \[ u = F_1 u_1 + F_2 u_2 + ... + F_{N+1} u_{N+1} \]

  \[ v = F_1 v_1 + F_2 v_2 + ... + F_{N+1} v_{N+1} \]

  \[ w = \underbrace{F_1 w_1} + \underbrace{F_2 w_2} + ... + \underbrace{F_{N+1} w_{N+1}} \]

  \[ \tau = 1 \quad \tau = 2 \quad \tau = N + 1 \]

- **Lagrange (LW-LGR)**: \( F_\tau(\zeta_k) = \prod_{i=0, i\neq s}^{N} \frac{\zeta_k - \zeta_{k_i}}{\zeta_{k_s} - \zeta_{k_i}}, -1 \leq \zeta_k \leq -1 \)

  **Note:** Each \( \zeta_{k_s} \) or \( \zeta_{k_i} \) indicates a “Sampling Surface” with physical displacements.

- **Legendre (LW-LGD)**: \( F_i = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2} \)

  **Note:** \( P_0 = 1, \quad P_1 = \zeta_k, \quad NP_N = (2N - 1)\zeta_k P_{N-1} - (N - 1)P_{N-2}, -1 \leq \zeta_k \leq -1 \)

- **Chebyshev (First Kind) (LW-CBS)**: \( F_i = \frac{T_0 + T_1}{2}, \quad F_b = \frac{T_0 - T_1}{2}, \quad F_r = T_r - T_{r-2} \)

  **Note:** \( T_0 = 1, \quad T_1 = \zeta_k, \quad T_N = 2\zeta_k T_{N-1} - T_{N-2}, -1 \leq \zeta_k \leq -1 \)
9-node Shell Finite Element

\[ \begin{align*}
N_1 &= \frac{1}{4}(\xi^2 - \xi)(\eta^2 - \eta), \\
N_2 &= \frac{1}{4}(1 - \xi^2)(\eta^2 - \eta), \\
N_3 &= \frac{1}{4}(\xi^2 + \xi)(\eta^2 - \eta), \\
N_4 &= \frac{1}{4}(\xi^2 + \xi)(1 - \eta^2), \\
N_5 &= \frac{1}{4}(\xi^2 + \xi)(\eta^2 + \eta), \\
N_6 &= \frac{1}{4}(1 - \xi^2)(\eta^2 + \eta), \\
N_7 &= \frac{1}{4}(\xi^2 - \xi)(\eta^2 + \eta), \\
N_8 &= \frac{1}{4}(\xi^2 - \xi)(1 - \eta^2), \\
N_9 &= (1 - \xi^2)(1 - \eta^2)
\end{align*} \]
Mixed Interpolation of Tensorial Components

Figure: Tying points for the MITC9 shell element.

\[ N_{m1} = [N_{A1}, N_{B1}, N_{C1}, N_{D1}, N_{E1}, N_{F1}] \]
\[ N_{m2} = [N_{A2}, N_{B2}, N_{C2}, N_{D2}, N_{E2}, N_{F2}] \]
\[ N_{m3} = [N_{P}, N_{Q}, N_{R}, N_{S}] \]

\[ \epsilon_p = \begin{bmatrix} \epsilon_{\alpha\alpha} \\ \epsilon_{\beta\beta} \\ \epsilon_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} N_{m1} & 0 & 0 \\ 0 & N_{m2} & 0 \\ 0 & 0 & N_{m3} \end{bmatrix} \begin{bmatrix} \epsilon_{\alpha\alpha m1} \\ \epsilon_{\beta\beta m2} \\ \epsilon_{\alpha\beta m3} \end{bmatrix} \]

\[ \epsilon_n = \begin{bmatrix} \epsilon_{\alpha z} \\ \epsilon_{\beta z} \\ \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} N_{m1} & 0 & 0 \\ 0 & N_{m2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{\alpha z m1} \\ \epsilon_{\beta z m2} \\ \epsilon_{zz} \end{bmatrix} \]
3D Constitutive Equations

Stress-strain relations for an orthotropic material:

\[
\sigma^k_p = C^k_{pp} \varepsilon^k_p + C^k_{pn} \varepsilon^k_n
\]

\[
\sigma^k_n = C^k_{np} \varepsilon^k_p + C^k_{nn} \varepsilon^k_n
\]

where

\[
C^k_{pp} = \begin{bmatrix}
C^k_{11} & C^k_{12} & C^k_{16} \\
C^k_{12} & C^k_{22} & C^k_{26} \\
C^k_{16} & C^k_{26} & C^k_{66}
\end{bmatrix}, \quad
C^k_{pn} = \begin{bmatrix}
0 & 0 & C^k_{13} \\
0 & 0 & C^k_{23} \\
0 & 0 & C^k_{36}
\end{bmatrix}
\]

\[
C^k_{np} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
C^k_{13} & C^k_{23} & C^k_{36}
\end{bmatrix}, \quad
C^k_{nn} = \begin{bmatrix}
C^k_{55} & C^k_{45} & 0 \\
C^k_{45} & C^k_{44} & 0 \\
0 & 0 & C^k_{33}
\end{bmatrix}
\]
Case 1†: Plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 2$ and 100, loaded on top surface

\[ p(x, y) = p_0 \cdot \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) \]

Figure: 2D FEM 1/4 model with symmetry for the composite plate.

Plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 2$ and $100$, $\bar{\sigma}_{xz}$ and $\bar{\sigma}_{yz}$ variation through $\bar{z}$, with LWN*

*Note: LW-LGRN, -LGDN, -CBSN lead to the same results within the numerical tolerance, uniformly denoted as LWN.

(a) $\bar{\sigma}_{xz}$, $a/h = 2$, Point B

(b) $\bar{\sigma}_{yz}$, $a/h = 2$, Point D

(c) $\bar{\sigma}_{xz}$, $a/h = 100$, Point B

(d) $\bar{\sigma}_{yz}$, $a/h = 100$, Point D
Plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 2$, contour plot of $\sigma_{xz}$, with LW\textsubscript{N} (LW-LGR\textsubscript{N}, -LGD\textsubscript{N}, -CBS\textsubscript{N})
Plate, \([0^\circ/90^\circ/0^\circ]\), \(a/h = 2\), variation of \(\bar{\sigma}_{xz}\) through \(\bar{z}\) at Point B, with ESLNZ.

(a) \(\bar{\sigma}_{xz}\), Taylor series.

(b) \(\bar{\sigma}_{xz}\), Exponential series.

(c) \(\bar{\sigma}_{xz}\), Hyperbolic series.

(d) \(\bar{\sigma}_{xz}\), Trigonometric series.
Plate, [0°/90°/0°], $a/h = 100$, variation of $\bar{\sigma}_{xz}$ through $\bar{z}$ at Point B, with ESLNZ

(a) $\bar{\sigma}_{xz}$, Taylor series.

(b) $\bar{\sigma}_{xz}$, Exponential series.

(c) $\bar{\sigma}_{xz}$, Hyperbolic series.

(d) $\bar{\sigma}_{xz}$, Trigonometric series.
Plate, $[0^\circ/90^\circ/0^\circ]$, $a/h = 100$, contour plot of $\sigma_{xz}$, with ESL, NLZ and LW4
A Summary of Case 1

(a) $a/h = 2$

<table>
<thead>
<tr>
<th>$a/h$</th>
<th>Kinematics</th>
<th>$\bar{u}_z$</th>
<th>$\bar{\sigma}_{xx}$</th>
<th>$\bar{\sigma}_{yy}$</th>
<th>$10\bar{\sigma}_{xy}$</th>
<th>$\bar{\sigma}_{xz}$</th>
<th>$10\bar{\sigma}_{yz}$</th>
<th>$\bar{\sigma}_{zz}$</th>
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<tbody>
<tr>
<td>2</td>
<td>ESL-TYL7Z</td>
<td>8.161</td>
<td>2.150</td>
<td>1.881</td>
<td>5.558</td>
<td>2.466</td>
<td>6.662</td>
<td>1.015</td>
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<td>ESL-EPN7Z</td>
<td>8.158</td>
<td>2.147</td>
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<td>8.163</td>
<td>2.150</td>
<td>1.900</td>
<td>5.559</td>
<td>2.542</td>
<td>6.796</td>
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<td>LW5</td>
<td>8.166</td>
<td>2.150</td>
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<td>5.571</td>
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<td>6.732</td>
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(b) $a/h = 100$

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<th>Kinematics</th>
<th>$\bar{u}_z$</th>
<th>$\bar{\sigma}_{xx}$</th>
<th>$\bar{\sigma}_{yy}$</th>
<th>$10\bar{\sigma}_{xy}$</th>
<th>$\bar{\sigma}_{xz}$</th>
<th>$10\bar{\sigma}_{yz}$</th>
<th>$\bar{\sigma}_{zz}$</th>
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<td>0.2537</td>
<td>0.8458</td>
<td>4.494</td>
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<td>0.6294</td>
<td>0.2544</td>
<td>0.8458</td>
<td>4.447</td>
<td>0.9407</td>
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<td>0.6294</td>
<td>0.2539</td>
<td>0.8458</td>
<td>4.496</td>
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<td>Pagano(1970)</td>
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<td>1.08</td>
<td>1.000</td>
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</table>
Case 2‡: Cylindrical shell, $[0^\circ/90^\circ]$, $R_\beta/h = 2$ and 100, loaded on bottom surface

Figure: Geometry feature of the cylindrical shell.

Figure: 2D FEM 1/16 model

Figure: Loading profile on cross section

Shell, $[0^\circ/90^\circ]$, $R_\beta/h = 2$ and 100, $\bar{\sigma}_{\alpha z}$ and $\bar{\sigma}_{\beta z}$ variation through $\bar{z}$, with LWN*

*Note: LW-LGRN, -LGDN, -CBSN lead to the same results within the numerical tolerance, uniformly denoted as LWN.

(a) $\bar{\sigma}_{\alpha z}$, $R_\beta/h = 2$, Point B

(b) $\bar{\sigma}_{\beta z}$, $R_\beta/h = 2$, Point D

(c) $\bar{\sigma}_{\alpha z}$, $R_\beta/h = 100$, Point B

(d) $\bar{\sigma}_{\beta z}$, $R_\beta/h = 100$, Point D
Shell, $[0^\circ/90^\circ]$, $R_\beta/h = 2$, contour plot of $\sigma_{\beta z}$, with LWN (LWLGRN, -LGDN, -CBSN)
Shell, $[0^\circ/90^\circ]$, $R_\beta/h = 2$, $\bar{\sigma}_{\beta z}$ variation through $\bar{z}$ at Point D, with ESLNZ

(a) $\bar{\sigma}_{\beta z}$, Taylor series.

(b) $\bar{\sigma}_{\beta z}$, Exponential series.

(c) $\bar{\sigma}_{\beta z}$, Hyperbolic series.

(d) $\bar{\sigma}_{\beta z}$, Trigonometric series.
Shell, $[0^\circ/90^\circ]$, $R_\beta/h = 2$, contour plot of $\sigma_{\beta z}$, with ESLNZ and LW5
Shell, $[0^\circ/90^\circ]$, $R_\beta/h = 100$, $\tilde{\sigma}_{\beta z}$ variation through $\tilde{z}$ at Point D, with ESLNZ

(a) $\tilde{\sigma}_{\beta z}$, Taylor series.

(b) $\tilde{\sigma}_{\beta z}$, Exponential series.

(c) $\tilde{\sigma}_{\beta z}$, Hyperbolic series.

(d) $\tilde{\sigma}_{\beta z}$, Trigonometric series.
Shell, $[0^\circ/90^\circ], R_\beta/h = 100$, contour plot of $\sigma_{\beta z}$, with ESLNZ and LW4

ESL-TYL7Z

ESL-EPN7Z

LW4

ESL-HPB7Z

ESL-TRG13Z
A Summary of Case 2

(a) $a/h = 2$

(b) $a/h = 100$

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<th>$R_{\beta}/h$</th>
<th>$\bar{u}_z$</th>
<th>$\bar{\sigma}_{\alpha\alpha}$</th>
<th>$\bar{\sigma}_{\beta\beta}$</th>
<th>$10\bar{\sigma}_{\alpha\beta}$</th>
<th>$\bar{\sigma}_{\alpha z}$</th>
<th>$10\bar{\sigma}_{\beta z}$</th>
<th>$\bar{\sigma}_{zz}$</th>
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<td>$A, z = 0$</td>
<td>$A, z = h/2$</td>
<td>$A, z = h/2$</td>
<td>$B, z = -h/4$</td>
<td>$B, z = -h/4$</td>
<td>$A, z = h/2$</td>
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Main Conclusions and Perspectives

1. CUF provides a general way to integrate various approximation theories to obtain refined FEM element with variable kinematics for analysis of multi-layered structures.

2. MITC9 shell element adopted is free of shear and membrane locking.

3. With sufficient number of expansions, all the thickness-functions studied can achieve good approximation of displacements and stresses, and the number of essential expansions depends on the specific situation.

4. In the case of LW approach, interpolation polynomials of Lagrange, Legendre and Chebyshev provide the same numerical results if the polynomials adopted are of the same order.

5. For the cases studied, LW4 is sufficient to capture the transverse shear stress variation, while LW5 is recommended for thick laminated structures.

6. In different cases, ESL models with various thickness functions show different convergence rate with the increase of expansion number, especially for thin laminated structures.

7. Trigonometric series tend to use more expansions to achieve convergence in the cases studied.

Future extensions

- Variable LW-ESL kinematics.
- Application in multi-field problems.
MITC9 Shell Finite Elements with Various Through-the-thickness Approximating Functions For the Analysis of Laminated Structures

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6 September 2016, Porto